Hypotheses Disambiguation in Retrospective
Supplementary Material

Ohad Shelly and Vadim Indelman

This document provides supplementary material to the paper [1]. Therefore, it should not be considered a self-contained document, but instead regarded as an appendix of [1].

1 Proof of Theorem 1

We now prove Theorem 1 from the main article,

Theorem 1 *The expression for* $\mathbb{P}(z_{m+j} \mid \gamma_m = i, H^-_{m+j})$ *for any* $j \in [2, p-1]$ *is given by,*

$$\mathbb{P}(z_{m+j} \mid \gamma_m = i, H^-_{m+j}) \approx \sum_{n=1}^{S} f(x^{n,i}_{m+j}, z_{m+j}) \triangleq \hat{\eta}^i_{m+j}. \tag{1}$$

1.1 Base Case: $j = p - 2$

We wish to prove that,

$$\mathbb{P}(z_{m+2} \mid \gamma_m = i, H^-_{m+2}) \approx \sum_{n=1}^{S} f(x^{m+2}_n, z_{m+2}). \tag{2}$$

Performing marginalization and chain rule yields,

$$\begin{aligned}\mathbb{P}(z_{m+2} \mid \gamma_m = i, H^-_{m+2}) = \quad & \sum_{g=1}^{N_L} \int_{x_{m+2}} \mathbb{P}(z_{m+2} \mid l_g, x_{m+2}) \mathbb{P}(\beta_{m+2} = g \mid x_{m+2}) \underbrace{\mathbb{P}(x_{m+2} \mid \gamma_m = i, H^-_{m+2})}_{b^{i-}_m[x_{m+2}]} dx_{m+2} = \\ & \sum_{g=1}^{N_L} \int_{x_{m+2} \in \Omega_{l_g}} \mathbb{P}(z_{m+2} \mid l_g, x_{m+2}) b^{i-}_m[x_{m+2}] dx_{m+2}\end{aligned} \tag{3}$$

Notice argument $b^{i-}_m[x_{m+2}]$ is conditioned by H^-_{m+2}. where H^-_{m+2} includes a new obtained measurement z_{m+1}, therefore its calculation in (3) is not received in a direct form, as we will show in the following section.

1.1.1 $b^{i-}_m[x_{m+2}]$ calculation

First, let us perform marginalization over x_{m+1} and chain rule,

$$b^{i-}_m[x_{m+2}] \equiv \mathbb{P}(x_{m+2} \mid \gamma_m = i, H^-_{m+2}) = \int_{x_{m+1}} \mathbb{P}(x_{m+2} \mid x_{m+1}, u_{m+1}) \cdot \mathbb{P}(x_{m+1} \mid \gamma_m = i, H^-_{m+2}) dx_{m+1} \tag{4}$$

Second, we take the argument $\mathbb{P}(x_{m+1} \mid \gamma_m = i, H^-_{m+2})$ from (4), perform Bayes rules, and marginalization over all possible landmarks to obtain the measurement model,

$$\begin{aligned}\mathbb{P}(x_{m+2} \mid \gamma_m = i, H^-_{m+2}) = \int_{x_{m+1} \in \Omega_{l_g}} \mathbb{P}(x_{m+2} \mid x_{m+1}, u_{m+1}) [\frac{\sum_{g=1}^{N_L} \mathbb{P}(z_{m+1} \mid l_g, x_{m+1}) b^{i-}[x_{m+1}]}{\mathbb{P}(z_{m+1} \mid \gamma_m = i, H^-_{m+1}, a_{m+1})}] dx_{m+1} = \\ \frac{1}{\mathbb{P}(z_{m+1} \mid \gamma_m = i, H^-_{m+1}, a_{m+1})} \int_{x_{m+1} \in \Omega_{l_g}} \mathbb{P}(x_{m+2} \mid x_{m+1}, u_{m+1}) \sum_{g=1}^{N_L} \mathbb{P}(z_{m+1} \mid l_g, x_{m+1}) b^{i-}[x_{m+1}] dx_{m+1}\end{aligned} \tag{5}$$

The authors are with the Department of TASP and Aerospace Engineering, Technion - Israel Institute of Technology, Haifa 32000, Israel. {ohad.shelly, vadim.indelman}@technion.ac.il.

$\mathbb{P}(\beta_{m+1} = g \mid x_{m+1})$, sets the boundaries of the integral where x_{m+1} in Ω_{l_g}. We start noticing a strong resemblance to the calculation done for $j = p - 1$. A direct approach would be to re-sample the propagated belief $b^{-}[x_{m+1}]$ and receive a set of samples x^n_{m+1} with $n \in [1..S]$. However, in such an approach we would need to *recalculate from scratch* the $f(x, z)$ values. In contrast we propose to re-use the previous taken samples of $b^i_[x_{m+1}]$ from the previous step, and by that, *re-use* the calculated values of $f(x^n_{m+1}, z_{m+1})$. By doing so (5) can be denoted as,

$$b^{i-}_m[x_{m+2}] \approx \frac{\sum_{n=1}^{S} \mathbb{P}(x_{m+2} \mid x^n_{m+1}, u_{m+1}) \cdot f(x^n_{m+1}, z_{m+1})}{\mathbb{P}(z_{m+1} \mid \gamma_m = i, H^{-}_{m+1}, a_{m+1})} \tag{6}$$

In the nominator we have two arguments per given sample x^n_{m+1}, the first is the motion model, and the second is the value of the f function for the given sample and measurement.

In order to calculate the denominator let us perform marginalization and chain rule over all possible landmarks and state at x_{m+1},

$$\begin{aligned}
&\mathbb{P}(z_{m+1} \mid \gamma_m = i, H^{-}_{m+1}, u_{m+1}) = \\
&\sum_{g=1}^{N_L} \int_{x_{m+1}} \mathbb{P}(z_{m+1} \mid x_{m+1}, l_g)\mathbb{P}(\beta_{m+1} = g \mid x_{m+1})\mathbb{P}(x_{m+1} \mid \gamma_m = i, H^{-}_{m+1})dx_{m+1} = \\
&\sum_{g=1}^{N_L} \int_{x_{m+1} \in \Omega_{l_g}} \mathbb{P}(z_{m+1} \mid x_{m+1}, l_g) b^{i-}[x_{m+1}]dx_{m+1}
\end{aligned} \tag{7}$$

let us notice that the received result is identical to the result of $j = m + 1$. As before we perform reuse of the sampled values of $b^{i-}[x_{m+1}]$ taken from the previous step, and calculated values of $f(x^n_{m+1}, z_{m+1})$, where $n \in [1..S]$. Therefore we denote,

$$\mathbb{P}(z_{m+1} | \gamma_m = i, H^{-}_{m+1}) \approx \sum_{n=1}^{S} f(x^n_{m+1}, z_{m+1}) \triangleq \eta^i_{m+1}. \tag{8}$$

By taking the denominator from (8) and placing it back in (6) we will get,

$$b^{i-}_m[x_{m+2}] \approx \sum_{n=1}^{S} \mathbb{P}(x_{m+2} \mid x^n_{m+1}, u_{m+1}) \underbrace{\frac{f(x^n_{m+1}, z_{m+1})}{\eta^i_{m+1}}}_{\zeta^{n,i}_{m+1}} \tag{9}$$

Since the motion model in (9) per a given sample of x^n_{m+1} is a Gaussian distribution we can address $b^{i-}[x_{m+2}]$ as a GMM with S components, where is fact $\zeta^{n,i}_{m+1}$ is the normalized weight of a given n component.

Now that we have shown that $b^{i-}_m[x_{m+2}]$ can be addressed as a valid GMM, let us return to (3). In resemblance to previous step of $j = m + 1$, $\mathbb{P}(\beta_{m+2} = g \mid x_{m+2})$ sets finite boundaries to the integral, therefore in order to calculate we will sample $b^{i-}_m[x_{m+2}]$ and receive a set of samples, $x^n_{m+2}, n \in [1, S]$.

$$\mathbb{P}(z_{m+2} \mid \gamma_m = i, H^{-}_{m+2}) = \frac{1}{S} \cdot \sum_{n=1}^{S} \sum_{g=1}^{N_L} \mathbb{P}(z_{m+2} \mid l_g, x^n_{m+2}) \tag{10}$$

As in previous sections we can replace the landmark associated index $\beta_{m+2} = g$, with the landmark's given coordinates, l_g. For the same motivation of calculation re-use as before we can present (10), as

$$\mathbb{P}(z_{m+2} | \gamma_m = i, H^{-}_{m+2}) \approx \cdot \sum_{n=1}^{S} f(x^n_{m+2}, z_{m+2}) \triangleq \hat{\eta}^i_{m+2} \tag{11}$$

In conclusion we have proven that for the base case of $j = m+2$, $\mathbb{P}(z_{m+2} \mid \gamma_m = i, H^{-}_{m+2}) \approx \sum_{n=1}^{S} f(x^{m+2}_n, z_{m+2})$.

1.2 Induction assumption

Let us make the induction assumption for $j = m + l - 1$,[Should I mention l boundaries?]

$$\mathbb{P}(z_{m+j-1} \mid \gamma_m = i, H^{-}_{m+j-1}) \approx \sum_{n=1}^{S} f(x^n_{m+j-1}, z_{m+j-1}) \tag{12}$$

Where $x^n_{m+j-1}, n \in [1..S]$ is a set of taken samples from $\mathbb{P}(x_{m+j-1} \mid \gamma_m = i, H^{-}_{m+j-1}) \doteq b^{i-}_m[x_{m+j-1}]$.

1.3 Inductive step for $j = m + l$

We wish to prove,

$$\mathbb{P}(z_{m+j} \mid \gamma_m = i, H^-_{m+j}) \approx \sum_{n=1}^{S} f(x^n_{m+j}, z_{m+j}) \tag{13}$$

Proof of the required induction step

We begin our induction proof by performing as before marginalization and chain rule over all given landmark index's at time $m + j$, and over the state at x_{m+j},

$$\mathbb{P}(z_{m+j} \mid \gamma_m = i, H^-_{m+j}) = \sum_{g=1}^{N_L} \int_{x_{m+j}} \mathbb{P}(z_{m+j} \mid l_g, x_{m+j}) \mathbb{P}(\beta_{m+j} = g \mid x_{m+j}) b^{i-}_m[x_{m+j}] dx_{m+j} \tag{14}$$

Where $b^{i-}_m[x_{m+j}] \doteq \mathbb{P}(x_{m+j} \mid \gamma_m = i, H^-_{m+j})$. We notice the calculation of $b^{i-}_m[x_{m+j}]$ in (14), resemblance to the calculation of $b^{i-}_m[x_{m+2}]$ in (3). Again, in order to calculate $b^{i-}_m[x_{m+j}]$ we use chain rule and marginalization over the previous state, via x_{m+j-1}, and landmark associations, perform Bayes rule to substract the observation model for measurement z_{m+j-1},

$$\begin{aligned} b^{i-}_m[x_{m+j}] = &\int_{x_{m+j-1}} \mathbb{P}(x_{m+j} \mid x_{m+j-1}, u_{m+j-1}) \cdot \\ &\frac{\left[\sum_{g=1}^{N_L} \mathbb{P}(z_{m+j-1} \mid x_{m+j-1}, l_g) \cdot \mathbb{P}(\beta_{m+j-1} = g \mid x_{m+j-1}) \cdot b^{i-}_m[x_{m+j-1}]\right] dx_{m+j-1}}{\mathbb{P}(z_{m+j-1} \mid \gamma_m = i, H^-_{m+j-1})} \end{aligned} \tag{15}$$

In order to calculate $b^{i-}_m[x_{m+j-1}]$ at (15) in the naive approach one needs to marginalize and perform Byes rule till retrieving the calculations to the time of the hypothesis we wish to reevaluate its weight. Instead let us look on the argument inside the brackets in (15), and see it resembles the value of $\mathbb{P}(z_{m+j-1} \mid \gamma_m = i, H^-_{m+j-1})$ from our induction assumption in 1.2. So instead of the direct approach that requires recalculation and re-sample of $b^{i-}_m[x_{m+j-1}]$, we will re-use the set of samples x^n_{m+j-1}, and by that the set of calculated values of the f function from the induction assumption. So (15) can appear as such,

$$b^{i-}_m[x_{m+j}] \approx \sum_{n=1}^{S} \mathbb{P}(x_{m+j} \mid x^{m+j-1}_n, u_{m+j-1}) \cdot \frac{f(x^n_{m+j-1}, z_{m+j-1})}{\mathbb{P}(z_{m+j-1} \mid \gamma_m = i, H^-_{m+j-1})} \tag{16}$$

For the denominator in resemble to 1.1 we marginalize over all given landmarks and x_{m+j-1}, and perform chain rule as before,

$$\begin{aligned} \mathbb{P}(z_{m+j-1} \mid \gamma_m = i, H^-_{m+j-1}) = &\int_{x_{m+j-1} \in \Omega_{l_g}} \sum_{j=1}^{N_L} \mathbb{P}(z_{m+j-1} \mid l_g, x_{m+j-1}) b^{i-}_m[x_{m+j-1}] dx_{m+j-1} \\ &\approx \sum_{n=1}^{S} \mathbb{P}(z_{m+j-1} \mid l_g, x^n_{m+j-1}) \doteq \hat{\eta}^i_{m+j-1} \end{aligned} \tag{17}$$

By placing (17) into (16) we get,

$$b^{i-}_m[x_{m+j}] \approx \sum_{n=1}^{S} \mathbb{P}(x_{m+j} \mid x^{m+j-1}_n, u_{m+j-1}) \cdot \underbrace{\frac{f(x^n_{m+j-1}, z_{m+j-1})}{\eta^i_{m+j-1}}}_{\zeta^{n,i}_{m+j-1}} \tag{18}$$

In resemble to (9), in (18) we get a GMM with S components, where $\mathbb{P}(x_{m+j} \mid x^{m+j-1}_n, u_{m+j-1})$ is a motion model with Gaussian distribution per a given sample, x^n_{m+j-1},nd $\zeta^{n,i}_{m+j-1}$ acts as the normalized weight for a given n hypothesis in the GMM.

Now after we have shown that $b_m^{i-}[x_{m+j}]$ is a valid GMM, we can take a set of samples x_{m+j}^n where $n \in [1..S]$. And by that (14) yields into,

$$\mathbb{P}(z_{m+j} \mid \gamma_m = i, H_{m+j}^-) \approx \frac{1}{S} \sum_{n=1}^{S} \sum_{g=1}^{N_L} \mathbb{P}(z_{m+j} \mid l_g, x_{m+j}^n) \doteq \sum_{n=1}^{S} f(x_{m+j}^n, z_{m+j}), \tag{19}$$

where (19) is what we wished to prove by induction.

References

[1] O. Shelly and V. Indelman. Hypotheses disambiguation in retrospective. *IEEE Robotics and Automation Letters (RA-L)*, 2022. Submitted.