1 Notation

$$\text{Position in body frame is } \mathbf{x}_B = [x_B \; y_B \; z_B]^T \tag{1}$$
$$\text{Position in inertial frame is } \mathbf{x}_I = [x_I \; y_I \; z_I]^T \tag{2}$$
$$\text{Attitude is } \Theta = [\phi \; \theta \; \psi]^T = [\text{roll pitch yaw}]^T \tag{3}$$
$$\text{Rotation from frame 1 to frame 2 is } H_1^2 \tag{4}$$
$$\text{Rotation from inertial to body frame is } H_I^B \tag{5}$$
$$\text{Velocity in body frame is } \mathbf{v}_B = [u \; v \; w]^T \tag{6}$$
$$\text{Angular rate in body frame is } \omega_B = [p \; q \; r]^T \tag{7}$$
$$\text{Total velocity in body frame is } V = \sqrt{u^2 + v^2 + w^2} \tag{8}$$
$$\text{Angle of attack in body frame is } \alpha = \tan^{-1}(w/u) \tag{9}$$
$$\text{Flight path angle is } \gamma = \theta - \alpha \tag{10}$$
$$\text{Sideslip angle in body frame is } \beta = \sin^{-1}(v/V) \tag{11}$$
$$\text{Flight path heading is } \xi = \psi + \beta \tag{12}$$
$$\text{Bank angle is } \beta \tag{13}$$
$$\text{Magnitude of the thrust vector is } T \tag{14}$$

If the thrust of vehicle is aligned with the centerline (x_B axis), then

$$[T_x \; T_y \; T_z]^T = \begin{bmatrix} T\cos\alpha \\ 0 \\ -T\sin\alpha \end{bmatrix} \tag{15}$$
$$\text{Aerodynamic forces are } \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_B = H_I^B \begin{bmatrix} -D \\ SF \\ -L \end{bmatrix} \tag{16}$$

where L is lift, D is drag, and SF is side force.

$$\tag{17}$$

$$\text{Aerodynamic moments are } \begin{bmatrix} L \\ M \\ N \end{bmatrix}_B = H_I^B \begin{bmatrix} L \\ M \\ N \end{bmatrix}_I \tag{18}$$

Yes, the roll moment and lift force are both denoted by L...

$$\text{Mass of vehicle is } m \tag{19}$$
$$\text{Reference area (e.g., wing area) is } S \tag{20}$$
$$\text{Wing span is } b \tag{21}$$
$$\text{Mean aerodynamic chord } \bar{c} \tag{22}$$
$$\text{Elevator deflection is } \delta E \tag{23}$$
$$\text{Aileron deflection is } \delta A \tag{24}$$
$$\text{Rudder deflection is } \delta R \tag{25}$$
$$\text{Dynamic pressure is } \bar{q} = \frac{1}{2}\rho V^2 \tag{26}$$

2 Longitudinal Model:

Longitudinal equations of motion are

$$\dot{x}_I = u\cos\theta + w\sin\theta \tag{27}$$
$$\dot{z}_I = -u\sin\theta + w\cos\theta \tag{28}$$
$$\dot{\theta} = q \tag{29}$$
$$\dot{u} = F_X/m - qw \tag{30}$$
$$\dot{w} = F_Z/m + qu \tag{31}$$
$$\dot{q} = M_m/I_{yy} \tag{32}$$

To get forces and moments (assuming zero wind)

$$\text{Moment of inertia about the } y \text{ axis is } I_{yy} \tag{33}$$
$$\text{Aerodynamic forces in stability frame: } \begin{bmatrix} -D \\ -L \end{bmatrix} \tag{34}$$
$$\text{Aerodynamic forces in body frame: } \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \times \begin{bmatrix} -D \\ -L \end{bmatrix} \tag{35}$$
$$\text{Force in } x_B \text{ direction } F_X = L\sin\alpha - D\cos\alpha + T - mg\sin\theta \tag{36}$$
$$\text{Force in } z_B \text{ direction } F_Z = -L\cos\alpha - D\sin\alpha + mg\cos\theta \tag{37}$$

We're going to simplify by ignoring the forces and moments from the fuselage and fuselage-wing interference, and compute the forces and moments as:

$$\text{Lift:} \quad L = C_L\bar{q}S \tag{38}$$
$$C_L = C_{L_0} + C_{L_\alpha}\alpha + C_{L_{\delta E}}\delta E \tag{39}$$
$$\text{Drag:} \quad D = C_D\bar{q}S \tag{40}$$
$$C_D = C_{D_0} + \epsilon C_L^2 \tag{41}$$
$$= (C_{D_0} + \epsilon C_{L_0}^2) + C_{D_\alpha} + C_{L_\alpha^2}\alpha^2 \tag{42}$$
$$\text{with induced drag factor } \epsilon \tag{43}$$
$$C_{D_\alpha} = 2\epsilon C_{L_0}C_{L_\alpha} \tag{44}$$
$$C_{D_\alpha^2} = \epsilon C_{L_\alpha^2} \tag{45}$$
$$\text{Pitch:} \quad M = C_M\bar{q}S\bar{c} \tag{46}$$
$$C_M = C_{M_0} + C_{M_\alpha}\alpha + C_{M_{\delta E}}\delta E \tag{47}$$
$$\text{where the elevator deflection is } \delta E \tag{48}$$

Quantities needed to implement this model:

Initial conditions:	$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}_I$, $\begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{bmatrix}_I$, Θ_0, ω_0
Vehicle properties:	mass m, wing area S, mean aerodynamic chord $\bar{c}$, inertial matrix I
Coefficients:	C_{L_0}, C_{L_α}, C_{D_0}, ϵ, $C_{L_\alpha^2}$,
Air density:	$\rho(x)$
Gravity:	g
Controls:	thrust T, elevator deflection δE

3 Linearized Longitudinal Model:

$$
\begin{aligned}
\dot{x}_I &= u\cos\theta + w\sin\theta \\
\dot{z}_I &= -u\sin\theta + w\cos\theta \\
\dot{\theta} &= q \\
\dot{u} &= -g\sin\theta + \frac{\rho V^2 S}{2m}\left[C_{X_0} + C_{X_\alpha}\alpha + C_{X_q}\frac{\bar{c}q}{2V} + C_{X_{\delta_e}\delta_e}\right] + (T+\delta T)/m - qw \\
\dot{w} &= -g\cos\theta + \frac{\rho V^2 S}{2m}\left[C_{Z_0} + C_{Z_\alpha}\alpha + C_{Z_q}\frac{\bar{c}q}{2V} + C_{Z_{\delta_e}\delta_e}\right] + qu \\
\dot{q} &= \frac{\rho V^2 \bar{c} S}{2I_{yy}}\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{\bar{c}q}{2V} + C_{m_{\delta e}}\delta e\right]
\end{aligned}
$$

$$
\begin{aligned}
w &= V\sin\alpha \\
\bar{w} &= V^*\cos\alpha^*\bar{\alpha}
\end{aligned}
$$

$$
\begin{bmatrix} \dot{\bar{x_I}} \\ \dot{\bar{z_I}} \\ \dot{\bar{\theta}} \\ \dot{\bar{u}} \\ \dot{\bar{\alpha}} \\ \dot{\bar{q}} \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & -u^*\sin\theta^* + w^*\cos\theta^* & \cos\theta^* & V^*\sin\theta^*\cos\alpha^* & 0 \\
0 & 0 & -u^*\cos\theta^* - w^*\sin\theta^* & -\sin\theta^* & V^*\cos\theta^*\cos\alpha^* & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -g\cos\theta^* & X_u & X_w V^*\cos\alpha & X_q \\
0 & 0 & \frac{-g\sin\theta^*}{V^*\cos\alpha^*} & \frac{Z_u}{V^*\cos\alpha^*} & Z_w & \frac{Z_q}{V^*\cos\alpha^*} \\
0 & 0 & 0 & M_u & M_w V^*\cos\alpha^* & M_q
\end{bmatrix}
\cdot
\begin{bmatrix} \bar{x_I} \\ \bar{z_I} \\ \bar{\theta} \\ \bar{u} \\ \bar{\alpha} \\ \bar{q} \end{bmatrix}
+
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
X_{\delta e} & X_{\delta t} \\
\frac{Z_{\delta e}}{V\cos\alpha} & 0 \\
M_{\delta e} & 0
\end{bmatrix}
\cdot
\begin{bmatrix} \bar{\delta e} \\ \bar{\delta T} \end{bmatrix}
$$

Coefficient	Formula
X_u	$\frac{u^*\rho S}{m}\left[C_{X_0} + C_{X_\alpha}\alpha^* + C_{X_{\delta e}}\delta e^*\right] - \frac{\rho S w^* C_{X_\alpha}}{2m} + \frac{\rho S\bar{c}C_{X_q}u^*q^*}{4mV^*}$
X_w	$-q^* + \frac{w^*\rho S}{m}\left[C_{X_0} + C_{X_\alpha}\alpha^* + C_{X_{\delta e}}\delta e^*\right] + \frac{\rho S u^* C_{X_\alpha}}{2m} + \frac{\rho S\bar{c}C_{X_q}w^*q^*}{4mV^*}$
X_q	$-w^* + \frac{\rho V^* S C_{X_q}\bar{c}}{4m}$
$X_{\delta e}$	$\frac{\rho V^{*2} S C_{X_{\delta e}}}{2m}$
$X_{\delta T}$	$\frac{1}{m}$
Z_u	$q^* + \frac{u^*\rho S}{m}\left[C_{Z_0} + C_{Z_\alpha}\alpha^* + C_{Z_{\delta e}}\delta e^*\right] - \frac{\rho S w^* C_{Z_\alpha}}{2m} + \frac{\rho S\bar{c}C_{Z_q}u^*q^*}{4mV^*}$
Z_w	$\frac{w^*\rho S}{m}\left[C_{Z_0} + C_{Z_\alpha}\alpha^* + C_{Z_{\delta e}}\delta e^*\right] + \frac{\rho S u^* C_{Z_\alpha}}{2m} + \frac{\rho S\bar{c}C_{Z_q}w^*q^*}{4mV^*}$
Z_q	$u^* + \frac{\rho V^* S C_{Z_q}\bar{c}}{4m}$
$Z_{\delta e}$	$\frac{\rho V^{*2} S C_{Z_{\delta e}}}{2m}$
M_u	$\frac{u^*\rho S\bar{c}}{I_{yy}}\left[C_{m_0} + C_{m_\alpha}\alpha^* + C_{m_{\delta e}}\delta e^*\right] - \frac{\rho S w^* C_{m_\alpha}}{2I_{yy}} + \frac{\rho S\bar{c}^2 C_{m_q}u^*q^*}{4I_{yy}V^*}$
M_w	$\frac{w^*\rho S\bar{c}}{I_{yy}}\left[C_{m_0} + C_{m_\alpha}\alpha^* + C_{m_{\delta e}}\delta e^*\right] - \frac{\rho S u^* C_{m_\alpha}}{2I_{yy}} + \frac{\rho S\bar{c}^2 C_{m_q}w^*q^*}{4I_{yy}V^*}$
M_q	$\frac{\rho V^* S\bar{c}^2 C_{m_{\delta e}}}{4I_{yy}}$
$M_{\delta e}$	$\frac{\rho V^{*2} S\bar{c} C_{m_{\delta e}}}{4I_{yy}}$

4 Lateral-Directional motion:

Lateral-directional equations of motion are

$$\dot{x}_I = u(\cos\theta\cos\psi) + v(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi) + w(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) \tag{49}$$
$$\dot{y}_I = u(\cos\theta\sin\psi) + v(\sin\phi\sin\theta\cos\psi + \cos\phi\cos\psi) + w(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi) \tag{50}$$
$$\dot{z}_I = -u\sin\theta + v\sin\phi\cos\theta + w\cos\phi\cos\theta \tag{51}$$
$$\dot{\phi} = p + r\cos\phi\tan\theta \tag{52}$$
$$\dot{\theta} = -r\sin\phi \tag{53}$$
$$\dot{\psi} = r\cos\phi\sec\theta \tag{54}$$
$$\dot{u} = -g\sin\theta + \frac{\rho V^2 S}{2m}\left[C_{X_0} + C_{X_\alpha}\alpha\right] + T/m + rv \tag{55}$$
$$\dot{v} = g\cos\theta\sin\phi + \frac{\rho V^2 S}{2m}\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{bp}{2V} + C_{Y_r}\frac{br}{2V} + C_{Y_{\delta_a}\delta_a} + C_{Y_{\delta_r}\delta_r}\right] + pw - ru \tag{56}$$
$$\dot{w} = g\cos\theta\cos\phi + \frac{\rho V^2 S}{2m}\left[C_{Z_0} + C_{Z_\alpha}\alpha\right] - pv \tag{57}$$
$$\dot{p} = \left(I_{zz}L + I_{xz}N\right)/(I_{xx}I_{zz} - I_{xz}^2) \tag{58}$$

remembering that L and N are the rolling and yawing moments.

$$\dot{q} = \frac{\rho V^2 \bar{c} S}{2I_{yy}}\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{\bar{c}q}{2V} + C_{m_{\delta e}}\delta e\right] \tag{59}$$
$$\dot{r} = \left(I_{xz}L + I_{xx}N\right)/(I_{xx}I_{zz} - I_{xz}^2) \tag{60}$$

And we can write the rolling (not lift) and yawing moments as

$$L = \frac{1}{2}\rho V^2 Sb\left[C_{l_0} + C_{l_\beta}\beta + C_{l_p}\frac{bp}{2V} + C_{l_r}\frac{br}{2V} + C_{l_{\delta a}}\delta a + C_{l_{\delta r}}\delta r\right] \tag{61}$$
$$N = \frac{1}{2}\rho V^2 Sb\left[C_{r_0} + C_{r_\beta}\beta + C_{r_p}\frac{bp}{2V} + C_{r_r}\frac{br}{2V} + C_{r_{\delta a}}\delta a + C_{r_{\delta r}}\delta r\right] \tag{62}$$
$$\tag{63}$$

If we are in level flight, with a small sideslip angle β, then our forces and moments are:

$$Y = C_{Y_\beta}\bar{q}S\beta + C_{Y_{\delta R}}\delta R \tag{64}$$

It's weird that the sideforce should be given with respect to the reference area S and dynamic pressure $\bar{q}$. Unpacking the co-efficient, it gets renormalised for the tail, as

$$C_{Y_\beta} = \left(\frac{\bar{q}_{vt}}{\bar{q}}\right)\left(1 + \frac{\partial\sigma}{\partial\beta}\right)\eta_{vt}\left(\frac{S_{vt}}{S}\right)\left(C_{Y_{\beta_{vt}}}\right). \tag{65}$$

where η_{vt} is a tail efficiency parameter, σ is the sidewash angle which we can ignore, and the parameters $(\cdot)_{vt}$ are the relevant parameters of the vertical tail. Similarly, the rolling and yawing moments get translated to the tail, for example as

$$C_{N_{\beta_{vt}}} = -C_{Y_{\beta_{vt}}}\eta_{vt}\frac{S_{vt}l_{vt}}{Sb} \tag{66}$$

where l_{vt} is the vertical tail length, i.e., the distance from centre of mass to tail centre of pressure. But if we are banked, then the longitudinal forces will have lateral-directional effect too.

(67)

Additional quantities needed to implement this model:

Coefficients:	C_{Y_0}, C_{Y_β}, C_{Y_p},C_{Y_r},$C_{Y_{\delta a}}$,$C_{Y_{\delta r}}$, C_{l_0}, C_{l_β}, C_{l_p},C_{l_r},$C_{l_{\delta a}}$,$C_{l_{\delta r}}$, C_{r_0}, C_{r_β}, C_{r_p},C_{r_r},$C_{r_{\delta a}}$,$C_{r_{\delta r}}$
Wing parameters:	b, S
Tail parameters:	S_{vt}, $\bar{q}_{vt}$, η_{vt}, l_{vt}.

5 Linearized Lateral-Directional Model:

Because we have the trim state and the longitudinal model, for the lateral dynamics case we can basically forget about the x, z and pitch variables in the state. Given a trim state that contains all the relevant variables, we can build and analyze a linear model that only contains the lateral velocity v, the roll and yaw ϕ and ψ, and the corresponding rates p and r.

$$
\begin{aligned}
\dot{v} &= g\cos\theta\sin\phi + \frac{\rho V^2 S}{2m}\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{bp}{2V} + C_{Y_r}\frac{br}{2V} + C_{Y_{\delta_a}\delta_a} + C_{Y_{\delta_r}\delta_r}\right] + pw - ru \\
\dot{p} &= \left(I_{zz}L + I_{xz}N\right)/\left(I_{xx}I_{zz} - I_{xz}^2\right) \\
\dot{r} &= \left(I_{xz}L + I_{xx}N\right)/\left(I_{xx}I_{zz} - I_{xz}^2\right) \\
\dot{\phi} &= p + r\cos\phi\tan\theta \\
\dot{\psi} &= r\cos\phi\sec\theta
\end{aligned}
$$

But

$$
v = V\sin\beta
$$

Linearizing around $\beta = \beta^*$:

$$
\begin{aligned}
\bar{v} &= V^*\cos\beta^*\bar{\beta} \\
\dot{\bar{\beta}} &= \frac{1}{V^*\cos\beta^*}\dot{\bar{v}}
\end{aligned}
$$

$$
\begin{bmatrix}\dot{\bar{\beta}}\\ \dot{\bar{p}}\\ \dot{\bar{r}}\\ \dot{\bar{\phi}}\\ \dot{\bar{\psi}}\end{bmatrix}
=
\begin{bmatrix}
Y_v & \frac{Y_p}{V^*\cos\beta^*} & \frac{Y_r}{V^*\cos\beta^*} & \frac{g\cos\theta^*\cos\phi^*}{V^*\cos\beta^*} & 0\\
L_v V^*\cos\beta^* & L_p & L_r & 0 & 0\\
N_v V^*\cos\beta^* & N_p & N_r & 0 & 0\\
0 & 1 & \cos\phi^*\tan\theta^* & \begin{array}{c} q^*\cos\phi^*\tan\theta^* \\ -r^*\sin\phi^*\tan\theta^* \end{array} & 0\\
0 & 0 & \cos\phi^*\sec\theta^* & \begin{array}{c} p^*\cos\phi^*\sec\theta^* \\ -r^*\sin\phi^*\sec\theta^* \end{array} & 0
\end{bmatrix}
\cdot
\begin{bmatrix}\bar{\beta}\\ \bar{p}\\ \bar{r}\\ \bar{\phi}\\ \bar{\psi}\end{bmatrix}
+
\begin{bmatrix}
\frac{Y_{\delta a}}{V^*\cos\beta^*} & \frac{Y_{\delta r}}{V^*\cos\beta^*}\\
L_{\delta a} & L_{\delta r}\\
N_{\delta a} & N_{\delta r}\\
0 & 0\\
0 & 0
\end{bmatrix}
\cdot
\begin{bmatrix}\bar{\delta}a\\ \bar{\delta}r\end{bmatrix}
$$

Aircraft are often symmetric about the plane spanned by x_b and z_b, which means that

$$
I = \begin{bmatrix} I_{xx} & 0 & -I_{xz}\\ 0 & I_{yy} & 0\\ -I_{xz} & 0 & I_{zz}\end{bmatrix}
$$

We can further define the following terms

$$
\begin{aligned}
\Gamma &= I_{xx}I_{zz} - I_{xz}^2 \\
\Gamma_1 &= \frac{I_{xz}(I_{xx} - I_{yy} + I_{zz})}{\Gamma} \\
\Gamma_2 &= \frac{I_{zz}(I_{zz} - I_{yy}) + I_{xz}^2}{\Gamma} \\
\Gamma_3 &= \frac{I_{zz}}{\Gamma} \\
\Gamma_4 &= \frac{I_{xz}}{\Gamma} \\
\Gamma_5 &= \frac{I_{zz} - I_{xx}}{I_{yy}} \\
\Gamma_6 &= \frac{I_{xz}}{I_{yy}} \\
\Gamma_7 &= \frac{(I_{xx} - I_{yy})I_{xx} + I_{xz}^2}{\Gamma} \\
\Gamma_8 &= \frac{I_{xx}}{\Gamma}
\end{aligned}
$$

We can therefore rewrite the angular rate derivatives as follows:

$$
\begin{aligned}
\dot{p} &= \Gamma_1 pq - \Gamma_2 qr \\
\dot{q} &= \Gamma_5 pr - \Gamma_6(p^2 - r^2) \\
\dot{r} &= \Gamma_7 pq - \Gamma_1 qr
\end{aligned}
$$

Coefficient	Formula
Y_v	$\frac{v^*\rho Sb}{4mV^*}\left[C_{Y_p}p^* + C_{Y_r}r^*\right] + \frac{v^*\rho S}{m}\left[C_{Y_0} + C_{Y_\beta}\beta^* + C_{Y_{\delta a}}\delta a^* + C_{Y_{\delta r}}\delta r^*\right] + \frac{\rho S C_{Y_\beta}}{2m}\sqrt{u^{*2} + w^{*2}}$
Y_p	$w^* + \frac{\rho V^* Sb}{4m}C_{Y_p}$
Y_r	$-u^* + \frac{\rho V^* Sb}{4m}C_{Y_r}$
$Y_{\delta a}$	$\frac{\rho V^{*2} S}{2m}C_{Y_{\delta a}}$
$Y_{\delta r}$	$\frac{\rho V^{*2} S}{2m}C_{Y_{\delta r}}$
L_v	$\frac{v^*\rho Sb^2}{4V^*}\left[C_{p_p}p^* + C_{p_r}r^*\right] + v^*\rho Sb\left[C_{p_0} + C_{p_\beta}\beta^* + C_{p_{\delta a}}\delta a^* + C_{p_{\delta r}}\delta r^*\right] + \frac{\rho SbC_{p_\beta}}{2}\sqrt{u^{*2} + w^{*2}}$
L_p	$\Gamma_1 q^* + \frac{\rho V^* Sb^2}{4}C_{p_p}$
L_r	$-\Gamma_2 q^* + \frac{\rho V^* Sb^2}{4}C_{p_r}$
$L_{\delta a}$	$\frac{\rho V^{*2} Sb}{2}C_{p_{\delta a}}$
$L_{\delta r}$	$\frac{\rho V^{*2} Sb}{2}C_{p_{\delta r}}$
N_v	$\frac{v^*\rho Sb^2}{4V^*}\left[C_{r_p}p^* + C_{r_r}r^*\right] + v^*\rho Sb\left[C_{r_0} + C_{r_\beta}\beta^* + C_{r_{\delta a}}\delta a^* + C_{r_{\delta r}}\delta r^*\right] + \frac{\rho SbC_{r_\beta}}{2}\sqrt{u^{*2} + w^{*2}}$
N_p	$\Gamma_7 q^* + \frac{\rho V^* Sb^2}{4}C_{r_p}$
N_r	$-\Gamma_1 q^* + \frac{\rho V^* Sb^2}{4}C_{r_r}$
$N_{\delta a}$	$\frac{\rho V^{*2} Sb}{2}C_{r_{\delta a}}$
$N_{\delta r}$	$\frac{\rho V^{*2} Sb}{2}C_{r_{\delta r}}$

6 Unified Model without Forces:

$$
\begin{aligned}
\dot{x_I} &= (\cos\theta\cos\psi)u + (\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)v + (\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)w \\
\dot{y_I} &= (\cos\theta\sin\psi)u + (\sin\phi\sin\theta\sin\psi - \cos\phi\cos\psi)v + (\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)w \\
\dot{z_I} &= u\sin\theta - v\sin\phi\cos\theta - w\cos\phi\cos\theta \\
\dot{u} &= rv - qw \\
\dot{v} &= pw - ru \\
\dot{w} &= qu - pv \\
\dot{\phi} &= p + (q\sin\phi + r\cos\phi)\tan\theta \\
\dot{\theta} &= q\cos\phi - r\sin\phi \\
\dot{\psi} &= (q\sin\phi + r\cos\phi)\sec\theta \\
\dot{p} &= \left(-[I_{xz}(I_{yy} - I_{xx} - I_{zz})p + [I_{xz}^2 + I_{zz}(I_{zz} - I_{yy})]r]q\right)/(I_{xx}I_{zz} - I_{xz}^2) \\
\dot{q} &= \frac{-(I_{xx} - I_{zz})pr - I_{xz}(p^2 - r^2)}{I_{22}} \\
\dot{r} &= \left(-[I_{xz}(I_{yy} - I_{xx} - I_{zz})r + [I_{xz}^2 + I_{xx}(I_{xx} - I_{yy})]p]q\right)/(I_{xx}I_{zz} - I_{xz}^2)
\end{aligned}
$$

7 Unified Model:

$$
\begin{aligned}
\dot{x_I} &= (\cos\theta\cos\psi)u + (\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi)v + (\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)w \\
\dot{y_I} &= (\cos\theta\sin\psi)u + (\sin\phi\sin\theta\sin\psi - \cos\phi\cos\psi)v + (\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)w \\
\dot{z_I} &= u\sin\theta - v\sin\phi\cos\theta - w\cos\phi\cos\theta \\
\dot{u} &= rv - qw - g\sin\theta + \frac{\bar{q}}{m}\left[C_X(\alpha) + C_{X_q}(\alpha)\frac{\bar{c}q}{2V_a} + C_{X_{\delta e}}(\alpha)\delta_e\right] + T \\
\dot{v} &= pw - ru + g\cos\theta\sin\phi + \frac{\bar{q}}{m}\left[C_{Y_0} + C_{Y_\beta}(\beta) + C_{Y_p}\frac{bp}{2V_a} + C_{Y_r}\frac{br}{2V_a} + C_{Y_{\delta a}}\delta_a + C_{Y_{\delta r}}\delta_r\right] \\
\dot{w} &= qu - pv - g\cos\theta\cos\phi + \frac{\bar{q}}{m}\left[C_Z(\alpha) + C_{Z_q}(\alpha)\frac{\bar{c}q}{2V_a} + C_{Z_{\delta e}}(\alpha)\delta_e\right] \\
\dot{\phi} &= p + (q\sin\phi + r\cos\phi)\tan\theta \\
\dot{\theta} &= q\cos\phi - r\sin\phi \\
\dot{\psi} &= (q\sin\phi + r\cos\phi)\sec\theta \\
\dot{p} &= \Gamma_1 pq - \Gamma_2 qr + \bar{q}b\left[C_{p_0} + C_{p_\beta}\beta + C_{p_p}\frac{bp}{2V_a} + C_{p_r}\frac{br}{2V_a} + C_{p_{\delta a}}\delta_a + C_{p_{\delta r}}\delta_r\right] \\
\dot{q} &= \Gamma_5 pr - \Gamma_6(p^2 - r^2) + \bar{q}\frac{\bar{c}}{I_{yy}}\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{\bar{c}q}{2V_a} + C_{m_{\delta e}}\delta_e\right] \\
\dot{r} &= \Gamma_7 pq - \Gamma_1 qr + \bar{q}b\left[C_{r_0} + C_{r_\beta}\beta + C_{r_p}\frac{bp}{2V_a} + C_{r_r}\frac{br}{2V_a} + C_{r_{\delta a}}\delta_a + C_{r_{\delta r}}\delta_r\right]
\end{aligned}
$$

where

$$\begin{aligned}
C_{p_0} &= \Gamma_3 C_{l_0} + \Gamma_4 C_{n_0} \\
C_{p_\beta} &= \Gamma_3 C_{l_\beta} + \Gamma_4 C_{n_\beta} \\
C_{p_p} &= \Gamma_3 C_{l_p} + \Gamma_4 C_{n_p} \\
C_{p_r} &= \Gamma_3 C_{l_r} + \Gamma_4 C_{n_r} \\
C_{p_{\delta_a}} &= \Gamma_3 C_{l_{\delta_a}} + \Gamma_4 C_{n_{\delta_a}} \\
C_{p_{\delta_r}} &= \Gamma_3 C_{l_{\delta_r}} + \Gamma_4 C_{n_{\delta_r}} \\
C_{r_0} &= \Gamma_4 C_{l_0} + \Gamma_8 C_{n_0} \\
C_{r_\beta} &= \Gamma_4 C_{l_\beta} + \Gamma_8 C_{n_\beta} \\
C_{r_p} &= \Gamma_4 C_{l_p} + \Gamma_8 C_{n_p} \\
C_{r_r} &= \Gamma_4 C_{l_r} + \Gamma_8 C_{n_r} \\
C_{r_{\delta_a}} &= \Gamma_4 C_{l_{\delta_a}} + \Gamma_8 C_{n_{\delta_a}} \\
C_{r_{\delta_r}} &= \Gamma_4 C_{l_{\delta_r}} + \Gamma_8 C_{n_{\delta_r}}
\end{aligned}$$

and we push the dependence on the angle of attack into the lift and drag co-efficients, so that

$$\begin{aligned}
C_X(\alpha) &= -C_D(\alpha)\cos\alpha + C_L(\alpha)\sin\alpha \\
C_{X_q}(\alpha) &= -C_{D_q}(\alpha)\cos\alpha + C_{L_q}(\alpha)\sin\alpha \\
C_{X_{\delta_e}}(\alpha) &= -C_{D_{\delta e}}(\alpha)\cos\alpha + C_{L_{\delta_e}}(\alpha)\sin\alpha \\
C_Z(\alpha) &= -C_D(\alpha)\sin\alpha - C_L(\alpha)\cos\alpha \\
C_{Z_q}(\alpha) &= -C_{D_q}(\alpha)\sin\alpha - C_{L_q}(\alpha)\cos\alpha \\
C_{Z_{\delta_e}}(\alpha) &= -C_{D_{\delta e}}(\alpha)\sin\alpha - C_{L_{\delta_e}}(\alpha)\cos\alpha
\end{aligned}$$

8 Cascaded Control Loops

- Roll attitude hold

$$\begin{aligned}
E_{\phi_t} &= \phi_t^c - \phi_t \\
D_t &= \frac{E_{\phi_t} - E_{\phi_{t-1}}}{\Delta t} \\
\delta a_t &= k_{p_\phi} E_t - k_{d_\phi} D_t
\end{aligned}$$

- Course hold

$$\begin{aligned}
E_{\chi_t} &= \chi_t^c - \chi_t \\
I_{\chi_t} &= I_{\chi_{t-1}} + E_{\chi_t}\Delta t \\
\phi^c &= k_{p_\chi} E_{\chi_t} + k_{i_\chi} I_{\chi_t}
\end{aligned}$$

- Sideslip Hold

$$\begin{aligned}
E_{\beta_t} &= \beta_t^c - \beta_t \\
I_{\beta_t} &= I_{\beta_{t-1}} + E_{\beta_t}\Delta t \\
\delta r &= -k_{p_\beta} E_{\beta_t} - k_{i_\beta} I_{\beta_t}
\end{aligned}$$

- Pitch attitude hold

$$
\begin{aligned}
E_{\theta_t} &= \theta_t^c - \theta_t \\
D_t &= \frac{E_{\theta_t} - E_{\theta_{t-1}}}{\Delta t} \\
\delta e &= k_{p_\theta} E_{\theta_t} + k_{d_\theta} D_{\theta_t}
\end{aligned}
$$

- Altitude hold

$$
\begin{aligned}
E_{z_t} &= z_t^c - z_t \\
I_{z_t} &= I_{z_{t-1}} + E_{z_t} \Delta t \\
\theta^c &= k_{p_z} E_{z_t} + k_{i_z} I_{z_t}
\end{aligned}
$$

- Airspeed hold using commanded pitch

$$
\begin{aligned}
E_{V_t} &= V_t^c - V_t \\
I_{V_t} &= I_{V_{t-1}} + E_{V_t} \Delta t \\
\theta^c &= k_{p_{V_2}} E_{V_t} + k_{i_{V_2}} I_{V_t}
\end{aligned}
$$

- Airspeed hold using commanded throttle

$$
\begin{aligned}
E_{V_t} &= V_t^c - V_t \\
I_{V_t} &= I_{V_{t-1}} + E_{V_t} \Delta t \\
\delta t &= \delta^* t + k_{p_V} E_{V_t} + k_{i_V} I_{V_t}
\end{aligned}
$$