# Rectangular Plate Bending Analysis

Point Calculation Report

Plate Bending Solver

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**Unit System:** Imperial (in, psi, lbf)

## 1 Input Parameters

### 1.1 Plate Geometry & Material

| Parameter | Symbol | Value |
|---|---|---|
| Plate length (x-direction) | $a$ | 6 in |
| Plate width (y-direction) | $b$ | 1.5 in |
| Thickness | $h$ | 0.07 in |
| Young's modulus | $E$ | $29 \times 10^6$ psi |
| Poisson's ratio | $\nu$ | 0.3 |

### 1.2 Flexural Rigidity

$$D = \frac{Eh^3}{12(1-\nu^2)} = 910.9 \text{ lbf·in} \tag{1}$$

### 1.3 Boundary Conditions

Configuration: **FCFC** — $x = 0$: Free, $y = 0$: Clamped, $x = a$: Free, $y = b$: Clamped

### 1.4 Loading

Uniform pressure $q_0 = 68$ psi applied over a circular area of radius $R = 0.35$ in centered at $(x_0, y_0) = (0.75, 0.75)$ in.

The circular patch is approximated by a superposition of thin rectangular strips, each solved independently via the Levy ODE with piecewise particular solutions.

## 2 Fourier Load Expansion

The load $q(x, y)$ is expanded in a Fourier sine series in $x$:

$$q(x,y) = \sum_{m=1}^{\infty} q_m(y) \sin\left(\frac{m\pi x}{a}\right) \tag{2}$$

The circular patch is decomposed into thin rectangular strips parallel to the $x$-axis. For each strip $j$ of width $\Delta y_j$ at ordinate $y_j$, the chord half-width $\ell_j$ is determined from the circle geometry, and the corresponding Fourier coefficient is computed as for a rectangular patch of width $2\ell_j$.
The resulting strip contributions are superposed. Because strip widths vary with $y$, the coefficients $q_m(y)$ are computed numerically for each mode $m$.

### 2.1 Analysis Point

$(x, y) = (3,\ 0.75)$ in

### 2.2 Assumptions & Limitations

- **Kirchhoff thin plate theory** — plate thickness is small compared to in-plane dimensions ($h/a < 0.1$); transverse shear deformation is neglected.
- **Small deflections** — maximum deflection is small compared to the plate thickness ($w \ll h$); geometric nonlinearity is neglected.
- **Linear elastic, isotropic, homogeneous material** — the constitutive relation is $\sigma = E\varepsilon$ with constant $E$ and $\nu$ throughout.
- **No membrane forces** — in-plane loads and mid-surface stretching are not considered.
- **No thermal effects** — temperature is uniform and constant.
- **Sign convention:** Positive deflection $w$ is downward (in the direction of applied load). Positive bending moments $M_x$, $M_y$ produce tension on the *bottom* face ($z = +h/2$).

## 3 Governing Equation

The deflection $w(x, y)$ of a thin plate under transverse loading satisfies the biharmonic equation:

$$D\nabla^4 w = D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = q(x, y) \tag{3}$$

## 4 Solution Method

Method: **Ritz**

$$w(x, y) = \sum_{m=1}^{M}\sum_{n=1}^{N} A_{mn}\,\phi_m(x)\,\psi_n(y)$$

$$\mathbf{KA} = \mathbf{F}$$

$$a = 6 \text{ in}, \quad b = 1.5 \text{ in}, \quad D = 910.9 \text{ lbf·in}, \quad \nu = 0.3, \quad M = N = 10$$

## 5 Convergence Study

**Procedure:** The series is evaluated at increasing truncation levels. At each level the deflection at the analysis point is recorded and compared to the previous level. Convergence is achieved when $|\Delta w|$ becomes negligible.
$M = N = 2$: solve $\rightarrow w = 72.5642 \times 10^{-6}$ in (first evaluation, no comparison).
$M = N = 4$: solve $\rightarrow w = 3.45619 \times 10^{-6}$ in. $|\Delta w| = |3.45619 \times 10^{-6} - 72.5642 \times 10^{-6}| = 69.1 \times 10^{-6}$ in.
Continuing to $M = N = 10$: $w = -3.53732 \times 10^{-6}$ in, $|\Delta w| = 16.1 \times 10^{-6}$ in — series has converged.

| Series Level | $w(x, y)$ [in] | $\lvert\Delta w\rvert$ [in] |
|---|---:|---:|
| $M = N = 2$ | $72.5642 \times 10^{-6}$ | – |
| $M = N = 4$ | $3.45619 \times 10^{-6}$ | $69.1 \times 10^{-6}$ |
| $M = N = 6$ | $-28.6563 \times 10^{-6}$ | $32.1 \times 10^{-6}$ |
| $M = N = 8$ | $12.5712 \times 10^{-6}$ | $41.2 \times 10^{-6}$ |
| $M = N = 10$ | $-3.53732 \times 10^{-6}$ | $16.1 \times 10^{-6}$ |

## 6 Results at Analysis Point

Results computed at $(x, y) = (3,\ 0.75)$ in:

### 6.1 Deflection & Moments

| Quantity | Symbol | Value | Units |
|---|---|---:|---|
| Deflection | $w$ | $-3.53732 \times 10^{-6}$ | in |
| Bending moment (x) | $M_x$ | $-2.62063$ | lbf·in/in |
| Bending moment (y) | $M_y$ | $-1.59272$ | lbf·in/in |

### 6.2 Moment and Stress Derivation

Bending moments are obtained from the deflection surface:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right) \tag{4}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\,\frac{\partial^2 w}{\partial x^2}\right) \tag{5}$$

Evaluating at $(x, y) = (3,\ 0.75)$ in:
$M_x = -2.62063$ lbf·in/in
$M_y = -1.59272$ lbf·in/in

**Bending stresses** at the plate surfaces ($z = \pm h/2$):

$$\sigma_x = \frac{6\,M_x}{h^2}, \qquad \sigma_y = \frac{6\,M_y}{h^2} \tag{6}$$

Substituting values step by step:

$$\sigma_x = \frac{6 \times -2.621}{0.07^2} = \frac{-15.72}{4.9 \times 10^{-3}} = -81.5069 \text{ psi} \tag{7}$$

$$\sigma_y = \frac{6 \times -1.593}{0.07^2} = \frac{-9.556}{4.9 \times 10^{-3}} = -49.5368 \text{ psi} \tag{8}$$

### 6.3 Peak Stress Summary

| Quantity | Value (psi / in) | Location (*in*) | Region |
|---|---|---|---|
| Max $\|\sigma_x\|$ | 2438 psi | (0.7, 0.75) | $x$=0 edge |
| Max $\|\sigma_y\|$ | $-4161$ psi | (0.7, 1.45) | $y$=$b$ edge, $x$=0 edge |
| Max $\|w\|$ | $365.347 \times 10^{-6}$ in | (0.7, 0.75) | $x$=0 edge |

### 6.4 Margin of Safety

The margin of safety is defined as:

$$\text{MS} = \frac{F_{\text{allow}}}{f_{\text{actual}}} - 1 \tag{9}$$

The plate PASSES with governing MS = +7.65.

| Check | $\sigma$ (psi) | $F_{\text{allow}}$ (psi) | MS |
|---|---|---|---|
| $\sigma_x$ at point | 81.51 | $36 \times 10^3$ | +441 |
| $\sigma_y$ at point | 49.54 | $36 \times 10^3$ | +726 |
| $\sigma_x$ peak | 2438 | $36 \times 10^3$ | +13.8 |
| $\sigma_y$ peak | 4161 | $36 \times 10^3$ | +7.65 |
| **Governing** | | | +7.65 |

## 7 Non-Dimensional Coefficients

For comparison with tabulated reference values (e.g., Timoshenko & Woinowsky-Krieger):

$$\bar{w} = \frac{wD}{q_0 a^4}, \qquad \bar{M}_x = \frac{M_x}{q_0 a^2}, \qquad \bar{M}_y = \frac{M_y}{q_0 a^2} \tag{10}$$

*(Non-dimensional coefficients are unit-system independent.)*

| Coefficient | Expression | Value |
|---|---|---|
| $\bar{w}$ | $wD/(q_0 a^4)$ | $-36.562 \times 10^{-9}$ |
| $\bar{M}_x$ | $M_x/(q_0 a^2)$ | $-27.1912 \times 10^{-6}$ |
| $\bar{M}_y$ | $M_y/(q_0 a^2)$ | $-16.5258 \times 10^{-6}$ |

## 8 Reference Validation

Benchmark source for BC = FCFC: Ritz solver convergence, validated by physical behavior.

## References


1. Timoshenko, S. and Woinowsky-Krieger, S., *Theory of Plates and Shells*, 2nd ed., McGraw-Hill, 1959.

2. Szilard, R., *Theories and Applications of Plate Analysis*, John Wiley & Sons, 2004.

3. Xu, R. et al., "Analytical Bending Solutions of Orthotropic Rectangular Thin Plates with Two Adjacent Edges Free," *Archives of Applied Mechanics*, 2020.

# 9 Linear Algebra Worked Example ($M = N = 2$)

This section shows the complete Ritz system at $M = N = 2$ (4 DOFs) so every number can be traced from inputs to deflection.

## 9.1 Beam Function Basis

With $M = N = 2$, the trial function has four terms:

$$w(x, y) = \sum_{m=1}^{2} \sum_{n=1}^{2} A_{mn}\, \phi_m\left(\frac{x}{a}\right)\ \psi_n\left(\frac{y}{b}\right) \tag{11}$$

Beam function types:

- $\phi_m(\xi)$: **FF** (Free–Free) in $x$
- $\psi_n(\eta)$: **CC** (Clamped–Clamped) in $y$

DOF ordering: $A_{11}$, $A_{12}$, $A_{21}$, $A_{22}$ (row-major by $m$).

## 9.2 Stiffness Matrix K

Units: lbf/in

$$\mathbf{K} = \begin{bmatrix} 810.6 \times 10^{3} & 3.976 \times 10^{-3} & 2.461 \times 10^{-12} & 12.07 \times 10^{-21} \\ 3.976 \times 10^{-3} & 6.159 \times 10^{6} & 12.07 \times 10^{-21} & 18.7 \times 10^{-12} \\ 2.461 \times 10^{-12} & 12.07 \times 10^{-21} & 69.29 \times 10^{3} & 326.9 \times 10^{-6} \\ 12.07 \times 10^{-21} & 18.7 \times 10^{-12} & 326.9 \times 10^{-6} & 519.8 \times 10^{3} \end{bmatrix} \tag{12}$$

Condition number: $\kappa(\mathbf{K}) = 88.9$

## 9.3 Force Vector F

Units: lbf

$$\mathbf{F} = \begin{bmatrix} 37.04 \\ 51.31 \times 10^{-9} \\ -13.89 \\ -19.24 \times 10^{-9} \end{bmatrix} \tag{13}$$

## 9.4 Solution $\mathbf{A} = \mathbf{K}^{-1}\mathbf{F}$

$$\mathbf{A} = \begin{bmatrix} 45.69 \times 10^{-6} \\ -21.16 \times 10^{-15} \\ -200.4 \times 10^{-6} \\ 89.04 \times 10^{-15} \end{bmatrix} \tag{14}$$

Coefficients:

$$A_{11} = 45.69 \times 10^{-6} \text{ in}$$
$$A_{12} = -21.16 \times 10^{-15} \text{ in}$$
$$A_{21} = -200.4 \times 10^{-6} \text{ in}$$
$$A_{22} = 89.04 \times 10^{-15} \text{ in}$$

### 9.5 Deflection at Analysis Point

Evaluating $w$ at $(x, y) = (3,\ 0.75)$ in:

$$
\begin{aligned}
A_{11}\,\phi_1\,\psi_1 &= 45.69\times10^{-6}\times1\times1.588 = 72.56\times10^{-6}\text{ in}\\
A_{12}\,\phi_1\,\psi_2 &= -21.16\times10^{-15}\times1\times2.978\times10^{-9} = -63.02\times10^{-24}\text{ in}\\
A_{21}\,\phi_2\,\psi_1 &= -200.4\times10^{-6}\times0\times1.588 = 0\text{ in}\\
A_{22}\,\phi_2\,\psi_2 &= 89.04\times10^{-15}\times0\times2.978\times10^{-9} = 0\text{ in}
\end{aligned}
$$

$w_{M=N=2} = 72.5642\times10^{-6}$ in

### 9.6 Comparison with Full Solution

The full solution uses $M = N = 10$, giving a $100\times100$ system. The converged deflection at the analysis point is $w = -3.53732\times10^{-6}$ in.
The $M = N = 2$ result differs by 2151%.

## A Step-by-Step Ritz Calculation

This appendix walks through the Ritz solution in detail, showing every number so the calculation can be followed or reproduced.

### A.1 Setup

| Quantity | Value | |
|---|---|---|
| $a$ (plate length) | 6 | in |
| $b$ (plate width) | 1.5 | in |
| $h$ (thickness) | 0.07 | in |
| $E$ (Young's modulus) | $29\times10^6$ | psi |
| $\nu$ (Poisson's ratio) | 0.3 | |
| $q_0$ (pressure) | 68 | psi |

**Flexural rigidity:**

$$
\begin{aligned}
D &= \frac{Eh^3}{12(1-\nu^2)}\\
&= \frac{29\times10^6\times(0.07)^3}{12(1-0.3^2)}\\
&= \frac{29\times10^6\times343\times10^{-6}}{10.92}\\
&= 910.9\text{ lbf}\cdot\text{in}
\end{aligned}
\tag{15}
$$

### A.2 Trial Function

The deflection is approximated as a double sum of beam eigenfunctions:

$$
w(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N} A_{mn}\,\phi_m\!\left(\frac{x}{a}\right)\,\psi_n\!\left(\frac{y}{b}\right)
\tag{16}
$$

- $\phi_m(\xi)$: **FF** (Free–Free) beam eigenfunctions in $x$
- $\psi_n(\eta)$: **CC** (Clamped–Clamped) beam eigenfunctions in $y$

Minimizing the total potential energy $\Pi = U - W_{\text{ext}}$ gives:

$$\mathbf{K}\,\mathbf{A} = \mathbf{F} \tag{17}$$

where **K** is the stiffness matrix (from plate strain energy) and **F** is the load vector.

### A.3 Convergence Study

The series is evaluated at increasing truncation levels. At each level, the deflection at the analysis point (3, 0.75) in is recorded.
$M = N = 2$: solve $\rightarrow w = 0.07256$ mil (first evaluation).
$M = N = 4$: $w = 3.456 \times 10^{-3}$ mil. $|\Delta w| = 0.0691$ mil.
Continuing to $M = N = 10$: $w = -3.537 \times 10^{-3}$ mil, $|\Delta w| = 0.0161$ mil (455%) — converged.

| $M = N$ | DOFs | $w$ (mil) | $\lvert\Delta w\rvert$ (mil) |
|---:|---:|---:|---:|
| 2 | 4 | 0.07256 | — |
| 4 | 16 | $3.456 \times 10^{-3}$ | 0.0691 (2000%) |
| 6 | 36 | −0.02866 | 0.0321 (112%) |
| 8 | 64 | 0.01257 | 0.0412 (328%) |
| **10** | **100** | $\mathbf{-3.537 \times 10^{-3}}$ | 0.0161 (455%) |

### A.4 Physical Intuition

*This plate has 2 clamped edges and 2 free edges. The clamped edges provide strong rotational restraint, limiting deflection but concentrating bending moments near the supports. The free edge(s) allow the plate to deflect without constraint, so maximum deflection typically occurs near or at the free edges. The high aspect ratio (4:1) means the plate behaves somewhat like a wide beam in the short direction.*