Sem11Synopsis

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Численное интегрирование

Пример 1

$$erfx = \frac{2}{\sqrt{\pi}} e^{-t^2} dt$$

Найти $\int$ ср. прямоуголь.

$$f(t) = \frac{2}{\sqrt{\pi}} e^{-t^2}$$

$$f'(t) = \frac{-4t}{\sqrt{\pi}} e^{-t^2}$$

$$f''(t) = \frac{2 \cdot (-2 + 4t^2)}{\sqrt{\pi}} e^{-t^2}$$

$$f'''(t) = \frac{2 \cdot (12t - 8t^3)}{\sqrt{\pi}} e^{-t^2}$$

$$f''(\sqrt{\frac{3}{2}}) < 2$$

$$f''(\theta) = \frac{4}{\sqrt{\pi}}$$

$$|R(erf\ 3)| < 1e - 4$$

$$|R(f)| \le \frac{M_2 \cdot (b-a) \cdot h^2}{24} = \frac{2.5 \cdot 3 \cdot h^2}{24} < 10^{-4}$$

$$M_2 = \max_{[} 0; 3] |f''| < 2.5$$

$$n = \frac{b-a}{h} = \frac{3}{1.6 \cdot 10^{-2}} = 167$$

Значит, получим $h \le 1.8 \cdot 10^{-2}$

Квадратурные формулы Гаусса

$$\int_a^b f(x)dx = \sum_{i=0}^n c_i f(x_i)$$

$$\int_a^b x^m dx = \sum_{i=0}^n c_i x^m$$

$$\begin{cases} c_1 + c_2 + ... + c_n = d_1 \\ c_1x_1 + c_2x_2 + ... + c_2x_n = d_2 \\ ... \\ c_1x_1^n + ...c_nx_n^n = d_n \end{cases}$$

$$P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$P_0 = 1$$

$$P_1 = x$$

$$P_2 = \frac{3}{2}x^2 - \frac{1}{2}$$

$$P_3 = \frac{5}{2}x^3 - \frac{3}{2}x$$

Для P_3:

$$\begin{cases} c_1 + c_2 + c_3 = \int_{-1}^1 dx \\ c_1\sqrt{\frac{3}{5}} + c_2 \cdot 0 + c_3\sqrt{\frac{3}{5}} = \int_{-1}^1 xdx \\ c_1\frac{3}{5} + c_2 \cdot 0 + c_3\frac{3}{5} = \int_{-1}^1 x^2dx \end{cases} \rightarrow \begin{cases} c_1 = c_3 = \frac{5}{9} \\ c_2 = \frac{8}{9} \\ x_2 = 0 \\ x_3 = -x_1 = \frac{3}{5} \end{cases}$$

Оказалось, что для подсчёта интеграла $\int_{-1}^1 f(x)dx$ мы можем взять формулу

$$\int_{-1}^1 f(x)dx \approx \frac{5}{9}f(-\sqrt{\frac{3}{5}}) + \frac{8}{9}f(0) + \frac{5}{9}f(\sqrt{\frac{3}{5}}) + \delta_3$$

$$\delta_n = M_{2n}\frac{2^{n+1}}{2n+1} \cdot \frac{(n!)^4}{((2n)!)^3}$$

$$M_{2n} = \max_{x\in[-1;1]} |f^{2n}|$$

Пример 1

Надо посчитать

$$\int_0^{-1} e^{-x^2}dx$$

Делаем замену

$$x = \frac{a+b}{2} + \frac{b-a}{2}t = \frac{1}{2} + \frac{1}{2}t$$

$$\int_0^{-1} e^{-x^2}dx = \frac{1}{2} \cdot (\int_{-1}^1 e^{-(\frac{1}{2}+\frac{1}{2}t)^2}dt) = \frac{1}{2} \cdot (e^{-(\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{\sqrt{3}})^2} + e^{-(\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2\sqrt{3}})^2}) \approx 0.7465$$

Пример 2

$$\int_{-1}^{1} \frac{x}{\sqrt{3}} arctg \frac{x}{\sqrt{3}} dx \approx f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}) \approx 0.209$$

Оптимизация

$$\int_{-1}^{1} \rho(x) f(x) dx$$

$$\rho(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\begin{cases} A_1 + A_2 = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx \\ A_1 x_1 + A_2 x_2 = \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} dx \\ A_1 x_1^2 + A_2 x_2^2 = \int_{-1}^{1} \frac{\rho(x) x^2}{\sqrt{1-x^2}} dx \\ A_1 x_1^2 + A_2 x_2^2 = \int_{-1}^{1} \frac{\rho(x) x^3}{\sqrt{1-x^2}} dx \end{cases} \to \begin{cases} A_1 = A_2 = \frac{\pi}{2} \\ x_1 = -\frac{1}{\sqrt{2}} \\ x_2 = \frac{1}{\sqrt{2}} \end{cases}$$

Пример 3

$$\int_{0}^{\infty} f(x) dx = \int_{0}^{\infty} e^{-x} [e^x f(x)] dx = \sum_{k=1}^{0} A_k e^{x_k} f(x_k)$$

A_k - квадратура Лагера.

$$I = \int_{0}^{\infty} e^{-x} f(x) dx$$

$$\begin{cases} A_1 = \frac{\sqrt{2}+1}{2\sqrt{2}} \\ A_2 = \frac{\sqrt{2}-1}{2\sqrt{2}} \\ x_1 = 2 - \sqrt{2} \\ x_2 = 2 + \sqrt{2} \end{cases}$$

$$I = \frac{\sqrt{2}+1}{2\sqrt{2}} f(2-\sqrt{2}) + \frac{\sqrt{2}-1}{2\sqrt{2}} f(2+\sqrt{2})$$

Пример

$$\int_{0}^{1} \frac{1+x^3}{1+x^5} dx = \frac{1}{2} \int_{-1}^{1} \frac{1+(\frac{1}{2}+\frac{1}{2}t)^3}{1+(\frac{1}{2}+\frac{1}{2}t)^5} dt = \frac{1}{2}(f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}) =$$

Квадратура Эрмита

$$\int_{-\infty}^{+\infty} f(x) dx = \int_{-\infty}^{+\infty} e^{-x^2} [e^{x^2} f(x)] dx = \sum_{k=1}^{n} A_k e^{x_k^2} f(x_k)$$