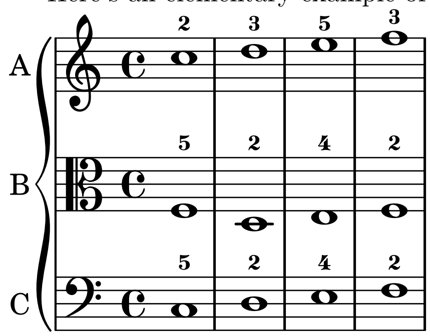
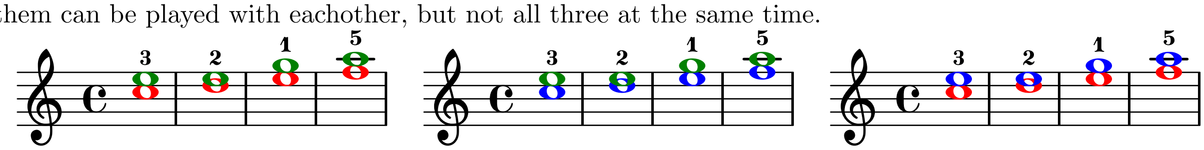
Going to use a toy model of species counterpoint for the purposes of explanation:

- Perfect consonances are fifth (4)[1] and octave (7).
- Imperfect consonances are third (2), fourth (3), and sixth (5).
- Perfect consonances must be approached by contrary or oblique motion.
- No restriction on melodic intervals.
- No unisons or crossing of voices.
- Harmonic intervals must be perfect or imperfect.

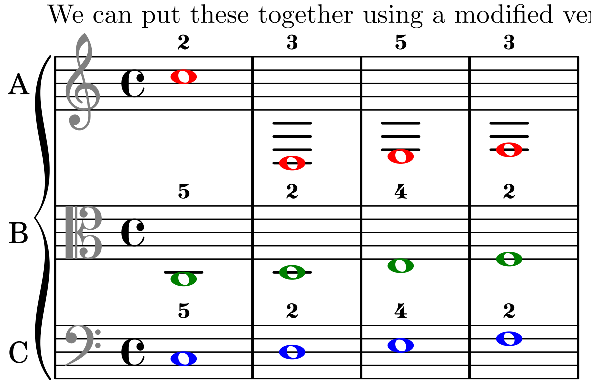
Here's an elementary example of 3-part species counterpoint:



Here are three melody shapes that can be transposed so that any two of them can be played with eachother, but not all three at the same time.



We can put these together using a modified version of the 3-part notation:



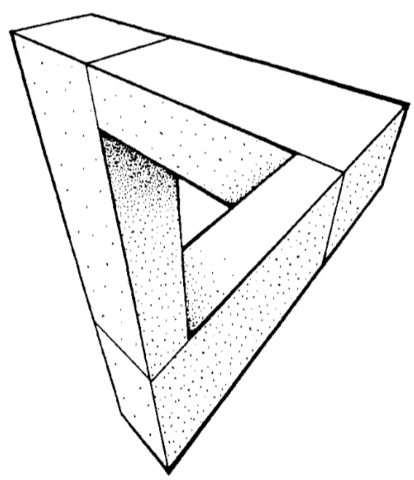
[1] I'm numbering intervals counting from 0 (=unison).

But the interval numbers no longer add up vertically! But they do have $bottom + medium + 3 = top$. We call 3 the obstruction. The clefs are a fiction to make comparing adjoining staves easier. To work out if the counterpoint is correct, you need to know the intervals and the shape of the melody - you ignore the clefs.

To write counterpoint of this form by hand, you use the following method.

- Pick a source melody to harmonize.
- Pick an obstruction.
- Write out all the chords that generate this obstruction[2].
- Fill in the rest of the counterpoint.

This is an impossible to perform composition in the same way that the Penrose triangle is impossible to realize[3]:

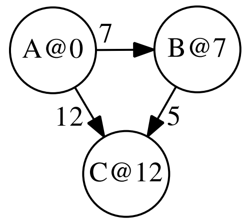


If any two pieces/melodies are present, the figure/composition can be realized (transposition of one clef might be needed), but all three parts cannot be combined together at once.

Take the first example given above of a vanilla counterpoint exercise. We can represent the starting pitches using integers - melody A starts on c, which we'll call 0 ($d = 1$, &c.), so let's notate it as $A@0$. And let's put the intervals on the edges.

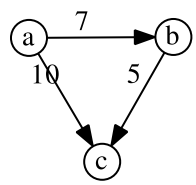
[2] I made a page here that generates them for you : `https://ded.increpare.com/~locus/comusic/chordgenerate.htmlgeneratesthem`.

[3] cf Penrose's paper for the geometrical analogue of what is to follow : `http://www.iri.upc.edu/people/ros/StructuralTopology/ST17/st17-05-a2-ocr.pdf`.

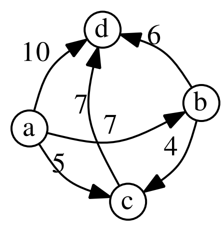


Note that if you travel in a loop and add up the nubmers as you go[4], you get $7 + 5 - 12 = 0$. It adds up to zero.

For the impossible example, we no longer have fixed base positions, so we don't label the vertices, but we can still label the edges, to get this:



Note that the triangle no longer adds to 0, but to 2. Here's a made-up graph of a more complicated system that could come from some specified system of voices:



Looking at this graph, the edges tell you what voices can be harmonized with what other voices, and if you want 3 at once, you're looking for triangles where the edges sum to zero[5].

Here's the semi-precise insufficiently-explained analogy with manifolds and cohomological language:

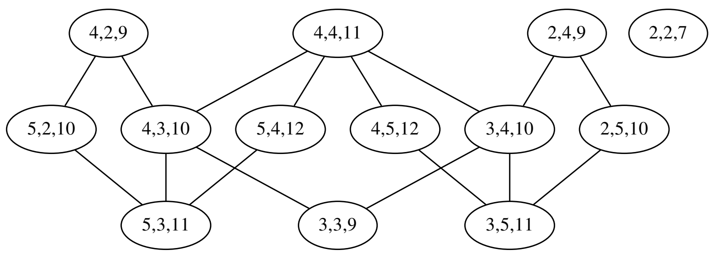
[4]negating them if they're on arrows facing the opposite direction

[5]This relates to a the fact that if all circuits sum to zero, you can assign numbers to the vertices such that an edge's value is the difference of its two vertices. This is why the obstruction can be called an obstruction: its presence stops you from realizing a graph with nicely-labelled vertices.

Manifold	Čech Cohomology	Counterpoint System
atlas		set of melodies
chart	vertex	melody
transition area	edges	harmonizing pair
	nerve	graph of melodies+intervals
	1-cocycle	interval choices
	1-coboundary	compatible intervals
	0-cocycle	assignment of pitches
	obstruction	obstruction

Definition $H(m_1, m_2, i)$ express the proposition "m_1 harmonizes with m_2 when at interval i."

For a given model of counterpoint, and a given obstruction, we can construct fun new voice-leading graphs. Here's one for a system with obstruction 3, showing ways of moving from one chord to another by changing a single note by a single step[6]:



It's possible (and fun) to just go and write 3-staff counterpoints by hand, but here's a more algorithmic approach:

- Start with a set M of melodies.
- Generate a graph G with a set of vertices M, with labelled edges:

$$E = \{(m_1 \xrightarrow{i} m_2) | m_1, m_2 \in M, i \in \mathbb{Z}, H(m_1, m_2, i)\}$$

  You can view this as a directed graph with a 1-cocycle.
- Complete subgraphs of G in which all triangles sum to 0 are realizable counterpoints.

`https://ded.increpare.com/~locus/comusic/graphgenerate.html` is an implementation of this algorithm. And here's one that does the same, but searches through canons instead: `https://ded.increpare.com/~locus/comusic/canongenerate.html`.

[6] Here's a program for calculating them : `https://ded.increpare.com/~locus/comusic/leadingsgenerate.html`.