MATHEMATICAL FINANCE

Ella Finstuen, Indiana Kretzschmar, Matthew Younce

APPM 4350, December 2024

Abstract

This project explores the Black-Scholes option pricing model, focusing on its derivation using partial differential equations and its connection to the 1D heat equation. We analyze its strengths and limitations, boundary conditions, and evaluate its effectiveness by comparing predicted option prices to real market data. This study highlights the model's importance in the real world use of PDEs and PDEs' role in financial mathematics.

Contents

1 Introduction **3**

2 Derivations of Black-Scholes Model **4**
- 2.1 Mathematical background . . . 4
- 2.2 Deriving Black-Scholes Boundary Value Problem . . . 5
- 2.3 Deriving the Solution . . . 7

3 Discussion of Model **10**
- 3.1 Strengths . . . 10
- 3.2 Limitations . . . 10
- 3.3 Exploration of the Boundary Conditions . . . 11

4 Real World Application **12**
- 4.1 Experimenting with the Model . . . 12
- 4.2 Analysis . . . 12

5 Conclusions **13**

References **14**

1 Introduction

The Black-Scholes equation is a powerful model that is used to calculate the theoretical value of an option, providing an important example of how partial differential equations can be used in the real world. The formula was derived by Fischer Black and Myron Scholes in 1973 as part of an effort to develop an accurate method for pricing options[4]. At the same time, Robert Merton was independently working on a similar model, and both arrived at roughly the same conclusion. A call option gives the buyer the right to buy the underlying asset, while a put option grants the right to sell it. Therefore, purchasing a call option reflects an expectation that the asset's price will rise, while purchasing a put option indicates an expectation of a price decline. If the market conditions are unfavorable, the buyer is not obligated to exercise the option and will only lose the premium paid for the option. An underlying asset is a financial concept or item that determines the value of a contract. This includes stocks, bonds, commodities, and currencies. Moreover, the price at which the option holder can buy or sell the asset is known as the strike price. The Black-Scholes equation predicts the fair market price for the option, which is the price where the buyer and the seller share the same risk from the transaction[4].

The fair price of an option is difficult to predict due to the uncertainty of stocks. To address this, the Black-Scholes model assumes that the fluctuation of stocks does not follow a definite pattern and are therefore described as random. This randomness is captured by modeling the stock price as a stochastic process, typically represented by Brownian motion[4]. The Brownian motion introduces uncertainty in determining an option's value at specific points in time. The Black-Scholes model deals with this by using the known payoff of the option at its expiration, rather than using an initial condition, making the PDE a final value problem.

This paper will walk through the derivation of the Black-Scholes equation, exploring its underlying assumptions and solution by relating it to the Heat Equation. It will also discuss the strengths and limitations of the model and evaluate its performance using real-world stock data. While European options only allow execution at expiration, American options provide the flexibility to exercise the option at any time before expiration. For simplicity, this paper will focus on the solution for European call options.

2 Derivations of Black-Scholes Model

2.1 Mathematical background

The Black-Scholes model is represented with the partial differential equation:

$$\frac{\partial C}{\partial t_1} + rS\frac{\partial C}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2} - rC = 0 \tag{1}$$

With the solution

$$C(S, t_1) = SN(d_1) - Xe^{-r(t_2-t_1)}N(d_2) \tag{2}$$

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + (r + \frac{\sigma^2}{2})(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}} \tag{3}$$

$$d_2 = \frac{\ln\left(\frac{S}{X}\right) + (r - \frac{\sigma^2}{2})(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}} = d_1 - \sigma\sqrt{t_2 - t_1} \tag{4}$$

We define the variables in the equations above to be:

$\boldsymbol{C(S, t_1)}$ - Value of option on stock $\boldsymbol{S}$ at time $\boldsymbol{t_1}$

$\boldsymbol{S(t_1)}$ - Stock price at $\boldsymbol{t_1}$

$\boldsymbol{t_1}$ - Current time, $\boldsymbol{t_1 \in [0, t_2]}$

$\boldsymbol{t_2}$ - Time of option expiration

$\boldsymbol{X}$ - Strike price

$\boldsymbol{r}$ - Risk free interest rate

$\boldsymbol{\sigma}$ - Volatilityof returns

$\boldsymbol{\mu}$ - Drift function

$\boldsymbol{\Pi(t_1)}$ - Portfolio(with one option)

$\boldsymbol{N(x)}$ - Normal distribution

Assumptions for the Black-Scholes Model[3]

i) The stock price $S(t_1)$ follows a stochastic process specifically the geometric Brownian motion. We use the standard Brownian motion function, $W(t)$, where $W(t)$ models the random and continuous changes in stock prices over time to model the stock process.

ii) Interest rate r and volatility σ are known.

iii) $C(S, t_1)$ is a smooth, continuous function, meaning it is infinitely differentiable.

iv) Short selling is permitted.

v) Arbitrage Opportunities do not exist

vi) The stock pays no dividends during the life of the option.

vii) Any number of the stock may be bought or sold, it does not need to be an integer.

viii) The model is for European options, because they do not allow for early exercise like American options do.

2.2 Deriving Black-Scholes Boundary Value Problem

To begin the derivation of Black-Scholes Boundary Value Problem, we will define a stock process with $W(t_1)$ on $[0, t_2]$. As this process is stochastic, it must follow the stochastic differential equation written below[4].

$$dS = \mu S dt_1 + \sigma S dW \tag{5}$$

As defined previously, $C(S, t_1)$ is the value of the option, and we wish to find an equation for the change of C due to both S and t_1. From the multi-variable Taylor Series expansion, because we are following the assumption that S is stochastic and C is smooth[2], we can write the following,

$$dC = \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial t_1} dt_1 + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(dS)^2 + \frac{\partial^2 C}{\partial S \partial t_1} dS dt_1 + \frac{1}{2}\frac{\partial^2 C}{\partial t_1^2}(dt_1)^2 + \dots \tag{6}$$

Itô's Lemma[2] states that if S is a stochastic process satisfying (5) and $C(S, t_1)$ is twice differentiable, then $C(S, t_1)$ inherits stochastic behavior and satisfies

$$dC = \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial t_1} dt_1 + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}^2 \tag{7}$$

There is also a useful corollary from the lemma:

$$(dS)^2 = \sigma^2 S^2 (dW)^2 = \sigma^2 S^2 dt \tag{8}$$

As mentioned previously, our assumptions that C is smooth thus it is twice differentiable and that S is a stochastic process make Itô's Lemma applicable here, thus we can assume (7). Now Implementing the stochastic differential equation (5) and the corollary in (8),

$$\begin{aligned} dC &= \frac{\partial C}{\partial t_1} dt_1 + \frac{\partial C}{\partial S}(\mu S dt_1 + \sigma S dW) + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} dt \\ &= \sigma S \frac{\partial C}{\partial S} dW + \Big(\mu S \frac{\partial C}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t_1}\Big) dt_1 \end{aligned} \tag{9}$$

We now introduce the portfolio $\Pi(t_1)$ with the one option $C(S, t_1)$ and number $-\frac{\partial C}{\partial S}$ of the Stock S. It can be represented as

$$\Pi = C - \frac{\partial C}{\partial S} S. \tag{10}$$

Now we can compute the change in the portfolio and use (5) and (9) to get a clean expression [9]:

$$\begin{aligned} d\Pi &= dC - \frac{\partial C}{\partial S} dS \\ &= \Big(\mu S \frac{\partial C}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t_1}\Big) dt_1 + \sigma S \frac{\partial C}{\partial S} dW - \frac{\partial C}{\partial S} dS \\ &= \Big(\frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t_1}\Big) dt_1 + \mu S \frac{\partial C}{\partial S} dt_1 + \sigma S \frac{\partial C}{\partial S} dW - \frac{\partial C}{\partial S} dS \\ &= \Big(\frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t_1}\Big) dt_1 + \frac{\partial C}{\partial S}\Big(\mu S dt_1 + \sigma S dW - dS\Big) \\ &= \Big(\frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t_1}\Big) dt_1 + \frac{\partial C}{\partial S}\Big(dS - dS\Big) \\ &= \Big(\frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t_1}\Big) dt_1 \end{aligned} \tag{11}$$

We also must consider that the Portfolio in (11) is risk-free, so it should earn the risk-free interest rate r [4]. Since the change in the portfolio is proportional to its current value over time with proportionality constant r, we can create the relationship:

$$\begin{aligned} d\Pi &= r\Pi dt_1 \\ &= r\big(C - \frac{\partial C}{\partial S}S\big)dt_1 \end{aligned} \tag{12}$$

Combining (11) and (12),

$$r\big(C - \frac{\partial C}{\partial S}S\big)dt_1 = \Big(\frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t_1}\Big)dt_1 \tag{13}$$

Moving the terms around produces

$$\frac{\partial C}{\partial t_1} + rS\frac{\partial C}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2} - rC = 0 \tag{14}$$

Which is identical to (1), thus we have derived the Black-Scholes partial differential equation. The first and third assumption are the most easily seen assumptions in the derivation, but all assumptions listed above are necessary for this model to be applicable.

Now that we have the PDE, it is necessary to find the final and boundary conditions to complete the derivation of the problem.

Final Condition
The final condition can be intuitively seen from our understanding of the call option. We have two cases to consider, when $S > X$ or when $S < X$ at $t_1 = t_2$. When $S > X$, $C(S, t_2) = S - X$ because the stock can be bought for X and immediately sold at S. When $S < X$, $C(S, t_2) = 0$ because the option will not be used since it would not be profitable. Thus we can express the final condition as

$$C(S, t_2) = \max(S - X, 0) \tag{15}$$

Boundary Condition
To find the boundary conditions for $S = 0$ and $S \to \infty$. If $S = 0$, the stock is worthless so the option is worthless thus the boundary condition is

$$C(0, t_1) = 0. \tag{16}$$

For $S \to \infty$, as S grows the worth of the option and the chance that the value is $S - X$ continues to increase. Once S grows past X and then becomes very much larger, X becomes negligible in $C = S - X$, so the boundary condition is

$$\lim_{S\to\infty} C(S, t_1) = S. \tag{17}$$

2.3 Deriving the Solution

Reducing to the 1D Heat Equation

To derive the solution to the Black-Scholes differential equation we can reduce the PDE above (14) to the well known 1D heat equation, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. To do this we will use the following transformations:

$$\begin{aligned} t_1 = t_2 - \frac{2\tau}{\sigma^2} \quad &\longrightarrow \quad \tau = \frac{\sigma^2}{2}(t_2 - t_1), \\ S = Xe^x \quad &\longrightarrow \quad x = \ln\Big(\frac{S}{X}\Big), \\ C(S, t_1) = Xc(x,\tau) \quad &\longrightarrow \quad c(x,\tau) = \frac{C(S,t_1)}{X} \end{aligned} \tag{18}$$

The change of variables t_1 changes the final condition of the Black-Scholes PDE to an initial condition which is necessary for the heat equation[2]. The change of variables on S is so that the PDE will not depend on S. The change of variables on $C(S, t_1)$ is to aid in the reduction to the heat equation as the new function is dependent on the new variables.
We will now alter the individual parts of the Black-Scholes boundary value problem so that we may transform it into the heat equation. First consider the individual parts of the PDE in (14).

$$\begin{aligned} \frac{\partial C}{\partial t_1} &= X\frac{\partial c}{\partial \tau}\frac{\partial \tau}{\partial t_1} = X\frac{\partial c}{\partial \tau}\Big(\frac{-\sigma^2}{2}\Big) = \frac{-\sigma^2 X}{2}\frac{\partial c}{\partial \tau} \\ \frac{\partial C}{\partial S} &= X\frac{\partial c}{\partial x}\frac{\partial x}{\partial S} = \frac{X}{S}\frac{\partial c}{\partial x} \\ \frac{\partial^2 C}{\partial S^2} &= \frac{\partial}{\partial S}\Big(X\frac{\partial c}{\partial x}\frac{1}{S}\Big) = K\frac{\partial c}{\partial x}\Big(\frac{-1}{S^2}\Big) + \frac{X}{S}\frac{\partial}{\partial S}\Big(\frac{\partial c}{\partial x}\Big) \\ &= -\frac{X}{S^2}\frac{\partial c}{\partial x} + \frac{X}{S}\frac{\partial^2 c}{\partial x^2}\frac{\partial x}{\partial S} = -\frac{X}{S^2}\frac{\partial c}{\partial x} + \frac{X}{S^2}\frac{\partial^2 c}{\partial x^2} \\ &= \frac{X}{S^2}\Big(\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial x}\Big) \end{aligned} \tag{19}$$

Now we will transform the final condition of $C(S, t_1)$ into the initial condition for $c(x, \tau)$. First consider the two equations

$$C(S, t_2) = \max(S - X, 0) = \max(Xe^x - X, 0) \quad \text{and} \quad C(S, t_2) = Xc(x, \tau(t_2)) = Xc(x, 0) \tag{20}$$

From these we can rearrange and establish

$$c(x, 0) = \max(e^x - 1, 0) \tag{21}$$

Coming back to the transformed PDEs in (19), put these equations into the Black Scholes PDE:

$$\begin{aligned} -\frac{\sigma^2 X}{2}\frac{\partial c}{\partial \tau} + rS\Big(\frac{X}{S}\frac{\partial c}{\partial x}\Big) + +\frac{\sigma^2}{2}S^2\Big(\frac{X}{S^2}\Big(\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial x}\Big)\Big) - rXc = 0 \\ -\frac{\sigma^2 X}{2}\frac{\partial c}{\partial \tau} + rX\frac{\partial c}{\partial x} + \frac{\sigma^2 X}{2}\Big(\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial x}\Big) - rXc = 0 \\ -\frac{\sigma^2}{2}\frac{\partial c}{\partial \tau} + r\frac{\partial c}{\partial x} + \frac{\sigma^2}{2}\Big(\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial x}\Big) - rc = 0 \\ \Big(\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial x}\Big) + \frac{r}{\frac{\sigma^2}{2}}\Big(\frac{\partial c}{\partial x} - c\Big) = \frac{\partial c}{\partial \tau} \end{aligned} \tag{22}$$

We will define the following constant:

$$h = \frac{r}{\frac{\sigma^2}{2}} \tag{23}$$

So that (22) now looks like:

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (h-1)\frac{\partial c}{\partial x} - hc \tag{24}$$

From this equation, we can perform a second change of variables where the constants a, b will be defined later on[2]:

$$c(x,\tau) = e^{ax+b\tau}u(x,\tau) \longrightarrow u(x,\tau) = c(x,\tau)e^{-ax-b\tau} \tag{25}$$

We will perform the same procedure as before, finding the transformed partial derivatives relevant to (24).

$$\begin{aligned} \frac{\partial c}{\partial \tau} &= be^{ax+b\tau}u + e^{ax+b\tau}\frac{\partial u}{\partial \tau} \\ \frac{\partial c}{\partial x} &= ae^{ax+b\tau}u + e^{ax+b\tau}\frac{\partial u}{\partial x} \\ \frac{\partial^2 c}{\partial x^2} &= a^2e^{ax+b\tau}u + 2ae^{ax+b\tau}\frac{\partial u}{\partial x} + e^{ax+b\tau}\frac{\partial^2 u}{\partial x^2} \end{aligned} \tag{26}$$

Now placing them into (24).

$$\begin{aligned} be^{ax+b\tau}u + e^{ax+b\tau}\frac{\partial u}{\partial \tau} &= a^2e^{ax+b\tau}u + 2ae^{ax+b\tau}\frac{\partial u}{\partial x} + e^{ax+b\tau}\frac{\partial^2 u}{\partial x^2} \\ &\quad + (h-1)\Big(ae^{ax+b\tau}u + e^{ax+b\tau}\frac{\partial u}{\partial x}\Big) - he^{ax+b\tau}u \\ bu + \frac{\partial u}{\partial \tau} &= a^2u + 2a\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (h-1)\Big(au + \frac{\partial u}{\partial x}\Big) - hu \\ \frac{\partial u}{\partial \tau} &= \frac{\partial^2 u}{\partial x^2} + (2a+h-1)\frac{\partial u}{\partial x} + (a^2 + (h-1)a - h - b)u \end{aligned} \tag{27}$$

We may now define the a, b as done below to guide us to the heat equation.

$$\begin{aligned} a &= -\frac{h-1}{2} \\ b &= a^2 + (h-1)a - h = -\frac{(h+1)^2}{4} \end{aligned} \tag{28}$$

When substituted into (27), we produce the one dimensional heat equation with the initial condition:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x) = \max\left(e^{(\frac{h+1}{2})} - e^{(\frac{h-1}{2})}, 0\right) \tag{29}$$

Solving for $C(S, t_1)$

Recall the solution to the heat equation [5]:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} f(y)e^{-\frac{(x-y)^2}{4\tau}}\,dy \tag{30}$$

With the following change of variable we may start working towards the solution with our original variables.

$$z = \frac{y-x}{\sqrt{2\tau}} \quad \longrightarrow \quad y = z\sqrt{2\tau} + x, dz = \frac{1}{\sqrt{2\tau}}dy \tag{31}$$

Substituting this into (30),

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z\sqrt{2\tau}+x)e^{-\frac{z^2}{2}}dy \tag{32}$$

We only need to integrate over the region where $y > 0$, because this is when $f(y) > 0$ and otherwise $f(y) = 0$. When $y > 0$,$z > -\frac{x}{\sqrt{2\tau}}$ and we can produce the following results

$$\begin{gathered} f(z\sqrt{2\tau}+x) = e^{(\frac{h+1}{2})(z\sqrt{2\tau}+x)} - e^{(\frac{h-1}{2})(z\sqrt{2\tau}+x)} \\ u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{(\frac{h+1}{2})(z\sqrt{2\tau}+x)}e^{-\frac{z^2}{2}}dy - \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{(\frac{h-1}{2})(z\sqrt{2\tau}+x)}e^{-\frac{z^2}{2}}dy \end{gathered} \tag{33}$$

Simplifying the integral by adjusting the exponent and setting $y = z - \sqrt{\frac{\tau}{2}}(h+1)$[2], we find the following expression of $u(x,\tau)$.

$$\begin{aligned} u(x,\tau) = \frac{1}{\sqrt{2\pi}}e^{(\frac{h+1}{2})x+\tau\frac{(h+1)^2}{4}} &\int_{-\infty}^{\frac{-x}{\sqrt{2\tau}}+\sqrt{\frac{\tau}{2}}(h+1)} e^{-\frac{y^2}{2}}dy \\ &+ \frac{1}{\sqrt{2\pi}}e^{(\frac{h-1}{2})x+\tau\frac{(h-1)^2}{4}} \int_{-\infty}^{\frac{-x}{\sqrt{2\tau}}+\sqrt{\frac{\tau}{2}}(h-1)} e^{-\frac{y^2}{2}}dy \quad [2] \end{aligned} \tag{34}$$

We notice that the respective integrals are identical to the Normal distribution of $N(d_1)$ and $N(d_2)$ when

$$d_1 = \frac{-x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(h+1), \quad d_2 = \frac{-x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(h-1) \tag{35}$$

Thus we can rewrite (34) as

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}}e^{(\frac{h+1}{2})x+\tau\frac{(h+1)^2}{4}}N(d_1) + \frac{1}{\sqrt{2\pi}}e^{(\frac{h-1}{2})x+\tau\frac{(h-1)^2}{4}}N(d_2) \tag{36}$$

From (25) we recall that

$$c(x,\tau) = u(x,\tau)e^{-\frac{h-1}{2}x-\frac{(h+1)^2}{4}\tau} \tag{37}$$

Therefore,

$$c(x,\tau) = e^{-x}N(d_1) - e^{-h\tau}N(d_2) \tag{38}$$

Reversing the change of variables done in (18),

$$\begin{aligned} C(S,t_1) &= Xe^{x}N(d_1) - Xe^{-h\tau}N(d_2) \\ &= SN(d_1) - Xe^{-r(t_2-t_1)}N(d_2) \end{aligned} \tag{39}$$

This is identical to the solution we desired from (2), thus we have solved the Black-Scholes PDE through reduction to the heat equation.

3 Discussion of Model

3.1 Strengths

Closed-Form Solution
One of the major strengths of the Black-Scholes model is that it provides a closed-form analytical solution for option prices. This makes it easy to calculate the theoretical value of European options and is computationally efficient compared to numerical methods. The formula for call and put options allows for quick implementation in financial software and trading algorithms [1].

Foundation of Modern Financial Derivatives
The Black-Scholes model laid the foundation for the modern theory of financial derivatives and option pricing. Its introduction in 1973 revolutionized the way financial markets approach options pricing and risk management. The model has led to the development of other sophisticated models and techniques such as risk-neutral valuation, pricing futures, and hedging portfolios [8].

3.2 Limitations

Assumption of Constant Volatility
One of the most significant limitations of the Black-Scholes model is its assumption of constant volatility. In reality, volatility is not constant and tends to vary over time (a phenomenon known as volatility clustering). This assumption simplifies the model but reduces its accuracy in calculating market dynamics [7].

No Dividends
The Black-Scholes model assumes that the underlying asset does not pay dividends during the life of the option. In practice, many stocks and other assets do pay dividends, which can significantly affect the option price. Although modifications to the model exist to account for dividends (e.g., adjusting the underlying asset price for the present value of dividends), this assumption reduces the model's applicability for dividend-paying assets [7].

Market Friction and Transaction Costs
The model assumes frictionless markets, meaning there are no transaction costs or taxes associated with buying or selling options. In real markets, however, transaction costs can be substantial, particularly for options with wide bid-ask spreads or low liquidity. This can cause the model to overestimate the profitability of certain strategies and undervalue the costs of others [6].

European-Style Options Only
The Black-Scholes model is designed specifically for European-style options, which can only be exercised at expiration. Many real-world options, such as American-style options, allow for early exercise. This introduces additional complexity that the Black-Scholes model does not account for. Although there are approximations and adjustments for American options, the model is less suitable for them [1].

No Consideration of Stochastic Interest Rates
The Black-Scholes model assumes that the risk-free interest rate is constant over the life of the option. However, in practice, interest rates are subject to fluctuations and may follow their own stochastic processes. This limitation is especially important in long-term options or when interest rates are volatile [8].

3.3 Exploration of the Boundary Conditions

As part of our discussion, we wanted to dive deeper into the boundary conditions of the PDE that solve the Black-Scholes equation. In the model, boundary conditions play a crucial role in determining the price of options over time. These conditions stem from the financial context of the problem and help to define the solution to the PDE. Proper boundary conditions ensure that the model provides meaningful and realistic results for option pricing.

Terminal Condition

The terminal condition is perhaps the most straightforward boundary condition in the Black-Scholes PDE. It defines the option's payoff at maturity, which is the value of the option at time t_2, the expiration date. For a European call option, the payoff at maturity is given by

$$C(S, t_2) = \max(S(t_2) - X, 0)$$

where $C(S, t_2)$ is the option price at time t_2, $S(t_2)$ is the stock price at maturity, and X is the strike price. For a European put option, the payoff is

$$C(S, t_2) = \max(X - S(t_2), 0)$$

This condition dictates the value of the option at the terminal time t_2, and it serves as the starting point for the solution of the PDE at maturity. The payoff function is piecewise linear, reflecting the nature of the option's value depending on whether the option is in-the-money or out-of-the-money[1].

Boundary Conditions at $S = 0$

At $S = 0$, the underlying asset has no value, and thus the option price must also be zero, reflecting the fact that the option cannot provide any positive payoff if the stock price is zero. Therefore, the boundary condition for $S = 0$ is

$$C(0, t_1) = 0 \quad \text{for all } t_1 \in [0, t_2]$$

This condition ensures that the option price at the origin is always zero, which is consistent with the idea that an option on a stock that has no value is worthless[1].

Boundary Condition as $S \to \infty$

As the stock price S becomes very large, the value of the strike price becomes irreverent. Therefore, the value of a call option approaches S, while the value of a put option approaches zero. Moreover, for large S, the option becomes essentially equivalent to a position in the underlying asset (for calls), or it becomes worthless (for puts). Thus, the boundary condition as $S \to \infty$ for a European call option is

$$\lim_{S \to \infty} C(S, t_1) = S(t_1)$$

For a put option, the boundary condition becomes

$$\lim_{S \to \infty} C(S, t_1) = 0$$

These conditions ensure that the option behaves as expected at extreme values of S, with the call option price growing linearly with S as the stock price increases, and the put option price asymptotically approaching zero[1].

Practical Implications of Boundary Conditions
These boundary conditions are vital for deriving the Black-Scholes model and for producing a solution that is both mathematically correct. In more complex or modified models, the boundary conditions may be adjusted to reflect factors such as dividends, early exercise in American options, or stochastic volatility. However, in the standard Black-Scholes model for European options, these boundary conditions provide a foundation for deriving the option price and analyzing its behavior under different market conditions[1].

4 Real World Application

4.1 Experimenting with the Model

To evaluate the performance of the Black-Scholes model, we applied it to historical stock data to calculate the fair market prices of various options. For this experiment, we assume the role of option writers, selling option contracts for NVIDIA and T-Mobile stocks. These contracts are offered on November 1, 2024, with an execution date of November 29, 2024. The contract prices are set using the fair market values determined by the Black-Scholes formula. For all calculations, we use a risk-free rate of 0.0437, the prevailing rate on the contract offering date.

The price of NVIDIA stock is \$135.4 and we are offering a strike price of \$138. The volatility is .3865, which gives us the solution to the Black-Scholes as:

$$d_1 = \frac{ln(\frac{135.4}{138}) + (.0437 + \frac{.3865^2}{2})(\frac{29}{365})}{.3865 * \sqrt{\frac{29}{365}}}$$
$$d_1 = -.088$$
$$N(d_1) = N(-.088) = .465$$
$$d_2 = d_1 - .3865 * \sqrt{\frac{29}{365}}$$
$$d_2 = -.197$$
$$N(d_2) = N(-.197) = .422$$
$$C(S, t_1) = 135.4 * .465 - 138 * e^{-.0437*(\frac{29}{100})} * .422$$
$$C(s, t_1) = 4.93$$

The solution the Black-Scholes equation tells us that we should price our stock option at \$4.93. This means that for a standard option of 100 stocks the contract would cost the buyer \$493.

The price of T-Mobile stock is \$223.28 and we are offering a strike price of \$227. The volatility is .2519. Following the same process that was used for the NVIDIA option, the Black-Schole equation gives the fair market price value of \$5.01. This costs the buyer \$501 for a standard option.

4.2 Analysis

After calculated the fair market price for the option, we analyzed whether the buyer or seller profited from the transaction. The actual price of NVIDIA at the expiration time was 138.25. This is above the strike price that was set, so it would be beneficial for the buyer to execute the contract. They would buy the 100 stocks at \$138 and then they would be able to sell them for \$138.25, resulting in a \$25 profit from the 100 stocks. However, since the buyer paid \$493 for the option contract, their net loss from the NVIDIA option was \$468. For T-Mobile, the actual price of the stock at the time of expiration was \$246.95. Because of this, the buyer would want to execute the option and would make a \$1994 profit from the stocks and \$1493 from the option contract.

5 Conclusions

The Black-Scholes model continues to serve as a necessary tool for financial professionals. Furthermore, it is a foundation in financial theory, providing an efficient and widely-used framework for pricing European options. This paper has delved into the derivation, assumptions, and strengths of the model, and examined its applicability in real-world scenarios, including an experiment with actual stock data to demonstrate its practical use. By reducing the Black-Scholes partial differential equation to a form of the heat equation, we were able to solve for the closed-form solution of the price of a European call option.

The Black-Scholes model's closed-form solution is one of its most significant advantages, allowing quick and accurate pricing of options under a set of simplifying assumptions. However, the model's reliance on constant volatility, the exclusion of dividends, and its assumption of a frictionless market are notable limitations that reduce its accuracy in some real-world settings.

Future work may explore possible improvements or extensions to the model that address its limitations, further increasing its applicability and accuracy in real-world financial problems.

References

[1] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[2] Ryan S. Brill. Deriving the black-scholes formula, 2018. Accessed: 2024-12-06.

[3] Raymond H. Chan. Black-scholes equations, Accessed: 2024-12-06. Lecture Notes for Math4210.

[4] Scott Guernsey. An introduction to the black-scholes pde model, 2013. Research Experience for Undergraduates (REU) Proposal.

[5] Richard Haberman. *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*. Pearson Modern Classic, 5th edition, 2012.

[6] E. G. Haug. *The Complete Guide to Option Pricing Formulas*. McGraw-Hill, 2007.

[7] J. C. Hull. *Options, Futures, and Other Derivatives*. Pearson, 9th edition, 2017.

[8] R. C. Merton. Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, 4(1):141–183, 1973.

[9] A. S. Shinde and K. C. Takale. Study of black-scholes model and its applications. *Procedia Engineering*, 38:270–279, 2012.