Qualitative Belief Space Planning via Compositions
Supplemenary Material

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This document provides supplementary material to [1]. Therefore, it should not be considered a self-contained document, but instead regarded as an appendix of [1]. Throughout this report, all notations and definitions are with compliance to the ones presented in [1]. Todo: revive [1]

1 Supplementary derivation of $\mathbb{P}(\beta_t|\mathcal{S}^{F_{t-1}:X_t},\mathcal{S}^{F_{t-1}},\mathcal{S}^{\beta_t},\mathcal{H}_t^-)$

We further develop the term $\mathbb{P}(\beta_t|\mathcal{S}^{F_{t-1}:X_t},\mathcal{S}^{F_{t-1}},\mathcal{S}^{\beta_t},\mathcal{H}_t)$ via marginaliztion over relevant metric realizations and considering dependencies:

$$\mathbb{P}(\beta_t|\mathcal{S}^{F_{t-1}:X_t},\mathcal{S}^{F_{t-1}},\mathcal{S}^{\beta_t},\mathcal{H}_t^-)=\iiint\limits_{x\in\mathcal{S}^{F_{t-1}:X_t},d\in\mathcal{S}^{F_{t-1}},\mathcal{L}\in\mathcal{S}^{\beta_t}}\mathbb{P}(\beta_t|x,d,\mathcal{L},F_{t-1})\mathbb{P}(x|\mathcal{S}^{F_{t-1}:X_t},\mathcal{H}_t^-)\mathbb{P}(d|\mathcal{S}^{F_{t-1}},\mathcal{H}_t^-)\mathbb{P}(\mathcal{L}|\mathcal{S}^{\beta_t},\mathcal{H}_t^-)\,dx\,dd\,d\mathcal{L}, \tag{1}$$

The term $\mathbb{P}(\beta_t|x,d,\mathcal{L},F_{t-1})$ is a deterministic geometric model that equals 1 if the metric hypotheses of β_t landmarks are inside the robot's sensing range, R (assumed to be a known hyperparameter), and 0 else, i.e.:

$$\mathbb{P}(\beta_t|x,d,\mathcal{L},F_{t-1})=\prod_{\mathcal{L}_i\in\{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}\}}\mathbb{1}\Big\{\|\mathcal{L}_i-x\|_2\leq\frac{R}{d}\Big\}, \tag{2}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are the local metric coordinate of the reference landmarks creating F_{t-1}. In most cases, $\mathcal{L}_1=(0,0)$ and $\mathcal{L}_2=(0,1)$. The metric priors can be further approximated as uniform distributions by neglecting the history term. Accordingly, (1) can be calculated offline.

2 Supplementary derivation of $\mathbb{P}(z_t|\mathcal{S}^{F_{t-1}:X_t},\mathcal{S}^{\beta_t},\beta_t,\mathcal{H}_t^-)$

We further develop the term $\mathbb{P}(z_t|\mathcal{S}^{F_{t-1}:X_t},\mathcal{S}^{\beta_t},\beta_t,\mathcal{H}_t^-)$ via marginaliztion over relevant metric realizations and considering dependencies:

$$\mathbb{P}(z_t|\mathcal{S}^{F_{t-1}:X_t},\mathcal{S}^{\beta_t},\beta_t,\mathcal{H}_t^-)=\iint\limits_{x\in\mathcal{S}^{F_{t-1}:X_t},\mathcal{L}\in\mathcal{S}^{\beta_t}}\mathbb{P}(z_t|x,\mathcal{L},F_{t-1})\mathbb{P}(x|\mathcal{S}^{F_{t-1}:X_t},\mathcal{H}_t^-)\mathbb{P}(\mathcal{L}|\mathcal{S}^{\beta_t},\mathcal{H}_t^-)\,dx\,d\mathcal{L}, \tag{3}$$

The term $\mathbb{P}(z_t|x,\mathcal{L},F_{t-1})$ is the metric measurement model. The metric priors $\mathbb{P}(x|\mathcal{S}^{F_{t-1}:X_t},\mathcal{H}_t^-)$ and $\mathbb{P}(\mathcal{L}|\mathcal{S}^{\beta_t},\mathcal{H}_t^-)$ can be further approximated as uniform distributions by neglecting the history term. Accordingly, (3) can be calculated offline.

3 Supplementary derivation of Eq. 10

$$
\begin{aligned}
J(b_k,a_{k+}) &= \mathop{\mathbb{E}}_{\beta_{k+1}}\Big[\mathop{\mathbb{E}}_{z_{k+1}|\beta_{k+1}}\Big[c_1(b_{k+1},a_k)+J(b_{k+1},a_{(k+1)+})\Big]\Big] \\
&= \sum_{\beta_{k+1}} \mathbb{P}(\beta_{k+1}|b_k,a_k) \int_{z_{k+1}} \mathbb{P}(z_{k+1}|\beta_{k+1},b_k,a_k)\cdot(c_1+J(b_{k+1},a_{(k+1)+}))\,dz_{k+1} \\
&\approx \sum_{m=1}^{N_\beta} \underbrace{\frac{\mathbb{P}(\beta^m_{k+1}|b_k,a_k)}{\sum_{q=1}^{N_\beta}\mathbb{P}(\beta^q_{k+1}|b_k,a_k)}}_{\tilde{w}^m} \int_{z_{k+1}} \mathbb{P}(z_{k+1}|\beta^m_{k+1},b_k,a_k)\cdot(c_1+J(b_{k+1},a_{(k+1)+}))\,dz_{k+1} \\
&\approx \sum_{i=1}^{n_\beta} \underbrace{\frac{\tilde{w}^i}{\sum_{q=1}^{n_{\beta_k}}\tilde{w}^q}}_{w^i} \int_{z_{k+1}} \mathbb{P}(z_{k+1}|\beta^i_{k+1},b_k,a_k)\cdot(c_1+J(b_{k+1},a_{(k+1)+}))\,dz_{k+1} \\
&\approx \sum_{i=1}^{n_\beta} \frac{w^i}{n_z}\sum_{j=1}^{n_z} \mathbb{P}(z^{i,j}_{k+1}|\beta^i_{k+1},b_k,a_k)\cdot(c_1+J(b_{k+1},a_{(k+1)+})),
\end{aligned}
\tag{4}
$$

where in the 1st approximation, we consider only the subset of N_β β_{k+1}'s realizations containing triplets that involve the current frame ($N_\beta{=}|\mathbb{L}|{-}2$), in the 2nd approximation, we further reduced this subset to the n_β triplets available in the belief ($n_\beta{\le}N_\beta$), and finally, in the 3rd approximation, we show how the inner expectation term can be evaluated via averaging over a finite set of z_{k+1} samples.

4 Supplementary derivation of $\mathbb{P}(\mathcal{S}^{F_t:X_t}|\mathcal{S}^{F_{t-1}:X_t},\mathcal{S}^\tau,\mathcal{H}_t)$

For each possible realization of the input variables described above, we marginalize over the corresponding metric state to calculate the model's outcome:

$$
\mathbb{P}(\mathcal{S}^{F_t:X_t}|\mathcal{S}^{F_{t-1}:X_t},\mathcal{S}^\tau,\mathcal{H}_t) = \iint\limits_{\mathcal{X}^{F_{t-1}:X_t}\in\mathcal{S}^{F_{t-1}:X_t},\,\mathcal{X}^\tau\in\mathcal{S}^\tau} \mathbb{P}(\mathcal{S}^{F_t:X_t},\mathcal{X}^{F_{t-1}:X_t},\mathcal{X}^\tau|\mathcal{S}^{F_{t-1}:X_t},\mathcal{S}^\tau,\mathcal{H}_t)\,d\mathcal{X}^{F_{t-1}:X_t}\,d\mathcal{X}^\tau. \tag{5}
$$

We continue developing the inner term using chain rule:

$$
\mathbb{P}(\mathcal{S}^{F_t:X_t},\mathcal{X}^{F_{t-1}:X_t},\mathcal{X}^\tau|\mathcal{S}^{F_{t-1}:X_t},\mathcal{S}^\tau,\mathcal{H}_t) = \mathbb{P}(\mathcal{S}^{F_t:X_t}|\mathcal{X}^{F_{t-1}:X_t},\mathcal{X}^\tau,a_t^{Link})\mathbb{P}(\mathcal{X}^{F_{t-1}:X_t}|\mathcal{S}^{F_{t-1}:X_t},\mathcal{H}_t)\mathbb{P}(\mathcal{X}^\tau|\mathcal{S}^\tau,\mathcal{H}_t), \tag{6}
$$

where $\mathbb{P}(\mathcal{S}^{F_t:X_t}|\mathcal{X}^{F_{t-1}:X_t},\mathcal{X}^\tau,a_t^{Link})$ is a geometric model that deterministically determines the new state, given a metric realization of the former one and of the related triplet. The metric priors $\mathbb{P}(\mathcal{X}^{F_{t-1}:X_t}|\mathcal{S}^{F_{t-1}:X_t},\mathcal{H}_t)$ and $\mathbb{P}(\mathcal{X}^\tau|\mathcal{S}^\tau,\mathcal{H}_t)$ can be further approximated as uniform distributions by neglecting the history term. Accordingly, (5) can be calculated offline.

5 Supplementary derivation of $\mathbb{P}(\mathcal{S}^{F_{t-1}}|\mathcal{S}^{\tau_1},\mathcal{S}^{\tau_2},\mathcal{H}_t^{\tau_1},\mathcal{H}_t^{\tau_2})$

We can further develop this posterior term via marginalization over the metric state of τ_1 and τ_2, followed by chain rule:

$$
\mathbb{P}(\mathcal{S}^{F_t}|\mathcal{S}^{\tau_1},\mathcal{S}^{\tau_2},\mathcal{H}_t^{\tau_1},\mathcal{H}_t^{\tau_2}) = \iint\limits_{\mathcal{X}^{\tau_1},\mathcal{X}^{\tau_2}} \mathbb{P}(\mathcal{S}^{F_t}|\mathcal{X}^{\tau_1},\mathcal{X}^{\tau_2})\mathbb{P}(\mathcal{X}^{\tau_1},\mathcal{X}^{\tau_2}|\mathcal{S}^{\tau_1},\mathcal{S}^{\tau_1},\mathcal{H}_t^{\tau_1},\mathcal{H}_t^{\tau_2})\,d\mathcal{X}^{\tau_1}\,d\mathcal{X}^{\tau_2},
$$

where $\mathbb{P}(\mathcal{S}^{F_t}|\mathcal{X}^{\tau_1},\mathcal{X}^{\tau_2})$ is a Dirac function equals to 1 if $\|\mathcal{X}^{\tau_1}{-}\mathcal{X}^{\tau_2}\|_2$ is in the interval represented by the value of $\mathcal{S}^{F_t}$ and to 0 otherwise. The metric prior term can be approximated via $\mathbb{P}(\mathcal{X}^{\tau_1},\mathcal{X}^{\tau_2}|\mathcal{S}^{\tau_1},\mathcal{S}^{\tau_1},\mathcal{H}_t^{\tau_1},\mathcal{H}_t^{\tau_2}) \approx \prod_{i=1}^2 \mathbb{P}(\mathcal{X}^{\tau_i}|\mathcal{S}^{\tau_i},\mathcal{H}_t^{\tau_i})$, where $\forall i{\in}\{1{,}2\}$ the individual prior term can be further approximated via $\mathbb{P}(\mathcal{X}^{\tau_i}|\mathcal{S}^{\tau_i},\mathcal{H}_t^{\tau_i}){\approx}\mathbb{P}(\mathcal{X}^{\tau_i}|\mathcal{S}^{\tau_i})$, i.e., assuming a uniform distribution.

6 *Composable* triplet sets

This section provides a reminder of the term *Composable* sets of triplets, which is only briefly (and informally) discussed in the paper. Moreover, the definitions given in this section are crucial to understanding the proof in Sec. 7.3.

In the following, we provide a series of definitions, originally formulated in [3], where the last one refers to the *Composable* set term.

Definition 1. Let $\mathcal{T}$ be a set of triplets. The *Landmark Space* of $\mathcal{T}$, denoted by $\mathcal{L}(\mathcal{T})$, is defined as:

$$\mathcal{L}(\mathcal{T})=\bigcup_{\tau\in\mathcal{T}}\tau$$

Note that the *Landmark Space* of a single triplet set is the triplet itself: $\mathcal{L}(\{\tau\})=\tau$.

Definition 2. Let $\mathcal{T}$ be a set of triplets. A *Cut* $C=(\mathcal{T}_L,\mathcal{T}_R)$ of $\mathcal{T}$, is a partition of $\mathcal{T}$ into two disjoint subsets, $\mathcal{T}_L$ and $\mathcal{T}_R$, s.t. $\forall\tau\in\mathcal{T}$, either $\tau\in\mathcal{T}_L$ or $\tau\in\mathcal{T}_R$, but not both.

Definition 3. Let $\mathcal{T}$ be a set of triplets and let $\alpha\in\mathbb{N}\cup\{0\}$. A *Cut* $C=(\mathcal{T}_L,\mathcal{T}_R)$ of $\mathcal{T}$ is called α-*common* if $|\mathcal{L}(\mathcal{T}_L)\cap\mathcal{L}(\mathcal{T}_R)|\geq\alpha$.

We are now ready to define the term of a *Composable* set of triplets.

Definition 4. Let $\mathcal{T}$ be a set of triplets and let $\mathcal{L}$ be a *Landmark Space*. We say that $\mathcal{T}$ is *Composable* under $\mathcal{L}$, if $\mathcal{L}\subseteq\mathcal{L}(\mathcal{T})$, and one of the following holds:

1. $|\mathcal{T}|=1$.
2. $|\mathcal{T}|>1$ and **there is** a 2-*common Cut* $C=(\mathcal{T}_L,\mathcal{T}_R)$ of $\mathcal{T}$, s.t. $\mathcal{T}_L$ is *Composable* under $\mathcal{L}(\mathcal{T}_L)$ and $\mathcal{T}_R$ is *Composable* under $\mathcal{L}(\mathcal{T}_R)$.

An illustration of a *Composable* set of triplets under the *Landmark Space* $\mathbb{L}$ can be found in Fig. 1a.s

7 Compositions and *Link-Graph*s

A *Link-Graph* is a topological graph representation for QRM. In this section, we use the *Link-Graph* and its properties to prove that in some scenarios, a plan can be found **exclusively** via compositions. We emphasize that we do not use the *Link-Graph* in our algorithm but rather exploit it for explanatory purposes alone.

7.1 *Link-Graph*

First defined in [2], the *Link-Graph* was used for generating high-level plans over a QRM as part of a more comprehensive planning architecture called *Q-Link*.

The *Link-Graph* encodes connectivity between triplets, represented by its nodes, and local frames, represented by its edges, as illustrated in Fig. 1b. Formally, the *Link-Graph* is defined as follows:

Definition 5. A *Link-Graph* is a graph $G=(V,E)$ where:

1. Each node $v\in V$ represents a triplet of landmarks, i.e., $v=\{L^1,L^2,L^3\}$.
2. There is an edge $e=(v_1,v_2)\in E$ if and only if $|v_1\bigcap v_2|=2$ (i.e., nodes v_1 and v_2 share exactly 2 landmarks in common).

7.2 *Link-Graph*'s connectivity and *Link*s

The *Link-Graph* is a good representation for links mobility, since each triplet node enables a transition, or *Link*, between any two frames' edges connected to it.

We provide the following example to clarify the above state. Consider a robot localized relative to frame AB and the triplets ABC and BCD to be the only available information. Suppose the robot aims to find landmark D. The location of D is only available relative to frame BC, through the triplet BCD. Consequently, the robot aims to link to BC next. To that end, it first deduces that the edge representing BC connects BCD with the triplet that includes its current frame, ABC. Then, the robot uses AB:C estimation to reach C and localizes itself relative to BC, i.e., it **links** from AB to BC. Finally, to accomplish its goal, it finds D via BC:D.

The topological rule illustrated in the example above is formulated as follows:

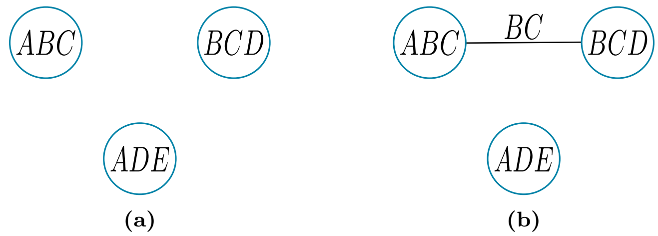

Figure 1: (a) An illustration of a *Composable* set under $\mathbb{L}$=$\{A,B,C,D,E\}$, consists of three tripelts; (b) An illustration of a *Link-Graph*, based on the set from (a). The graph has a single edge connecting ABC with BCD, as B and C are mutual landmarks; There is an Invertible transformation between the two.

Lemma 1. A direct *Link* from F_1 to F_2 is feasible based on a triplet τ, if $F_i{\subseteq}\tau$, $\forall i{\in}\{1{,}2\}$, or, in ***Link-Graph*'s terms**, if the edges representing F_1 and F_2 are connected to the node representing τ.

One can further conclude from Lemma 1 that a *Link-Graph*'s path encodes a feasible sequence of link actions, where the edges along the path are the different frames, and the in-between nodes are the triplets the robot relies on to execute the *Link*s.

7.3 Compositions' necessity in sparse scenarios - A *Link-Graph*'s based proof

Using the insight from Sec. 7.2, we now aim to prove that in some cases, a plan can be found only via compositions.

Before approaching the formal proof, we provide some intuition. The key point of our explanation is simple. Via compositions, the robot can link to more frames than it could before. According to the conclusion from Lemma 1, the robot is allowed to link based on a path of a *Link-Graph*, whose nodes represent the set of available triplets, $\mathcal{M}_k$. Thus, without compositions, links are possible only based on existing paths. In contrast, using composition, we can create new triplets, i.e., augment the graph with new nodes, thus creating additional paths. Consequently, in cases where there is no path in the *Link-Graph* at planning time to a target triplet without compositions, we cannot find a valid plan towards the triplet.

Suppose that the robot's initial map, $\mathcal{M}_k$, is *Composable* under $\mathbb{L}$ (see Fig. 1a for illustration). Alternatively, we could assume that a *Link-Graph* whose nodes represent $\mathcal{M}_k$ is *Composable* under $\mathbb{L}$, considering the following definition:

Definition 6. Let G=(V,E) be a *Link-Graph*. We say that G is *Composable* under $\mathbb{L}$ if V represents a *Composable* set of triplets under $\mathbb{L}$.

We aim to prove that any connected Link-Graph is, in particular, *Composable*, but not the other way around:

Theorem 2. Let $\mathbb{G}_{cn}$ and $\mathbb{G}_{cm}$ be the sets of all connected and *Composable Link-Graph*s under the landmark space $\mathbb{L}$, respectively. Then $\mathbb{G}_{cn}{\subsetneq}\mathbb{G}_{cm}$.

Proof. First we show that $\mathbb{G}_{cn}{\subseteq}\mathbb{G}_{cm}$.
Let G=(V,E) be a connected Link graph under $\mathbb{L}$. We prove that G is also *Composable* under $\mathbb{L}$ by induction on number of vertices in G, $|V|$.
Base step: When $|V|$=1, V is *Composable* under $\mathbb{L}$ by definition. Thus, G is also *Composable* under $\mathbb{L}$ by definition.
Induction step: Suppose G is *Composable* under $\mathbb{L}$ for all $1{\leq}|V|{\leq}n$. We show that G is *Composable* under $\mathbb{L}$ for $|V|$=n+1. We choose a cut in G, C=(S,T), s.t. $G_S{\triangleq}\big(S,\big\{(u,v){\in}E|(u,v){\in}S^2\big\}\big)$ and $G_T{\triangleq}\big(T,\big\{(u,v){\in}E|(u,v){\in}T^2\big\}\big)$ are both connected graphs, where $S,T{\neq}\emptyset$. Note that such choise always exists for any $|V|{>}1$, since G is connected. Let us now observe the set of edges in G connecting S with T, that is, $E_{S,T}{\triangleq}\{(u,v){\in}E|u{\in}S \wedge v{\in}T\}$. Since G is connected, we are guaranteed that $E_{S,T}{\neq}\emptyset$. Thus, C is a 2-common cut in G. Finally, since G_S,G_T are both connected subgraphs of G, they are both connected *Link-Graph*s, and since $1{\leq}|S|,|T|{\leq}n$, we further conclude that they are both *Composable* under $\mathbb{L}$, according to the assumption. That is to say, we showed by definition that for $|V|$=n+1, G is *Composable* under $\mathbb{L}$.
Conclusion: $\mathbb{G}_{cn}{\subseteq}\mathbb{G}_{cm}$.
We are left to show an instance of a *Composable Link-Graph* under $\mathbb{L}$ that is not connected. To that end, consider the landmark space $\{A,B,C,D,E\}$, and the *Link-Graph* from Fig. 1b.
Final conclusion: $\mathbb{G}_{cn}{\subsetneq}\mathbb{G}_{cm}$
■

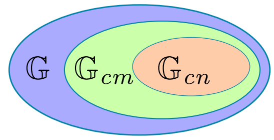

Figure 2: Relationships between general *Link-Graphs* ($\mathbb{G}$), *Composable Link-Graphs* ($\mathbb{G}_{cm}$), and connected *Link-Graphs*, all under the same landmark space, ($\mathbb{G}_{cn}$) are described through a Venn diagram.

Meaning, in some scenarios, where $\mathcal{M}_k$ creates a *Composable Link-Graph* that is disconnected, compositions are necessary to allow the robot to plan towards its goal.

8 Supplementary derivation of a single composition operation

We directly compose the triplet τ_3 using the source triplets τ_1 and τ_2, using the following probabilistic formulation, based on [1]:

$$\mathbb{P}(\mathcal{S}^{\tau_3}|\mathcal{S}^{\tau_1},\mathcal{S}^{\tau_2},\mathcal{H}^1,\mathcal{H}^2)=\int\limits_{\mathcal{X}^{\tau_1}\in\mathcal{S}^{\tau_1}}\int\limits_{\mathcal{X}^{\tau_2}\in\mathcal{S}^{\tau_2}}\mathbb{P}(\mathcal{S}^{\tau_3}|\mathcal{X}^{\tau_1},\mathcal{X}^{\tau_2})\mathbb{P}(\mathcal{X}^{\tau_1},\mathcal{X}^{\tau_2}|\mathcal{S}^{\tau_1},\mathcal{S}^{\tau_1},\mathcal{H}_t^{\tau_1},\mathcal{H}_t^{\tau_2})\,d\mathcal{X}^{\tau_1}\,d\mathcal{X}^{\tau_2}, \quad (7)$$

where $\mathbb{P}(\mathcal{S}^{\tau_3}|\mathcal{X}^{\tau_1},\mathcal{X}^{\tau_2})$ is a simple deterministic geometric model. The metric prior term can be approximated via $\mathbb{P}(\mathcal{X}^{\tau_1},\mathcal{X}^{\tau_2}|\mathcal{S}^{\tau_1},\mathcal{S}^{\tau_1},\mathcal{H}_t^{\tau_1},\mathcal{H}_t^{\tau_2}) \approx \prod_{i=1}^{2}\mathbb{P}(\mathcal{X}^{\tau_i}|\mathcal{S}^{\tau_i},\mathcal{H}_t^{\tau_i})$, where $\forall i\in\{1,2\}$ the individual prior term can be further approximated via $\mathbb{P}(\mathcal{X}^{\tau_i}|\mathcal{S}^{\tau_i},\mathcal{H}_t^{\tau_i})\approx\mathbb{P}(\mathcal{X}^{\tau_i}|\mathcal{S}^{\tau_i})$, i.e., assuming a uniform distribution.

References

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[2] Jennifer Padgett and Mark Campbell. Q-link: A general planning architecture for navigation with qualitative relational information. *Robotics and Autonomous Systems*, 108:51–65, 2018.

[3] Itai Zilberman, Ehud Rivlin, and Vadim Indelman. Incorporating compositions in qualitative approaches. *IEEE Robotics and Automation Letters*, 7(2):2660–2667, 2022.