$\texttt{iX-BSP}$: Incremental Belief Space Planning with Selective Resampling
Supplementary Material

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This document provides supplementary material to the paper [1]. Therefore, it should not be considered a self-contained document, but instead regarded as an appendix of [1].

Throughout this report, all notations and definitions are with compliance to the ones presented in [1].

This report covers the formulation of Multiple Importance Sampling - Appendix A, and a toy example for our novel paradigm iX-BSP- Appendix B.

Appendix A: Multiple Importance Sampling Formulation

Let us assume we wish to express expectation over some function $f(x)$ with respect to distribution $p(x)$, by sampling x from a different distribution $q(x)$,

$$\mathbb{E}_p f(x) = \int f(x) \cdot p(x) dx = \int \frac{f(x) \cdot p(x)}{q(x)} q(x) dx = \mathbb{E}_q \left(\frac{f(x) \cdot p(x)}{q(x)} \right). \tag{10}$$

Eq. (10) presents the basic importance sampling problem, where $\mathbb{E}_q$ denotes expectation for $x \sim q(x)$. The probability ratio between the nominal distribution p and the importance sampling distribution q is usually referred to as the likelihood ratio. Our problem is more complex, since our samples are potentially taken from M different distributions while $M \in [1, (n_x \cdot n_z)^L]$, i.e. a multiple importance sampling problem

$$\mathbb{E}_p f(x) = \tilde{\mu}(x) \sim \sum_{m=1}^{M} \frac{1}{n_m} \sum_{i=1}^{n_m} w_m(x_{im}) \frac{f(x_{im}) p(x_{im})}{q_m(x_{im})}, \tag{11}$$

where $w_m(.)$ are weight functions satisfying $\Sigma_{m=1}^M w_m(x) = 1$. For $q_m(x) > 0$ whenever $w_m(x)p(x)f(x) \neq 0$, Eq. (11) forms an unbiased estimator

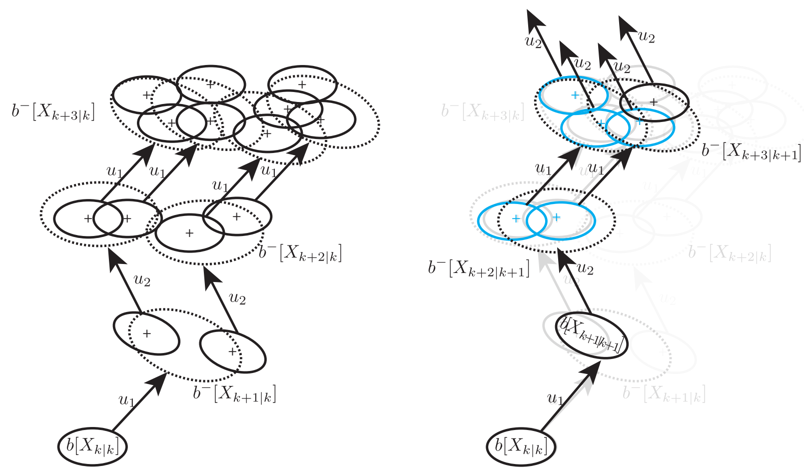
$$\mathbb{E}\left[\tilde{\mu}(x)\right] = \sum_{m=1}^{M} \mathbb{E}_{q_m} \left[\frac{1}{n_m} \sum_{i=1}^{n_m} w_m(x_{im}) \frac{f(x_{im}) p(x_{im})}{q_m(x_{im})} \right] = \tilde{\mu}(x). \tag{12}$$

Although there are numerous options for weight functions satisfying $\Sigma_{m=1}^M w_m(x) = 1$, we chose to consider the Balance Heuristic [2], considered to be nearly optimal in the sense of estimation variance,

$$w_m(x) = w_m^{BH}(x) = \frac{n_m q_m(x)}{\Sigma_{s=1}^M n_s q_s(x)}. \tag{13}$$

Using (13) in (11) produces the multiple importance sampling with the balance heuristic

$$\mathbb{E}_p f(x) \sim \frac{1}{n} \sum_{m=1}^{M} \sum_{i=1}^{n_m} \frac{p(x_{im})}{\Sigma_{s=1}^M \frac{n_s}{n} q_s(x_{im})} f(x_{im}). \tag{14}$$



(a) (b)

Figure 4: Consider BSP presented in a spatial belief propagation diagram, with propagated beliefs denoted by dashed line ellipse, samples denoted by $+$, actions denoted by arrows, beliefs denoted by solid line ellipse and with $n_x = 2$ and $n_z = 1$. (a) Presents `X-BSP` at planning time k for the action sequence $u_1 \to u_2 \to u_1$, as such all measurements were sampled from their original distributions. Assuming action u_1 have been executed, (b) presents `iX-BSP` for the succeeding planning time $k+1$, for the action sequence $u_2 \to u_1 \to u_2$, where re-used samples and corresponding re-used beliefs are denoted in blue, and freshly sampled measurements and corresponding freshly calculated beliefs are denoted in black.

Appendix B: Toy example for `iX-BSP`

To better understand the problem let us consider a toy example. We have access to all calculations from planning time k, in-which we performed `X-BSP` (or `iX-BSP`) for a horizon of 3 steps, with $n_x = 2$ and $n_z = 1$, while considering 2 candidate actions u_1 and u_2. Figure 4a illustrates a specific action sequence, $u_1 \to u_2 \to u_1$, considered as part of planning at time k. We are currently at time k+1, performing planning using `iX-BSP` with the same horizon length and number of samples per action, for the action sequence $u_2 \to u_1 \to u_2$, see Figure 4b.

Following Alg. 1 line 1, out of the two available beliefs from planning time k, $\{b[X_{k+1|k}]\}_1^2$, the left one is closer to $b[X_{k+1|k+1}]$, so we consider all its descendants as the set $\mathcal{B}_{k+1|k}$, and denote the difference between $b[X_{k+1|k}]$ and $b[X_{k+1|k+1}]$ as $\{\delta\}$. We continue with re-using the beliefs in the set $\mathcal{B}_{k+1|k}$ (Alg. 1 line 3). First we check whether the two available samples from planning time k constitute an adequate representation for $b^-[X_{k+2|k+1}]$ (Alg. 2 line 7); since they are, we flag them both for re-use, and update $\{b[X_{k+2|k}]\}_1^2$ into $\{b[X_{k+2|k+1}]\}_1^2$ (Alg. 2 line 11). Now for the next future time step, we propagate $\{b[X_{k+2|k+1}]\}_1^2$ with action u_1 to obtain $\{b^-[X_{k+3|k+1}]\}_1^2$ (Alg. 2 line 3), and check whether the four available samples from planning time k constitute an adequate representation for $\{b^-[X_{k+3|k+1}]\}_1^2$ (Alg. 2 line 7); since only three of them are, we flag them for re-use and sample the forth one (black colored belief at $k+3|k+1$ in Figure 4b) from $b^-[X_{k+3|k+1}]$. We then update $\{b[X_{k+3|k}]\}_1^3$ into $\{b[X_{k+3|k+1}]\}_1^3$, and $b^-[X_{k+3|k+1}]$ into $b^4[X_{k+3|k+1}]$ using the newly sampled measurement (Alg. 2 line 11). The last step of the horizon $k+4|k+1$ is calculated using `X-BSP` (Alg. 1 line 4).

At this point we have all inference results for all beliefs along the action sequence $u_2 \to u_1 \to u_2$, so we can calculate all reward(cost) values for this action sequence for planning at time k+1. For the look ahead at time $k+2$ of planning session at time $k+1$, i.e. $k+2|k+1$, we have two reward(cost) values, $\{r_{k+2|k+1}(b[X_{k+2|k+1}], u_2)\}_1^2$, each calculated with a different belief $b[X_{k+2|k+1}]$ considering a different sample $z_{k+2|k}$. Calculating the expected reward(cost) value for future time step $k+2|k+1$ would mean in this case, using measurements sampled from $\mathbb{P}(z_{k+2|k}|H_{k+1|k}, u_2)$ rather then from $\mathbb{P}(z_{k+2|k+1}|H_{k+1|k+1}, u_2)$. This problem, of performing estimation using forced samples is called importance sampling. Since for a single time step we might have samples from multiple different distributions, e.g. future time $k+3|k+1$ in Figure 4b, our problem falls within the special case of Multiple Importance Sampling (see Appendix A). Using the formulation of multiple importance sampling using the balance heuristic (14)

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we can write down the estimation for the expected reward value at planning time $k+2|k+1$,

$$\mathbb{E}\left[r_{k+2|k+1}(.)\right] \sim \frac{1}{2}\frac{p_1(z^1_{k+2|k})}{\frac{2}{2}q_1(z^1_{k+2|k})}\cdot r^1_{k+2|k+1}(.)+\frac{1}{2}\frac{p_1(z^2_{k+2|k})}{\frac{2}{2}q_1(z^2_{k+2|k})}\cdot r^2_{k+2|k+1}(.), \tag{15}$$

where $p_1(.) \doteq \mathbb{P}(z_{k+2|k+1}|H_{k+1|k+1}, u_2)$ and $q_1(.) \doteq \mathbb{P}(z_{k+2|k}|H_{k+1|k}, u_2)$. In the same manner, following (14), we can also write down the estimation for the expected reward(cost) value at look ahead step $k+3$ from planning session at time $k+1$, i.e. $k+3|k+1$,

$$\begin{aligned}\mathbb{E}\left[r_{k+3|k+1}(.)\right] \sim{}& \frac{1}{4}\frac{p_2(z^1_{k+2:k+3|k})}{\frac{3}{4}q_2(z^1_{k+2:k+3|k})+\frac{1}{4}p_2(z^1_{k+2:k+3|k})}r^1_{k+3|k+1}(.)+\frac{1}{4}\frac{p_2(z^2_{k+2:k+3|k})}{\frac{3}{4}q_2(z^2_{k+2:k+3|k})+\frac{1}{4}p_2(z^2_{k+2:k+3|k})}r^2_{k+3|k+1}(.)+\\ &\frac{1}{4}\frac{p_2(z^3_{k+2:k+3|k})}{\frac{3}{4}q_2(z^3_{k+2:k+3|k})+\frac{1}{4}p_2(z^3_{k+2:k+3|k})}r^3_{k+3|k+1}(.)+\frac{1}{4}\frac{p_2(z^4_{k+2:k+3|k+1})}{\frac{3}{4}q_2(z^4_{k+2:k+3|k+1})+\frac{1}{4}p_2(z^4_{k+2:+3|k+1})}r^4_{k+3|k+1}(.),\end{aligned} \tag{16}$$

where $p_2(.) \doteq \mathbb{P}(z_{k+2:k+3|k+1}|H_{k+1|k+1}, u_2, u_1)$ and $q_2(.) \doteq \mathbb{P}(z_{k+2:k+3|k}|H_{k+1|k}, u_2, u_1)$. When considering

$$\mathbb{P}(z_{k+1:k+L|k}|H_{k|k}, u_{k:k+L-1}) = \prod_{i=k+1}^{k+L}\mathbb{P}(z_{i|k}|H^-_{i|k}) \tag{17}$$

we can re-write the measurement likelihood from (16) into a product of measurement likelihoods per look ahead step, e.g. $p_2(z^1_{k+2:k+3|k}) = p_1(z^1_{k+2|k})\tilde{p}_2(z^1_{k+3|k})$, when $p_1(.)$ need not be calculated at look ahead step $k+3$, since it is given from (15).

References

[1] E. Farhi and V. Indelman. ix-bsp: Incremental belief space planning with selective resampling. In *Proc. of the Intl. Symp. of Robotics Research (ISRR)*, October 2019. Submitted.

[2] Eric Veach and Leonidas J Guibas. Optimally combining sampling techniques for monte carlo rendering. In *Proceedings of the 22nd annual conference on Computer graphics and interactive techniques*, pages 419–428. ACM, 1995.