The quantum circuit model

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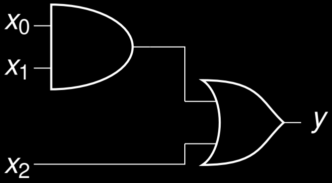
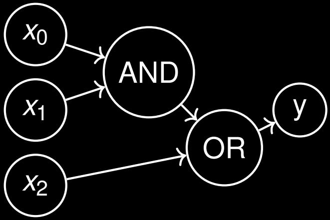
Circuit model

Boolean circuits

A Boolean circuit is a directed acyclic graph whose vertices are either inputs, outputs, or computational nodes representing the logical gates AND, OR, NOT.





Boolean functions

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Which Boolean functions can we compute?

You can compute *any* Boolean function f using a circuit composed of AND, OR, and NOT gates.

Decision problem

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Can we use the circuit framework to solve a decision problem?

Tentative:

A *circuit family* $\mathcal{C} = \{C_n\}_{n \in \mathbb{N}}$ is a sequence of circuits, one for each input size n.

$\mathcal{C} = \{C_n\}_{n \in \mathbb{N}}$ solves L if

- for all $x \in L, |x| = n$ we have $C_n(x) = 1$
- for all $x \notin L, |x| = n$ we have $C_n(x) = 0$

Uniformity

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What if we have a single, finite program that can generate all the circuits?

A circuit family $\mathcal{C} = \{C_n\}_{n\in\mathbb{N}}$ is *uniformly polynomial* if there exists a Turing machine T that for input n gives a description of C_n in $\mathcal{O}(\text{poly}\log n)$ space.

P

A decision problem L is in the complexity class P if there exists a uniformly polynomial circuit family $\mathcal{C} = \{C_n\}_{n \in \mathbb{N}}$ such that

- for every $x \in L$ with $|x| = n$ we have $C_n(x) = 1$
- for every $x \notin L$ with $|x| = n$ we have $C_n(x) = 0$

BPP

A randomized circuit family $\mathcal{C} = \{C_n\}_{n\in\mathbb{N}}$ is a sequence of circuits, one for each input size n, such that each circuit is provided with $\mathcal{O}(\text{poly}\, n)$ random bits in addition to the input.

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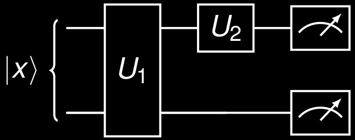
We can improve the value of this threshold.

Quantum circuits

A quantum circuit:



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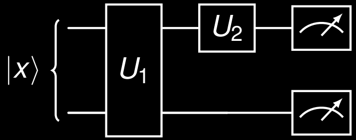

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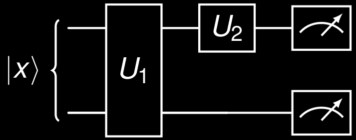

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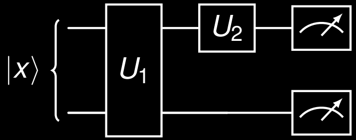

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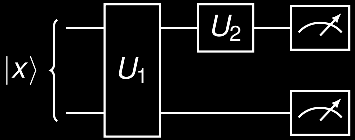

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- Reversibility: the number of fan-in and fan-out is the same in each layer
- Parallel operations are composed via the tensor product
- Sequential operations are composed by multiplication

Quantum gates

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$$

$$\text{CNOT} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{I} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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Every unitary *U* over *n*-qubit
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Every 1- and 2-qubit unitary
can be *approximately* implemented
using only gates from a small, finite set
e.g. $\mathcal{G} = \{\textsc{cnot}, H, T\}$

BQP

A decision problem $L \subseteq \{0,1\}^*$ is in the complexity class BQP if there exists a uniformly polynomial quantum circuit family $\mathcal{C} = \{Q_n\}_{n \in \mathbb{N}}$ such that

- for all $x \in L, |x| = n$ we have $Pr[Q_n(x) = 1] \geq 2/3$;
- for all $x \notin L, |x| = n$ we have $Pr[Q_n(x) = 0] \leq 1/3$.

Query model

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2. The input $x \in \{0,1\}^N$, with $N = 2^n$, is provided as a black-box oracle O_x:

$$O_x |i\rangle |b\rangle = |i\rangle |b \oplus x_i\rangle .$$

Each use or call of the oracle is referred to as a *query*.

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   Each use or call of the oracle is referred to as a *query*.
3. The quantum circuit interleaves the queries to the oracle with standard, non-query operations.

Motivation

It is difficult to prove lower bounds on the complexity of computing some functions over explicit input data.

In contrast, we can often demonstrate that many queries are required to compute some given function of the black-box input.

Phase Oracle

The input $x \in \{0,1\}^N$, with $N = 2^n$, is provided as a black-box *phase oracle* $O_x^{\pm}$ if:

$$O_x^{\pm} |i\rangle = (-1)^{x_i} |i\rangle .$$

Equivalence between oracle formats

The circuit implementing the phase oracle given a traditional oracle is $O_x^{\pm} = (\mathbb{I} \otimes H) O_x (\mathbb{I} \otimes HX)$.

[whiteboard]

Whiteboard

$$
\begin{aligned}
2^{-1/2} O_x \left|i\right\rangle (\left|0\right\rangle - \left|1\right\rangle) &= 2^{-1/2} \left|i\right\rangle (\left|0 \oplus x_i\right\rangle - \left|1 \oplus x_i\right\rangle) \\
&= 2^{-1/2} \begin{cases} \left|i\right\rangle (\left|0\right\rangle - \left|1\right\rangle), & x_i = 0 \\ \left|i\right\rangle (-1)(\left|0\right\rangle - \left|1\right\rangle), & x_i = 1 \end{cases} \\
&= 2^{-1/2} \begin{cases} \left|i\right\rangle (-1)^0(\left|0\right\rangle - \left|1\right\rangle), & x_i = 0 \\ \left|i\right\rangle (-1)^1(\left|0\right\rangle - \left|1\right\rangle), & x_i = 1 \end{cases} \\
&= 2^{-1/2}(-1)^{x_i} \left|i\right\rangle (\left|0\right\rangle - \left|1\right\rangle)
\end{aligned}
$$

Deutsch-Jozsa algorithm

BALANCED-OR-CONSTANT problem

Input: oracle access O_f to a Boolean function $f : \{0,1\}^n \to \{0,1\}$

Promise: f is either constant or balanced ($|f^{-1}(0)| = |f^{-1}(1)|$).

Output: 0 if f is constant and 1 if f is balanced.

The classical query complexity for the BALANCED-OR-CONSTANT problem is $N/2 + 1 \in \mathcal{O}(2^n)$.

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The quantum query complexity for the BALANCED-OR-CONSTANT problem is 1. This is achieved using the DEUTSCH-JOZSA algorithm.

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$$|\psi_0\rangle = |0^n\rangle$$

$$|\psi_1\rangle = H^{\otimes n}|0^n\rangle = 2^{-n/2}\sum_{i\in\{0,1\}^n}|i\rangle$$

$$|\psi_2\rangle = O_f^{\pm}\left(2^{-n/2}\sum_{i\in\{0,1\}^n}|i\rangle\right) = 2^{-n/2}\sum_{i\in\{0,1\}^n}(-1)^{f(i)}|i\rangle$$

Whiteboard

$$
\begin{aligned}
|\psi_3\rangle &= H^{\otimes n}\left(2^{-n/2}\sum_{i\in\{0,1\}^n}(-1)^{f(i)}\,|i\rangle\right)\\
&= 2^{-n/2}\sum_{i\in\{0,1\}^n}(-1)^{f(i)}H^{\otimes n}\,|i\rangle\\
&= 2^{-n}\sum_{i\in\{0,1\}^n}(-1)^{f(i)}\sum_{j\in\{0,1\}^n}(-1)^{i\cdot j}\,|j\rangle
\end{aligned}
$$

Here,

$$
\begin{aligned}
H^{\otimes n}\,|i\rangle &= (H\,|i_0\rangle)\otimes\cdots\otimes(H\,|i_{n-1}\rangle)\\
&= \frac{|0\rangle+(-1)^{i_0}\,|1\rangle}{\sqrt{2}}\otimes\cdots\otimes\frac{|0\rangle+(-1)^{i_{n-1}}\,|1\rangle}{\sqrt{2}}\\
&= \frac{1}{\sqrt{2}^n}\sum_{j\in\{0,1\}^n}(-1)^{i\cdot j}\,|j\rangle
\end{aligned}
$$

Whiteboard

$$|\psi_3\rangle = \frac{1}{2^n} \sum_{i \in \{0,1\}^n} (-1)^{f(i)} (-1)^{i \cdot 0} |0\rangle + \cdot ...$$

$$= \frac{1}{2^n} \sum_{i \in \{0,1\}^n} (-1)^{f(i)} |0\rangle + \cdot ...$$

Here,

$$\frac{1}{2^n} \sum_{i \in \{0,1\}^n} (-1)^{f(i)} = 0 \text{ if } f \text{ is balanced}$$

$$\frac{1}{2^n} \sum_{i \in \{0,1\}^n} (-1)^{f(i)} = 1 \text{ if } f \text{ is constant one}$$

$$\frac{1}{2^n} \sum_{i \in \{0,1\}^n} (-1)^{f(i)} = -1 \text{ if } f \text{ is constant zero}$$

Note: If we allow a small constant probability of error, the BALANCED-OR-CONSTANT problem can be solved efficiently with a constant number of queries.

Bernstein-Vazirani algorithm

SECRET-STRING problem

Input: oracle access O_f to a Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$.

Promise: $f(x) = (s \cdot x) \bmod 2$

Output: the n-bit string s

The classical query complexity for the SECRET-STRING problem is n:

$$f(100\cdots0) = s_0$$
$$f(010\cdots0) = s_1$$
$$f(001\cdots0) = s_2$$
$$f(000\cdots1) = s_{n-1}$$

Each query returns one bit of information. Therefore, we cannot do better than $\mathcal{O}(n)$.

The quantum query complexity for the SECRET-STRING problem is 1, as demonstrated by the BERNSTEIN-VAZIRANI algorithm.

[whiteboard]

Whiteboard

$$|\psi_2\rangle = O_f^{\pm} H^{\otimes n} \left|0^n\right\rangle = 2^{-n/2} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot s} |x\rangle$$

Recall that $H^{\otimes n} |s\rangle = 2^{-n/2} \sum_{y \in \{0,1\}^n} (-1)^{s \cdot y} |y\rangle$.

To recover s we just need to invert $H^{\otimes n}$. After that, we get $|s\rangle$ in the computational basis.

Thank you!