Decision Making and Planning in Sparse (Conservative) Belief Space

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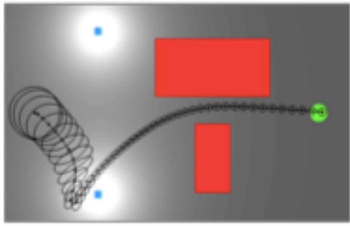
Introduction

- Decision making under uncertainty - fundamental problem in autonomous systems and artificial intelligence
- Examples
  - **Belief space planning** in uncertain/unknown environments (e.g. for autonomous navigation)
  - **Active simultaneous localization and mapping** (SLAM)
  - **Informative planning, active sensing**
  - **Sensor selection, sensor deployment**
  - **Multi-agent** informative planning and active SLAM
  - **Graph sparsification** for long-term autonomy



Introduction

- Information-theoretic decision making
  - **Objective**: find action(s) that minimizes an information-theoretic objective function (e.g. entropy)
  - Extensively investigated, e.g., in the context of sensor selection

- Decision making over **high-dimensional** state spaces is expensive!

State vector: $\mathbf{x} \in \mathbb{R}^n$

Covariance matrix: $\Sigma \doteq \mathbb{E}\left[\left(\mathbf{x} - \mathbb{E}\left[\mathbf{x}\right]\right)\left(\mathbf{x} - \mathbb{E}\left[\mathbf{x}\right]\right)^T\right] \in \mathbb{R}^{n \times n}$

- Evaluating impact of a candidate action typically involves determinant calculation - $O(n^3)$ in the general case

Motivating Example I – Belief Space Planning

- **Joint** state vector $$X_k \doteq \{\underbrace{x_0, \ldots, x_k}_{\text{Past \& current robot states}}, \underbrace{L_k}_{\text{Mapped environment}}\}$$

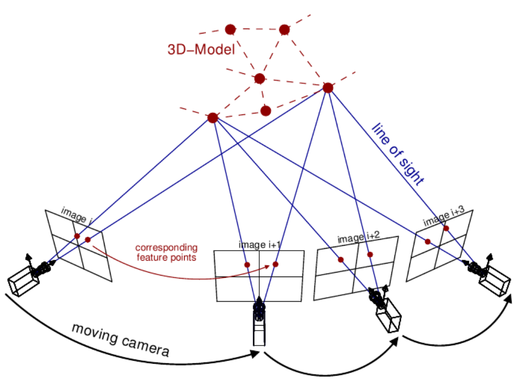
- **Joint** probability distribution function $p\left(X_k | \mathcal{Z}_k, \mathcal{U}_{k-1}\right)$

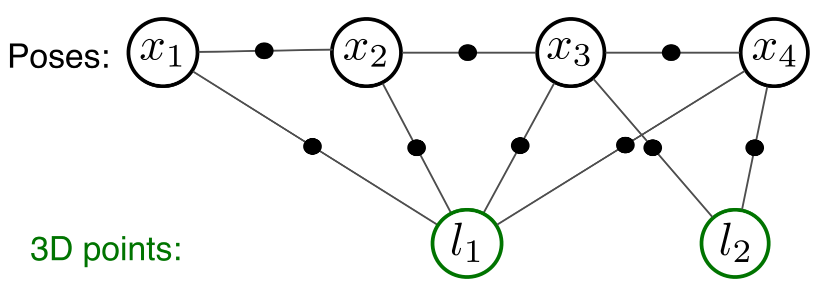
$$p\left(X_k | \mathcal{Z}_k, \mathcal{U}_{k-1}\right) = priors \cdot \prod_{i=1}^{k} p\left(x_i | x_{i-1}, u_{i-1}\right) \underline{p\left(z_i | X_i^o\right)}$$

General observation model $X_i^o \subseteq X_i$

- Computationally-efficient maximum a posteriori inference e.g. [Kaess et al. 2012]

$$p\left(X_k | \mathcal{Z}_k, \mathcal{U}_{k-1}\right) \sim N\left(X_k^*, \Sigma_k\right)$$

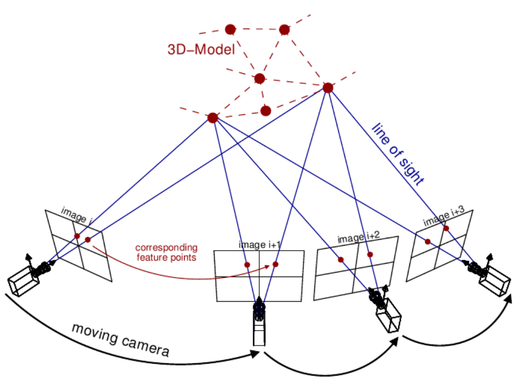


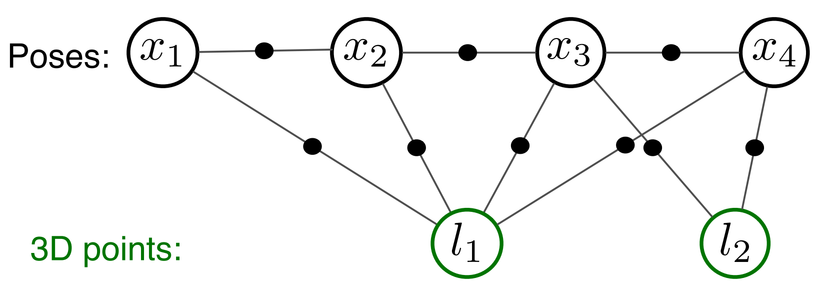


Motivating Example I – Belief Space Planning

- How to autonomously determine future action(s)?
- Involves reasoning, for different candidate actions, about belief evolution

$$p\left(X_k | \mathcal{Z}_k, \mathcal{U}_{k-1}\right) \sim N\left(X_k^*, \Sigma_k\right)$$





Motivating Example I – Belief Space Planning

- How to autonomously determine future action(s)?
- Involves reasoning, for different candidate actions, about belief evolution

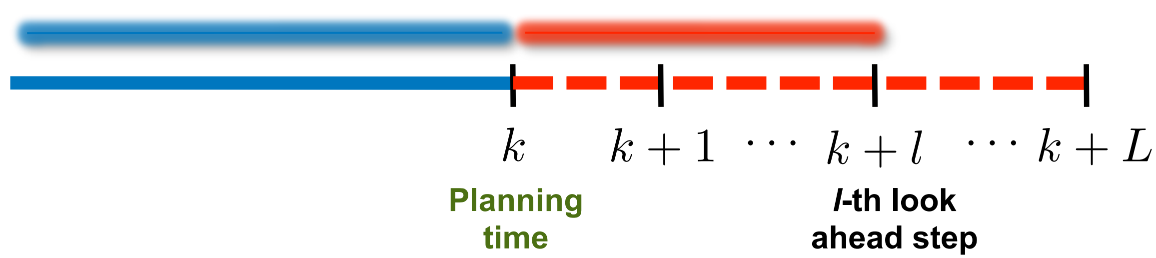
- Belief at the l-th look-ahead step

$$b\left(X_{k+l}\right) \doteq p\left(X_{k+l} | \mathcal{Z}_k, \mathcal{U}_{k-1}, Z_{k+1:k+l}, u_{k:k+l-1}\right)$$

Joint state at the *l*-th look ahead step

Past controls & measurements

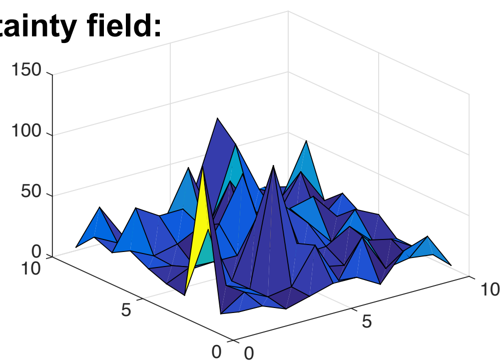
Controls & measurements at the first *l* look-ahead steps



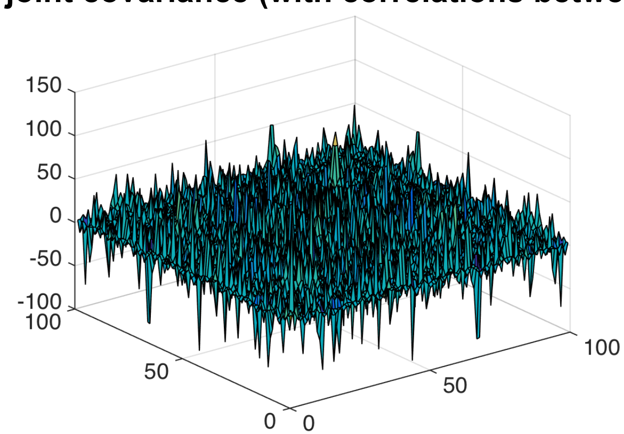
Motivating Example II - Sensor Deployment

- **Objective**: deploy k sensors in an $N \times N$ area
  - provide localization
  - monitor spatial-temporal field (e.g. temperature)
  - …

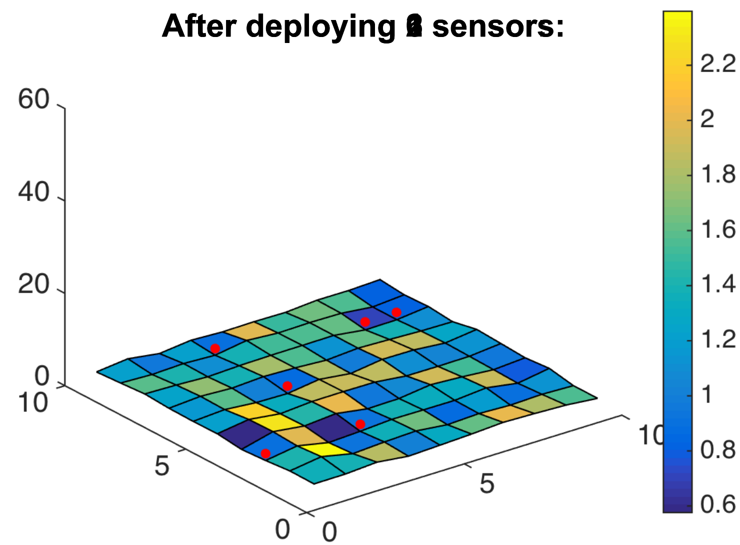
Prior uncertainty field:



A priori joint covariance (with correlations between cells)



After deploying 8 sensors:



Introduction

- More generally, decision making over multiple look-ahead steps
  - A partially observable Markov decision process (POMDP), NP-hard
  - Different sub-optimal approaches exist (greedy, sampling, …)

- **This work:**
  - Resort to **conservative information fusion** techniques for **information-theoretic decision making**

- **Conservative information fusion** approaches
  - Allow to fuse information from multiple correlated sources, **without** knowing the correlation
  - Guarantee **consistent** estimation
  - Pioneered by Julier & Uhlmann [ACC 1997]: **Covariance intersection**

Introduction

- More generally, decision making over multiple look-ahead steps
  - A partially observable Markov decision process (POMDP), NP-hard
  - Different sub-optimal approaches exist (greedy, sampling, …)

- **This work:**
  - Resort to **conservative information fusion** techniques for **information-theoretic decision making**
  - **Motivation**: these techniques allow correlation terms to be unknown!
  - **Key idea:**
    - Reduce computational complexity by (appropriately) dropping correlations
    - Extreme case: drop all correlations; computational complexity becomes

$$O(n^3) \longrightarrow O(n)$$

  - **Do we get the same performance??**

Problem Formulation

- Probability distribution function (pdf) at time t_k : $p\left(x_k | z_{0:k}, u_{0:k-1}\right)$
- Transition/motion model $p\left(x_{k+1} | x_k, u_k\right)$
- Observation model (unary) $p\left(z_k | x_k\right)$

- Given control u_k and new observation(s) z_{k+1}, pdf becomes

$$p\left(x_{k+1} | z_{0:k+1}, u_{0:k}\right) = \eta p\left(z_k | x_k\right) \cdot \int p\left(x_k | z_{0:k}, u_{0:k-1}\right) p\left(x_{k+1} | x_k, u_k\right) dx_k$$

- Entropy: $\mathcal{H}\left(p\left(x\right)\right) = -\mathbb{E}\left[\log p\left(x\right)\right] = -\int p\left(x\right) \log p\left(x\right) dx$

- **Information-theoretic objective function** (single look-ahead step):

$$J\left(u_k\right) = \mathbb{E}_{z_{k+1}}\left[\mathcal{H}\left(p\left(x_{k+1} | z_{0:k+1}, u_{0:k}\right)\right)\right]$$

- **Optimal control:** $u_k^\star = \underset{u_k}{\arg\min}\, J\left(u_k\right).$

Problem Formulation

- **Assumptions**:
  - Gaussian distributions $p\left(x_k | z_{0:k}, u_{0:k-1}\right) = N\left(\mu_k, I_k^{-1}\right)$
  - Unary observation models $z_i = h_i\left(x_i\right) + v_i \quad , \quad v_i \sim N\left(0, \Sigma_{vi}\right)$
  - No dynamics, deterministic control
- Entropy becomes $\mathcal{H}\left(p\left(x_{k+1} | z_{0:k+1}, u_{0:k}\right)\right) = -\frac{1}{2}\log\left[\left(2\pi e\right)^n \left|I_{k+1}^+\right|\right]$
- A posteriori information matrix: $I_{k+1}^+ = I_k + H^T \Sigma_v^{-1} \underbrace{H}_{\text{Jacobian}}$
- Best action = highest information gain
- Impact evaluation for a candidate action is in the general case: $O(n^3)$

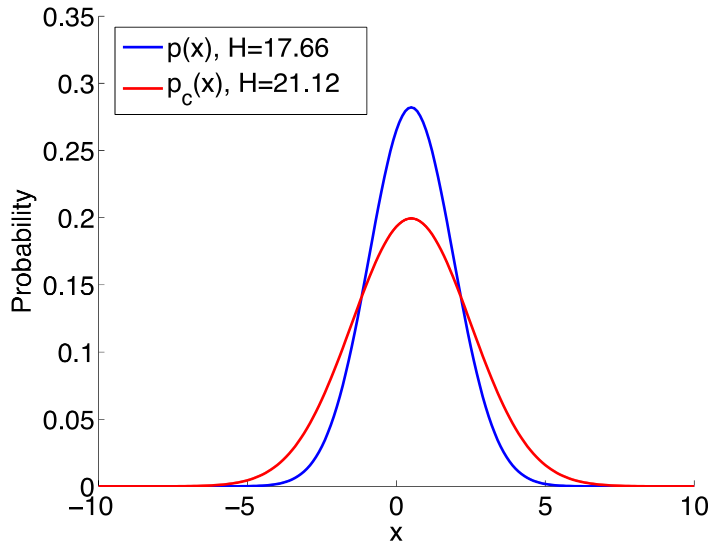
Conservative Information Space

- Conservative approximation of a pdf – sufficient conditions **[Bailey et al. 2012 Fusion]**:
  - Entropy: $\mathcal{H}\left(p\left(x\right)\right) \leq \mathcal{H}\left(p_c\left(x\right)\right)$
  - Order preserving (same shape):

$$\forall x_i, x_j \quad p_c(x = x_i) \leq p_c(x = x_j) \ \text{ iff } \ p(x = x_i) \leq p(x = x_j)$$

- Gaussian case:

$$|I_c| \leq |I|$$



Concept

Decision Making Over a Conservative Information Space - **1D Case**

- Consider some two actions **a** and **b** with measurement models

$$z_a = h_a(x) + v_a \qquad z_b = h_b(x) + v_b$$

- **Theorem** - for the 1D case:

$$\boxed{\left|I^{a+}\right| \leq \left|I^{b+}\right| \text{ iff } \left|I_c^{a+}\right| \leq \left|I_c^{b+}\right|}$$

  – where the a posteriori information matrices are calculated using

	Action **a**	Action **b**
• **original** information matrix:	$I^{a+} = I + H_a^T \Sigma_v^{-1} H_a$	$I^{b+} = I + H_b^T \Sigma_v^{-1} H_b$
• **conservative** information matrix:	$I_c^{a+} = I_c + H_a^T \Sigma_v^{-1} H_a$	$I_c^{b+} = I_c + H_b^T \Sigma_v^{-1} H_b$

Concept

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- In words:

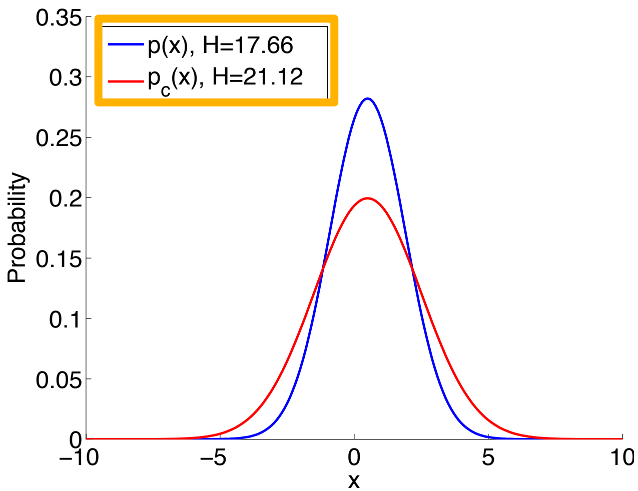
  the impact of any two candidate actions has the same trend regardless if it is calculated based on the original or conservative information space

- Therefore: decision making can be done considering a conservative information space

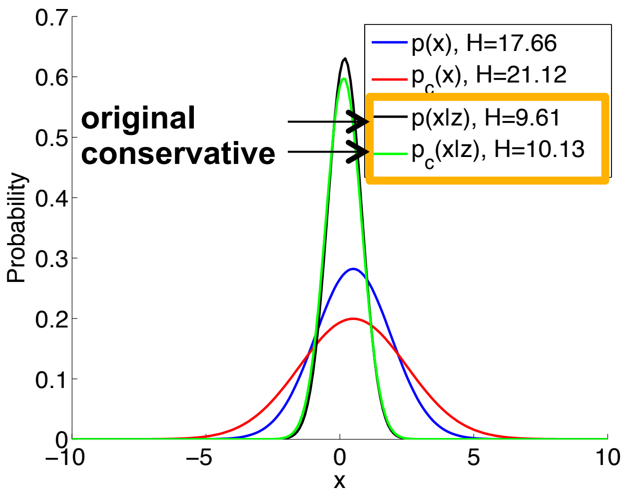
Basic Example – 1D Case

$$\left|I^{a+}\right| \leq \left|I^{b+}\right| \text{ iff } \left|I_c^{a+}\right| \leq \left|I_c^{b+}\right|$$

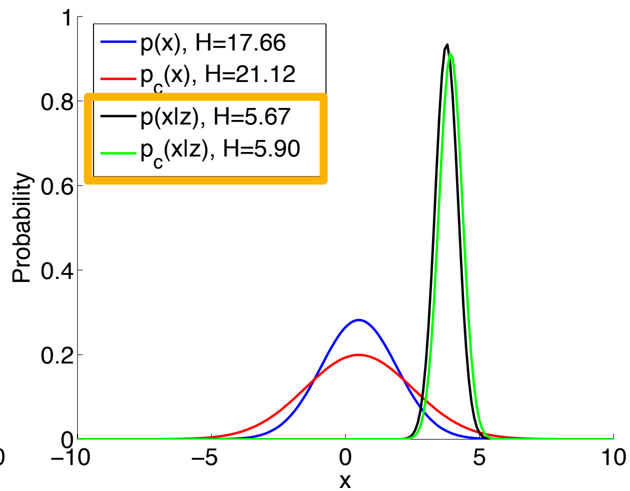
Entropy values are shown in legend



I, I_c



Action a



Action b

$$z_a = h_a(x) + v_a \qquad z_b = h_b(x) + v_b$$

$$\Sigma_v = 0.5^2 \qquad \Sigma_v = 0.2^2$$

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High Dimensional State Space

Recall: $\mathcal{H}\left(p\left(x_{k+1}|z_{0:k+1},u_{0:k}\right)\right)=-\frac{1}{2}\log\left[(2\pi e)^{n}\left|I_{k+1}^{+}\right|\right]$

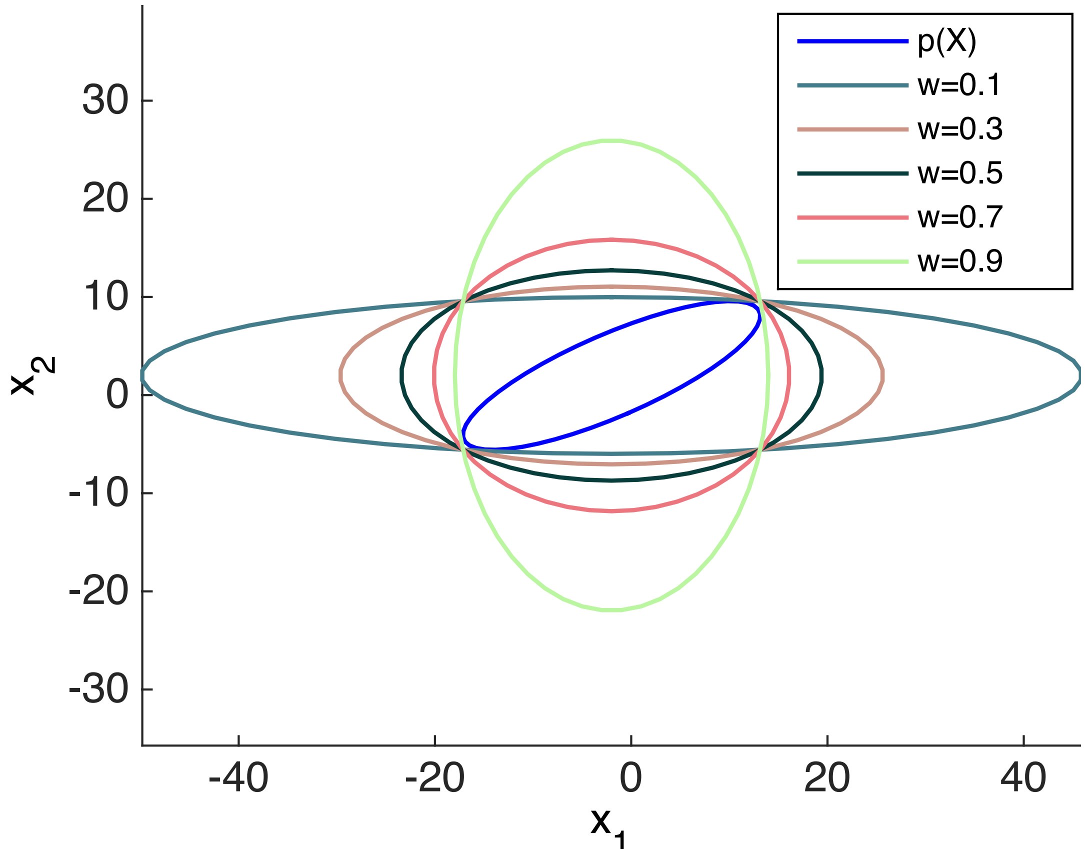
- Is the concept valid also for high-dimensional spaces?
- Why is it interesting?
  - Consider an information matrix $I\in\mathbb{R}^{n\times n}$
  - Calculating $|I|$ is often expensive ($O(n^3)$, in the general case)
  - **Instead**
    - Calculate a conservative **sparse** information matrix I_c
    - Evaluating $|I_c|$ can be done very efficiently
    - If concept applies, same performance is guaranteed!
- Next: Going to the extreme – appropriately **drop all correlation terms**
  - I_c is diagonal
  - Complexity is reduced to $O(n)$

"Decoupled" Conservative PDF

- Definition:

$$p_c(X) \doteq \eta \prod_i p^{w_i}(x_i) \qquad \forall x_i \in X \qquad \sum_i w_i = 1$$

- 2D case:



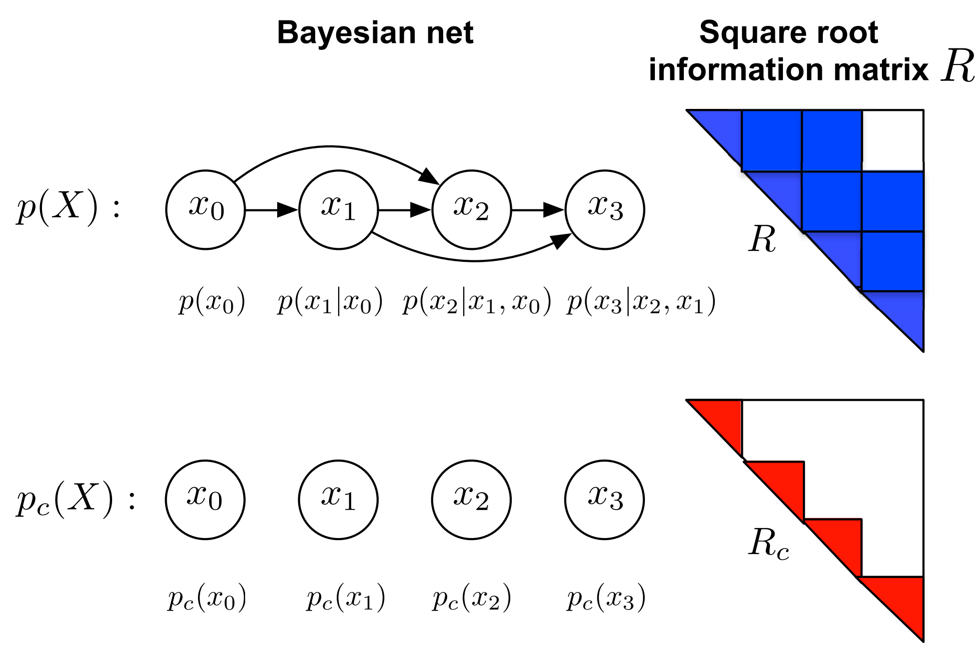
“Decoupled” Conservative PDF

- Definition:

$$p_c(X) \doteq \eta \prod_i p^{w_i}(x_i) \qquad \forall x_i \in X \qquad \sum_i w_i = 1$$

- Example - $X \in \mathbb{R}^4$:

$$p(X) = p(x_0)\, p(x_1|x_0)\, p(x_2|x_1, x_0)\, p(x_3|x_2, x_1) \Longrightarrow p_c(X) = p_c(x_0)\, p_c(x_1)\, p_c(x_2)\, p_c(x_3)\,.$$



$$I = R^T R$$

“Decoupled” Conservative PDF

- Definition:

$$p_c(X) \doteq \eta \prod_i p^{w_i}(x_i) \qquad \forall x_i \in X \qquad \sum_i w_i = 1$$

- How to calculate?
  - Need marginal covariance Σ_{ii} for each state x_i
  - Requires marginalization: $$p(x_i) = \int_{\neg x_i} p(X) = N(\hat{x}_i, \Sigma_{ii})$$
  - Then: $$\eta_i p^{w_i}(x_i) = N(\hat{x}_i, \Sigma_{ii}/w_i)$$

- Can be efficiently calculated directly from the (**sparse**) square root information matrix R , with $\Sigma = \left(R^T R\right)^{-1}$:

$$\Sigma_{ll} = \frac{1}{r_{ll}}\left(\frac{1}{r_{ll}} - \sum_{j=l+1}^{n} r_{lj}\Sigma_{jl}\right) \qquad \Sigma_{il} = \frac{1}{r_{ii}}\left(-\sum_{j=i+1}^{l} r_{ij}\Sigma_{jl} - \sum_{j=l+1}^{n} r_{ij}\Sigma_{lj}\right)$$

High Dimensional State Space

- Is the concept valid also for high-dimensional spaces?
  - In particular, in conjunction with the **decoupled** conservative pdf

- Valid (at least) in the following cases:
  - Observation models include the same arbitrary states, possibly with different measurement noise covariance

$$z_i = h\left(X'\right) + v_i \qquad X' \subset X$$

  - Unary observation models, possibly involving different states

$$z_i = h_i\left(x_i\right) + v_i \qquad x_i \in X$$

  - Pairwise observation models with the same uncorrelated state

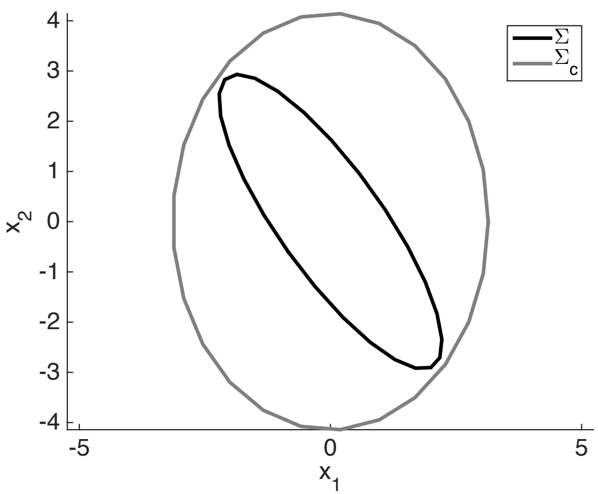
$$z_i = h_i\left(x, x_i\right) + v_i \qquad x, x_i \in X$$

    - Here, x is not correlated with other states

Example

$$|I^{a+}| \leq |I^{b+}| \text{ iff } |I_c^{a+}| \leq |I_c^{b+}|$$

- Unary observation models, possibly involving different states $z_i = h_i(x_i) + v_i$



- Original covariance: $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix}$
- Conservative covariance: $\Sigma_c = \begin{bmatrix} \Sigma_{c,11} & 0 \\ 0 & \Sigma_{c,22} \end{bmatrix}$

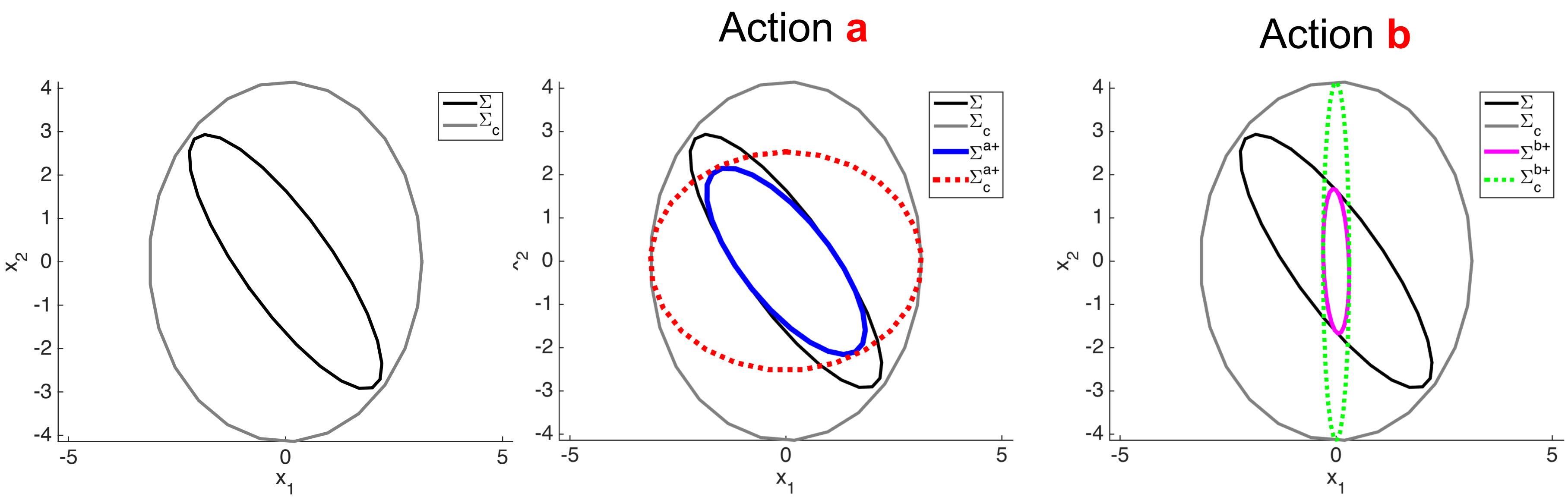
- Consider two actions/sensors:
  - Action **a**: 2nd state is measured
  - Action **b**: 1st state is measured
- Recall - a posteriori information matrix:

$$I^+ = I + H^T \Sigma_v^{-1} H$$

Example

$$\left|I^{a+}\right| \leq \left|I^{b+}\right| \text{ iff } \left|I_c^{a+}\right| \leq \left|I_c^{b+}\right|$$

- Unary observation models, possibly involving different states $z_i = h_i\left(x_i\right) + v_i$



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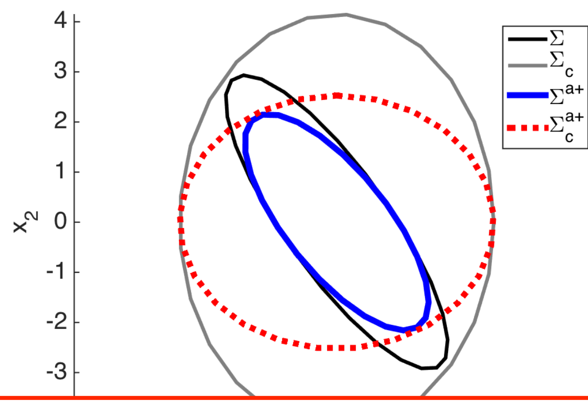
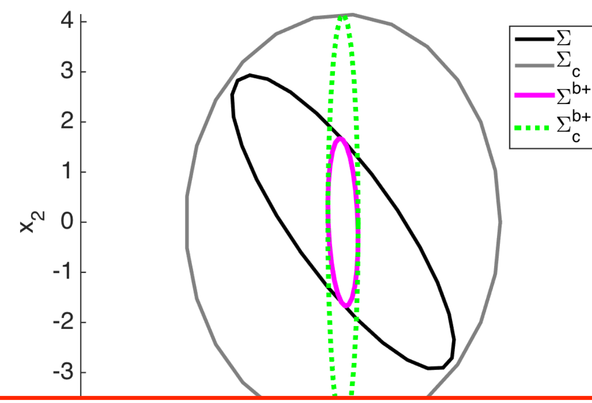
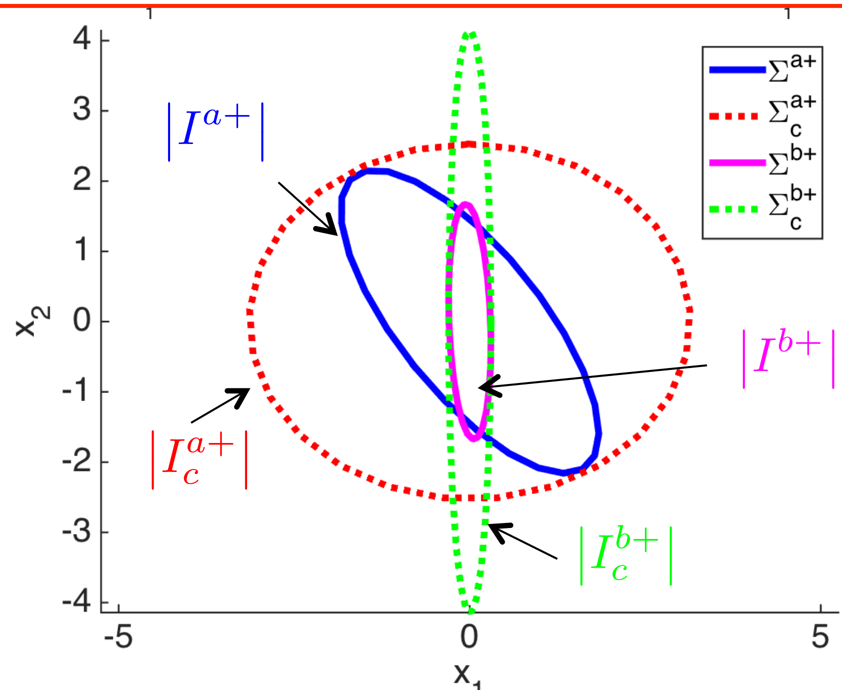
Action **a**



Action **b**



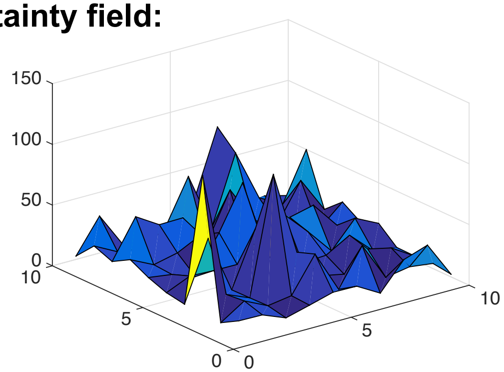
Do **not** need correlations to decide which action is better



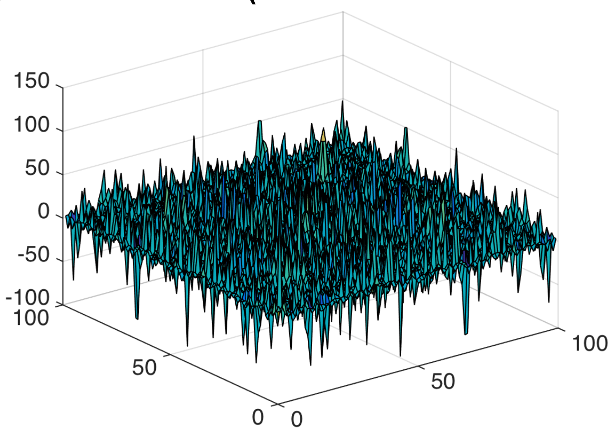
Application to Sensor Deployment Problems

- **Objective**: deploy k sensors in an $N \times N$ area
- e.g., provide localization, monitor spatial-temporal field

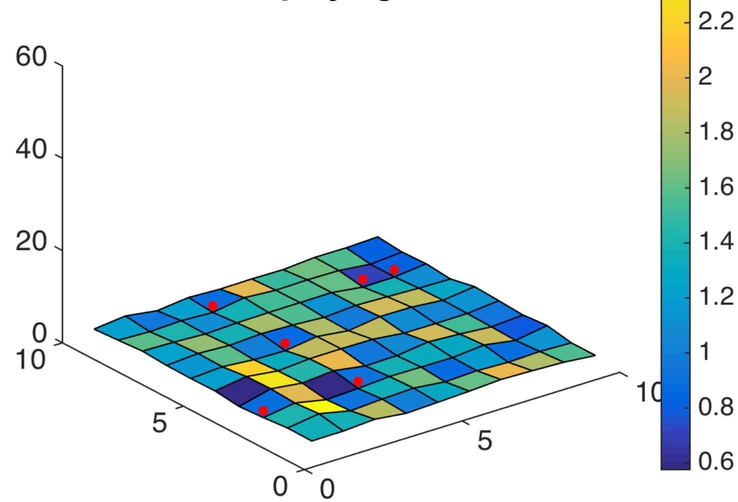
Prior uncertainty field:



A priori joint covariance (with correlations between cells)



After deploying 8 sensors:



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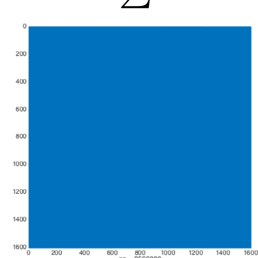
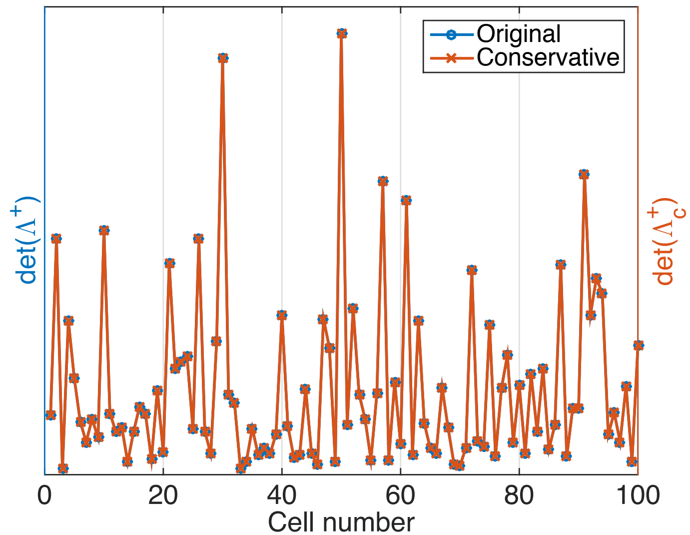
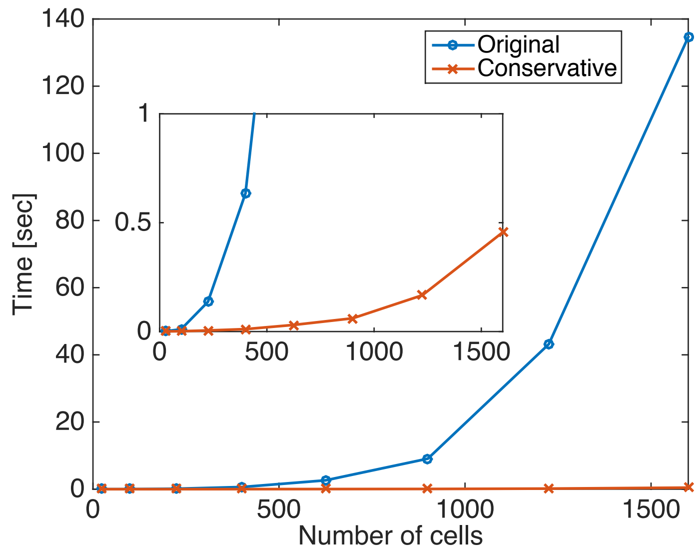
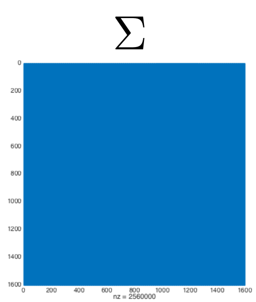


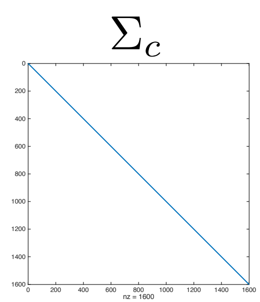
Application to Sensor Deployment Problems

- Same trend → same decisions!
- Timing is significantly reduced (conservative info. matrix is diagonal)









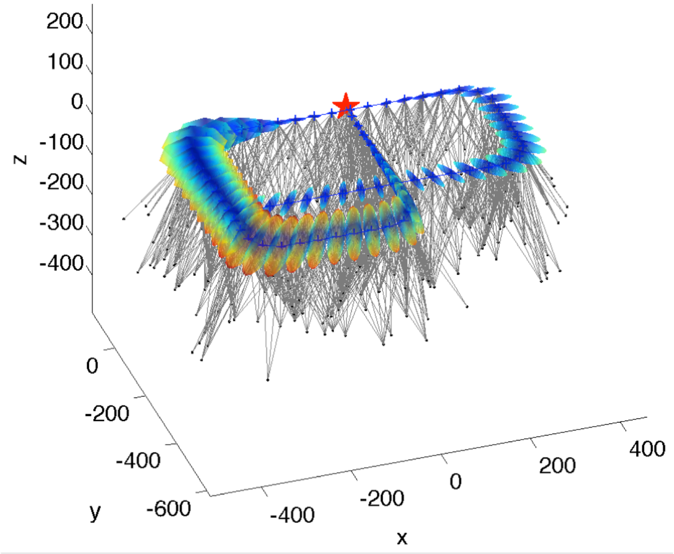
Example II

$$|I^{a+}| \leq |I^{b+}| \text{ iff } |I_c^{a+}| \leq |I_c^{b+}|$$

- Pairwise observation models with the same uncorrelated state

$$z_i = h_i(x, x_i) + v_i$$

- Aerial visual SLAM scenario



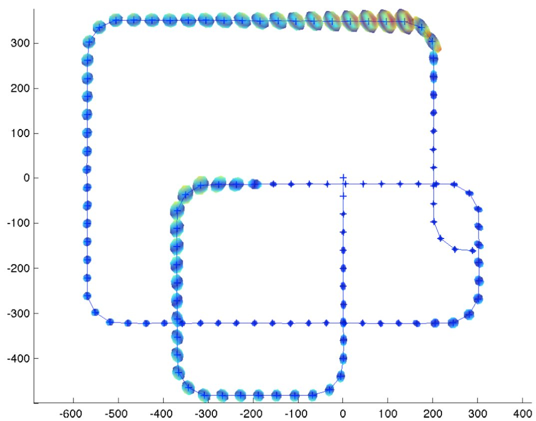
- Objective - each time a new image is received:
  - Decide what image observations to use
  - Identify most informative visual observations

- Remarks:
  - New camera pose x remains **uncorrelated** as long as no image observations have been incorporated
  - Note: can still add a prior $p(x)$

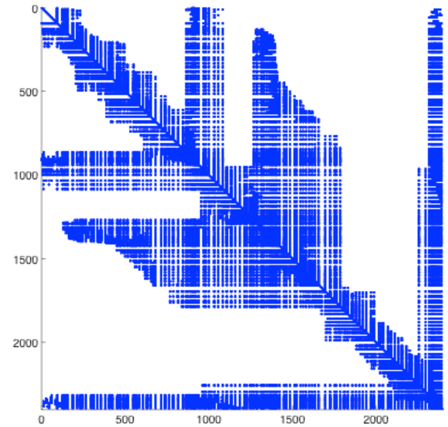
Example II

$$|I^{a+}| \leq |I^{b+}| \ \text{iff} \ |I_c^{a+}| \leq |I_c^{b+}|$$

Trajectory (top view)

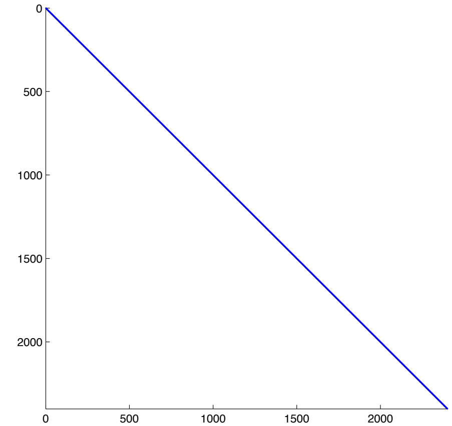


I



nnz = 411652

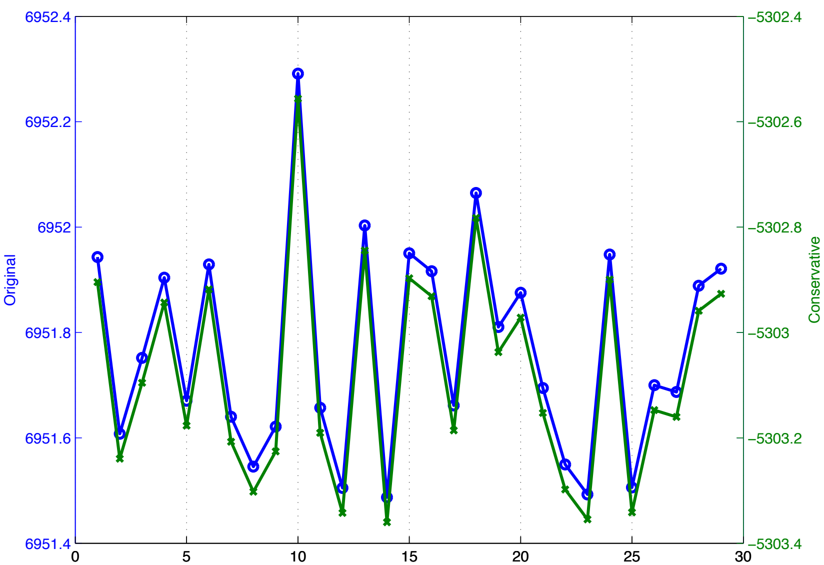
I_c



nnz = 2400

- Same trend!

$\log(|I|)$



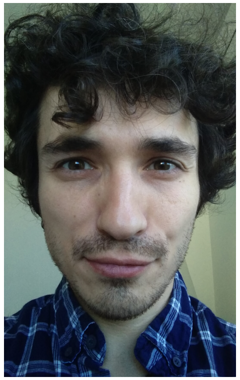
$\log(|I_c|)$

Candidate observation number

Key Question - Can the Concept be Generalized?

- Recall assumptions:
  - Gaussian distributions $p\left(x_k | z_{0:k}, u_{0:k-1}\right) = N\left(\mu_k, I_k^{-1}\right)$
  - No dynamics
  - Unary observation model $z_i = h_i\left(x_i\right) + v_i \quad , \quad v_i \sim N\left(0, \Sigma_{vi}\right)$
  - Myopic (single look ahead step)
- Is the concept applicable also to more general cases?
- For example, to belief space planning and active SLAM (stochastic control, pairwise observation models, non-myopic)

Key Idea II (Dmitry Kopitkov)



- Existing approaches calculate $|\Lambda^+|$ for each candidate action (or sequence of actions)
- Instead – use **matrix determinant lemma** for calculating **action impact**

$$|\Lambda^+| = |\Lambda + \Delta\Lambda| \qquad \Delta\Lambda = A^T \Sigma_v^{-1} \underline{A}$$

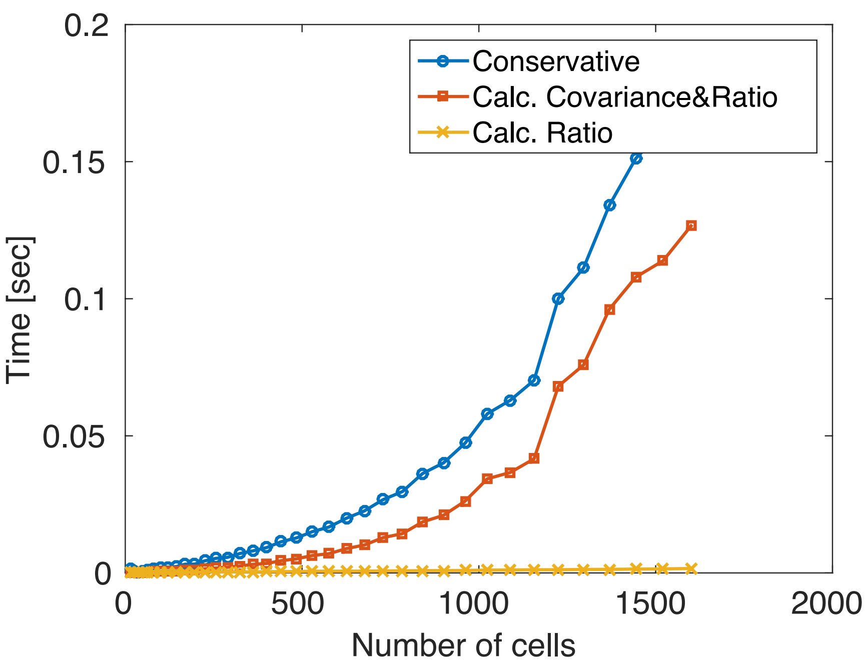
Sparse, mainly zeros

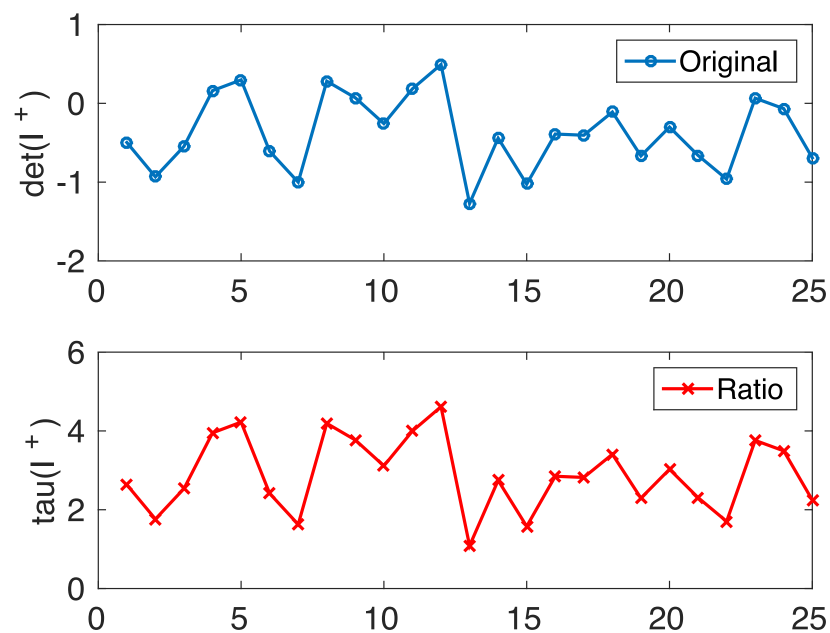
$$\boxed{\tau = \frac{|\Lambda^+|}{|\Lambda|} = \left|I + A\Lambda^{-1}A^T\right|}$$

- Why?
  - Fast (do need to calculate determinants of large matrices)
  - Exact
  - General (supports stochastic control, general observation models, multiple look ahead steps)

Key Idea II (Dmitry Kopitkov)

- Existing approaches calculate $|\Lambda^+|$ for each candidate action (or sequence of actions)
- Instead – use **matrix determinant lemma** for calculating **action impact**
- Preliminary results:





Conclusions

- **Decision making in the conservative information space**
  - Use a **sparse conservative** information matrix to greatly reduce computational complexity
  - In particular:
    - **Decoupled** conservative pdf – diagonal information matrix
    - Computational complexity is reduced by 2 orders of magnitude
    - Concept was proved to yield **the same performance** (decisions) in several scenarios of interest
- Multiple extensions to be investigated

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