

Research Article

Analytical Bending Solutions of Orthotropic Rectangular Thin Plates with Two Adjacent Edges Free and the Others Clamped or Simply Supported Using Finite Integral Transform Method

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For the first time, the finite integral transform method is introduced to explore the accurate bending analysis of orthotropic rectangular thin plates with two adjacent edges free and the others clamped or simply supported. Previous solutions mostly focused on plates with simply supported and clamped edges, but the existence of free corner makes the solution procedure much complex to solve by conventional inverse/semi-inverse methods. Compared with the conventional methods, the employed method eliminates the need to preselect the deflection function, which makes it more reasonable and theoretical for calculating the mechanical responses of the plates. Moreover, the approach used can also analyze static problems of moderately thick plates and thick plates with the same boundary conditions investigated in this article. Finally, comprehensive analytical results obtained in this paper illuminate the validity of the proposed approach by comparing with the previous literature and finite element method by using (ABAQUS) software.

1. Introduction

Due to the better structural performance such as superior strength-to-weight ratios and stiffness-to-weight ratios, composite materials are widely applied in engineering fields such as mechanical engineering, civil and structural engineering, naval, and aerospace. Orthotropic rectangular plate is considered as the fundamental structural application of composite materials. Thus, the research on the mechanical behavior of orthotropic plate aroused the interest of scientists and engineers for more than a century.

Literature surveys reveal that numerical methods such as finite difference method [1], spline element method [2], boundary element method [3], meshless method [4], finite element method (FEM) [5], boundary particle method [6], isogeometric collocation method [7], discrete singular convolution method [8–11], and differential quadrature method [12, 13] are competent to analyze the bending of plates with different edge conditions, loading patterns, and material properties. However, the aforementioned

numerical methods satisfy the engineering requirements with acceptable error, but approximate solution is obtained, which is the main disadvantage of the numerical methods.

Compared with numerical methods, analytical solution is relatively sparse, which is due to the mathematical complexity in choosing proper trial function to satisfy the governing equation and the boundary conditions simultaneously. It is known that the classic traditional semi-inverse methods [14–17] such as Navier solution and Levy solution can only deal with the static problem of plate with two opposite simply supported edges. Superposition method [18] can be applied for static problems of plates with various boundary conditions. However, the method involves complex solving procedure.

Among the combinations of boundary conditions of rectangular plate, it is very difficult to obtain analytical solution for plate with at least two adjacent edges free. Most previous research studies dealt with the plate with simply or clamped boundary conditions; however, the existence of two adjacent free edges creates a free corner, which eventually

makes the solution procedure much complex. The classical superposition method cannot deal with the title problem because of requiring the plate deflections to vanish at the free corner. The existing solutions are very few, and the solution procedure is much more difficult which needs a thorough knowledge of mathematics and mechanics. Recently, a Green's function approach is utilized to solve the free vibration problems of circular thin plates [19–21]. This approach allows obtaining the analytical frequency equations as power series fast convergent to exact eigenvalues for different number of nodal diameters. The symplectic superposition method is developed which is the combination of superposition method as stated above and symplectic elasticity approach [22–25], and applied systematically to the bending [26, 27], buckling [28, 29], and free vibration [30, 31] problems of plate. This method has attracted wide attention, including plate; it is also applicable to solve shell problems [32]. However, the method involves complex mathematical manipulations, which require skilled personnel in the fields of mathematics and mechanics. Therefore, researchers are still exploring new analytical methods to analyze the title problem with a more effective way and to develop accurate analytical solutions for validating other numerical/approximate solutions.

Recently, the finite integral transform method [33], an effective mathematical method, is developed which is successfully implemented to solve the bending [34–37] and free vibration [38–40] plate problems with different boundary conditions. Unfortunately, there is no report available which presents the solution of the title problem using finite integral transform. For the reason, this study adopts a simpler and more general, finite integral transformation to investigate the title problem. In the solution process, after finite integral transformation, using some inherent properties of the integral kernel, the bending governing equation is transformed into a fully regular infinite system of simultaneous linear algebraic equations with the unknowns determined by satisfying associated boundary conditions. Then, through some mathematical manipulation, the analytical solution is elegantly achieved in a straightforward procedure. “Compared with the traditional semi-inverse methods (e.g., the Navier method, Levy method, and superposition method), the present one is simpler and more rational. A semi-inverse method normally fails to yield a unified solution procedure since it requires case-by-case trial functions to satisfy both the governing equation and boundary conditions. By using finite integral transformation, the high-order partial differential equation is transformed into a system of linear algebraic equations, and the solution of these equations is achieved in a straightforward way. The present results are believed to present a benchmark for validation of other numerical and analytical methods and can be useful for engineers and scientists for academic and practical applications. The succinct but effective technique presented in this study may provide an easy-to-implement theoretical tool to seek more analytic solutions of buckling and free vibration problems of thin plates.”

The three complex boundary value problems, i.e., CCFF, CSFF, and SSFF, are studied, where F denotes free edge, S

denotes simply supported edge, and C denotes clamped edge, and the boundary conditions are taken in clockwise direction.

2. Application of Finite Integral Transformation for Bending Analysis of Orthotropic Rectangular Thin Plates

Figure 1 shows the orthotropic thin plate with dimensions of $a \times b \times h$. The bending governing equation of classical Kirchhoff plate theory is expressed as follows [37]:

$$D_x \frac{\partial^4 W(x, y)}{\partial x^4} + 2H \frac{\partial^4 W(x, y)}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W(x, y)}{\partial y^4} = q(x, y), \quad (1)$$

where $W(x, y)$ and $q(x, y)$ are the deflection of plate midplane and the distributed transverse load, respectively; D_x and D_y are the flexural rigidities in the x and y directions, respectively; $H = D_1 + 2D_{xy}$ is defined as the effective torsional rigidity, where D_{xy} is the torsional rigidity, $D_1 = \mu_y D_x = \mu_x D_y$ is defined with Poisson's ratios μ_x and μ_y ; the internal forces of orthotropic plates can be expressed in terms of $W(x, y)$ as follows:

$$\begin{aligned} M_x &= -D_x \left(\frac{\partial^2 W}{\partial x^2} + \mu_y \frac{\partial^2 W}{\partial y^2} \right), \\ M_y &= -D_y \left(\frac{\partial^2 W}{\partial y^2} + \mu_x \frac{\partial^2 W}{\partial x^2} \right), \\ M_{xy} &= -2D_{xy} \frac{\partial^2 W}{\partial x \partial y}, \end{aligned} \quad (2)$$

$$\begin{aligned} Q_x &= -\frac{\partial}{\partial x} \left(D_x \frac{\partial^2 W}{\partial x^2} + H \frac{\partial^2 w}{\partial y^2} \right), \\ Q_y &= -\frac{\partial}{\partial y} \left(D_y \frac{\partial^2 W}{\partial y^2} + H \frac{\partial^2 w}{\partial x^2} \right), \\ V_x &= -\frac{\partial}{\partial x} \left[D_x \frac{\partial^2 W}{\partial x^2} + (H + 2D_{xy}) \frac{\partial^2 W}{\partial y^2} \right], \\ V_y &= -\frac{\partial}{\partial y} \left[D_y \frac{\partial^2 W}{\partial y^2} + (H + 2D_{xy}) \frac{\partial^2 W}{\partial x^2} \right]. \end{aligned} \quad (3)$$

The CCFF rectangular plate is clamped at two adjacent edges $x = 0$ and $y = 0$, and the other edges are free; the boundary conditions of the plate are as follows:

$$W|_{x=0} = W|_{y=0} = 0, \quad 2D_{xy} \frac{\partial W}{\partial x} \Big|_{y=b} = 0, \quad (4a)$$

$$\begin{aligned} \frac{\partial W}{\partial x} \Big|_{x=0} &= 0, \quad M_x|_{x=a} = V_x|_{x=a} = 0, \\ \frac{\partial W}{\partial y} \Big|_{y=0} &= 0, \quad M_y|_{y=b} = V_y|_{y=b} = 0. \end{aligned} \quad (4b)$$

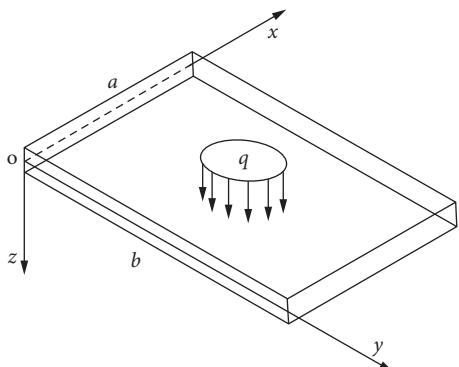


FIGURE 1: Schematic illustration of thin plate.

If $W(x, y)$ is a function of the two independent variables x and y , defined in a rectangle $0 \leq x \leq a, 0 \leq y \leq b$, the double finite sine integral transform is described as follows:

$$W_{mn} = \int_0^a \int_0^b W(x, y) \sin \frac{\alpha_m x}{2} \sin \frac{\beta_n y}{2} dx dy \quad (5)$$

$$(m = 1, 3, 5, \dots, n = 1, 3, 5, \dots).$$

The inversion is expressed as

$$W(x, y) = \frac{4}{ab} \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} W_{mn} \sin \frac{\alpha_m x}{2} \sin \frac{\beta_n y}{2}, \quad (6)$$

where $\alpha_m = m\pi/a$ and $\beta_n = n\pi/b$.

The high-order partial derivatives of $W(x, y)$ in equation (1) are derived as follows:

$$\begin{aligned} & \int_0^a \int_0^b \frac{\partial^4 W}{\partial x^4} \sin \frac{\alpha_m x}{2} \sin \frac{\beta_n y}{2} dx dy \\ &= \left(\frac{\alpha_m}{2} \right)^4 W_{mn} + \int_0^b \left[\begin{array}{l} (-1)^{(m-1/2)} \frac{\partial^3 W}{\partial x^3} \Big|_{x=a} + \frac{\alpha_m}{2} \frac{\partial^2 W}{\partial x^2} \Big|_{x=0} \\ -(-1)^{(m-1/2)} \left(\frac{\alpha_m}{2} \right)^2 \frac{\partial W}{\partial x} \Big|_{x=a} - \left(\frac{\alpha_m}{2} \right)^3 W \Big|_{x=0} \end{array} \right] \sin \frac{\beta_n y}{2} dy. \end{aligned} \quad (7)$$

$$\begin{aligned} & \int_0^a \int_0^b \frac{\partial^4 W}{\partial y^4} \sin \frac{\alpha_m x}{2} \sin \frac{\beta_n y}{2} dx dy \\ &= \left(\frac{\beta_n}{2} \right)^4 W_{mn} + \int_0^a \left[\begin{array}{l} (-1)^{(n-1/2)} \frac{\partial^3 W}{\partial y^3} \Big|_{y=b} + \frac{\beta_n}{2} \frac{\partial^2 W}{\partial y^2} \Big|_{y=0} \\ -(-1)^{(n-1/2)} \left(\frac{\beta_n}{2} \right)^2 \frac{\partial W}{\partial y} \Big|_{y=b} - \left(\frac{\beta_n}{2} \right)^3 W \Big|_{y=0} \end{array} \right] \sin \frac{\alpha_m x}{2} dx. \end{aligned} \quad (8)$$

The second term of equation (1) can be divided into two parts. In the first part, we consider the partial derivative of x first:

$$\begin{aligned} & \int_0^a \int_0^b \frac{\partial^4 W}{\partial x^2 \partial y^2} \sin \frac{\alpha_m x}{2} \sin \frac{\beta_n y}{2} dx dy \\ &= \left(\frac{\alpha_m}{2} \right)^2 \left(\frac{\beta_n}{2} \right)^2 W_{mn} + \int_0^b \left[(-1)^{(m-1/2)} \frac{\partial^3 W}{\partial x \partial y^2} \Big|_{x=a} \right. \\ & \quad \left. + \frac{\alpha_m}{2} \frac{\partial^2 W}{\partial y^2} \Big|_{x=0} \right] \sin \frac{\beta_n y}{2} dy \\ & \quad - \left(\frac{\alpha_m}{2} \right)^2 \int_0^a \left[(-1)^{(n-1/2)} \frac{\partial W}{\partial y} \Big|_{y=b} + \frac{\beta_n}{2} W \Big|_{y=0} \right] \sin \frac{\alpha_m x}{2} dx. \end{aligned} \quad (9)$$

In the second part, we first consider the partial derivative of y :

$$\begin{aligned} & \int_0^a \int_0^b \frac{\partial^4 W}{\partial x^2 \partial y^2} \sin \frac{\alpha_m x}{2} \sin \frac{\beta_n y}{2} dx dy \\ &= \left(\frac{\alpha_m}{2} \right)^2 \left(\frac{\beta_n}{2} \right)^2 W_{mn} + \int_0^a \left[(-1)^{(n-1/2)} \frac{\partial^3 W}{\partial x^2 \partial y} \Big|_{y=b} \right. \\ & \quad \left. + \frac{\beta_n}{2} \frac{\partial^2 W}{\partial x^2} \Big|_{y=0} \right] \sin \frac{\alpha_m x}{2} dx \\ & \quad - \left(\frac{\beta_n}{2} \right)^2 \int_0^b \left[(-1)^{(m-1/2)} \frac{\partial W}{\partial x} \Big|_{x=a} + \frac{\alpha_m}{2} W \Big|_{x=0} \right] \sin \frac{\beta_n y}{2} dy. \end{aligned} \quad (10)$$

By applying one-dimensional finite sine integral transform on the effective shearing forces V_x of edge $x = a$, we will obtain

$$\begin{aligned}
& D_x \int_0^b \frac{\partial^3 W}{\partial x^3} \Big|_{x=a} \sin \frac{\beta_n y}{2} dy \\
&= -(H + 2D_{xy}) \int_0^b \frac{\partial^3 W}{\partial x \partial y^2} \Big|_{x=a} \sin \frac{\beta_n y}{2} dy \\
&= (H + 2D_{xy}) \left(\frac{\beta_n}{2} \right)^2 \int_0^b \frac{\partial W}{\partial x} \Big|_{x=a} \sin \frac{\beta_n y}{2} dy.
\end{aligned} \tag{11}$$

Similarly, after employing one-dimensional finite sine integral transform on the effective shearing forces V_y of edge $y=b$, we will obtain

$$\begin{aligned}
& D_y \int_0^a \frac{\partial^3 W}{\partial y^3} \Big|_{y=b} \sin \frac{\alpha_m x}{2} dx \\
&= -(H + 2D_{xy}) \int_0^a \frac{\partial^3 W}{\partial x^2 \partial y} \Big|_{y=b} \sin \frac{\alpha_m x}{2} dx \\
&= (H + 2D_{xy}) \left(\frac{\alpha_m}{2} \right)^2 \int_0^a \frac{\partial W}{\partial y} \Big|_{y=b} \sin \frac{\alpha_m x}{2} dx.
\end{aligned} \tag{12}$$

q_{mn} is defined as the transform of the load function $q(x, y)$:

$$q_{mn} = \int_0^a \int_0^b q(x, y) \sin \frac{\alpha_m x}{2} \sin \frac{\beta_n y}{2} dx dy. \tag{13}$$

By substituting equations (7)–(13) and boundary condition of equation (4a) into equation (1), we can obtain the following equation:

$$\begin{aligned}
& D_x \frac{\alpha_m}{2} \int_0^b \frac{\partial^2 W}{\partial x^2} \Big|_{x=0} \sin \frac{\beta_n y}{2} dy + D_y \frac{\beta_n}{2} \int_0^a \frac{\partial^2 W}{\partial y^2} \Big|_{y=0} \sin \frac{\alpha_m x}{2} dx \\
& - (-1)^{\frac{m-1}{2}} D_x \left[\left(\frac{\alpha_m}{2} \right)^2 + \mu_y \left(\frac{\beta_n}{2} \right)^2 \right] \int_0^b \frac{\partial W}{\partial x} \Big|_{x=a} \sin \frac{\beta_n y}{2} dy \\
& - (-1)^{\frac{n-1}{2}} D_y \left[\mu_x \left(\frac{\alpha_m}{2} \right)^2 + \left(\frac{\beta_n}{2} \right)^2 \right] \int_0^a \frac{\partial W}{\partial y} \Big|_{y=b} \sin \frac{\alpha_m x}{2} dx \\
& + \left[D_x \frac{\alpha_m^4}{16} + H \frac{\alpha_m^2 \beta_n^2}{8} + D_y \frac{\beta_n^4}{16} \right] W_{mn} = q_{mn}.
\end{aligned} \tag{14}$$

Some parts of equation (14) are definite integral, which are constants. Let

$$\begin{cases} I_m = \int_0^a \frac{\partial W}{\partial y} \Big|_{y=b} \sin \frac{\alpha_m x}{2} dx, J_m = \int_0^a \frac{\partial^2 W}{\partial y^2} \Big|_{y=0} \sin \frac{\alpha_m x}{2} dx, \\ K_n = \int_0^b \frac{\partial W}{\partial x} \Big|_{x=a} \sin \frac{\beta_n y}{2} dy, L_n = \int_0^b \frac{\partial^2 W}{\partial x^2} \Big|_{x=0} \sin \frac{\beta_n y}{2} dy. \end{cases} \tag{15}$$

The unidentified constants J_m and L_n have evident physical meaning. When the plate is clamped, or simply supported at edges $x=0, y=0$, we can easily obtain the following equation:

$$\frac{\partial^2 W}{\partial y^2} \Big|_{x=0} = 0, \quad \frac{\partial^2 W}{\partial x^2} \Big|_{y=0} = 0. \tag{16}$$

Substitution of equation (16) into J_m and L_n leads to

$$\begin{aligned}
J_m &= \int_0^a \left(\frac{\partial^2 W}{\partial y^2} + \mu_x \frac{\partial^2 W}{\partial x^2} \right) \Big|_{y=0} \sin \frac{\alpha_m x}{2} dx, \\
L_n &= \int_0^b \left(\frac{\partial^2 W}{\partial x^2} + \mu_y \frac{\partial^2 W}{\partial y^2} \right) \Big|_{x=0} \sin \frac{\beta_n y}{2} dy.
\end{aligned} \tag{17}$$

Obviously, the integrands of $-D_y J_m$ and $-D_x L_n$ are Fourier coefficients of the bending moments of edges $y=0$ and $x=0$, respectively. Similarly, I_m and K_n are Fourier coefficients of the slopes of free edges $y=b$ and $x=a$, respectively. As to the simply supported edges, the corresponding unknowns will be zero. Accordingly, equation (14) is expressed by an unidentified constants I_m, J_m, K_n , and L_n as follows:

$$\begin{aligned}
W_{mn} &= C_{mn} \left[q_{mn} + (-1)^{(n-1)/2} D_y R_{mn} I_m - D_y \frac{\beta_n}{2} J_m \right. \\
&\quad \left. + (-1)^{(m-1)/2} D_x P_{mn} K_n - \frac{\alpha_m}{2} D_x L_n \right], \tag{18}
\end{aligned}$$

where $C_{mn} = (1/(D_x (\alpha_m^4/16) + H (\alpha_m^2 \beta_n^2/8) + D_y (\beta_n^4/16)))$, $R_{mn} = \mu_x (\alpha_m^2/4) + (\beta_n^2/4)$, and $P_{mn} = (\alpha_m^2/4) + \mu_y (\beta_n^2/4)$.

By substituting equation (18) into equation (6), the expression for $W(x, y)$ is obtained as follows for $m = 1, 3, 5, \dots \infty$ and $n = 1, 3, 5, \dots \infty$:

$$W(x, y) = \frac{4}{ab} \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} C_{mn} \left[\begin{aligned} & q_{mn} + (-1)^{(n-1)/2} D_y R_{mn} I_m - D_y \frac{\beta_n}{2} J_m \\ & + (-1)^{(m-1)/2} D_x P_{mn} K_n - \frac{\alpha_m}{2} D_x L_n \end{aligned} \right] \sin \frac{\alpha_m x}{2} \sin \frac{\beta_n y}{2}. \tag{19}$$

Case 1. For the CCFF rectangular plate, equation (19) has satisfied the boundary conditions $W|_{x=0} = W|_{y=0} = 0$, $V_x|_{x=a} = V_y|_{y=b} = 0$, and $\partial W^2/\partial x \partial y|_{x=a, y=b} = 0$. By substituting $W(x, y)$ into the other boundary conditions

$\partial W/\partial x|_{x=0} = \partial W/\partial y|_{y=0} = 0$ and $M_x|_{x=a} = M_y|_{y=b} = 0$, and taking the differentiation technique of trigonometric series [41], we can obtain

$$\sum_{m=1,3}^{\infty} (-1)^{(n-1/2)} D_y C_{mn} R_{mn} \alpha_m I_m - \sum_{m=1,3}^{\infty} D_y C_{mn} \alpha_m \frac{\beta_n}{2} J_m + \sum_{m=1,3}^{\infty} (-1)^{(m-1/2)} D_x C_{mn} P_{mn} \alpha_m K_n \\ (20)$$

$$- \sum_{m=1,3}^{\infty} D_x C_{mn} \alpha_m \frac{\alpha_m}{2} L_n = - \sum_{m=1,3}^{\infty} C_{mn} \alpha_m q_{mn} \quad n = 1, 3, 5 \dots,$$

$$\sum_{m=1,3}^{\infty} (-1)^{(m-1/2)} (-1)^{(n-1/2)} [\mu_y - D_y C_{mn} R_{mn} P_{mn}] I_m + \sum_{m=1,3}^{\infty} (-1)^{(m-1/2)} D_y C_{mn} P_{mn} \frac{\beta_n}{2} J_m \\ + \sum_{m=1,3}^{\infty} (-1)^{m-1} [1 - D_x C_{mn} P_{mn}^2] K_n \\ (21)$$

$$+ \sum_{m=1,3}^{\infty} (-1)^{(m-1/2)} D_x C_{mn} P_{mn} \frac{\alpha_m}{2} L_n = \sum_{m=1,3}^{\infty} (-1)^{(m-1/2)} C_{mn} P_{mn} q_{mn} \quad n = 1, 3, 5 \dots,$$

$$\sum_{n=1,3}^{\infty} (-1)^{(n-1/2)} D_y C_{mn} R_{mn} \beta_n I_m - \sum_{n=1,3}^{\infty} D_y C_{mn} \beta_n \frac{\beta_n}{2} J_m + \sum_{n=1,3}^{\infty} (-1)^{(m-1/2)} D_x C_{mn} P_{mn} \beta_n K_n \\ (22)$$

$$- \sum_{n=1,3}^{\infty} D_x C_{mn} \beta_n \frac{\alpha_m}{2} L_n = - \sum_{n=1,3}^{\infty} C_{mn} \beta_n q_{mn} \quad m = 1, 3, 5 \dots,$$

$$\sum_{n=1,3}^{\infty} (-1)^{n-1} [1 - D_y C_{mn} R_{mn}^2] I_m + \sum_{n=1,3}^{\infty} (-1)^{(n-1/2)} D_y C_{mn} R_{mn} \frac{\beta_n}{2} J_m \\ + \sum_{n=1,3}^{\infty} (-1)^{(n-1/2)} (-1)^{(m-1/2)} [\mu_x - D_x C_{mn} R_{mn} P_{mn}] K_n \\ + \sum_{n=1,3}^{\infty} (-1)^{(n-1/2)} D_x C_{mn} R_{mn} \frac{\alpha_m}{2} L_n = \sum_{n=1,3}^{\infty} (-1)^{(n-1/2)} C_{mn} R_{mn} q_{mn} \quad m = 1, 3, 5 \dots, \\ (23)$$

The constants I_m , J_m , K_n , and L_n ($m, n = 1, 3, 5, \dots$) can be obtained by solving the infinite linear simultaneous equations described by equations (20)–(23). In calculation, a finite number of terms are taken in each set of equations and solved for a finite number of constants, i.e., $m = 1, 3, 5, \dots, M$, and $n = 1, 3, 5, \dots, N$, where M and N are any positive integers. Here, the same term t is chosen for m and n , with

their upper limit taken as $M = (2t - 1)/2$ and $N = (2t - 1)/2$. By substituting the above constant solutions into equation (19), we finally get the analytical bending solutions for plate with two adjacent edges free and the other two edges clamped.

The bending moment along the clamped edge can be easily found by the following equation:

$$M_x = -D_x \left(\frac{\partial^2 W}{\partial x^2} + \mu_y \frac{\partial^2 w}{\partial y^2} \right) \Big|_{x=0} = -D_x \frac{\partial^2 W}{\partial x^2} \Big|_{x=0} = -D_x \frac{2}{b} \sum_{n=1,3}^{\infty} L_n \sin \frac{\beta_n y}{2}, \\ (24)$$

$$M_y = -D_y \left(\frac{\partial^2 W}{\partial y^2} + \mu_x \frac{\partial^2 W}{\partial x^2} \right) \Big|_{y=0} = -D_y \frac{\partial^2 W}{\partial y^2} \Big|_{y=0} = -D_y \frac{2}{a} \sum_{n=1,3}^{\infty} J_m \sin \frac{\alpha_m x}{2}.$$

Case 2. For the rectangular plate clamped at edge $x = 0$, simply supported at edge $y = 0$, and free at edges $x = a$ and

$y = b$, the undetermined unknowns J_m will be zero, and the deflection of the plate reduces to

$$W(x, y) = \frac{4}{ab} \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} C_{mn} \left[\begin{array}{l} q_{mn} + (-1)^{(n-1/2)} D_y R_{mn} I_m \\ + (-1)^{(m-1/2)} D_x P_{mn} K_n - \frac{\alpha_m}{2} D_x L_n \end{array} \right] \sin \frac{\alpha_m x}{2} \sin \frac{\beta_n y}{2}. \\ (25)$$

By substituting $W(x, y)$ into the other boundary conditions $\partial W/\partial x|_{x=0} = 0$ and $M_x|_{x=a} = M_y|_{y=b} = 0$, and taking

$$\begin{aligned} & \sum_{m=1,3}^{\infty} (-1)^{(n-1/2)} D_y C_{mn} R_{mn} \alpha_m I_m + \sum_{m=1,3}^{\infty} (-1)^{(m-1/2)} D_x C_{mn} P_{mn} \alpha_m K_n - \sum_{m=1,3}^{\infty} D_x C_{mn} \alpha_m \frac{\alpha_m}{2} L_n \\ &= - \sum_{m=1,3}^{\infty} C_{mn} \alpha_m q_{mn} \quad n = 1, 3, 5 \dots, \end{aligned} \quad (26)$$

$$\begin{aligned} & \sum_{m=1,3}^{\infty} (-1)^{(m-1/2)} (-1)^{(n-1/2)} [\mu_y - D_y C_{mn} R_{mn} P_{mn}] I_m + \sum_{m=1,3}^{\infty} (-1)^{m-1} [1 - D_x C_{mn} P_{mn}^2] K_n \\ &+ \sum_{m=1,3}^{\infty} (-1)^{(m-1/2)} D_x C_{mn} P_{mn} \frac{\alpha_m}{2} L_n = \sum_{m=1,3}^{\infty} (-1)^{(m-1/2)} C_{mn} P_{mn} q_{mn} \quad n = 1, 3, 5 \dots, \end{aligned} \quad (27)$$

$$\begin{aligned} & \sum_{n=1,3}^{\infty} (-1)^{n-1} [1 - D_y C_{mn} R_{mn}^2] I_m + \sum_{n=1,3}^{\infty} (-1)^{(n-1/2)} (-1)^{(m-1/2)} [\mu_x - D_x C_{mn} R_{mn} P_{mn}] K_n \\ &+ \sum_{n=1,3}^{\infty} (-1)^{(n-1/2)} D_x C_{mn} R_{mn} \frac{\alpha_m}{2} L_n = \sum_{n=1,3}^{\infty} (-1)^{(n-1/2)} C_{mn} R_{mn} q_{mn} \quad m = 1, 3, 5 \dots \end{aligned} \quad (28)$$

The constants I_m , K_n , and L_n ($m, n = 1, 3, 5, \dots$) can be obtained by solving the infinite linear simultaneous equations described by equations (26)–(28) in a similar way to that in the Case 1.

the differentiation technique of trigonometric series [41], we achieved

Case 3. For the rectangular plate simply supported at edges $x = 0$ and $y = 0$ and free at edges $x = a$ and $y = b$, the undetermined unknowns I_m and L_n will be zero, and the deflection of the plate reduces to

$$W(x, y) = \frac{4}{ab} \sum_{m=1,3}^{\infty} \sum_{n=1,3}^{\infty} C_{mn} \left[q_{mn} + (-1)^{(n-1/2)} D_y R_{mn} I_m + (-1)^{(m-1/2)} D_x P_{mn} K_n \right] \sin \frac{\alpha_m x}{2} \sin \frac{\beta_n y}{2}. \quad (29)$$

Equation (29) has to satisfy the remaining boundary conditions $M_x|_{x=a} = M_y|_{y=b} = 0$, and by taking the differentiation technique of trigonometric series [41], we can get

$$\begin{aligned} & \sum_{m=1,3}^{\infty} (-1)^{(m-1/2)} (-1)^{(n-1/2)} [\mu_y - D_y C_{mn} R_{mn} P_{mn}] I_m + \sum_{m=1,3}^{\infty} (-1)^{m-1} [1 - D_x C_{mn} P_{mn}^2] K_n \\ &= \sum_{m=1,3}^{\infty} (-1)^{(m-1/2)} C_{mn} P_{mn} q_{mn} \quad n = 1, 3, 5 \dots, \end{aligned} \quad (30)$$

$$\begin{aligned} & \sum_{n=1,3}^{\infty} (-1)^{n-1} [1 - D_y C_{mn} R_{mn}^2] I_m + \sum_{n=1,3}^{\infty} (-1)^{(n-1/2)} (-1)^{(m-1/2)} [\mu_x - D_x C_{mn} R_{mn} P_{mn}] K_n \\ &= \sum_{n=1,3}^{\infty} (-1)^{(n-1/2)} C_{mn} R_{mn} q_{mn} \quad m = 1, 3, 5 \dots \end{aligned} \quad (31)$$

The unknown constants I_m and K_n ($m, n = 1, 3, 5, \dots$) can be obtained by solving equations (30)–(31) in a similar way to that in the Case 1 section.

3. Numerical Results and Discussion

Comprehensive examinations on the isotropic/orthotropic rectangular plates under three different support conditions are

TABLE 1: Deflections and bending moments of CCFF isotropic square plate subjected to uniform loading with various Poisson's ratios.

Poisson's ratio μ	Method	Number of terms	$W(q_0 a^4/D), y = b$				$M_y(q_0 a^2), y = 0$			
			$x = a$	$x = 0.75a$	$x = 0.50a$	$x = 0.25a$	$x = 0.75a$	$x = 0.50a$	$x = 0.25a$	
0	Present	$t = 10$	0.034972	0.027382	0.017034	0.0060965	-0.20029	-0.12256	-0.041199	
		$t = 30$	0.035778	0.027647	0.017274	0.0062238	-0.19607	-0.12255	-0.042982	
		$t = 50$	0.035943	0.027730	0.017321	0.0062381	-0.19730	-0.12254	-0.042474	
		$t = 100$	0.036067	0.027780	0.017356	0.0062543	-0.19689	-0.12254	-0.042645	
		$t = 200$	0.036129	0.027807	0.017374	0.0062618	-0.19689	-0.12254	-0.042644	
		$t = 300$	0.036150	0.027816	0.017380	0.0062643	-0.19689	-0.12254	-0.042644	
		$t = 500$	0.036166	0.027823	0.017384	0.0062660	-0.19689	-0.12254	-0.042644	
	Liu and Li [42]		0.036237	—	—	—	-0.19284	—	-0.042656	
	Huang and Conway [43]		0.03619	—	—	—	—	—	—	
	FEM		0.036196	0.027836	0.017392	0.0062687	-0.19499	-0.12166	-0.042779	
0.3	Present	$x = a$	—	$x = 0.75a$	$x = 0.50a$	$x = 0.25a$	—	—	—	
		$t = 10$	0.041986	0.032229	0.019507	0.0066292	-0.23929	-0.16565	-0.095009	
		$t = 30$	0.043062	0.032583	0.019801	0.0067455	-0.23910	-0.16552	-0.094859	
		$t = 50$	0.043278	0.032691	0.019858	0.0067528	-0.23898	-0.16549	-0.094845	
		$t = 100$	0.043441	0.032756	0.019901	0.0067648	-0.23892	-0.16548	-0.094837	
		$t = 200$	0.043523	0.032791	0.019922	0.0067700	-0.23889	-0.16547	-0.094835	
		$t = 300$	0.043550	0.032803	0.019930	0.0067717	-0.23889	-0.16547	-0.094834	
		$t = 500$	0.043572	0.032812	0.019935	0.0067728	-0.23889	-0.16547	-0.094834	
	Liu and Li [42]		0.043601	0.032828	0.019945	0.006774	-0.242318	—	-0.096808	
	Fo-van [44]		0.043678	0.032872	0.019993	0.006771	-0.23400	—	-0.09102	
1/3	Present	FEM		0.043611	0.032829	0.019945	0.0067745	-0.23650	-0.16426	-0.094810
		$x = a$	—	$x = 0.75a$	$x = 0.50a$	$x = 0.25a$	—	—	—	
		$t = 10$	0.043044	0.032965	0.019878	0.0067015	-0.21694	-0.13156	-0.045939	
		$t = 30$	0.044156	0.033329	0.020178	0.0068145	-0.22696	-0.13164	-0.041677	
		$t = 50$	0.044379	0.033441	0.020236	0.0068203	-0.21987	-0.13165	-0.044603	
		$t = 100$	0.044548	0.033508	0.020280	0.0068314	-0.22272	-0.13167	-0.043406	
		$t = 200$	0.044632	0.033544	0.020302	0.0068362	-0.22274	-0.13166	-0.043410	
		$t = 300$	0.044660	0.033556	0.020309	0.0068378	-0.22274	-0.13166	-0.043409	
		$t = 500$	0.044668	0.033566	0.020315	0.0068391	-0.22274	-0.13166	-0.043409	
	Liu and Li [42]		0.044665	0.033579	0.020316	0.006787	-0.225449	—	-0.042035	
Leissa and Niedenfuhr [45]	FEM		0.04055	—	0.018969	—	-0.20753	—	-0.04332	
			0.044709	0.033573	0.020319	0.0068382	-0.22071	-0.13103	-0.042797	

conducted to validate the present approach and examine the accuracy of the obtained results. The selected aspect ratio b/a for rectangular plates ranges from 0.5 to 4. Parametric analysis (effects of aspect ratios, Poisson's ratio, and boundary restraint) on bending characteristics of CCFF, CSFF, and SSFF isotropic/orthotropic rectangular plates is also conducted. It is noteworthy that, for an isotropic plate, $H = D_x = D_y = D$, $D_{xy} = (1 - \mu)D/2$, and $\mu_x = \mu_y = \mu$ where μ and D are Poisson's ratio and flexural stiffness of plate, respectively.

- (1) An isotropic square CCFF plate with various Poisson ratios, under uniform load ($q(x, y) = q_0$)
- (2) An orthotropic rectangular CCFF plate under linearly varying load ($q(x, y) = q_0 \cdot x/a$), where $D_y = 4D_x$, $D_{xy} = 0.85D_x$, $\mu_x = 0.075$, and $\mu_y = 0.3$.
- (3) An isotropic rectangular CSFF plate with Poisson ratio $\mu = 1/3$, under uniform load ($q(x, y) = q_0$).
- (4) An orthotropic rectangular CSFF plate under central concentrate load ($q(x, y) = P$), where $D_y = 4D_x$, $D_{xy} = 0.85D_x$, $\mu_x = 0.075$, and $\mu_y = 0.3$.

(5) An isotropic rectangular SSFF plate with Poisson ratio $\mu = 1/3$, subjected to a uniform load ($q(x, y) = q_0$).

(6) An orthotropic rectangular SSFF plate under sinusoidal load ($q(x, y) = q_0 \sin(\pi x/a)$), where $D_y = 4D_x$, $D_{xy} = 0.85D_x$, $\mu_x = 0.075$, and $\mu_y = 0.3$.

As shown in Table 1, the obtained nondimensional bending solutions (deflections and bending moments at specific points) of CCFF square isotropic rectangular plates increased with the increase of Poisson's ratio. It is clear that the obtained bending solutions of SSFF plate are always larger than those of CSFF and CCFF plates for each aspect ratio; the bending solutions of CSFF plate is always larger than those of CCFF plate, which indicates plates with clamped edges possess lower deform capability than plates with simply supported edges. Finally, it is also observed that the bending solutions of isotropic/orthotropic plates under three different support conditions decreased with the decreasing aspect ratio. The above parametric studies reveal that aspect ratios, Poisson's ratio, and boundary restraint have a significant influence on the bending characteristics of plates.

TABLE 2: Deflections and bending moments of CCFF orthotropic rectangular plate under linear loading ($D_y = 4D_x$, $D_{xy} = 0.85D_x$, $\mu_x = 0.075$, $\mu_y = 0.3$).

Method	b/a	$W(q_0 a^4/D_x)$, $y = b$					$M_x(q_0 a^2)$, $x = 0$		
		$x = a$	$x = 0.75a$	$x = 0.50a$	$x = 0.25a$	$y = 0.75b$	$y = 0.50b$	$y = 0.25b$	
Present	0.5	0.0014045	0.0011974	0.00081803	0.00033817	-0.012265	-0.0052803	-0.00077096	
FEM		0.0014051	0.0011978	0.00081832	0.00033825	-0.012287	-0.0052763	-0.00079197	
Present	1.0	0.012793	0.010023	0.0062899	0.0022584	-0.065799	-0.032091	-0.0064661	
FEM		0.012804	0.010028	0.0062931	0.0022593	-0.065586	-0.031816	-0.0064944	
Present	2.0	0.051082	0.035948	0.020085	0.0062465	-0.19256	-0.12485	-0.039702	
FEM		0.051098	0.035953	0.020082	0.0062464	-0.19107	-0.12423	-0.039649	
Present	3.0	0.075008	0.050613	0.026856	0.0078044	-0.27312	-0.21195	-0.089978	
FEM		0.075010	0.050615	0.026855	0.0078025	-0.27087	-0.21034	-0.090173	
Present	4.0	0.084869	0.056324	0.029256	0.0082630	-0.30949	-0.26530	-0.13949	
FEM		0.084838	0.056321	0.029253	0.0082595	-0.30702	-0.26318	-0.13898	

TABLE 3: Deflections and bending moments of CSFF isotropic square plate under uniform loading with Poisson's ratio $\mu = 1/3$.

Method	b/a	$W(q_0 a^4/D_x)$, $y = b$					$M_x(q_0 a^2)$, $x = 0$		
		$x = a$	$x = 0.75a$	$x = 0.50a$	$x = 0.25a$	$y = 0.75b$	$y = 0.50b$	$y = 0.25b$	
Present	0.5	0.029137	0.023545	0.015447	0.0057435	-0.21714	-0.14227	-0.074437	
FEM		0.029145	0.023548	0.015448	0.0057428	-0.21621	-0.14211	-0.074837	
Present	1.0	0.073524	0.052971	0.030484	0.0096223	-0.34599	-0.25601	-0.14954	
FEM		0.073563	0.052976	0.030485	0.0096193	-0.34440	-0.25567	-0.14891	
Present	2.0	0.11539	0.077393	0.040624	0.011296	-0.47278	-0.41166	-0.27960	
FEM		0.11540	0.077372	0.040600	0.011286	-0.46779	-0.40738	-0.27711	
Present	3.0	0.12262	0.081040	0.041686	0.011250	-0.50257	-0.46955	-0.35885	
FEM		0.12261	0.081010	0.041663	0.011234	-0.49749	-0.46475	-0.35525	
Present	4.0	0.12347	0.081403	0.041722	0.011201	-0.50624	-0.48994	-0.41077	
FEM		0.12347	0.081369	0.041695	0.011182	-0.50116	-0.48499	-0.40652	

TABLE 4: Deflections and bending moments of CSFF orthotropic square plate subjected to central concentrated load ($D_y = 4D_x$, $D_{xy} = 0.85D_x$, $\mu_x = 0.075$, $\mu_y = 0.3$).

Method	b/a	$W(Pa^2/D_x)$, $y = b$					$M_x(P)$, $x = 0$		
		$x = a$	$x = 0.75a$	$x = 0.50a$	$x = 0.25a$	$y = 0.75b$	$y = 0.50b$	$y = 0.25b$	
Present	0.5	0.025644	0.024551	0.019873	0.0084041	-0.33149	-0.21000	-0.10442	
FEM		0.025655	0.024562	0.019880	0.0084065	-0.33043	-0.21026	-0.10339	
Present	1.0	0.035436	0.029660	0.020200	0.0073935	-0.28753	-0.22909	-0.12999	
FEM		0.035458	0.029670	0.020204	0.0073936	-0.28347	-0.22620	-0.12963	
Present	2.0	0.033743	0.023512	0.012874	0.0037013	-0.20785	-0.25596	-0.14027	
FEM		0.033746	0.023510	0.012864	0.0036989	-0.20600	-0.25154	-0.13967	
Present	3.0	0.023279	0.014747	0.0070937	0.0016934	-0.14649	-0.26115	-0.11393	
FEM		0.023271	0.014741	0.0070888	0.0016910	-0.14595	-0.25654	-0.11378	
Present	4.0	0.014617	0.0087457	0.0038585	0.00079266	-0.10166	-0.25972	-0.084991	
FEM		0.014602	0.0087393	0.0038539	0.00079059	-0.10103	-0.25506	-0.085080	

Through Tables 1–6, it can be seen that the obtained results of deflections and bending moments agree very well with those from the literature and, especially, those by FEM results. The validity of the employed method and the accuracy of the obtained results are illuminated through the above examples, which means that the present approach is competent for analyzing the bending problems of orthotropic rectangular thin plates with the title boundary condition.

Table 1 shows the convergence study and results for deflection and bending moment obtained with the accuracy of five significant figures. It is theoretically exact for the

results when $t \rightarrow \infty$, while in the present paper the obtained analytical results suffice for the title problem by taking the series terms ($t = 500$). It is beyond the expectation that convergence rates of the bending moments are faster than ones of the transverse deflection. This is attributing to the fact that the present deflection results are at free edges. The slower convergent results are due to the final solution expressed by double sine Fourier series. However, the linear simultaneous equations can be easily calculated via Mathematica software, and above all, the value of the employed method lies in its merits of rational solution procedure and simple mathematical manipulation that

TABLE 5: Deflections and bending moments of SSFF isotropic square plate with Poisson's ratio subjected to uniform loading.

Method	b/a	$W(q_0 a^4/D), y = b$				$W(q_0 a^4/D)$		
		$x = a$	$x = 0.75a$	$x = 0.50a$	$x = 0.25a$	($0.75a, 0.75b$)	($0.50a, 0.50b$)	($0.25a, 0.25b$)
Present	0.5	0.046862	0.040536	0.030864	0.017032	0.030728	0.015757	0.0044281
FEM		0.046879	0.040547	0.030872	0.017038	0.030740	0.015758	0.0044300
Present	1.0	0.18750	0.150924	0.107884	0.056845	0.119112	0.059318	0.016342
Liu, 2010		0.18750	0.150894	0.107868	0.056839	0.119112	0.059318	0.016342
Leissa, 1963	1.0	0.19276	—	0.110819	—	0.122306	0.060857	0.016746
FEM		0.18749	0.150885	0.107861	0.056836	0.119103	0.059318	0.016340
Present	2.0	0.74979	0.57559	0.39293	0.20013	0.49165	0.25212	0.070849
FEM		0.74984	0.57559	0.39294	0.20013	0.49159	0.25206	0.070841
Present	3.0	1.6871	1.2791	0.86226	0.43487	1.1372	0.59568	0.16980
FEM		1.6871	1.2790	0.86222	0.43486	1.1370	0.59560	0.16978
Present	FEM	2.9993	2.2635	1.5186	0.76303	2.0525	1.0850	0.31153
FEM		2.9992	2.2633	1.5184	0.76298	2.0522	1.0848	0.31150

TABLE 6: Deflections of SSFF orthotropic square plate subjected to sinusoidal load ($D_y = 4D_x, D_{xy} = 0.85D_x, \mu_x = 0.075, \mu_y = 0.3$).

Method	b/a	$W(q_0 a^4/D_x), y = b$				$W(q_0 a^4/D_x)$		
		$x = a$	$x = 0.75a$	$x = 0.50a$	$x = 0.25a$	($0.75a, 0.75b$)	($0.50a, 0.50b$)	($0.25a, 0.25b$)
Present	0.5	0.011700	0.010881	0.0088150	0.0050191	0.0082315	0.0045234	0.0013139
FEM		0.011708	0.010888	0.0088204	0.0050225	0.0082376	0.0045251	0.0013150
Present	1.0	0.046785	0.040375	0.030744	0.016855	0.031381	0.016617	0.0047686
FEM		0.046825	0.040401	0.030764	0.016866	0.031389	0.016621	0.0047697
Present	2.0	0.18719	0.14885	0.10524	0.054918	0.12382	0.064574	0.018470
FEM		0.18729	0.14890	0.10528	0.054937	0.12384	0.064584	0.018474
Present	3.0	0.42118	0.32521	0.22339	0.11420	0.28311	0.14924	0.043046
FEM		0.42139	0.32531	0.22346	0.11424	0.28317	0.14927	0.043054
Present	4.0	0.74878	0.57114	0.38747	0.19629	0.51004	0.27077	0.078462
FEM		0.74912	0.57130	0.38758	0.19635	0.51014	0.27083	0.078478

enable the method to serve as an easy-to-implement theoretical tool in exploring analytical bending solutions of a plate.

4. Conclusion

This paper investigates the analytical bending solution of orthotropic rectangular thin plates with two adjacent edges free and the others clamped or simply supported. The significant merits of the adopted method differing from typical semi-inverse ones are as follows: (1) it provides a rigorous and theoretical solution procedure for precise plate bending analysis without assuming the solution forms; (2) based on integral transform theory, it reduces the mathematical difficulty of plate problem by converting the boundary value problems of higher-order partial differential equation into solving linear algebra equations; (3) the present approach allows more precise analytical solutions for static problems of moderately thick and thick plates under similar boundary conditions. The obtained results are expected to serve as an accurate solution for validation of other numerical methods.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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