Show that

$$a_3 = \frac{f_3 - \frac{f_2-f_1}{x_2-x_1}(x_3-x_1) - f_1}{(x_3-x_1)(x_3-x_2)} = f[x_1,x_2,x_3] = \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1} = \frac{\frac{f_3-f_2}{x_3-x_2} - \frac{f_2-f_1}{x_2-x_1}}{x_3-x_1}$$

Divided Differences Formulas

Solution:
Starting with the following,

$$\frac{f_3 - \frac{f_2-f_1}{x_2-x_1}(x_3-x_1) - f_1}{(x_3-x_1)(x_3-x_2)}$$

we want to end with this formula,

$$\frac{\frac{f_3-f_2}{x_3-x_2} - \frac{f_2-f_1}{x_2-x_1}}{x_3-x_1}$$

Now from the first equation we are going we are going to divide everything by $(x_3 - x_2)$ so we end with the following,

$$\frac{\frac{f_3}{x_3-x_2} - \frac{f_2-f_1}{(x_2-x_1)(x_3-x_2)}(x_3-x_1) - \frac{f_1}{(x_3-x_2)}}{(x_3-x_1)}$$

Now we can focus on the numerator of this fraction;

$$\frac{\frac{f_3}{x_3-x_2} - \frac{(f_2-f_1)(x_3-x_1)}{(x_2-x_1)(x_3-x_2)} - \frac{f_1}{(x_3-x_2)}}{(x_3-x_1)}$$

Now we can move the f_3 and the f_1 together,

$$\frac{\frac{f_3}{x_3-x_2} - \frac{f_1}{(x_3-x_2)} - \frac{(f_2-f_1)(x_3-x_1)}{(x_2-x_1)(x_3-x_2)}}{(x_3-x_1)}$$

I can add and subtract a multiple of f_2 so that we have,

$$\frac{\frac{f_3-f_2}{x_3-x_2} + \frac{f_2-f_1}{(x_3-x_2)} - \frac{(f_2-f_1)(x_3-x_1)}{(x_2-x_1)(x_3-x_2)}}{(x_3-x_1)}$$

Now we can, keep the one fraction together as that is part of the formula that we want to get to at the end. We can combine the other two fractions and simplify to get.

$$\frac{\frac{f_3-f_2}{x_3-x_2} + \frac{(f_2-f_1)(x_2-x_1) - (f_2-f_1)(x_3-x_1)}{(x_2-x_1)(x_3-x_2)}}{(x_3-x_1)}$$

For the sake of space, I am going to focus on simplifying the one fraction on the top $\frac{(f_2-f_1)(x_2-x_1)-(f_2-f_1)(x_3-x_1)}{(x_2-x_1)(x_3-x_2)}$ and the simplification.

$$\frac{f_2x_2-f_2x_1-f_1x_2+f_1x_1-(f_2x_3-f_2x_1-f_1x_3+f_1x_1)}{(x_2-x_1)(x_3-x_2)}$$

$$\frac{f_2x_2-f_2x_1-f_1x_2+f_1x_1-f_2x_3+f_2x_1+f_1x_3-f_1x_1)}{(x_2-x_1)(x_3-x_2)}$$

$$\frac{f_2x_2-f_1x_2-f_2x_3+f_1x_3}{(x_2-x_1)(x_3-x_2)}$$

$$\frac{-f_2(x_3-x_2)+f_1(x_3-x_2)}{(x_2-x_1)(x_3-x_2)}$$

$$\frac{(f_1-f_2)(x_3-x_2)}{(x_2-x_1)(x_3-x_2)}$$

$$\frac{(f_1-f_2)}{(x_2-x_1)}$$

$$\frac{-(f_2-f_1)}{(x_2-x_1)}$$

Plugging it back into the equation that we were working on we get,

$$\frac{\dfrac{f_3-f_2}{x_3-x_2}+\dfrac{-(f_2-f_1)}{(x_2-x_1)}}{(x_3-x_1)}$$

$$\frac{\dfrac{f_3-f_2}{x_3-x_2}-\dfrac{f_2-f_1}{x_2-x_1}}{(x_3-x_1)}$$

Therefore showing the that

$$\frac{f_3-\frac{f_2-f_1}{x_2-x_1}(x_3-x_1)-f_1}{(x_3-x_1)(x_3-x_2)}=\frac{\frac{f_3-f_2}{x_3-x_2}-\frac{f_2-f_1}{x_2-x_1}}{x_3-x_1}.$$