1 General approach

Here the general idea of the constrained effective potential (CEP) will be described. It is basically what is written in Philipp's thesis (chapter 4 and 6.3). The only real difference is, that in the thesis a general factor of N_f drawn in front of the action to consider the remaining part as independent from N_f. For that, some redefinitions of the coupling constants are performed. The reason to do that was, to investigate the phase structure of the model in the large N_f-limit. Further I will only quote the fermionic contributions, since I don't understand their derivation.

The general idea do consider a potential that only depends on an assumed ground state of the system. In the here considered model we consider the ground state to consist of a zero momentum mode and a so called staggered mode:

$$\Phi_x^g = m_\Phi \cdot \hat{\Phi}_1 + s_\Phi \cdot e^{ip_s \cdot x} \cdot \hat{\Phi}_2 \qquad p_s = (\pi, \pi, \pi, \pi), \tag{1.1}$$

with $\hat{\Phi}_{1/2}$ being constant vectors. Since the (relative) orientation of those vectors does not matter, one can assume them to be identical. The magnetization m_Φ and the staggered magnetization s_Φ will later be the order parameters of the model.

The scalar field wil be considered in momentum space. As a reminder, Fourier transformation looks like:

$$\tilde{\Phi}_p = \frac{1}{\sqrt{V}} \sum_x e^{-i\,p\cdot x} \Phi_x, \quad \Phi_x = \frac{1}{\sqrt{V}} \sum_x e^{i\,p\cdot x} \tilde{\Phi}_p. \tag{1.2}$$

The magnetizations are defined as follows:

$$m_\Phi = \left| \frac{1}{V} \sum_x \Phi_x \right| \to \tilde{\Phi}_0 = \sqrt{V} m_\Phi, \quad s_\Phi = \left| \frac{1}{V} \sum_x e^{i\,p_s\cdot x} \Phi_x \right| \to \tilde{\Phi}_{p_s} = \sqrt{V} s_\Phi. \tag{1.3}$$

With that, the CEP $U(m, s)$ is defined via:

$$V \cdot U(m, s) = -\log \left(\int D\Psi D\bar{\Psi} \left[\prod_{0 \neq p \neq p_s} d\tilde{\Phi}_p \right] e^{-S[\Psi,\bar{\Psi},\Phi]} \Bigg|_{\substack{\tilde{\Phi}_0 = \sqrt{V} m_\Phi \\ \tilde{\Phi}_{p_s} = \sqrt{V} s_\Phi}} \right), \tag{1.4}$$

with the action being composed of a fermionic (S_F) and a purely bosonic part (S_B). The bosonic action can be written as follows:

$$S_B[\Phi] = -\kappa \sum_{x,\mu} \Phi_x^\dagger \left[\Phi_{x+\hat{\mu}} + \Phi_{x-\hat{\mu}} \right] + \sum_x \Phi_x^\dagger \Phi_x + \hat{\lambda} \sum_x \left(\Phi_x^\dagger \Phi_x - N_f \right)^2, \tag{1.5}$$

and the Fourier transformation of the individual parts are:

$$\begin{aligned} \sum_x \Phi_x^\dagger \Phi_x &= \frac{1}{V} \sum_x \sum_{p,q} \left(e^{-i\,p\cdot x} \tilde{\Phi}_p^\dagger \right) \left(e^{i\,q\cdot x} \tilde{\Phi}_q \right), \\ &= \frac{1}{V} \sum_{p,q,} \underbrace{\sum_x e^{-i(p-q)\cdot x}}_{V\cdot\delta_{p,q}} \tilde{\Phi}_p^\dagger \tilde{\Phi}_q, \\ &= \sum_p \tilde{\Phi}_p^\dagger \tilde{\Phi}_p, \end{aligned} \tag{1.6}$$

$$\begin{aligned} \sum_{x,\mu} \Phi_x^\dagger \left[\Phi_{x+\hat{\mu}} + \Phi_{x-\hat{\mu}} \right] &= \frac{1}{V} \sum_{x,\mu} \sum_{p,q} e^{-i\,p\cdot x} \tilde{\Phi}_p^\dagger \left[e^{i\,q\cdot(x+\hat{\mu})} \tilde{\Phi}_q + e^{i\,q\cdot(x-\hat{\mu})} \tilde{\Phi}_q \right], \\ &= \frac{1}{V} \sum_{p,q,\mu} \sum_x e^{-i(p-q)\cdot x} \tilde{\Phi}_p^\dagger \tilde{\Phi}_q \underbrace{\left(e^{i\,q_\mu} + e^{-i\,q_\mu} \right)}_{2\cdot\cos(q_\mu)}, \\ &= \sum_p \tilde{\Phi}_p^\dagger \left[2 \cdot \sum_\mu \cos(p_\mu) \right] \tilde{\Phi}_p, \end{aligned} \tag{1.7}$$

$$\sum_x \left(\Phi_x^\dagger \Phi_x\right)^2 = \frac{1}{V^2} \sum_x \sum_{p,q,r,s} \left(e^{-i\,p\cdot x}\tilde{\Phi}_p^\dagger\right) \left(e^{i\,q\cdot x}\tilde{\Phi}_q\right) \left(e^{-i\,r\cdot x}\tilde{\Phi}_r^\dagger\right) \left(e^{i\,s\cdot x}\tilde{\Phi}_s\right),$$
$$= \frac{1}{V^2} \sum_{p,q,r,s} \sum_x e^{-i(p+r-q-s)\cdot x} \tilde{\Phi}_p^\dagger \tilde{\Phi}_q \tilde{\Phi}_r^\dagger \tilde{\Phi}_s,$$
$$= \frac{1}{V} \sum_{p,q,r,s} \delta_{p+r,q+s} \tilde{\Phi}_p^\dagger \tilde{\Phi}_q \tilde{\Phi}_r^\dagger \tilde{\Phi}_s. \tag{1.8}$$

Now, in momentum space, the bosonic action can be written as:

$$S_B[\Phi] = -\kappa \sum_{x,\mu} \Phi_x^\dagger \left[\Phi_{x+\hat{\mu}} + \Phi_{x-\hat{\mu}}\right] + \sum_x \Phi_x^\dagger \Phi_x + \hat{\lambda} \sum_x \left(\Phi_x^\dagger \Phi_x - N_f\right)^2,$$
$$= \frac{1}{2} \sum_p \tilde{\Phi}_p^\dagger \left[2 - 4\hat{\lambda} N_f - 4\kappa \sum_\mu \cos(p_\mu)\right] \tilde{\Phi}_p + \frac{\hat{\lambda}}{V} \sum_{p,q,r,s} \delta_{p+r,q+s} \tilde{\Phi}_p^\dagger \tilde{\Phi}_q \tilde{\Phi}_r^\dagger \tilde{\Phi}_s. \tag{1.9}$$

A constant term was ignored here, since we are only interested in terms depending on m and s.

If one could compute the CEP, one obtains some information: First of all, the absolute minimum of the CEP will be the ground state and if the minimum occures at zero or non-zero (staggered) magnetization determines the phase the system is in. Further, any observable that only depends on the $\tilde{\Phi}_0$ and/or Φ_{p_s} can be computed from a two-dimensional integral:

$$\left\langle \mathcal{O}(\tilde{\Phi}_0, \tilde{\Phi}_{p_s}) \right\rangle = \mathcal{Z}^{-1} \int d\tilde{\Phi}_0 \, d\tilde{\Phi}_{p_s} \left(\mathcal{O}(\tilde{\Phi}_0, \tilde{\Phi}_{p_s})\, e^{-V\cdot U(\tilde{\Phi}_0, \tilde{\Phi}_{p_s})} \right) \tag{1.10}$$

2 Expansion of the action

To compute the CEP perturbatively, one decomposes the action into a part, that only depends on the assumed ground state (S_{B,Φ^g}), a gaussian part ($S_{B,0}$) where the fields only appear quadratically and an interaction part (S_I) that consiste of the rest. The treelevel part part can be computed directly [1]. The gaussian part can be integrated out. It gives a determinant and determines the propagators of the bosonic field. The determinant will be ignored in the following, since for what we need it does not matter, since it is indeppendent of m and s. The decomposition of the bosonic part is:

$$S_{B,\Phi^g} = V\left(-8\kappa\left(m_\Phi^2 - s_\Phi^2\right) + m_\Phi^2 + s_\Phi^2 + \hat\lambda\left(m_\Phi^4 + s_\Phi^4 + 6m_\Phi^2 s_\Phi^2 - 2N_f\left(m_\Phi^2 + s_\Phi^2\right)\right)\right), \tag{2.1}$$

$$S_{B,0} = \frac{1}{2}\sum_{0\neq p\neq p_s} \tilde\Phi_p^\dagger\left[2 - 4\hat\lambda N_f - 4\kappa\sum_\mu \cos(p_\mu)\right]\tilde\Phi_p, \tag{2.2}$$

$$S_{B,I} = \frac{\hat\lambda}{V}\widehat{\sum_{p,q,r,s}}\delta_{p+r,q+s}\tilde\Phi_p^\dagger\tilde\Phi_q\tilde\Phi_r^\dagger\tilde\Phi_s, \tag{2.3}$$

with $\hat\sum$ meaning, that at leas one of the summed momenta is not in $\{0,p_s\}$. For the prefactor $(6m_\Phi^2 s_\Phi^2)$ in the treelevel expression one has to take into account, that the $\delta_{p,q}$ has respect the periodicity, meaning that p and q are equal up to a difference of multiples of 2π.

From S_0 on gets the contraction of the scala field:

$$\overbracket{\Phi_p^\dagger\Phi_q} = \frac{4\cdot\delta_{p,q}}{2 - 4\hat\lambda N_f - 4\kappa\sum_\mu\cos(p_\mu)}. \tag{2.4}$$

The factor 4 in the numerator comes from the fact, that this contraction is performed on every component of the scalar field. **Since this is a crucial statement, find someone to check this!**

For the exponential of the CEP we can write now (including the fermionic Part, that will contribute):

$$e^{-V\cdot U(m,s)} = e^{-S_{B,\Phi^g}[\Phi^g]}e^{-N_f\log\det\mathcal{M}[\Phi^g]}\int\left[\prod_{0\neq p\neq p_s}d\tilde\Phi_p\right]e^{-S_{B,I}[\Phi]+S_{B,0}[\Phi]}\Bigg|_{\substack{\tilde\Phi_0=\sqrt{V}m_\Phi\\ \tilde\Phi_{p_s}=\sqrt{V}s_\Phi}}, \tag{2.5}$$

If one expands the exponential from $S_{B,I}$ to first order, the integral in (2.5) we end up with:

$$\begin{aligned}
\int\left[\prod_{0\neq p\neq p_s}d\tilde\Phi_p\right]e^{-S_{B,I}[\Phi]+S_{B,0}[\Phi]} &= \int\left[\prod_{0\neq p\neq p_s}d\tilde\Phi_p\right]\left(1 - \frac{\hat\lambda}{V}\widehat{\sum_{p,q,r,s}}\delta_{p+r,q+s}\tilde\Phi_p^\dagger\tilde\Phi_q\tilde\Phi_r^\dagger\tilde\Phi_s\right)e^{-S_B,0}\\
&=\sqrt{\frac{V}{\det\mathcal{B}}}\cdot\left(1 - \frac{\hat\lambda}{V}\left(C_{p,q} + C_{m,p} + C_{s,p}\right)\right)\\
&=\sqrt{\frac{V}{\det\mathcal{B}}}\cdot e^{-\frac{\hat\lambda}{V}\left(C_{p,q}+C_{m,p}+C_{s,p}\right)}
\end{aligned} \tag{2.6}$$

In the above equations, $\det\mathcal{B}$ is the bosonic determinant. The matrix $\mathcal{B}$ is given by:

$$\mathcal{B}^{i,j}(p,q) = \delta_{i,j}\delta_{p,q}\left(2 - 4\hat\lambda N_f - 4\kappa\sum_\mu\cos(p_\mu)\right) \tag{2.7}$$

Check, if determinant occurs in the correct power! The indices i and j label the components of the scalar field. It is independet of m_Φ/s_Φ and will, when the log of the r.h.s. of (2.6) is taken, only contributa a constant and can be neglected. The C come from the different possibilities to contract the four fields. The first one, $C_{p,q}$ comes from a full contraction:

$$\overbracket{\tilde\Phi_p^\dagger\tilde\Phi_q}\overbracket{\tilde\Phi_r^\dagger\tilde\Phi_s} = \frac{\delta_{p,q}}{2 - 4\hat\lambda N_f - 4\kappa\sum_\mu\cos(p_\mu)}\cdot\frac{\delta_{r,s}}{2 - 4\hat\lambda N_f - 4\kappa\sum_\mu\cos(r_\mu)}, \tag{2.8}$$

and gives some kind of vacuum bubble which is independent of zero or staggered mode and will therefore be neglected here. The other two contributions in the exponantial origin from those terms in the sum, where some

[1] The computation of the fermionic contribution is somehow complicated, but in principle free of any perturbative arguments **check that**

(but not all) occuring momenta are either 0 or p_s:

$$
\begin{aligned}
C_{p,m} =& \widehat{\sum_{p,q}} \left(\Phi_p^\dagger \Phi_q \Phi_0^\dagger \Phi_0 + \Phi_p^\dagger \Phi_0 \Phi_0^\dagger \Phi_q + \Phi_0^\dagger \Phi_p \Phi_q^\dagger \Phi_0 + \Phi_0^\dagger \Phi_0 \Phi_p^\dagger \Phi_q \right) \\
=& 16 \cdot V \cdot m_\Phi^2 \sum_{0 \neq p \neq p_s} \frac{1}{2 - 4\hat{\lambda} N_f - 4\kappa \sum_\mu \cos(p_\mu)} \qquad (2.9) \\
C_{p,s} =& 16 \cdot V \cdot s_\Phi^2 \sum_{0 \neq p \neq p_s} \frac{1}{2 - 4\hat{\lambda} N_f - 4\kappa \sum_\mu \cos(p_\mu)}. \qquad (2.10)
\end{aligned}
$$

For later use, the propagator sum will be defined:

$$
P_B \equiv \frac{1}{V} \sum_{0 \neq p \neq p_s} \frac{1}{2 - 4\hat{\lambda} N_f - 4\kappa \sum_\mu \cos(p_\mu)}. \qquad (2.11)
$$

3 Determination of the phase structure

To determine the phase structure of the model, i.e. determine the phase the system is in depending on the choice of parameters, one has to find out, what the minimum of the CEP is w.r.t. m_Φ and s_Φ. The simplest ansatz, would be the tree-level (again with fermions):

$$\begin{aligned}
U(m_\Phi, s_\Phi) =& -8\kappa\left(m_\Phi^2 - s_\Phi^2\right) + \left(m_\Phi^2 + s_\Phi^2\right) + \hat{\lambda}\left(m_\Phi^4 + s_\Phi^4 + 6m_\Phi^2 s_\Phi^2 - 2N_f\left(m_\Phi^2 + s_\Phi^2\right)\right)\\
&- \frac{2N_f}{V}\sum_p \log\left[\left(|\nu^+(p)||\nu^+(\wp)| + \hat{y}^2\left(m_\Phi^2 - s_\Phi^2\right)|\gamma^+(p)||\gamma^+(\wp)|\right)^2\right.\\
&\left.+ m_\Phi^2\hat{y}^2\left(|\gamma^+(p)||\nu^+(\wp)| - |\nu^+(p)||\gamma^+(\wp)|\right)^2\right]
\end{aligned} \tag{3.1}$$

Here $\nu^+(p)$ labels the eigenvalue of the overlap operator[2]. It expression and the other abreviations used are $(a = 1)$:

$$\nu^\pm(p) = \rho\left(1 + \frac{\pm i\sqrt{\tilde{p}^2} + r\hat{p}^2 - \rho}{\sqrt{\tilde{p}^2 + \left(r\hat{p}^2 - \rho\right)^2}}\right), \qquad \gamma^\pm(p) = 1 - \frac{1}{2\rho}\nu^\pm(p), \tag{3.2}$$

$$\hat{p}^2 = 4\sum_\mu \sin^2\left(\frac{p_\mu}{2}\right), \qquad \tilde{p}^2 = \sum_\mu \sin^2\left(p_\mu\right), \qquad \wp = p + p_s. \tag{3.3}$$

This is however just a very crude approximation. In Philipp's thesis, he draw a common factor of N_f in front of the action and redefined the couplings, magnetizations and the field according to:

$$\hat{y} = \frac{y_N}{\sqrt{N_f}}, \qquad \hat{\lambda} = \frac{\lambda_N}{N_f}, \qquad m_\Phi = \sqrt{N_f}\,\breve{m}_\Phi, \qquad s_\Phi = \sqrt{N_f}\,\breve{s}_\Phi, \qquad \Phi = \sqrt{N_f}\,\breve{\Phi}. \tag{3.4}$$

The reason for that is, that the action becomes independet of N_f. When keeping now λ_N and y_N constant when performing the limit $N_f \to \infty$, everything but the treelevel get's suppressed by powers of $\frac{1}{N_f}$. To reproduce the expression used by Philipp, also draw a factor in front of U:

$$\begin{aligned}
\tilde{U}(\breve{m}_\Phi, \breve{s}_\Phi) =& \frac{U}{N_f}\\
=& -8\kappa\left(\breve{m}_\Phi^2 - \breve{s}_\Phi^2\right) + \left(\breve{m}_\Phi^2 + \breve{s}_\Phi^2\right) + \lambda_N\left(\breve{m}_\Phi^4 + \breve{s}_\Phi^4 + 6\breve{m}_\Phi^2\breve{s}_\Phi^2 - 2\left(\breve{m}_\Phi^2 + s_\Phi^2\right)\right)\\
&- \frac{2}{V}\sum_p \log\left[\left(|\nu^+(p)||\nu^+(\wp)| + \hat{y}^2\left(\breve{m}_\Phi^2 - \breve{s}_\Phi^2\right)|\gamma^+(p)||\gamma^+(\wp)|\right)^2\right.\\
&\left.+ \breve{m}_\Phi^2 y_N^2\left(|\gamma^+(p)||\nu^+(\wp)| - |\nu^+(p)||\gamma^+(\wp)|\right)^2\right].
\end{aligned} \tag{3.5}$$

The addition of the first order term in $\hat{\lambda}$ is straight foward:

$$U(m_\Phi, s_\Phi) \to U(m_\Phi, s_\Phi) + 16\,\hat{\lambda}\,\left(m_\Phi^2 + s_\Phi^2\right)P_B, \qquad \tilde{U}(\breve{m}_\Phi, \breve{s}_\Phi) \to \tilde{U}(\breve{m}_\Phi, \breve{s}_\Phi) + 16\,\frac{\lambda_N}{N_f}\,\left(\breve{m}_\Phi^2 + \breve{s}_\Phi^2\right)P_B. \tag{3.6}$$

Limits of the CEP and problems with first order in λ

The whole approach of integrating out the Gaussian contribution is only valid, if the bosonic determinant (2.7) is positive definit. This limits the range of values for $\hat{\lambda}$ and κ:

$$0 \overset{!}{<} 2 - 4N_f - 4\kappa\sum_\mu \cos(p_\mu) \quad \Rightarrow \quad \hat{\lambda} < \frac{1}{2N_f}, \qquad |\kappa| < \frac{1 - 2N_f\hat{\lambda}}{8}. \tag{3.7}$$

While this might only be a somehow estitical problem for the tree-level, it gets serious when including the the first order in λ ((2.9) and (2.10)), since the denominator becomes negative or (close to) zero, spoiling everything. This is extremely unpleasant, since the first order in λ has a huge effect. To see this, remove the fermions and assume, you are in a phase, where there should be either the symmetric or the ferromagnetic phase, i.e. $s_\Phi = 0$. Then the CEP reduces to:

$$U(m_\Phi) = \Big(\underbrace{1 - 8\kappa - 2N_f\hat{\lambda}}_{\mathcal{O}(0\ldots2)} + \underbrace{16\,\hat{\lambda}\,P_B}_{\mathcal{O}(10\hat{\lambda})}\Big)\cdot m_\Phi^2 + \hat{\lambda}m_\Phi^4. \tag{3.8}$$

[2]In the definition of this eigenvalue and how it is implemeted in the code, there is a difference in the prefactor of r, the Wilson parameter. I think, its origin may lie in an ambiguity of the definition of the Wilson parameter. What is written hiere, is how it is in the code and how I implemented it

For this potential, to have a minimum at $m_\Phi \neq 0$, the coefficient of the quadratic term must be negative. Without the first order term, this happens when κ violates the condition (3.7). With fermions turned on, they help, that a non-zero minimum is found without violating this condition. However, the effect of the fermions is only small. If one now compares the order of magnitude, the first order term in $\hat{\lambda}$ give a very strong contribution which cannot be compensated by the fermions anymore (with resonable couplings at least).

4 Inclusion of a higher dimensional operator

Here the addition of a higher dimensional operator should be discussed. The new bosonic action in the lattice notation is then:

$$S_B[\Phi] = -\kappa \sum_{x,\mu} \Phi_x^\dagger \left[\Phi_{x+\hat\mu} + \Phi_{x-\hat\mu}\right] + \sum_x \Phi_x^\dagger \Phi_x + \hat\lambda \sum_x \left(\Phi_x^\dagger \Phi_x - N_f\right)^2 + \hat\lambda_6 \sum_x \left(\Phi_x^\dagger \Phi_x\right)^3 \tag{4.1}$$

With that, the CEP changes. With the given notation however, only the tree level part changes, since the new parameter does not enter the gaussion contribution. The Fourier transorm of the new term looks like:

$$\begin{aligned}
\sum_x \left(\Phi_x^\dagger \Phi_x\right)^3 &= \frac{1}{V^3} \sum_x \sum_{p_1,\ldots p6} \left(e^{-i\,p_1\cdot x}\tilde\Phi^\dagger_{p_1}\right)\left(e^{i\,p_2\cdot x}\tilde\Phi_{p_2}\right)\left(e^{-i\,p_3\cdot x}\tilde\Phi^\dagger_{p_3}\right)\left(e^{i\,p_4\cdot x}\tilde\Phi_{p_4}\right)\left(e^{-i\,p_5\cdot x}\tilde\Phi^\dagger_{p_5}\right)\left(e^{i\,p_6\cdot x}\tilde\Phi_{p_6}\right), \\
&= \frac{1}{V^3} \sum_{p_1,\ldots p6} \sum_x e^{-i(p_1+p_3+p_5-p_2-p_4-p_6)\cdot x} \tilde\Phi^\dagger_{p_1}\tilde\Phi_{p_2}\tilde\Phi^\dagger_{p_3}\tilde\Phi_{p_4}\tilde\Phi^\dagger_{p_5}\tilde\Phi_{p_6}, \\
&= \frac{1}{V^2} \sum_{p_1,\ldots p6} \delta_{p_1+p_3+p_5,p_2+p_4+p_6} \tilde\Phi^\dagger_{p_1}\tilde\Phi_{p_2}\tilde\Phi^\dagger_{p_3}\tilde\Phi_{p_4}\tilde\Phi^\dagger_{p_5}\tilde\Phi_{p_6}.
\end{aligned} \tag{4.2}$$

The contribution do the tree-level of the CEP is then:

$$\begin{aligned}
U^{\text{tree}}(m_\Phi, s_\Phi) = -8\kappa\left(m_\Phi^2 - s_\Phi^2\right) + \left(m_\Phi^2 + s_\Phi^2\right) + \hat\lambda\left(m_\Phi^4 + s_\Phi^4 + 6m_\Phi^2 s_\Phi^2 - 2N_f\left(m_\Phi^2 + s_\Phi^2\right)\right) \\
+ \hat\lambda_6\left(m_\Phi^6 + s_\Phi^6 + 15m_\Phi^4 s_\Phi^2 + 15m_\Phi^2 s_\Phi^4\right).
\end{aligned} \tag{4.3}$$

let this check! If one changes to the notation in Philipp's thesis, with the common factor N_f drawn out, one has to reparametrize $\hat\lambda_6$:

$$\hat\lambda_6 = \frac{\lambda_{6N}}{N_f^2}, \tag{4.4}$$

leading to an equivalent contribution to $\tilde U(\check m_\Phi, \check s_\Phi)$.

Higher diemensional operators and 1st order in $\hat\lambda_6$

If one wants to compute the first order contribution in $\hat\lambda_6$, analogous to (3.6), one must consider all possible contractions. As for the first order in $\hat\lambda$, the full contraction of all six fields does not contribute, since it is independent of m_Φ and s_Φ. Since texing the contractions is annoying, the combinatorics can be found in tab. 1. **In addition to thoase combinatoric factors a factor of 4 has to be taken into account for each contraction of momenta coming from the nominator in eq.** (2.4)**.** With this, the contribution to the potential is:

$$U(m_\Phi, s_\Phi) \to U(m_\Phi, s_\Phi) + \hat\lambda_6\,\left[36\left(m_\Phi^4 + s_\Phi^4 + 6m_\Phi^2 s_\Phi^2\right)P_B + 576\left(m_\Phi^2 + s_\Phi^2\right)P_B^2\right], \tag{4.5}$$

$$\tilde U(\check m_\Phi, \check s_\Phi) \to \tilde U(\check m_\Phi, \check s_\Phi) + \hat\lambda_6\,\left[\frac{36}{N_f}\left(\check m_\Phi^4 + \check s_\Phi^4 + 6\check m_\Phi^2 \check s_\Phi^2\right)P_B + \frac{576}{N_f^2}\left(\check m_\Phi^2 + \check s_\Phi^2\right)P_B^2\right], \tag{4.6}$$

coefficient	p_1	p_3	p_5	p_2	p_4	p_6	factor
m_Φ^2	0	p	q	0	p	q	$6\times 6 = 36$
s_Φ^2	p_s	p	q	p_s	p	q	$6\times 6 = 36$
m_Φ^4	0	0	p	0	0	p	$3\times 3 = 9$
s_Φ^4	p_s	p_s	p	p_s	p_s	p	$3\times 3 = 9$
$m_\Phi^2\, m_\Phi^2$	0	0	p	p_s	p_s	p	$3\times 3\times 2 = 18$
$m_\Phi^2\, m_\Phi^2$	0	p_s	p	0	p_s	p	$6\times 6 = 36$

Table 1: Combinatoric factors for the inclusion of the first order contribution of $\hat\lambda_6$

5 Alternative expansion of the bosonic action

Here another way to expand the action should be discussed. More accurate: More terms will contribute to the Gaussian part of the action $S_{B,0}$ which might improve the predictive power of the CEP without the ened to include the one loop term. What will be done is: Take from the interactive part those term, that only have two of the bosonic fields being not the zero or staggered mode. With that, they can be considered as contribution to the Gaussioan part. The drawback from this is, that one has to include the bosonic determinant, since it then depends on m_Φ and s_Φ. Further, if one considers further orders in $\hat{\lambda}$, the propagator sums will be more complicated and also vacuum bubbles are not independent of m_Φ and s_Φ anymore.

Starting point is the expression for the $\hat{\lambda}\left(\Phi^\dagger\Phi\right)^2$ term. So far, the decomposition of this term looked like:

$$\frac{\hat{\lambda}}{V}\sum_{p,q,r,s}\delta_{p+r,q+s}\tilde{\Phi}_p^\dagger\tilde{\Phi}_q\tilde{\Phi}_r^\dagger\tilde{\Phi}_s = V\left(\hat{\lambda}\left(m_\Phi^4+s_\Phi^4+6m_\Phi^2 s_\Phi^2\right)\right)+\frac{\hat{\lambda}}{V}\widehat{\sum_{p,q,r,s}}\delta_{p+r,q+s}\tilde{\Phi}_p^\dagger\tilde{\Phi}_q\tilde{\Phi}_r^\dagger\tilde{\Phi}_s, \tag{5.1}$$

with the hat above the sum indicating that not *all* occuring momenta are eigther 0 and/or p_s. The terms in the sum, where two of the momenta are eigther 0 or p_s are then taken care of in the first order expansion of $\hat{\lambda}$. Another possibility would be:

$$\begin{aligned}\frac{\hat{\lambda}}{V}\sum_{p,q,r,s}\delta_{p+r,q+s}\tilde{\Phi}_p^\dagger\tilde{\Phi}_q\tilde{\Phi}_r^\dagger\tilde{\Phi}_s &= V\left(\hat{\lambda}\left(m_\Phi^4+s_\Phi^4+6m_\Phi^2 s_\Phi^2\right)\right)\\ &+\frac{1}{2}\sum_{0\neq p\neq p_s}\tilde{\Phi}_p^\dagger\left[8\hat{\lambda}\left(m_\Phi^2+s_\Phi^2\right)\right]\tilde{\Phi}_p\\ &+\frac{\hat{\lambda}}{V}\widetilde{\sum_{p,q,r,s}}\delta_{p+r,q+s}\tilde{\Phi}_p^\dagger\tilde{\Phi}_q\tilde{\Phi}_r^\dagger\tilde{\Phi}_s.\end{aligned} \tag{5.2}$$

Here, the tilde above the some means, that *none* of the occuring momenta is 0 and/or p_s.

Then, the total bosonic Gaussian contribution is:

$$S_{B,0}=\frac{1}{2}\sum_{0\neq p\neq p_s}\tilde{\Phi}_p^\dagger\left[2-4\hat{\lambda}N_f+8\hat{\lambda}\left(m_\Phi^2+s_\Phi^2\right)-4\kappa\sum_\mu\cos(p_\mu)\right]\tilde{\Phi}_p. \tag{5.3}$$

If one then expands the interaction part to zeroth order, the calculation for the CEP looks like the following:

$$\begin{aligned}e^{-V\cdot U(m_\Phi,s_\Phi)} &= e^{-V\left(-8\kappa\left(m_\Phi^2-s_\Phi^2\right)+m_\Phi^2+s_\Phi^2+\hat{\lambda}\left(m_\Phi^4+s_\Phi^4+6m_\Phi^2 s_\Phi^2-2N_f\left(m_\Phi^2+s_\Phi^2\right)\right)\right)}\\ &\times e^{-N_f\log\det\mathcal{M}[\Phi^g]}\\ &\times\int\left[\prod_{0\neq p\neq p_s}d\tilde{\Phi}_p\right]e^{\frac{1}{2}\sum\limits_{0\neq p\neq p_s}\tilde{\Phi}_p^\dagger\left[2-4\hat{\lambda}N_f+8\hat{\lambda}\left(m_\Phi^2+s_\Phi^2\right)-4\kappa\sum_\mu\cos(p_\mu)\right]\tilde{\Phi}_p}, \end{aligned}\tag{5.4}$$

$$\begin{aligned}&= e^{-VU^{\text{tree}}(m_\Phi,s_\Phi)}\times e^{-N_f\log\det\mathcal{M}[\Phi^g]}\\ &\times\sqrt{\frac{V}{\prod\limits_{0\neq p\neq p_s}\left(2-4\hat{\lambda}N_f+8\hat{\lambda}\left(m_\Phi^2+s_\Phi^2\right)-4\kappa\sum_\mu\cos(p_\mu)\right)}},\end{aligned}\tag{5.5}$$

$$\begin{aligned}&= e^{-VU^{\text{tree}}(m_\Phi,s_\Phi)}\times e^{-N_f\log\det\mathcal{M}[\Phi^g]}\\ &\times e^{\sum\limits_{0\neq p\neq p_s}\frac{1}{2}\log\left(2-4\hat{\lambda}N_f+8\hat{\lambda}\left(m_\Phi^2+s_\Phi^2\right)-4\kappa\sum_\mu\cos(p_\mu)\right)}\times e^{\frac{1}{2}\log V}.\end{aligned}\tag{5.6}$$

The factor 1/2 in the exponents in the last line (coming from the squareroot in the line before) was missing in earlier results. However, it was correctly considered in the final result (5.7)
There might be an additional factor of 4 coming from the components of the bosonic field. this factor however can be absorbed in a constant that is neglected for the potential anyway. Is this correct?

So finally the *improved* zero-Order potential is given by:

$$
\begin{aligned}
U(m_\Phi, s_\Phi) = & -8\kappa\left(m_\Phi^2 - s_\Phi^2\right) + \left(m_\Phi^2 + s_\Phi^2\right) + \hat{\lambda}\left(m_\Phi^4 + s_\Phi^4 + 6m_\Phi^2 s_\Phi^2 - 2N_f\left(m_\Phi^2 + s_\Phi^2\right)\right) \\
& - \frac{2N_f}{V}\sum_p \log\Big[\left(|\nu^+(p)||\nu^+(\wp)| + \hat{y}^2\left(m_\Phi^2 - s_\Phi^2\right)|\gamma^+(p)||\gamma^+(\wp)|\right)^2 \\
& + m_\Phi^2 \hat{y}^2\left(|\gamma^+(p)||\nu^+(\wp)| - |\nu^+(p)||\gamma^+(\wp)|\right)^2\Big] \\
& - \frac{1}{2V}\sum_{0\neq p\neq p_s}\log\left(2 - 4\hat{\lambda}N_f + 8\hat{\lambda}\left(m_\Phi^2 + s_\Phi^2\right) - 4\kappa\sum_\mu \cos(p_\mu)\right)
\end{aligned} \tag{5.7}
$$

Alternative approach with a higher dimensional operator

Here I will show what enters the bosonic determinant if a $\lambda_6(\Phi^\dagger\Phi)^3$-term is included. The decomposition with the naive approach looks like:

$$
\begin{aligned}
\frac{\hat{\lambda}_6}{V^2}\sum_{p_1,\ldots p6}\delta_{p_1+p_3+p_5,p_2+p_4+p_6}\tilde{\Phi}_{p_1}^\dagger\tilde{\Phi}_{p_2}\tilde{\Phi}_{p_3}^\dagger\tilde{\Phi}_{p_4}\tilde{\Phi}_{p_5}^\dagger\tilde{\Phi}_{p_6} = & \, V\hat{\lambda}_6\cdot\left(m_\Phi^6 + s_\Phi^6 + 15\cdot\left(m_\Phi^4 s_\Phi^2 + m_\Phi^2 s_\Phi^4\right)\right) \\
& + \frac{\hat{\lambda}_6}{V^2}\widehat{\sum_{p_1,\ldots p6}}\delta_{p_1+p_3+p_5,p_2+p_4+p_6}\tilde{\Phi}_{p_1}^\dagger\tilde{\Phi}_{p_2}\tilde{\Phi}_{p_3}^\dagger\tilde{\Phi}_{p_4}\tilde{\Phi}_{p_5}^\dagger\tilde{\Phi}_{p_6}
\end{aligned} \tag{5.8}
$$

Here again one can take out a Gaussian contribution:

$$
\begin{aligned}
dummy = & \frac{\hat{\lambda}_6}{V^2}\sum_{p_1,\ldots p6}\delta_{p_1+p_3+p_5,p_2+p_4+p_6}\tilde{\Phi}_{p_1}^\dagger\tilde{\Phi}_{p_2}\tilde{\Phi}_{p_3}^\dagger\tilde{\Phi}_{p_4}\tilde{\Phi}_{p_5}^\dagger\tilde{\Phi}_{p_6} \\
= & \, V\hat{\lambda}_6\left(m_\Phi^6 + s_\Phi^6 + 15\left(m_\Phi^4 s_\Phi^2 + m_\Phi^2 s_\Phi^4\right)\right) \\
& + \frac{1}{2}\sum_{0\neq p\neq p_s}\tilde{\Phi}_p^\dagger\left[\hat{\lambda}_6\left(18\left(m_\Phi^4 + s_\Phi^4\right) + 108\, m_\Phi^2 s_\Phi^2\right)\right]\tilde{\Phi}_p \\
& + \frac{\hat{\lambda}_6}{V}\left(9\left(m_\Phi^2 + s_\Phi^2\right)\right)\widetilde{\sum_{p_1,\ldots p4}}\delta_{p_1+p_3,p_2+p_4}\tilde{\Phi}_{p_1}^\dagger\tilde{\Phi}_{p_2}\tilde{\Phi}_{p_3}^\dagger\tilde{\Phi}_{p_4} \\
& + \frac{\hat{\lambda}_6}{V^2}\widetilde{\sum_{p_1,\ldots p6}}\delta_{p_1+p_3+p_5,p_2+p_4+p_6}\tilde{\Phi}_{p_1}^\dagger\tilde{\Phi}_{p_2}\tilde{\Phi}_{p_3}^\dagger\tilde{\Phi}_{p_4}\tilde{\Phi}_{p_5}^\dagger\tilde{\Phi}_{p_6}.
\end{aligned} \tag{5.9}
$$

As before, this gives a non-trivial contribution to the bosonic determinant. So finally, the CEP tree-level is given by:

$$
\begin{aligned}
U(m_\Phi, s_\Phi) = & -8\kappa\left(m_\Phi^2 - s_\Phi^2\right) + \left(m_\Phi^2 + s_\Phi^2\right) + \hat{\lambda}\left(m_\Phi^4 + s_\Phi^4 + 6m_\Phi^2 s_\Phi^2 - 2N_f\left(m_\Phi^2 + s_\Phi^2\right)\right) \\
& + \hat{\lambda}_6\left(m_\Phi^6 + s_\Phi^6 + 15\left(m_\Phi^4 s_\Phi^2 + m_\Phi^2 s_\Phi^4\right)\right) \\
& - \frac{2N_f}{V}\sum_p \log\Big[\left(|\nu^+(p)||\nu^+(\wp)| + \hat{y}^2\left(m_\Phi^2 - s_\Phi^2\right)|\gamma^+(p)||\gamma^+(\wp)|\right)^2 \\
& + m_\Phi^2 \hat{y}^2\left(|\gamma^+(p)||\nu^+(\wp)| - |\nu^+(p)||\gamma^+(\wp)|\right)^2\Big] \\
& - \frac{1}{2V}\sum_{0\neq p\neq p_s}\log\left(2 - 4\hat{\lambda}N_f - 4\kappa\sum_\mu\cos(p_\mu) + 8\hat{\lambda}\left(m_\Phi^2 + s_\Phi^2\right) + 18\,\hat{\lambda}_6\left(m_\Phi^4 + s_\Phi^4 + 6\, m_\Phi^2 s_\Phi^2\right)\right)
\end{aligned} \tag{5.10}
$$

The expression with the reparametrizations (3.4) and (4.4) for the rescaled potential $\tilde{U} = U/N_f$ is straight foward.

$$
\begin{aligned}
\tilde{U}(\breve{m}_\Phi, \breve{s}_\Phi) = & -8\kappa \left(\breve{m}_\Phi^2 - \breve{s}_\Phi^2\right) + \left(\breve{m}_\Phi^2 + \breve{s}_\Phi^2\right) + \lambda_N \left(\breve{m}_\Phi^4 + \breve{s}_\Phi^4 + 6\breve{m}_\Phi^2 \breve{s}_\Phi^2 - 2\left(\breve{m}_\Phi^2 + \breve{s}_\Phi^2\right)\right) \\
& + \lambda_{6N} \left(\breve{m}_\Phi^6 + \breve{s}_\Phi^6 + 15\left(\breve{m}_\Phi^4 \breve{s}_\Phi^2 + \breve{m}_\Phi^2 \breve{s}_\Phi^4\right)\right) \\
& - \frac{2}{V} \sum_p \log \Big[\left(|\nu^+(p)||\nu^+(\wp)| + \hat{y}_N^2 \left(\breve{m}_\Phi^2 - \breve{s}_\Phi^2\right) |\gamma^+(p)||\gamma^+(\wp)|\right)^2 \\
& + \breve{m}_\Phi^2 y_N^2 \left(|\gamma^+(p)||\nu^+(\wp)| - |\nu^+(p)||\gamma^+(\wp)|\right)^2 \Big] \\
& - \frac{1}{2VN_f} \sum_{0 \neq p \neq p_s} \log \left(2 - 4\lambda_N - 4\kappa \sum_\mu \cos(p_\mu) + 8\lambda_N \left(\breve{m}_\Phi^2 + \breve{s}_\Phi^2\right) + 18\,\lambda_{6N} \left(\breve{m}_\Phi^4 + \breve{s}_\Phi^4 + 6\,\breve{m}_\Phi^2 \breve{s}_\Phi^2\right) \right)
\end{aligned}
\tag{5.11}
$$

6 First order in $\hat{\lambda}$ and $\hat{\lambda}_6$

Here I want do add the first order in $\hat{\lambda}$ and $\hat{\lambda}_6$ with the inclusion of the bosonic determinant as discussed in 5. In that case, the bosonic interaction part of the action is given by (collecting the necessary parts of eq. (5.2) and (5.9)):

$$
\begin{aligned}
S_{B,I} =& \frac{\hat{\lambda}}{V} \widetilde{\sum_{p_1 \ldots p_4}} \delta_{p+r,q+s} \tilde{\Phi}_p^\dagger \tilde{\Phi}_q \tilde{\Phi}_r^\dagger \tilde{\Phi}_s \\
&+ \frac{\hat{\lambda}_6}{V} \left(9\left(m_\Phi^2 + s_\Phi^2\right)\right) \widetilde{\sum_{p_1,\ldots p4}} \delta_{p_1+p_3,p_2+p_4} \tilde{\Phi}_{p_1}^\dagger \tilde{\Phi}_{p_2} \tilde{\Phi}_{p_3}^\dagger \tilde{\Phi}_{p_4} + \frac{\hat{\lambda}_6}{V^2} \widetilde{\sum_{p_1,\ldots p6}} \delta_{p_1+p_3+p_5,p_2+p_4+p_6} \tilde{\Phi}_{p_1}^\dagger \tilde{\Phi}_{p_2} \tilde{\Phi}_{p_3}^\dagger \tilde{\Phi}_{p_4} \tilde{\Phi}_{p_5}^\dagger \tilde{\Phi}_{p_6}, \\
=& \frac{\hat{\lambda} + \hat{\lambda}_6 \left(9\left(m_\Phi^2 + s_\Phi^2\right)\right)}{V} \widetilde{\sum_{p_1,\ldots p4}} \delta_{p_1+p_3,p_2+p_4} \tilde{\Phi}_{p_1}^\dagger \tilde{\Phi}_{p_2} \tilde{\Phi}_{p_3}^\dagger \tilde{\Phi}_{p_4} \\
&+ \frac{\hat{\lambda}_6}{V^2} \widetilde{\sum_{p_1,\ldots p6}} \delta_{p_1+p_3+p_5,p_2+p_4+p_6} \tilde{\Phi}_{p_1}^\dagger \tilde{\Phi}_{p_2} \tilde{\Phi}_{p_3}^\dagger \tilde{\Phi}_{p_4} \tilde{\Phi}_{p_5}^\dagger \tilde{\Phi}_{p_6}.
\end{aligned}
\tag{6.1}
$$

As a reminder: The tilde above the sum indicates, that none of the momenta in the sum is the zero or staggered mode. The contraction of the scalar fields is given by:

$$
\Phi_p^\dagger \Phi_q = \frac{4 \cdot \delta_{p,q}}{2 - 4\hat{\lambda}N_f - 4\kappa \sum_\mu \cos(p_\mu) + 8\,\hat{\lambda}\,(m_\Phi^2 + s_\Phi^2) + 18\,\hat{\lambda}_6\,(m_\Phi^4 + s_\Phi^4 + 6\,m_\Phi^2 s_\Phi^2)}. \tag{6.2}
$$

To clean the notation a little bit, the propagator sum will now be given by:

$$
\tilde{P}_B \equiv \frac{1}{V} \sum_{0 \neq p \neq p_s} \frac{1}{2 - 4\hat{\lambda}N_f - 4\kappa \sum_\mu \cos(p_\mu) + 8\,\hat{\lambda}\,(m_\Phi^2 + s_\Phi^2) + 18\,\hat{\lambda}_6\,(m_\Phi^4 + s_\Phi^4 + 6\,m_\Phi^2 s_\Phi^2)}. \tag{6.3}
$$

If one now repeats the steps leading to (2.6), one finds:

$$
\begin{aligned}
\int \left[\prod_{0 \neq p \neq p_s} d\tilde{\Phi}_p\right] e^{-S_{B,I}[\Phi] + S_{B,0}[\Phi]} =& \sqrt{\frac{V}{\det \mathcal{B}}} \cdot \left(1 - \frac{\hat{\lambda} + \hat{\lambda}_6 \left(9\left(m_\Phi^2 + s_\Phi^2\right)\right)}{V} \left(\tilde{C}_{p,q}\right) - \frac{\hat{\lambda}_6}{V^2} \left(\tilde{C}_{p,q,r}\right)\right) \\
=& \sqrt{\frac{V}{\det \mathcal{B}}} \cdot e^{-\left\{\frac{\hat{\lambda} + \hat{\lambda}_6 \left(9\left(m_\Phi^2 + s_\Phi^2\right)\right)}{V} \left(\tilde{C}_{p,q}\right) + \frac{\hat{\lambda}_6}{V^2} \left(\tilde{C}_{p,q,r}\right)\right\}}.
\end{aligned}
\tag{6.4}
$$

The $\tilde{C}$ are given by: **Here was a serious mistake! The factor of 4 for the contractions was only counted once instead of $4^{\#\text{contractions}}$**

$$
\begin{aligned}
\tilde{C}_{p,q} &= \widetilde{\sum_{p,q}} \left(\tilde{\Phi}_p^\dagger \tilde{\Phi}_p \tilde{\Phi}_q^\dagger \tilde{\Phi}_q + \tilde{\Phi}_p^\dagger \tilde{\Phi}_q \tilde{\Phi}_q^\dagger \tilde{\Phi}_p\right) \\
&= 2 \cdot 4^2\, V^2\, \tilde{P}_B^2,
\end{aligned}
\tag{6.5}
$$

$$
\begin{aligned}
\tilde{C}_{p,q,r} =& \widetilde{\sum_{p,q,r}} \Big(\tilde{\Phi}_p^\dagger \tilde{\Phi}_p \tilde{\Phi}_q^\dagger \tilde{\Phi}_q \tilde{\Phi}_r^\dagger \tilde{\Phi}_r + \tilde{\Phi}_p^\dagger \tilde{\Phi}_p \tilde{\Phi}_q^\dagger \tilde{\Phi}_r \tilde{\Phi}_r^\dagger \tilde{\Phi}_q + \tilde{\Phi}_p^\dagger \tilde{\Phi}_q \tilde{\Phi}_q^\dagger \tilde{\Phi}_p \tilde{\Phi}_r^\dagger \tilde{\Phi}_r \\
&+ \tilde{\Phi}_p^\dagger \tilde{\Phi}_q \tilde{\Phi}_r^\dagger \tilde{\Phi}_p \tilde{\Phi}_q^\dagger \tilde{\Phi}_r + \tilde{\Phi}_p^\dagger \tilde{\Phi}_q \tilde{\Phi}_q^\dagger \tilde{\Phi}_r \tilde{\Phi}_r^\dagger \tilde{\Phi}_p + \tilde{\Phi}_p^\dagger \tilde{\Phi}_q \tilde{\Phi}_r^\dagger \tilde{\Phi}_r \tilde{\Phi}_q^\dagger \tilde{\Phi}_p\Big) \\
=& 6 \cdot 4^3\, V^3\, \tilde{P}_B^3.
\end{aligned}
\tag{6.6}
$$

If one also includes the contribution coming fro the first order in $\hat{\lambda}$ and $\hat{\lambda}_6$ to the CEP (5.10) or (5.11)the potential changes to:

$$
U(m_\Phi, s_\Phi) \to U(m_\Phi, s_\Phi) + 32\left(\hat{\lambda} + \hat{\lambda}_6\left(9\left(m_\Phi^2 + s_\Phi^2\right)\right)\right)\tilde{P}_B^2 + 384\,\hat{\lambda}_6 \tilde{P}_B^3, \tag{6.7}
$$

$$
\tilde{U}(\check{m}_\Phi, \check{s}_\Phi) \to \tilde{U}(\check{m}_\Phi, \check{s}_\Phi) + \frac{32}{N_f^2}\left(\lambda_N + \lambda_{6N}\left(9\left(\check{m}_\Phi^2 + \check{s}_\Phi^2\right)\right)\right)\tilde{P}_B^2 + \frac{384}{N_f^3}\lambda_{6N}\tilde{P}_B^3. \tag{6.8}
$$

7 The potential and its derivatives

Here I just summerize the potential and it's derivatives for future use. Starting point is the expression (5.10), with the possible addition of the first order in $\hat{\lambda}$ and $\hat{\lambda}_6$ given in (6.7) **There is a mistake for the 1st order contribution in λ and λ_6. There is a factor of 4 missing for the $\tilde{P}_B^2$ and a factor of 16 in the $\tilde{P}_B^3$ term missing**

$$
\begin{aligned}
U(m_\Phi, s_\Phi) &= U^{\text{tree}}(m_\Phi, s_\Phi) + U^{\text{ferm}}(m_\Phi, s_\Phi) + U^{\text{BosDet}}(m_\Phi, s_\Phi) + U^{\text{1st}}(m_\Phi, s_\Phi) && (7.1)\\
U^{\text{tree}}(m_\Phi, s_\Phi) &= -8\kappa\left(m_\Phi^2 - s_\Phi^2\right) + \left(m_\Phi^2 + s_\Phi^2\right) + \hat{\lambda}\left(m_\Phi^4 + s_\Phi^4 + 6m_\Phi^2 s_\Phi^2 - 2N_f\left(m_\Phi^2 + s_\Phi^2\right)\right)\\
&\quad + \hat{\lambda}_6\left(m_\Phi^6 + s_\Phi^6 + 15\left(m_\Phi^4 s_\Phi^2 + m_\Phi^2 s_\Phi^4\right)\right) && (7.2)\\
U^{\text{ferm}}(m_\Phi, s_\Phi) &= -\frac{2N_f}{V}\sum_p \log\left[\left(|\nu^+(p)||\nu^+(\wp)| + \hat{y}^2\left(m_\Phi^2 - s_\Phi^2\right)|\gamma^+(p)||\gamma^+(\wp)|\right)^2\right.\\
&\quad \left. + m_\Phi^2 \hat{y}^2\left(|\gamma^+(p)||\nu^+(\wp)| - |\nu^+(p)||\gamma^+(\wp)|\right)^2\right] && (7.3)\\
U^{\text{BosDet}}(m_\Phi, s_\Phi) &= -\frac{1}{2V}\sum_{0\neq p\neq p_s} \log\Big(2 - 4\hat{\lambda}N_f - 4\kappa\sum_\mu \cos(p_\mu) + 8\hat{\lambda}\left(m_\Phi^2 + s_\Phi^2\right) + 18\,\hat{\lambda}_6\left(m_\Phi^4 + s_\Phi^4 + 6\,m_\Phi^2 s_\Phi^2\right)\Big)\\
&&& (7.4)\\
U^{\text{1st}}(m_\Phi, s_\Phi) &= 8\,\left(\hat{\lambda} + \hat{\lambda}_6\left(9\left(m_\Phi^2 + s_\Phi^2\right)\right)\right)\tilde{P}_B^2 + 24\,\hat{\lambda}_6 \tilde{P}_B^3, && (7.5)
\end{aligned}
$$

Tree-level

In the follwoing the 1st and 2nd derivatives w.r.t. m_Φ and s_Φ are given. First the tree-level:

$$
\begin{aligned}
\partial_{m_\Phi} U^{\text{tree}}(m_\Phi, s_\Phi) &= -16\,\kappa\, m_\Phi + 2\,m_\Phi + \hat{\lambda}\left(4\,m_\Phi^3 + 12\,m_\Phi\,s_\Phi^2 - 4\,N_f\,m_\Phi\right)\\
&\quad + \hat{\lambda}_6\left(30\,m_\Phi^5 + 60\,m_\Phi^3\,s_\Phi^2 + 30\,m_\Phi\,s_\Phi^4\right) && (7.6)\\
\partial_{s_\Phi} U^{\text{tree}}(m_\Phi, s_\Phi) &= +16\,\kappa\, s_\Phi + 2\,s_\Phi + \hat{\lambda}\left(4\,s_\Phi^3 + 12\,m_\Phi^2\,s_\Phi - 4\,N_f\,s_\Phi\right)\\
&\quad + \hat{\lambda}_6\left(30\,s_\Phi^5 + 30\,m_\Phi^4\,s_\Phi + 60\,m_\Phi^2\,s_\Phi^3\right) && (7.7)\\
\left(\partial_{m_\Phi}\right)^2 U^{\text{tree}}(m_\Phi, s_\Phi) &= -16\,\kappa + 2 + \hat{\lambda}\left(12\,m_\Phi^2 + 12\,s_\Phi^2 - 4\,N_f\right) + \hat{\lambda}_6\left(30\,m_\Phi^4 + 180\,m_\Phi^2\,s_\Phi^2 + 30\,s_\Phi^4\right) && (7.8)\\
\left(\partial_{s_\Phi}\right)^2 U^{\text{tree}}(m_\Phi, s_\Phi) &= +16\,\kappa + 2 + \hat{\lambda}\left(12\,s_\Phi^2 + 12\,m_\Phi^2 - 4\,N_f\right) + \hat{\lambda}_6\left(30\,m_\Phi^4 + 180\,m_\Phi^2\,s_\Phi^2 + 30\,s_\Phi^4\right) && (7.9)\\
\partial_{m_\Phi}\partial_{s_\Phi} U^{\text{tree}}(m_\Phi, s_\Phi) &= \hat{\lambda}\left(24\,m_\Phi\,s_\Phi\right) + \hat{\lambda}_6\left(120\left(m_\Phi^3\,s_\Phi + m_\Phi\,s_\Phi^3\right)\right) && (7.10)
\end{aligned}
$$

Fermionic contribution

For the derivatives of the fermionic contributions some shortcuts will be used:

$$
\begin{aligned}
U^{\text{ferm}}(m_\Phi, s_\Phi) &= -\frac{2N_f}{V}\sum_p \log\left[A_p^{m,s}\right], && (7.11)\\
A_p^{m,s} &= a_p^{m,s\,2} + m_\Phi^2 \hat{y}^2 \cdot b_p^2, && (7.12)\\
a_p^{m,s} &= |\nu^+(p)||\nu^+(\wp)| + \hat{y}^2\left(m_\Phi^2 - s_\Phi^2\right)|\gamma^+(p)||\gamma^+(\wp)|, && (7.13)\\
b_p &= |\gamma^+(p)||\nu^+(\wp)| - |\nu^+(p)||\gamma^+(\wp)|, && (7.14)\\
g_p &= |\gamma^+(p)||\gamma^+(\wp)| && (7.15)
\end{aligned}
$$

so tht the they can be written in a more compact form:

$$
\begin{aligned}
\partial_{m_\Phi} U^{\text{ferm}}(m_\Phi, s_\Phi) &= -\frac{2N_f}{V}\sum_p \frac{4m_\Phi \hat{y}^2 \cdot g_p \cdot a_p^{m,s} + 2m_\Phi \hat{y}^2 \cdot b_p^2}{A_p^{m,s}}\\
&= -\frac{2N_f}{V} \cdot 2m_\Phi \hat{y}^2 \cdot \sum_p \frac{2 \cdot g_p \cdot a_p^{m,s} + b_p^2}{A_p^{m,s}} && (7.16)
\end{aligned}
$$

$$
\begin{aligned}
\partial_{s_\Phi} U^{\text{ferm}}(m_\Phi, s_\Phi) &= -\frac{2N_f}{V}\sum_p \frac{-4s_\Phi \hat{y}^2 \cdot g_p \cdot a_p^{m,s}}{A_p^{m,s}}\\
&= -\frac{2N_f}{V} \cdot \left(-4s_\Phi \hat{y}^2\right) \cdot \sum_p \frac{g_p \cdot a_p^{m,s}}{A_p^{m,s}} && (7.17)
\end{aligned}
$$

$$
\begin{aligned}
(\partial_{m_\Phi})^2 U^{\text{ferm}}(m_\Phi, s_\Phi) &= -\frac{2N_f}{V} \cdot 2\hat{y}^2 \sum_p \left[\frac{2 \cdot g_p \cdot a_p^{m,s} + b_p^2}{A_p^{m,s}} + m_\Phi \cdot \frac{4m_\Phi \hat{y}^2 \cdot (g_p)^2}{A_p^{m,s}} \right. \\
&\quad \left. - m_\Phi \cdot \frac{2m_\Phi \hat{y}^2 \cdot \left(2 \cdot g_p \cdot a_p^{m,s} + b_p^2\right)^2}{A_p^{m,s2}} \right] \\
&= -\frac{2N_f}{V} \cdot 2\hat{y}^2 \sum_p \left[\frac{2 \cdot g_p \cdot a_p^{m,s} + b_p^2}{A_p^{m,s}} \cdot \left(1 - \frac{2m_\Phi^2 \hat{y}^2}{A_p^{m,s}}\right) + \frac{4m_\Phi^2 \hat{y}^2 \cdot g_p^2}{A_p^{m,s}} \right] \qquad (7.18)
\end{aligned}
$$

$$
\begin{aligned}
(\partial_{s_\Phi})^2 U^{\text{ferm}}(m_\Phi, s_\Phi) &= -\frac{2N_f}{V} \cdot (-4\hat{y}^2) \sum_p \left[\frac{g_p \cdot a_p^{m,s}}{A_p^{m,s}} + s_\Phi \frac{-2s_\Phi \hat{y}^2 \cdot g_p^2}{A_p^{m,s}} - s_\Phi \frac{-4s_\Phi \hat{y}^2 \cdot \left(g_p \cdot a_p^{m,s}\right)^2}{A_p^{m,s2}} \right] \\
&= -\frac{2N_f}{V} \cdot (-4\hat{y}^2) \sum_p \left[\frac{g_p}{A_p^{m,s}} \cdot \left(a_p^{m,s} + 2s_\Phi^2 \hat{y}^2 g_p \cdot \left(\frac{2 \cdot a_p^{m,s2}}{A_p^{m,s}} - 1 \right) \right) \right] \qquad (7.19)
\end{aligned}
$$

$$
\begin{aligned}
\partial_{m_\Phi} \partial_{s_\Phi} U^{\text{ferm}}(m_\Phi, s_\Phi) &= -\frac{2N_f}{V} \cdot (-4s_\Phi \hat{y}^2) \sum_p \left[\frac{2m_\Phi \hat{y}^2 g_p^2}{A_p^{m,s}} \right. \\
&\quad \left. - \frac{2m_\Phi \hat{y}^2 \cdot g_p \cdot a_p^{m,s} \cdot \left(2 \cdot g_p \cdot a_p^{m,s} + b_p^2\right)}{A_p^{m,s2}} \right] \\
&= -\frac{2N_f}{V} \cdot (-8m_\Phi s_\Phi \hat{y}^4) \sum_p \left[\frac{g_p}{A_p^{m,s2}} \cdot \left(g_p - \frac{a_p^{m,s} \cdot \left(2 \cdot g_p \cdot a_p^{m,s} + b_p^2\right)}{A_p^{m,s}} \right) \right] \qquad (7.20)
\end{aligned}
$$

Bosonic determinant

For the part coming from the bosonic determinant I use the shortcut:

$$
\begin{aligned}
U^{\text{BosDet}}(m_\Phi, s_\Phi) &= -\frac{1}{2V} \sum_{0 \neq p \neq p_s} \log\left[D_p^{m,s}\right], \\
D_p^{m,s} &= 2 - 4\hat{\lambda} N_f - 4\kappa \sum_\mu \cos(p_\mu) + 8\,\hat{\lambda}\left(m_\Phi^2 + s_\Phi^2\right) + 18\,\hat{\lambda}_6\left(m_\Phi^4 + s_\Phi^4 + 6\,m_\Phi^2 s_\Phi^2\right). \qquad (7.21) \\
\partial_{m_\Phi} D_p^{m,s} &= 16\,\hat{\lambda}\, m_\Phi + 72\,\hat{\lambda}_6\left(m_\Phi^3 + 3\,m_\Phi s_\Phi^2\right), \qquad (7.22) \\
(\partial_{m_\Phi})^2 D_p^{m,s} &= 16\,\hat{\lambda} + 216\,\hat{\lambda}_6\left(m_\Phi^2 + s_\Phi^2\right), \qquad (7.23) \\
\partial_{m_\Phi} \partial_{s_\Phi} D_p^{m,s} &= 432\,\hat{\lambda}_6\, m_\Phi\, s_\Phi. \qquad (7.24)
\end{aligned}
$$

Note, that the corresponding derivatives of $D_p^{m,s}$ with respect to s_Φ can be obtained by exchanging m_Φ and $_\Phi$. Further, a generalization of the bosonic propagator sum (6.3) might be useful:

$$
\tilde{P}_{B^n} \equiv \frac{1}{V} \sum_{0 \neq p \neq p_s} \left(\frac{1}{D_p^{m,s}}\right)^n \qquad (7.25)
$$

$$
\partial_{m_\Phi / s_\Phi} \tilde{P}_{B^n} = n\left(\partial_{m_\Phi / s_\Phi} D_p^{m,s}\right) \tilde{P}_{B^{n+1}}, \qquad (7.26)
$$

Since U^{BosDet} is completely symmetric in m_Φ and s_Φ, so I'll only write the ones w.r.t. m_Φ:

$$
\begin{aligned}
\partial_{m_\Phi} U^{\text{BosDet}}(m_\Phi, s_\Phi) &= -\frac{1}{2}\left(\partial_{m_\Phi} D_p^{m,s}\right) \tilde{P}_B \qquad (7.27) \\
(\partial_{m_\Phi})^2 U^{\text{BosDet}}(m_\Phi, s_\Phi) &= -\frac{1}{2}\left[\left((\partial_{m_\Phi})^2 D_p^{m,s}\right) \tilde{P}_B - \left(\partial_{m_\Phi} D_p^{m,s}\right)^2 \tilde{P}_{B^2}\right] \qquad (7.28) \\
\partial_{m_\Phi} \partial_{s_\Phi} U^{\text{BosDet}}(m_\Phi, s_\Phi) &= -\frac{1}{2}\left[\left(\partial_{m_\Phi} \partial_{s_\Phi} D_p^{m,s}\right) \tilde{P}_B - \left(\partial_{m_\Phi} D_p^{m,s}\right)\left(\partial_{s_\Phi} D_p^{m,s}\right) \tilde{P}_{B^2}\right] \qquad (7.29)
\end{aligned}
$$

First order Contribution

$$U^{1\mathrm{st}}(m_\Phi, s_\Phi) \equiv \alpha^{m,s}\,\tilde{P}_B^2 + \beta\,\tilde{P}_B^3, \tag{7.30}$$

$$\alpha^{m,s} = 8\left(\hat{\lambda} + \hat{\lambda}_6\left(9\left(m_\Phi^2 + s_\Phi^2\right)\right)\right) \tag{7.31}$$

$$\partial_{m_\Phi}\alpha^{m,s} = 144\,\hat{\lambda}_6\,m_\Phi \tag{7.32}$$

$$\left(\partial_{m_\Phi}\right)^2\alpha^{m,s} = 144\,\hat{\lambda}_6 \tag{7.33}$$

$$\beta = 24\,\hat{\lambda}_6 \tag{7.34}$$

Since the first order contribution is symmetric in m_Φ and s_Φ, only the derivatives w.r.t. $m_\Phi$are given.

$$\partial_{m_\Phi}U^{1\mathrm{st}}(m_\Phi, s_\Phi) = 144\,\hat{\lambda}_6\,m_\Phi\,\tilde{P}_B^2 - 2\,\alpha^{m,s}\left(\partial_{m_\Phi}D_p^{m,s}\right)\tilde{P}_B\,\tilde{P}_{B^2} - 3\,\beta\left(\partial_{m_\Phi}D_p^{m,s}\right)\tilde{P}_B^2\,\tilde{P}_{B^2} \tag{7.35}$$

$$\begin{aligned}\left(\partial_{m_\Phi}\right)^2U^{1\mathrm{st}}(m_\Phi, s_\Phi) = {}& 144\,\hat{\lambda}_6\,\tilde{P}_B^2 - \left[576\,\hat{\lambda}_6\,m_\Phi\left(\partial_{m_\Phi}D_p^{m,s}\right) + 2\,\alpha^{m,s}\left(\left(\partial_{m_\Phi}\right)^2D_p^{m,s}\right)\right]\tilde{P}_B\,\tilde{P}_{B^2}\\ &+ 2\,\alpha^{m,s}\left(\partial_{m_\Phi}D_p^{m,s}\right)^2\left[\tilde{P}_{B^2}^2 + 2\,\tilde{P}_B\,\tilde{P}_{B^3}\right]\\ &- 3\,\beta\left(\left(\partial_{m_\Phi}\right)^2D_p^{m,s}\right)\tilde{P}_B^2\,\tilde{P}_{B^2} + 6\,\beta\left(\partial_{m_\Phi}D_p^{m,s}\right)^2\left[\tilde{P}_B\,\tilde{P}_{B^2}^2 + \tilde{P}_B^2\,\tilde{P}_{B^3}\right]\end{aligned} \tag{7.36}$$

$$\begin{aligned}\partial_{m_\Phi}\partial_{s_\Phi}U^{1\mathrm{st}}(m_\Phi, s_\Phi) = {}& \left[-288\,\hat{\lambda}_6\left(m_\Phi\left(\partial_{s_\Phi}D_p^{m,s}\right) + s_\Phi\left(\partial_{m_\Phi}D_p^{m,s}\right)\right) - 2\,\alpha^{m,s}\left(\partial_{m_\Phi}\partial_{s_\Phi}D_p^{m,s}\right)\right]\tilde{P}_B\,\tilde{P}_{B^2}\\ &+ 2\,\alpha^{m,s}\left(\partial_{m_\Phi}D_p^{m,s}\right)\left(\partial_{s_\Phi}D_p^{m,s}\right)\left[\tilde{P}_{B^2}^2 + 2\,\tilde{P}_B\,\tilde{P}_{B^3}\right]\\ &- 3\,\beta\left(\partial_{m_\Phi}\partial_{s_\Phi}D_p^{m,s}\right)\tilde{P}_B^2\,\tilde{P}_{B^2} + 6\,\beta\left(\partial_{m_\Phi}D_p^{m,s}\right)\left(\partial_{s_\Phi}D_p^{m,s}\right)\left[\tilde{P}_B\,\tilde{P}_{B^2}^2 + \tilde{P}_B^2\,\tilde{P}_{B^3}\right]\end{aligned} \tag{7.37}$$

8 The potential and its derivatives in the rescaled formulation

To get rid of some confusion, I will here write the corresponding equations to those from 7 with the reparametrizations (3.4) and (4.4) included. The potential itself is then given by: **There is a mistake for the 1st order contribution in λ and λ_6. There is a factor of 4 missing for the $\tilde{P}_B^2$ and a factor of 16 in the $\tilde{P}_B^3$ term missing**

$$\tilde{U}(\breve{m}_\Phi, \breve{s}_\Phi) = \tilde{U}^{\mathrm{tree}}(\breve{m}_\Phi, \breve{s}_\Phi) + \tilde{U}^{\mathrm{ferm}}(\breve{m}_\Phi, \breve{s}_\Phi) + \tilde{U}^{\mathrm{BosDet}}(\breve{m}_\Phi, \breve{s}_\Phi) + \tilde{U}^{\mathrm{1st}}(\breve{m}_\Phi, \breve{s}_\Phi) \tag{8.1}$$

$$\begin{aligned}\tilde{U}^{\mathrm{tree}}(\breve{m}_\Phi, \breve{s}_\Phi) &= -8\,\kappa\left(\breve{m}_\Phi^2 - \breve{s}_\Phi^2\right) + \left(\breve{m}_\Phi^2 + \breve{s}_\Phi^2\right) + \lambda_N\left(\breve{m}_\Phi^4 + \breve{s}_\Phi^4 + 6\breve{m}_\Phi^2\breve{s}_\Phi^2 - 2\left(\breve{m}_\Phi^2 + \breve{s}_\Phi^2\right)\right) \\ &\quad + \lambda_{6N}\left(\breve{m}_\Phi^6 + \breve{s}_\Phi^6 + 15\left(\breve{m}_\Phi^4\breve{s}_\Phi^2 + \breve{m}_\Phi^2\breve{s}_\Phi^4\right)\right)\end{aligned} \tag{8.2}$$

$$\begin{aligned}\tilde{U}^{\mathrm{ferm}}(\breve{m}_\Phi, \breve{s}_\Phi) &= -\frac{2}{V}\sum_p \log\Big[\left(|\nu^+(p)||\nu^+(\wp)| + y_N^2\left(\breve{m}_\Phi^2 - \breve{s}_\Phi^2\right)g_p\right)^2 \\ &\quad + \breve{m}_\Phi^2 y_N^2\left(|\gamma^+(p)||\nu^+(\wp)| - |\nu^+(p)||\gamma^+(\wp)|\right)^2\Big]\end{aligned} \tag{8.3}$$

$$\begin{aligned}\tilde{U}^{\mathrm{BosDet}}(\breve{m}_\Phi, \breve{s}_\Phi) &= -\frac{1}{2\,N_f\,V}\sum_{0\neq p\neq p_s}\log\Big[2 - 4\lambda_N - 4\kappa\sum_\mu\cos(p_\mu) \\ &\quad + 8\lambda_N\left(\breve{m}_\Phi^2 + \breve{s}_\Phi^2\right) + 18\,\lambda_{6N}\left(\breve{m}_\Phi^4 + \breve{s}_\Phi^4 + 6\,\breve{m}_\Phi^2\breve{s}_\Phi^2\right)\Big]\end{aligned} \tag{8.4}$$

$$\tilde{U}^{\mathrm{1st}}(\breve{m}_\Phi, \breve{s}_\Phi) = \frac{8}{N_f^2}\left(\lambda_N + \lambda_{6N}\left(9\left(\breve{m}_\Phi^2 + \breve{s}_\Phi^2\right)\right)\right)\tilde{P}_B^2 + 24\,\frac{\lambda_{6N}}{N_f^3}\tilde{P}_B^3, \tag{8.5}$$

Tree-level

$$\begin{aligned}\partial_{\breve{m}_\Phi}\tilde{U}^{\mathrm{tree}}(\breve{m}_\Phi, \breve{s}_\Phi) &= -16\,\kappa\,\breve{m}_\Phi + 2\,\breve{m}_\Phi + \lambda_N\left(4\,\breve{m}_\Phi^3 + 12\,\breve{m}_\Phi\,\breve{s}_\Phi^2 - 4\,\breve{m}_\Phi\right) \\ &\quad + \lambda_{N6}\left(30\,\breve{m}_\Phi^5 + 60\,\breve{m}_\Phi^3\,\breve{s}_\Phi^2 + 30\,\breve{m}_\Phi\,\breve{s}_\Phi^4\right)\end{aligned} \tag{8.6}$$

$$\begin{aligned}\partial_{\breve{s}_\Phi}\tilde{U}^{\mathrm{tree}}(\breve{m}_\Phi, \breve{s}_\Phi) &= +16\,\kappa\,\breve{s}_\Phi + 2\,\breve{s}_\Phi + \lambda_N\left(4\,\breve{s}_\Phi^3 + 12\,\breve{m}_\Phi^2\,\breve{s}_\Phi - 4\,\breve{s}_\Phi\right) \\ &\quad + \lambda_{N6}\left(30\,\breve{s}_\Phi^5 + 30\,\breve{m}_\Phi^4\,\breve{s}_\Phi + 60\,\breve{m}_\Phi^2\,\breve{s}_\Phi^3\right)\end{aligned} \tag{8.7}$$

$$\left(\partial_{\breve{m}_\Phi}\right)^2\tilde{U}^{\mathrm{tree}}(\breve{m}_\Phi, \breve{s}_\Phi) = -16\,\kappa + 2 + \lambda_N\left(12\,\breve{m}_\Phi^2 + 12\,\breve{s}_\Phi^2 - 4\right) + \lambda_{N6}\left(30\,\breve{m}_\Phi^4 + 180\,\breve{m}_\Phi^2\,\breve{s}_\Phi^2 + 30\,\breve{s}_\Phi^4\right) \tag{8.8}$$

$$\left(\partial_{\breve{s}_\Phi}\right)^2\tilde{U}^{\mathrm{tree}}(\breve{m}_\Phi, \breve{s}_\Phi) = +16\,\kappa + 2 + \lambda_N\left(12\,\breve{s}_\Phi^2 + 12\,\breve{m}_\Phi^2 - 4\right) + \lambda_{N6}\left(30\,\breve{m}_\Phi^4 + 180\,\breve{m}_\Phi^2\,\breve{s}_\Phi^2 + 30\,\breve{s}_\Phi^4\right) \tag{8.9}$$

$$\partial_{\breve{m}_\Phi}\partial_{\breve{s}_\Phi}\tilde{U}^{\mathrm{tree}}(\breve{m}_\Phi, \breve{s}_\Phi) = \lambda_N\left(24\,\breve{m}_\Phi\,\breve{s}_\Phi\right) + \lambda_{N6}\left(120\left(\breve{m}_\Phi^3\,\breve{s}_\Phi + \breve{m}_\Phi\,\breve{s}_\Phi^3\right)\right) \tag{8.10}$$

Fermionic contribution

For the derivatives of the fermionic contributions the shortcuts change slightly and N_f does not appear in the overall numerator anymore. Further, the number for $A_p^{m,s}$ is th same, since the rescaling of $\hat{y}$ and m_Φ/s_Φ cancel:

$$\tilde{U}^{\mathrm{ferm}}(\breve{m}_\Phi, \breve{s}_\Phi) = -\frac{2}{V}\sum_p\log\left[A_p^{m,s}\right], \tag{8.11}$$

$$A_p^{m,s} = a_p^{m,s\,2} + \breve{m}_\Phi^2\hat{y}^2\cdot b_p^2, \tag{8.12}$$

$$a_p^{m,s} = |\nu^+(p)||\nu^+(\wp)| + \hat{y}^2\left(\breve{m}_\Phi^2 - \breve{s}_\Phi^2\right)|\gamma^+(p)||\gamma^+(\wp)|, \tag{8.13}$$

For b_p and g_p nothing changes at all.

$$\begin{aligned}\partial_{\breve{m}_\Phi}\tilde{U}^{\mathrm{ferm}}(\breve{m}_\Phi, \breve{s}_\Phi) &= -\frac{2}{V}\sum_p\frac{4\breve{m}_\Phi\hat{y}^2\cdot g_p\cdot a_p^{m,s} + 2\breve{m}_\Phi\hat{y}^2\cdot b_p^2}{A_p^{m,s}} \\ &= -\frac{2}{V}\cdot 2\breve{m}_\Phi\hat{y}^2\cdot\sum_p\frac{2\cdot g_p\cdot a_p^{m,s} + b_p^2}{A_p^{m,s}}\end{aligned} \tag{8.14}$$

$$\begin{aligned}\partial_{\breve{s}_\Phi}\tilde{U}^{\mathrm{ferm}}(\breve{m}_\Phi, \breve{s}_\Phi) &= -\frac{2}{V}\sum_p\frac{-4\breve{s}_\Phi\hat{y}^2\cdot g_p\cdot a_p^{m,s}}{A_p^{m,s}} \\ &= -\frac{2}{V}\cdot\left(-4\breve{s}_\Phi\hat{y}^2\right)\cdot\sum_p\frac{g_p\cdot a_p^{m,s}}{A_p^{m,s}}\end{aligned} \tag{8.15}$$

$$
\begin{aligned}
\left(\partial_{\breve{m}_\Phi}\right)^2 \tilde{U}^{\text{ferm}}(\breve{m}_\Phi, \breve{s}_\Phi) &= -\frac{2}{V} \cdot 2\hat{y}^2 \sum_p \left[\frac{2 \cdot g_p \cdot a_p^{m,s} + b_p^2}{A_p^{m,s}} + \breve{m}_\Phi \cdot \frac{4\breve{m}_\Phi \hat{y}^2 \cdot (g_p)^2}{A_p^{m,s}} \right. \\
&\qquad \left. - \breve{m}_\Phi \cdot \frac{2\breve{m}_\Phi \hat{y}^2 \cdot \left(2 \cdot g_p \cdot a_p^{m,s} + b_p^2\right)^2}{A_p^{m,s2}} \right] \\
&= -\frac{2}{V} \cdot 2\hat{y}^2 \sum_p \left[\frac{2 \cdot g_p \cdot a_p^{m,s} + b_p^2}{A_p^{m,s}} \cdot \left(1 - \frac{2\breve{m}_\Phi^2 \hat{y}^2}{A_p^{m,s}}\right) + \frac{4\breve{m}_\Phi^2 \hat{y}^2 \cdot g_p^2}{A_p^{m,s}} \right] \qquad (8.16)
\end{aligned}
$$

$$
\begin{aligned}
\left(\partial_{\breve{s}_\Phi}\right)^2 \tilde{U}^{\text{ferm}}(\breve{m}_\Phi, \breve{s}_\Phi) &= -\frac{2}{V} \cdot (-4\hat{y}^2) \sum_p \left[\frac{g_p \cdot a_p^{m,s}}{A_p^{m,s}} + \breve{s}_\Phi \frac{-2\breve{s}_\Phi \hat{y}^2 \cdot g_p^2}{A_p^{m,s}} - \breve{s}_\Phi \frac{-4\breve{s}_\Phi \hat{y}^2 \cdot \left(g_p \cdot a_p^{m,s}\right)^2}{A_p^{m,s2}} \right] \\
&= -\frac{2}{V} \cdot (-4\hat{y}^2) \sum_p \left[\frac{g_p}{A_p^{m,s}} \cdot \left(a_p^{m,s} + 2\breve{s}_\Phi^2 \hat{y}^2 g_p \cdot \left(\frac{2 \cdot a_p^{m,s2}}{A_p^{m,s}} - 1 \right) \right) \right] \qquad (8.17)
\end{aligned}
$$

$$
\begin{aligned}
\partial_{\breve{m}_\Phi} \partial_{\breve{s}_\Phi} \tilde{U}^{\text{ferm}}(\breve{m}_\Phi, \breve{s}_\Phi) &= -\frac{2}{V} \cdot (-4\breve{s}_\Phi \hat{y}^2) \sum_p \left[\frac{2\breve{m}_\Phi \hat{y}^2 g_p^2}{A_p^{m,s}} \right. \\
&\qquad \left. - \frac{2\breve{m}_\Phi \hat{y}^2 \cdot g_p \cdot a_p^{m,s} \cdot \left(2 \cdot g_p \cdot a_p^{m,s} + b_p^2\right)}{A_p^{m,s2}} \right] \\
&= -\frac{2}{V} \cdot (-8\breve{m}_\Phi \breve{s}_\Phi \hat{y}^4) \sum_p \left[\frac{g_p}{A_p^{m,s2}} \cdot \left(g_p - \frac{a_p^{m,s} \cdot \left(2 \cdot g_p \cdot a_p^{m,s} + b_p^2\right)}{A_p^{m,s}} \right) \right] \qquad (8.18)
\end{aligned}
$$

Bosonic determinant

Here, for the shortcut $D_p^{m,s}$ nothing changes, but the replacement, but its derivatives change, so let's redefine it:

$$
\begin{aligned}
\tilde{U}^{\text{BosDet}}(\breve{m}_\Phi, \breve{s}_\Phi) &= -\frac{1}{2\, N_f\, V} \sum_{0 \neq p \neq p_s} \log\left[\tilde{D}_p^{m,s}\right], \\
\tilde{D}_p^{m,s} &= 2 - 4\lambda_N - 4\kappa \sum_\mu \cos(p_\mu) + 8\,\lambda_N \left(\breve{m}_\Phi^2 + \breve{s}_\Phi^2\right) + 18\,\lambda_{6N} \left(\breve{m}_\Phi^4 + \breve{s}_\Phi^4 + 6\,\breve{m}_\Phi^2 \breve{s}_\Phi^2\right). \qquad (8.19) \\
\partial_{\breve{m}_\Phi} \tilde{D}_p^{m,s} &= 16\,\lambda_N\, \breve{m}_\Phi + 72\,\lambda_{6N} \left(\breve{m}_\Phi^3 + 3\,\breve{m}_\Phi \breve{s}_\Phi^2\right), \qquad (8.20) \\
\left(\partial_{\breve{m}_\Phi}\right)^2 \tilde{D}_p^{m,s} &= 16\,\lambda_N + 216\,\lambda_{6N} \left(\breve{m}_\Phi^2 + \breve{s}_\Phi^2\right), \qquad (8.21) \\
\partial_{\breve{m}_\Phi} \partial_{\breve{s}_\Phi} \tilde{D}_p^{m,s} &= 432\,\lambda_{6N}\, \breve{m}_\Phi\, \breve{s}_\Phi. \qquad (8.22)
\end{aligned}
$$

For the generalized Propagatorsums nothing special happens. Formally, it looks completely equal:

$$
\tilde{P}_{B^n} \equiv \frac{1}{V} \sum_{0 \neq p \neq p_s} \left(\frac{1}{\tilde{D}_p^{m,s}} \right)^n \qquad (8.23)
$$

$$
\partial_{\breve{m}_\Phi / \breve{s}_\Phi} \tilde{P}_{B^n} = n \left(\partial_{\breve{m}_\Phi / \breve{s}_\Phi} \tilde{D}_p^{m,s} \right) \tilde{P}_{B^{n+1}}, \qquad (8.24)
$$

$$
\begin{aligned}
\partial_{\breve{m}_\Phi} \tilde{U}^{\text{BosDet}}(\breve{m}_\Phi, \breve{s}_\Phi) &= -\frac{1}{2\,N_f} \left(\partial_{\breve{m}_\Phi} \tilde{D}_p^{m,s} \right) \tilde{P}_B \qquad (8.25) \\
\left(\partial_{\breve{m}_\Phi}\right)^2 \tilde{U}^{\text{BosDet}}(\breve{m}_\Phi, \breve{s}_\Phi) &= -\frac{1}{2\,N_f} \left[\left(\left(\partial_{\breve{m}_\Phi}\right)^2 \tilde{D}_p^{m,s} \right) \tilde{P}_B - \left(\partial_{\breve{m}_\Phi} \tilde{D}_p^{m,s} \right)^2 \tilde{P}_{B^2} \right] \qquad (8.26) \\
\partial_{\breve{m}_\Phi} \partial_{\breve{s}_\Phi} \tilde{U}^{\text{BosDet}}(\breve{m}_\Phi, \breve{s}_\Phi) &= -\frac{1}{2\,N_f} \left[\left(\partial_{\breve{m}_\Phi} \partial_{\breve{s}_\Phi} \tilde{D}_p^{m,s} \right) \tilde{P}_B - \left(\partial_{\breve{m}_\Phi} \tilde{D}_p^{m,s} \right) \left(\partial_{\breve{s}_\Phi} \tilde{D}_p^{m,s} \right) \tilde{P}_{B^2} \right] \qquad (8.27)
\end{aligned}
$$

First order Contribution

Here some factors of N_f appear:

$$\tilde{U}^{1\text{st}}(\breve{m}_\Phi, \breve{s}_\Phi) \equiv \tilde{\alpha}^{m,s}\, \tilde{P}_B^2 + \tilde{\beta}\, \tilde{P}_B^3, \tag{8.28}$$

$$\tilde{\alpha}^{m,s} = \frac{8}{N_f^2}\left(\lambda_N + \lambda_{6N}\left(9\left(\breve{m}_\Phi^2 + \breve{s}_\Phi^2\right)\right)\right) \tag{8.29}$$

$$\partial_{\breve{m}_\Phi}\tilde{\alpha}^{m,s} = 144\,\frac{\lambda_{6N}}{N_f^2}\,\breve{m}_\Phi \tag{8.30}$$

$$\left(\partial_{\breve{m}_\Phi}\right)^2\tilde{\alpha}^{m,s} = 144\,\frac{\lambda_{6N}}{N_f^2} \tag{8.31}$$

$$\tilde{\beta} = 24\,\frac{\lambda_{6N}}{N_f^3} \tag{8.32}$$

Since the first order contribution is symmetric in $\breve{m}_\Phi$ and $\breve{s}_\Phi$, only the derivatives w.r.t. $\breve{m}_\Phi$are given.

$$\partial_{\breve{m}_\Phi}\tilde{U}^{1\text{st}}(\breve{m}_\Phi, \breve{s}_\Phi) = 144\,\frac{\lambda_{6N}}{N_f^2}\,\breve{m}_\Phi\,\tilde{P}_B^2 - 2\,\tilde{\alpha}^{m,s}\left(\partial_{\breve{m}_\Phi}\tilde{D}_p^{m,s}\right)\tilde{P}_B\,\tilde{P}_{B^2} - 3\,\tilde{\beta}\left(\partial_{\breve{m}_\Phi}\tilde{D}_p^{m,s}\right)\tilde{P}_B^2\,\tilde{P}_{B^2} \tag{8.33}$$

$$\begin{aligned}\left(\partial_{\breve{m}_\Phi}\right)^2\tilde{U}^{1\text{st}}(\breve{m}_\Phi, \breve{s}_\Phi) = {}& 144\,\frac{\lambda_{6N}}{N_f^2}\,\tilde{P}_B^2 - \left[576\,\frac{\lambda_{6N}}{N_f^2}\,\breve{m}_\Phi\left(\partial_{\breve{m}_\Phi}\tilde{D}_p^{m,s}\right) + 2\,\tilde{\alpha}^{m,s}\left(\left(\partial_{\breve{m}_\Phi}\right)^2\tilde{D}_p^{m,s}\right)\right]\tilde{P}_B\,\tilde{P}_{B^2}\\ &+ 2\,\tilde{\alpha}^{m,s}\left(\partial_{\breve{m}_\Phi}\tilde{D}_p^{m,s}\right)^2\left[\tilde{P}_{B^2}^2 + 2\,\tilde{P}_B\,\tilde{P}_{B^3}\right]\\ &- 3\,\tilde{\beta}\left(\left(\partial_{\breve{m}_\Phi}\right)^2\tilde{D}_p^{m,s}\right)\tilde{P}_B^2\,\tilde{P}_{B^2} + 6\,\tilde{\beta}\left(\partial_{\breve{m}_\Phi}\tilde{D}_p^{m,s}\right)^2\left[\tilde{P}_B\,\tilde{P}_{B^2}^2 + \tilde{P}_B^2\,\tilde{P}_{B^3}\right]\end{aligned} \tag{8.34}$$

$$\begin{aligned}\partial_{\breve{m}_\Phi}\partial_{\breve{s}_\Phi}\tilde{U}^{1\text{st}}(\breve{m}_\Phi, \breve{s}_\Phi) = {}& \left[-288\,\frac{\lambda_{6N}}{N_f^2}\left(\breve{m}_\Phi\left(\partial_{\breve{s}_\Phi}\tilde{D}_p^{m,s}\right) + \breve{s}_\Phi\left(\partial_{\breve{m}_\Phi}\tilde{D}_p^{m,s}\right)\right) - 2\,\tilde{\alpha}^{m,s}\left(\partial_{\breve{m}_\Phi}\partial_{\breve{s}_\Phi}\tilde{D}_p^{m,s}\right)\right]\tilde{P}_B\,\tilde{P}_{B^2}\\ &+ 2\,\tilde{\alpha}^{m,s}\left(\partial_{\breve{m}_\Phi}\tilde{D}_p^{m,s}\right)\left(\partial_{\breve{s}_\Phi}\tilde{D}_p^{m,s}\right)\left[\tilde{P}_{B^2}^2 + 2\,\tilde{P}_B\,\tilde{P}_{B^3}\right]\\ &- 3\,\tilde{\beta}\left(\partial_{\breve{m}_\Phi}\partial_{\breve{s}_\Phi}\tilde{D}_p^{m,s}\right)\tilde{P}_B^2\,\tilde{P}_{B^2} + 6\,\tilde{\beta}\left(\partial_{\breve{m}_\Phi}\tilde{D}_p^{m,s}\right)\left(\partial_{\breve{s}_\Phi}\tilde{D}_p^{m,s}\right)\left[\tilde{P}_B\,\tilde{P}_{B^2}^2 + \tilde{P}_B^2\,\tilde{P}_{B^3}\right]\end{aligned} \tag{8.35}$$

9 Lower mass bound from the CEP

To determin the lower Higgs boson mass in the CEP one cann assume to be in the broken phase wie non-zero magnetization and a staggered mode being zero. Further one can decompose the scalar into a Higgs mode (h) and three Goldstone modes ($g^\alpha, \alpha = 1, 2, 3$). Further we are interested in the effect, the addition of a $\lambda_6 \cdot (\phi^\dagger \phi)^3$) term might have on this bound.

In this chaper we use the continuum parameters ($m_0^2, \lambda, \lambda_6$), however, they are still considered as dimensionless.

In this approach, the cutoff is an input and m_0^2 is tuned to obtain a minimum at the desired value of the field. The potential is given by:

$$\begin{aligned} U(\hat{v}) = U_f(\hat{v}) + \frac{m_0^2}{2}\hat{v}^2 + \lambda\hat{v}^4 + \lambda_6\hat{v}^6 \\ + \lambda \cdot \hat{v}^2 \cdot 6(P_H + P_G) + \lambda_6 \cdot \left(\hat{v}^2 \cdot (45P_H^2 + 54P_GP_H + 45P_G^2) + \hat{v}^4 \cdot (15P_H + 9P_G)\right), \end{aligned} \tag{9.1}$$

with:

$$U_f(\hat{v}) = -\frac{2\,N_f}{V}\left[\sum_p \log\left|\nu(p) + y_t \cdot \hat{v} \cdot \left(1 - \frac{1}{2\rho}\nu(p)\right)\right|^2 + \sum_p \log\left|\nu(p) + y_b \cdot \hat{v} \cdot \left(1 - \frac{1}{2\rho}\nu(p)\right)\right|^2\right] \tag{9.2}$$

$$P_{H/G} = \frac{1}{V}\sum_{p\neq 0}\frac{1}{\hat{p}^2 + m_{H/G}^2} \tag{9.3}$$

With the cutoff Λ being fixed, the location of the minimum is fixed and m_0^2 has to be coosen according to:

$$\left.\frac{\mathrm{d}U}{\mathrm{d}\hat{v}}\right|_{\hat{v}=v} \stackrel{!}{=} 0, \tag{9.4}$$

$$\begin{aligned} U'(\hat{v}) = & U_f'(\hat{v}) + m_0^2\hat{v} + 4\lambda\hat{v}^3 + 6\lambda_6\hat{v}^5 + \\ & + \lambda \cdot \hat{v} \cdot 12(P_H + P_G) + \lambda_6 \cdot \left(2\hat{v} \cdot (45P_H^2 + 54P_GP_H + 45P_G^2) + 4\hat{v}^3 \cdot (15P_H + 9P_G)\right) \end{aligned} \tag{9.5}$$

$$\begin{aligned} \Rightarrow m_0^2 = & -\frac{U_f'(v)}{v} - 4\lambda v^2 - 6\lambda_6 v^4 + \\ & \lambda \cdot 12(P_H + P_G) - \lambda_6 \cdot \left(2 \cdot (45P_H^2 + 54P_GP_H + 45P_G^2) + 4v^2 \cdot (15P_H + 9P_G)\right) \end{aligned} \tag{9.6}$$

$$\begin{aligned} U(\hat{v}) = & U_f(\hat{v}) - 2\frac{U_f'(v)}{v}\hat{v}^2 + \lambda\left(\hat{v}^4 - 2\hat{v}^2(v^2)\right) \\ & + \lambda_6\left(\hat{v}^6 + \hat{v}^4(15P_H + 9P_G) - 2v^2\hat{v}^2(15P_H + 9P_G) - 3v^4\hat{v}^2\right) \end{aligned} \tag{9.7}$$

The Higgs boson mass m_H is then obtained by the curvature of the potential in its minimum:

$$\begin{aligned} U''(\hat{v}) = U_f''(\hat{v}) + m_0^2 + 12\lambda\hat{v}^2 + 30\lambda_6\hat{v}^4 + \\ + \lambda \cdot (12(P_H + P_G)) + \lambda_6 \cdot \left(2(45P_H^2 + 54P_HP_G + 45P_G^2) + 12\hat{v}^2(15P_H + 9P_G)\right) \end{aligned} \tag{9.8}$$

10 Phase structure and Higgs mass in the broken phase

Since it turned out, that the approach where one assumes to be in te broken phase with the potential given in (9.1) works pretty well in determining the phase structure in presence of small but negative λ and small positive λ_6 we continue from this approach. Since there are still deviation between simulation data and prediction from the CEP, David suggested, to use a similar approach as in chapter 5, where the gaussian contributions from the interaction part are actually considered for the gaussian part with the drawback of having a vev-dependent bosonic determinant and more complicated expressions for the propagators. However, in this ansatz no iterative scheme is needed, since the Higgs mass should only be taken from the curvature of the potential in its minimum. Still, the minimization has to be performed, which will be slightly more complicated.

Starting point here is the bosonic action, where the scalar field was already decomposed into Higgs and Goldstone modes. The only difference in treating those two kinds of fields is in the assumtion of the zero mode. For the Higgs field it is proportional to the vev, while it is zero for the Goldstones:

$$\tilde{h}_0 = \sqrt{V}\hat{v}, \qquad \tilde{g}_0^\alpha = 0. \tag{10.1}$$

The action is:

$$\begin{aligned} S = &\frac{1}{2}\sum_p \left\{ \tilde{h}_{-p}\left(\hat{p}^2 + m_0^2\right)\tilde{h}_p + \sum_\alpha \tilde{g}^\alpha_{-p}\left(\hat{p}^2 + m_0^2\right)\tilde{g}^\alpha_p \right\} \\ &+ \frac{\lambda}{V}\sum_{p_1\dots p_4} \delta_{p_1+\dots+p_4,0}\left(\tilde{h}_{p_1}\tilde{h}_{p_2} + \sum_\alpha \tilde{g}^\alpha_{p_1}\tilde{g}^\alpha_{p_2}\right)\left(\tilde{h}_{p_3}\tilde{h}_{p_4} + \sum_\alpha \tilde{g}^\alpha_{p_3}\tilde{g}^\alpha_{p_4}\right) \\ &+ \frac{\lambda_6}{V^2}\sum_{p_1\dots p_6} \delta_{p_1+\dots+p_6,0}\left(\tilde{h}_{p_1}\tilde{h}_{p_2} + \sum_\alpha \tilde{g}^\alpha_{p_1}\tilde{g}^\alpha_{p_2}\right)\left(\tilde{h}_{p_3}\tilde{h}_{p_4} + \sum_\alpha \tilde{g}^\alpha_{p_3}\tilde{g}^\alpha_{p_4}\right)\left(\tilde{h}_{p_5}\tilde{h}_{p_6} + \sum_\alpha \tilde{g}^\alpha_{p_5}\tilde{g}^\alpha_{p_6}\right). \end{aligned} \tag{10.2}$$

With this, a decomposition of this action can be done by decomposing it into a tree-level, a gaussian and an interaction part. Tree-level is easy, simply collect all terms that only contain zero modes.

$$S_B^{\text{tree}} = V\left(\frac{m_0^2}{2}\hat{v}^2 + \lambda\hat{v}^4 + \lambda_6\hat{v}^6\right). \tag{10.3}$$

For th gaussian term, we collect all terms, where all but two fields are set to their zero modes. This should lead to:

$$S_B^{\text{gauss}} = \frac{1}{2}\sum_{p\neq 0}\left\{ \tilde{h}_{-p}\left(\hat{p}^2 + m_0^2 + 12\hat{v}^2\lambda + 30\hat{v}^4\lambda_6\right)\tilde{h}_p + \sum_\alpha \tilde{g}^\alpha_{-p}\left(\hat{p}^2 + m_0^2 + 4\hat{v}^2\lambda + 6\hat{v}^4\lambda_6\right)\tilde{g}^\alpha_p \right\} \tag{10.4}$$

The interaction part is simply given by the rest:

$$\begin{aligned} S_B^{\text{int}} = &\frac{\lambda}{V}\widetilde{\sum_{p_1\dots p_4}} \delta_{p_1+\dots+p_4,0}\left(\tilde{h}_{p_1}\tilde{h}_{p_2} + \sum_\alpha \tilde{g}^\alpha_{p_1}\tilde{g}^\alpha_{p_2}\right)\left(\tilde{h}_{p_3}\tilde{h}_{p_4} + \sum_\alpha \tilde{g}^\alpha_{p_3}\tilde{g}^\alpha_{p_4}\right) \\ &+ \frac{\lambda_6}{V}\hat{v}^2\widetilde{\sum_{p_1\dots p_4}} \delta_{p_1+\dots+p_4,0}\left(15\,\tilde{h}_{p_1}\tilde{h}_{p_2}\tilde{h}_{p_3}\tilde{h}_{p_4} + 18\,\tilde{h}_{p_1}\tilde{h}_{p_2}\sum_\alpha \tilde{g}^\alpha_{p_3}\tilde{g}^\alpha_{p_4} + 3\sum_{\alpha,\beta}\tilde{g}^\alpha_{p_1}\tilde{g}^\alpha_{p_2}\tilde{g}^\beta_{p_3}\tilde{g}^\beta_{p_4}\right) \\ &+ \frac{\lambda_6}{V^2}\widetilde{\sum_{p_1\dots p_6}} \delta_{p_1+\dots+p_6,0}\left(\tilde{h}_{p_1}\tilde{h}_{p_2} + \sum_\alpha \tilde{g}^\alpha_{p_1}\tilde{g}^\alpha_{p_2}\right)\left(\tilde{h}_{p_3}\tilde{h}_{p_4} + \sum_\alpha \tilde{g}^\alpha_{p_3}\tilde{g}^\alpha_{p_4}\right)\left(\tilde{h}_{p_5}\tilde{h}_{p_6} + \sum_\alpha \tilde{g}^\alpha_{p_5}\tilde{g}^\alpha_{p_6}\right) \end{aligned} \tag{10.5}$$

Resorting gives:

$$\begin{aligned} S_B^{\text{int}} = &\frac{\lambda}{V}\widetilde{\sum_{p_1\dots p_4}} \delta_{p_1+\dots+p_4,0}\left(\tilde{h}_{p_1}\tilde{h}_{p_2}\tilde{h}_{p_3}\tilde{h}_{p_4} + 2\,\tilde{h}_{p_1}\tilde{h}_{p_2}\sum_\alpha \tilde{g}^\alpha_{p_3}\tilde{g}^\alpha_{p_4} + \sum_{\alpha,\beta}\tilde{g}^\alpha_{p_1}\tilde{g}^\alpha_{p_2}\tilde{g}^\alpha_{p_3}\tilde{g}^\alpha_{p_4}\right) \\ &+ \frac{\lambda_6}{V}\hat{v}^2\widetilde{\sum_{p_1\dots p_4}} \delta_{p_1+\dots+p_4,0}\left(15\,\tilde{h}_{p_1}\tilde{h}_{p_2}\tilde{h}_{p_3}\tilde{h}_{p_4} + 18\,\tilde{h}_{p_1}\tilde{h}_{p_2}\sum_\alpha \tilde{g}^\alpha_{p_3}\tilde{g}^\alpha_{p_4} + 3\sum_{\alpha,\beta}\tilde{g}^\alpha_{p_1}\tilde{g}^\alpha_{p_2}\tilde{g}^\beta_{p_3}\tilde{g}^\beta_{p_4}\right) \\ &+ \frac{\lambda_6}{V^2}\widetilde{\sum_{p_1\dots p_6}} \delta_{p_1+\dots+p_6,0}\Bigg(\tilde{h}_{p_1}\tilde{h}_{p_2}\tilde{h}_{p_3}\tilde{h}_{p_4}\tilde{h}_{p_5}\tilde{h}_{p_6} + 3\,\tilde{h}_{p_1}\tilde{h}_{p_2}\tilde{h}_{p_3}\tilde{h}_{p_4}\sum_\alpha \tilde{g}^\alpha_{p_5}\tilde{g}^\alpha_{p_6} \\ &\qquad + 3\,\tilde{h}_{p_1}\tilde{h}_{p_2}\sum_{\alpha,\beta}\tilde{g}^\alpha_{p_3}\tilde{g}^\alpha_{p_4}g^\beta_{p_5}\tilde{g}^\beta_{p_6} + \sum_{\alpha,\beta,\gamma}\tilde{g}^\alpha_{p_1}\tilde{g}^\alpha_{p_2}\tilde{g}^\beta_{p_3}\tilde{g}^\beta_{p_4}g^\gamma_{p_5}\tilde{g}^\gamma_{p_6}\Bigg). \end{aligned} \tag{10.6}$$

Note, in the second line, in former versions it was $30hhhh$, which is wrong.
From the gaussian part (10.4), we get the propagators:

$$\tilde{h}_p\tilde{h}_q = \frac{\delta_{p+q,0}}{\hat{p}^2+m_0^2+12\hat{v}^2\lambda+30\hat{v}^4\lambda_6} \quad \text{and} \quad \tilde{g}_p^\alpha\tilde{g}_q^\beta = \frac{\delta_{p+q,0}\delta_{\alpha,\beta}}{\hat{p}^2+m_0^2+4\hat{v}^2\lambda+6\hat{v}^4\lambda_6}. \tag{10.7}$$

As before, for convenience let's define the propagator sums for later use:

$$\tilde{P}_H = \frac{1}{V}\sum_{p\neq 0}\frac{1}{\hat{p}^2+m_0^2+12\hat{v}^2\lambda+30\hat{v}^4\lambda_6} \quad \text{and} \quad \tilde{P}_G = \frac{1}{V}\sum_{p\neq 0}\frac{1}{\hat{p}^2+m_0^2+4\hat{v}^2\lambda+6\hat{v}^4\lambda_6}. \tag{10.8}$$

In this approach, the bosonic determinant has to be considered for the CEP as well as the first order contributions in λ and λ_6

$$U(\hat{v}) = U^F(\hat{v}) + U^T(\hat{v}) + U^D(\hat{v}) + U^{(1)}(\hat{v}) \tag{10.9}$$

$$U^F(\hat{v}) = -\frac{2\,N_f}{V}\sum_p\left\{\log\left|\nu(p)+y_t\cdot\hat{v}\cdot\left(1-\frac{1}{2\rho}\nu(p)\right)\right|^2 + \log\left|\nu(p)+y_b\cdot\hat{v}\cdot\left(1-\frac{1}{2\rho}\nu(p)\right)\right|^2\right\} \tag{10.10}$$

$$U^T(\hat{v}) = \frac{m_0^2}{2}\hat{v}^2 + \lambda\hat{v}^4 + \lambda_6\hat{v}^6 \tag{10.11}$$

For the part coming from teh bosonic determinant, do keep the prefacor somewhat free, I'll assume:

$$\int\left[\prod_{p\neq 0} dh_p\,dg_p^\alpha\right] e^{-h\mathcal{A}h}e^{-g^\alpha\mathcal{B}g^\alpha}\cdot\{\text{smth}\} = \frac{V^a}{\left(\det\mathcal{A}\cdot\det^3\mathcal{B}\right)^b}\cdot\{\text{smth else}\} \tag{10.12}$$

$$= e^{a\cdot\log V - b(\mathrm{Tr}\log\mathcal{A}+3\mathrm{Tr}\log\mathcal{B})+\log\{\text{smth else}\}} \tag{10.13}$$

$$U^D(\hat{v}) = \frac{b}{V}\sum_{p\neq 0}\left\{\log\left(\hat{p}^2+m_0^2+12\hat{v}^2\lambda+30\hat{v}^4\lambda_6\right) + 3\,\log\left(\hat{p}^2+m_0^2+4\hat{v}^2\lambda+6\hat{v}^4\lambda_6\right)\right\} \tag{10.14}$$

To get the first order contribution, one might be careful with the combinatorics:

- for the $hhhh$ term, there are 3 combinations
- the $hhg^\alpha g^\alpha$ term also generates 3 contractions (α)
- for the $g^\alpha g^\alpha g^\beta g^\beta$ term, we have 6 contributions for $\alpha\neq\beta$ and $3\cdot 3 = 9$ contarctions for $\alpha=\beta$. Total is 15.
- $hhhhhhh$ has $\frac{6\cdot 5}{2}\frac{4\cdot 3}{2}\frac{1}{6} = 15$ contractions
- $hhhhg^\alpha g^\alpha$ has 9 contractions(3 (from the $hhhh$) times 3 (α))
- $hhg^\alpha g^\alpha g^\beta g^\beta$ gives 15 from ($gggg$) contractions. the hh is fixed.
- $g^\alpha g^\alpha g^\beta g^\beta g^\gamma g^\gamma$ gives 105 (6 from $\{\alpha,\beta,\gamma\}$ being a permutation of $\{1,2,3\}$ giving only 1 possibility each., 3 times 6 when two indizes are equal giving 3 contractions each (3*6*3=54) and 3 times 15 if $\alpha=\beta=\gamma$).

This leads to a total first order contribution:

$$U^{(1)}(\hat{v}) = \lambda\left(3\,\tilde{P}_H^2 + 6\,\tilde{P}_H\tilde{P}_G + 15\,\tilde{P}_G^2\right) + \lambda_6\hat{v}^2\left(45\,\tilde{P}_H^2 + 54\,\tilde{P}_H\tilde{P}_G + 45\,\tilde{P}_G^2\right) + \lambda_6\left(15\,\tilde{P}_H^3 + 27\,\tilde{P}_H^2\tilde{P}_G + 45\,\tilde{P}_H\tilde{P}_G^2 + 105\,\tilde{P}_G^3\right) \tag{10.15}$$

Second derivatives

For an overview I'll give the second derivativ of the gaussian and the 1st order contribution:

$$U^{D''}(\hat{v}) = \frac{b}{V}\sum_{p\neq 0}\left(\frac{D_H''D_H - D_H'^2}{D_H^2} + 3\frac{D_G''D_G - D_G'^2}{D_G^2}\right) \tag{10.16}$$

with

$$D_H(p) = \hat{p}^2 + m_0^2 + 12\hat{v}^2\lambda + 30\hat{v}^4\lambda_6 \tag{10.17}$$

$$D_G(p) = \hat{p}^2 + m_0^2 + 4\hat{v}^2\lambda + 6\hat{v}^4\lambda_6 \tag{10.18}$$

$$D'_H = 24\hat{v}\lambda + 120\hat{v}^3\lambda_6 \tag{10.19}$$

$$D'_G = 8\hat{v}\lambda + 24\hat{v}^3\lambda_6 \tag{10.20}$$

$$D''_H = 24\lambda + 360\hat{v}^2\lambda_6 \tag{10.21}$$

$$D''_G = 8\lambda + 72\hat{v}^2\lambda_6 \tag{10.22}$$

I ommit the tilde for the prop sums, since its annoying to tex...

$$\begin{aligned}
U^{(1)''}(\hat{v}) &= \lambda\left(6P_H'^2 + 6P_HP_H'' + 6P_H''P_G + 12P_H'P_G' + 6P_HP_G'' + 30P_G'^2 + 30P_GP_G''\right)\\
&+ \lambda_6\hat{v}^2\left(90P_H'^2 + 90P_HP_H'' + 54P_H''P_G + 108P_H'P_G' + 54P_HP_G'' + 90P_G'^2 + 90P_GP_G''\right)\\
&+ 4\lambda_6\hat{v}\left(90P_HP_H' + 54P_H'P_G + 54P_HP_G' + 90P_GP_G'\right)\\
&+ 2\lambda_6\left(45P_H^2 + 54P_HP_G + 45P_G^2\right)\\
&+ \lambda_6\left(90P_HP_H'^2 + 45P_H^2P_H'' + 54P_H'^2P_G + 54P_HP_H''P_G + 108P_HP_H'P_G' + 27P_H^2P_G''\right.\\
&\qquad\left. +45P_H''P_G^2 + 180P_H'P_GP_G' + 90P_HP_G'^2 + 90P_HP_GP_G'' + 630P_GP_G'^2 + 315P_G^2P_G''\right)
\end{aligned} \tag{10.23}$$