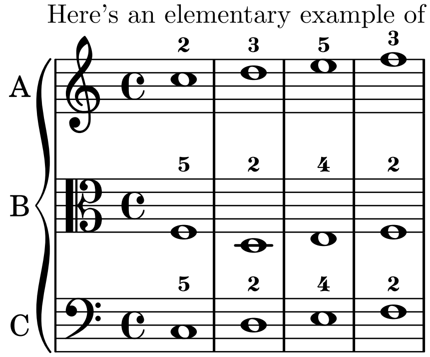
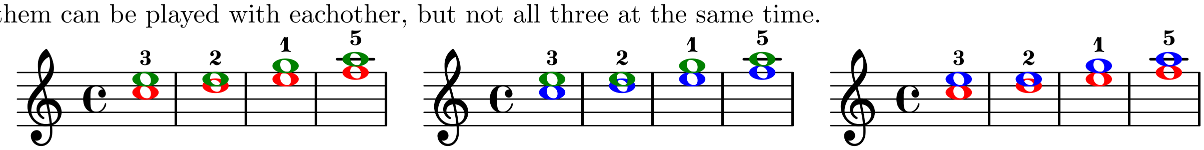
First, I'm numbering intervals from 0 (unison is 0, third is 2, &c.).
Going to use a toy model of species counterpoint for this[1]:

- Perfect consonances are fifth (4) and octave (7).
- Imperfect consonances are third (2) and sixth (5).
- Perfect consonances must be approached by contrary or oblique motion.
- No restriction on melodic intervals.
- No unisons or crossing of voices.
- No more than two imperfect consonances in a row.
- Harmonic intervals must be perfect or imperfect.

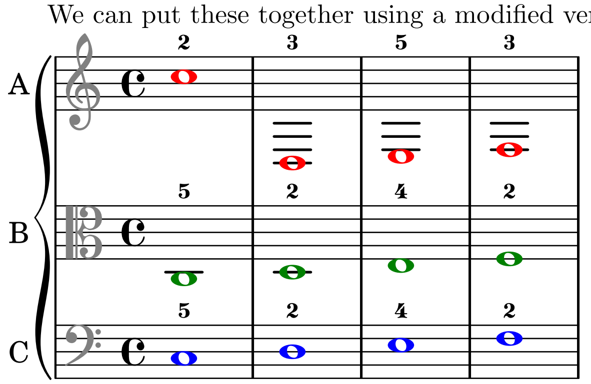
Here's an elementary example of 3-part species counterpoint:




Here are three melody shapes that can be transposed so that any two of them can be played with eachother, but not all three at the same time.



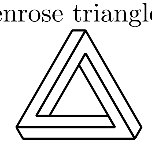

We can put these together using a modified version of the 3-part notation:




[1]we could use a more detailed one as well, though not the full set of species rules...

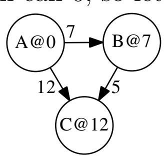
Note that the interval numbers no longer add up correctly; the clefs are a fiction to make comparing adjoining staves easier. To work out if the counterpoint is correct, you need to know the intervals and the shape of the melody - you ignore the clefs.

This is an impossible to perform composition in the same way that this penrose triangle is impossible to realize[2]:



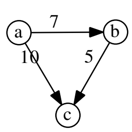
If any two staves are present, the composition can be realized (transposition of one clef might be needed), but all three parts cannot be combined together.

Take the first example given above of a vanilla counterpoint exercise. We can represent the starting pitches using integers - melody $A$ starts on $c$, which we'll call 0, so let's notate it as $A$@0. And let's put the intervals on the edges.

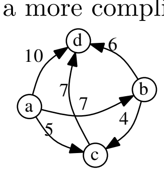



note that if you travel in a loop, adding numbers as you go (or subtracting if you're going against an arrow), you get $7 + 5 - 12 = 0$. It adds up to zero.

For our other graph, we no longer have fixed base positions, so we don't label the vertices, but we can still label the edges, to get this:




Note that the triangle no longer adds to 0, but to 2. Here's a made up graph of a more complicated system:




If we're looking at a graph that shows what intervals are compatible between melodies, and we're looking for triangles where the intervals sum to zero. (For four-part counterpoints, we're looking for tetrahedra where all the constituent triangles sum to zero).

If I have a graph of intervals where all the circular paths sum to zero, I can assign definite pitches to each melody.

Here's the analogy with manifolds and cohomological language:

[2]cf penrose's paper for the geometrical analogue of what is to follow http://www.iri.upc.edu/people/ros/StructuralTopology/ST17/st17-05-a2-ocr.pdf

| Manifold | Čech Cohomology | Counterpoint System |
|---|---|---|
| atlas | | set of melodies |
| chart | vertex | melody |
| transition area | edges | harmonizing pair |
| | nerve | graph of melodies+intervals |
| | 1-cocycle | interval choice |
| | 1-coboundary | compatible intervals |
| | 0-cocycle | assignment of pitches |

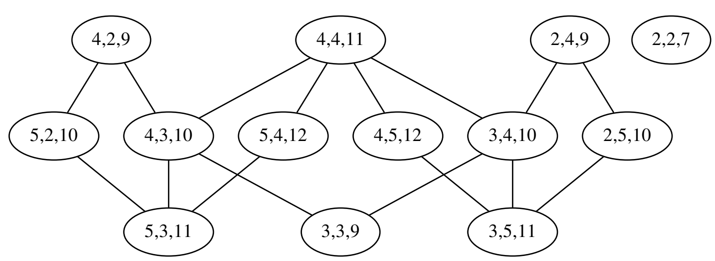
**Definition** $H(m_1, m_2, i)$ express the proposition "$m_1$ harmonizes with $m_2$ when at interval $i$."

It's possible (and fun) to just go and write 3-staff counterpoints by hand, but here's a more algorithmic approach:

- Start with a set $M$ of melodies.
- Generate a graph $G$ with a set of vertices $M$, with labelled edges:

  $$E = \{(m_1 \xrightarrow{i} m_2) | m_1, m_2 \in M, i \in \mathbb{Z}, H(m_1, m_2, i)\}$$

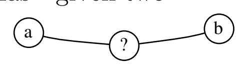
  You can view this as a directed graph with a 1-cocycle.
- Complete subgraphs of $G$ in which all triangles sum to 0 are realizable counterpoints.




Notes:

- Mostly what appeals to me about this idea is the idea of an "impossible" species counterpont.
- In terms of compositional usefulness, I think having a tool that you could input melodies into and that would output you a graph of possible ways of putting them together might be nice to have.
- 
- You can leave the diatonic world and take a set of chromatic constraints instead - it no longer is so nicely notatable with regular clefs, but computationally it's much the same, and can distinguish major/minor/diminished intervals properly.

- Also, there's no reason this wouldn't work for higher species.
- There's a balance to be struck between trying to adhere to mathematical formalism and trying to make it clear in and of itself. I don't know how deeply the relationship goes - I don't think cohomology is the natural end-point for it - it branches off from the ech cohomology construction before the cohomology group has been even defined (Penrose's paper basically did as well). I'd probably be more inclined to take a computational direction first, implement something where if you give it a bunch of melody shapes it'll plot a nice graph of them, and see if anything interesting pops up from that. You could also ask questions like "what melody has the most/fewest harmonizations"? The answer would be an artefact of the countrapuntal moda sd
- You could do a reverse search where you specify a graph and it fills it out with melodies. That could be used to produce some contrapuntal curiosities, perhaps - maybe if you have two melodies that don't harmonize with eachother, you could search to see if you could find some set of accompanying melodies that can be combined with either of them.
- There's probably some category theory flavoured version of this possible as well (melodies are objects, valid harmonizations are morphisms).

  A direction would be to ask what properties this category has - given two melodies, is there always a third that harmonizes with both? 
  

  Any interesting functors or natural transformations? I haven't thought this through at all, just thinking out loud.
- this is mostly me throwing associations at it seeing what sticks. Someone sees a black car and, remembering that they once had a black cat, asks if it meows.
- have a look at other papers that reference that Penrose paper, maybe there are some further fun tools that could be applicable there.