Risk Aware Belief-dependent Constrained Simplified POMDP Planning - Supplementary Material

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This document provides supplementary material to the paper [1]. Therefore, it should not be considered a self-contained document, but instead regarded as an appendix of [1]. Throughout this report, all notations and definitions are with compliance to the ones presented in [1].

This supplementary document contains:

1. Discussion on applying the simplification paradigm in the context of the second form of the constraint;
2. The proofs of the claims of the main manuscript.

1 Application of the simplification on the second form of the constraint

$$P\left(\prod_{\ell=k}^{k+L-1}\mathbf{1}\left\{\phi(b_{\ell+1},b_\ell)\geq\delta\right\}|b_k,\pi^*\right)\geq P\left(\prod_{\ell=k}^{k+L-1}\mathbf{1}\left\{l(b_{\ell+1},b_\ell)\geq\delta\right\}|b_k,\pi^*\right), \tag{1}$$

$$P\left(\prod_{\ell=k}^{k+L-1}\mathbf{1}\left\{\phi(b_{\ell+1},b_\ell)\geq\delta\right\}|b_k,\pi^*\right)\leq P\left(\prod_{\ell=k}^{k+L-1}\mathbf{1}\left\{u(b_{\ell+1},b_\ell)\geq\delta\right\}|b_k,\pi^*\right), \tag{2}$$

The proof is similar to the proof of Theorem 1.

2 Proof of Lemma 1 (representation of our general constraint).

$$\begin{aligned}&P\left(c(b_{k:k+L})|b_k,\pi,a_k\right)=\int_{\substack{b_{k+1:k+L}\\ z_{k+1:k+L}}}P\left(c(b_{k:k+L})|b_k,\pi,a_k,b_{k+1:k+L}\right)\\&\mathbb{P}(b_{k+1:k+L}|b_k,\pi,a_k,z_{k+1:k+L})p(z_{k+1:k+L}|b_k,\pi,a_k)= &(3)\\&\mathbb{E}_{z_{k+1:k+L}}\left[c(b_{k:k+L})\middle|b_k,\pi,a_k\right]. &(4)\end{aligned}$$

We used the fact that $p(b_{k+1:k+L}|b_k,\pi,a_k,z_{k+1:k+L})$ is Dirac's delta function. ■

3 Proof of Theorem 1

Since it holds that $l(\omega)\leq\phi(\omega)\ \forall\omega\in\Omega$ so the following relation over the sets holds

$$\{\omega:l(\omega)>\delta\}\subseteq\{\omega:\phi(\omega)>\delta\}\quad\forall\delta\in\mathbb{R}. \tag{5}$$

Hence

$$P\Big(\{\omega:\ell(\omega)>\delta\}\Big)\leq P\Big(\{\omega:\phi(\omega)>\delta\}\Big)\quad\forall\delta\in\mathbb{R}. \tag{6}$$

This is known as *usual stochastic order*. ■

4 Proof of Lemma 2 (decomposition of the safe trajectory probability)

$$P(\tau \in \times_{i=1}^{L} X_i^{\text{safe}} | b_k, \pi_{k+1:k+L-1}, a_k) = \mathbb{E}_{\tau}\left[\mathbf{1}\left\{\tau \in \times_{i=1}^{L} X^{\text{safe}}\right\}\middle| b_k, \pi_{k+1:k+L-1}, a_k\right]$$

$$= \int_{\tau} \mathbf{1}\left\{\tau \in \times_{i=1}^{L} X^{\text{safe}}\right\} \mathbb{P}(\tau | b_k, \pi_{k+1:k+L-1}, a_k) \tag{7}$$

Paying attention that the event $\mathbf{1}\left\{\tau \in \times_{i=1}^{L} X_i^{\text{safe}}\right\} = \prod_{i=k}^{k+L} \mathbf{1}\left\{x_i \in X^{\text{safe}}\right\}$; we can write

$$P(\tau \in \times_{i=1}^{L} X_i^{\text{safe}} | b_k, \pi_{k+1:k+L-1}, a_k) = \int_{\tau} \prod_{i=k}^{k+L} \mathbf{1}\left\{x_i \in X^{\text{safe}}\right\} \mathbb{P}(\tau | b_k, \pi_{k+1:k+L-1}, a_k) \tag{8}$$

■

5 Proof of Lemma 3 (probability of the trajectory).

$$\begin{aligned}
&\mathbb{P}(x_{k:k+L} | b_k, \pi_{k+1:k+L-1}, a_k) = \\
&\int_{z_{k+L-1}} \mathbb{P}(x_{k+L} | z_{k+L-1}, x_{k:k+L-1}, b_k, \pi_{k+1:k+L-1}, a_k) \\
&\mathbb{P}(z_{k+L-1}, x_{k:k+L-1} | b_k, \pi_{k+1:k+L-1}, a_k) =
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\int_{z_{k+L-1}} \mathbb{P}_T(x_{k+L} | x_{k+L-1}, a_{k+L-1}) \mathbb{P}_Z(z_{k+L-1} | x_{k+L-1}) \\
&\mathbb{P}(x_{k:k+L-1} | b_k, \pi_{k+1:k+L-2}, a_k) = \\
&\int_{z_{k+L-1}} \mathbb{P}_T(x_{k+L} | x_{k+L-1}, a_{k+L-1}) \mathbb{P}_Z(z_{k+L-1} | x_{k+L-1}) \\
&\int_{z_{k+L-2}} \mathbb{P}_T(x_{k+L-1} | x_{k+L-2}, a_{k+L-2}) \mathbb{P}_Z(z_{k+L-2} | x_{k+L-2}) \\
&\mathbb{P}(x_{k:k+L-2} | b_k, \pi_{k+1:k+L-3}, a_k)
\end{aligned} \tag{10}$$

Overall

$$\begin{aligned}
&\mathbb{P}(\tau | b_k, \pi_{k+1:k+L-1}, a_k) = \\
&\mathbb{P}_T(x_{k+1} | x_k, a_k) b_k(x_k) \int_{z_{k+1:k+L-1}} \prod_{i=k+1}^{k+L-1} \mathbb{P}_T(x_{i+1} | x_i, \pi(b_i(b_{i-1}, a_{i-1}, z_i))) \mathbb{P}_Z(z_i | x_i)
\end{aligned} \tag{11}$$

■

6 Proof of Lemma 4 Decomposition of the safe belief sequence

The following relation is a direct consequence of the definition of indicator function

$$P(x_{k:k+L} \in \times_{i=1}^{L} X_i^{\text{safe}} | b_{k:k+L}) = \mathbb{E}_{x_{k+} \sim b_{k+}}[\mathbf{1}\left\{x_{k:k+L} \in \times_{i=1}^{L} X^{\text{safe}}\right\}] \tag{12}$$

Paying attention on the following

$$\mathbf{1}\left\{x_{k:k+L} \in \times_{i=1}^{L} X^{\text{safe}}\right\} = \prod_{i=k}^{k+L} \mathbf{1}\left\{x_i \in X^{\text{safe}}\right\}, \tag{13}$$

we obtain

$$\mathbb{E}_{x_{k+} \sim b_{k+}}[\mathbf{1}\left\{x_{k:k+L} \in \times_{i=1}^{L} X^{\text{safe}}\right\}] = \tag{14}$$

$$\int_{x_{k:k+L}} \mathbf{1}\left\{x_{k:k+L} \in \times_{i=1}^{L} X^{\text{safe}}\right\} \mathbb{P}(x_{k:k+L} | b_{k:k+L}) = \tag{15}$$

$$P(\mathbf{1}\left\{x_k \in X^{\text{safe}}\right\} | b_k) \int_{x_{k+1:k+L}} \prod_{i=k+1}^{k+L} \mathbf{1}\left\{x_i \in X^{\text{safe}}\right\} \mathbb{P}(x_i | x_{k:k+i-1}, b_{k:k+i}) = \tag{16}$$

$$P(\mathbf{1}\left\{x_k \in X^{\text{safe}}\right\} | b_k) \prod_{i=k+1}^{k+L} \int_{x_i} \mathbf{1}\left\{x_i \in X^{\text{safe}}\right\} \mathbb{P}(x_i | x_{k:k+i-1}, b_{k:k+i}) \tag{17}$$

Observe that given the belief distribution any additional conditioning is redundant, so $p(x_i|x_{k:k+i-1}, b_{k:k+i}) = p(x_i|b_i)$. We obtain that

$$\underset{x_{k+}\sim b_{k+}}{\mathbb{E}}[\mathbf{1}\left\{x_{k:k+L} \in \times_{i=1}^{L} X^{\text{safe}}\right\}] = \prod_{i=k}^{k+L} P(\mathbf{1}\left\{x_i \in X^{\text{safe}}\right\}|b_i). \quad (18)$$

This concludes the proof. ■

References

1. A. Zhitnikov and V. Indelman. Risk aware belief-dependent constrained simplified pomdp planning. In *Proc. of the Intl. Symp. of Robotics Research (ISRR)*, 2022. submitted.