A COMPLEX INVESTIGATION OF THE RIEMANN ZETA FUNCTION

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Contents

1 Abstract

The Riemann Zeta function is famous for its connection to the distribution of prime numbers and inexplicable non-trivial zeros where it's real part is 1/2. This project explores the Riemann Zeta function, emphasizing its properties such as its analyticity, singularities, zeros, contour integrals, and applications of the function. This will be done by utilizing skills from complex analysis such as the argument principle, knowledge of analyticity, contour integration, and knowledge of singular points and divergence. The functions properties will also be considered in the scope of how they relate to specific applications, such as quantum mechanics and number theory.

2 Introduction

The Riemann Zeta function, defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

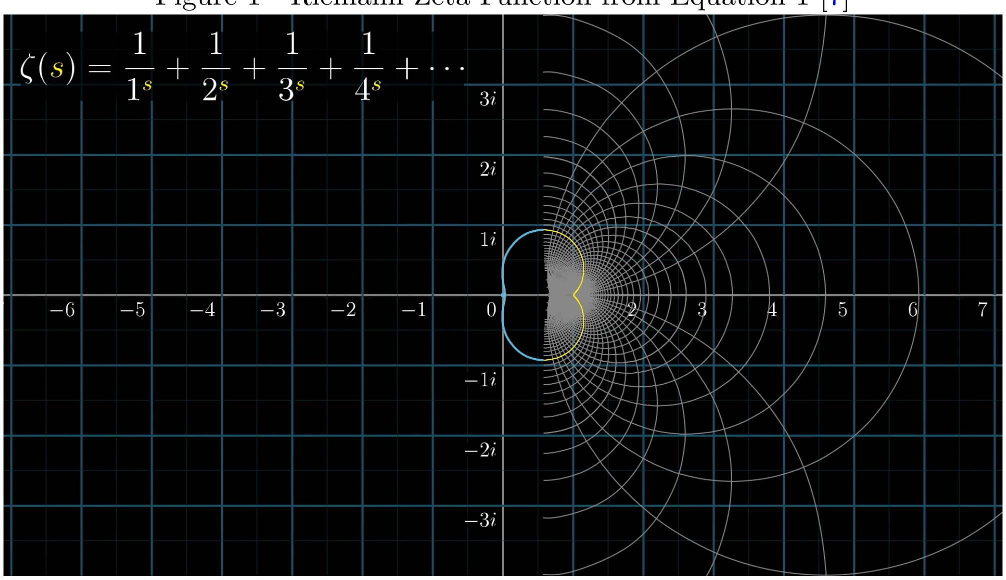
is a function containing the complex variable s, $s = \sigma + it$. The above summation is valid when $Re(s) = \sigma > 1$. Beyond this domain, the function diverges and must be analytically continued. Analytic continuation is "the process of extending the range of validity of a representation or more generally extending the region of definition of an analytic function" [1]. Because the function is undefined where $s <= 1$, we seek a continuation of the function in this region. With analytic continuation, the continuation of the function must be analytic everywhere which leads to there being only one valid extension. From this continuation we can explain equations such as

$$\zeta(-1) = \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = -\frac{1}{12} \tag{2}$$

which may otherwise seem nonsensical as it as a sum of natural numbers. A way that we can guarantee that the extension is analytic everywhere is by considering the angle preserving property. For a function to have a derivative everywhere, or is analytic, it must be true that for any two lines in the input space intersecting at an angle θ, after the transformation of the extended function, the two lines will still intersect at θ.

For the Riemann Zeta function, its summation representation seen in Equation 1 is only defined for $\sigma > 1$, which is shown graphically in Figure 1.

Figure 1 - Riemann Zeta Function from Equation 1 [7]

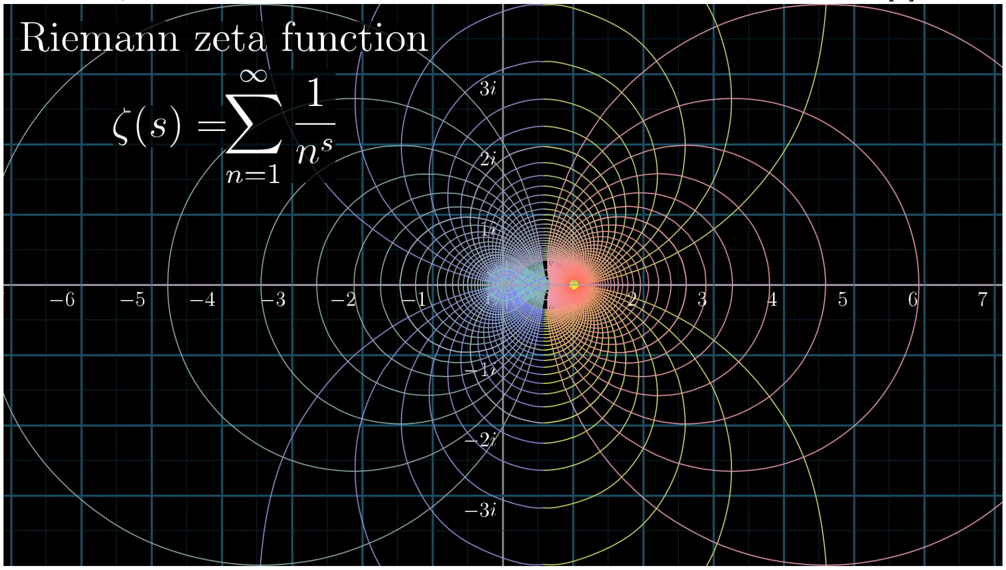


In this paper we look to investigate its analytic continuation which is defined on the whole complex plane as

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2}) \Gamma(1-s) \zeta(1-s) \tag{3}$$

Using this function definition, which is known as the completed Riemann Zeta function, the Riemann Zeta function can be continued for $\sigma \leq 1$ as shown in Figure 2.

Figure 2 - Continued Riemann Zeta function from Equation 3 [7]



One of the curious thing about the valid analytic continuation is the behavior of the zeros of the function. All of the negative even numbers are mapped to the origin through this extension, which have the name of "trivial zeros." The other "nontrivial zeros" are known to lie in the critical strip, where $0 < Re(s) < 1$, and believed to lie on the critical line s, where $Re(s) = \frac{1}{2}$. The distribution of the nontrivial zeros gives us insight on prime numbers as well which will be discussed further on. This behavior is only believed to be true and has been deemed as the "1,000,000 dollar question" due to the 1 million dollar reward offered to anyone who can prove that all the nontrivial zeros lie on the critical line.

In this project we will further investigate the continuation shown in Equation 3 through further properties of the Riemann Zeta function such as its zeros, contour integrals, singularities, and applications like quantum mechanics and number theory.

3 Investigation

3.1 Zeros and Contour Integrals of Riemann-Zeta

The Riemann-Zeta function has two types of zeros, trivial and non-trivial. The trivial zeros are found where s is a negative even integer, meaning that $t = 0$ and σ is a negative even integer. From Equation 3, the function $\zeta(s)$ evaluates to 0 as $\sin(\frac{\pi s}{2}) = 0$ for any negative even integer s. This does not apply for positive even integers as the gamma function's pole cancel out these zeros [9].The non-trivial zeros of the Riemann-Zeta function are the true mystery. Although it is known that all zeros lie on the critical strip where $0 < Re(s) < 1$, it is theorized that all non-trivial zeros of the Riemann-Zeta function lie on the critical line where $s = \frac{1}{2} + it$. This theory is known as the Riemann Hypothesis [8].

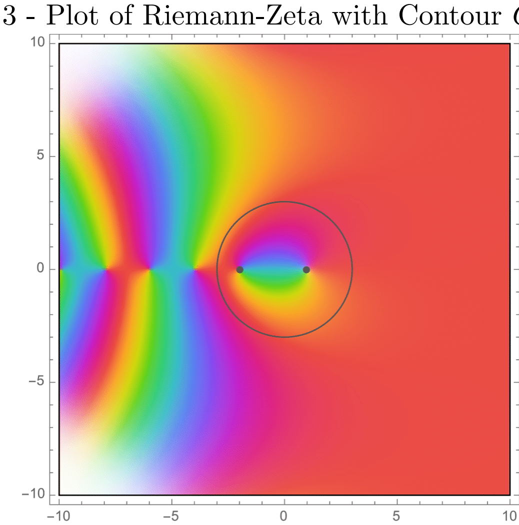
By using the Argument Principle, we can investigate the contour integral of the Riemann-zeta over certain regions to determine the number of zeros in a region. The Argument Principle states that for a meromorphic function $f(z)$ defined inside and on a simple closed contour C, with no zeros or poles on C,

$$I = \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz = N - P \tag{4}$$

where N is the number of zeros of $f(z)$ and P is the number of poles of $f(z)$ inside C, accounting for multiplicity. [1] This principle is helpful as we do not have to integrate over a contour for the completed Riemann Zeta function and its derivative, which would be very complicated.Since we know there is only one pole,$s = 1$, the amount of zeros within a certain contour can be determined using this principle.

For example, using the contour $C : |s| = 3$, we can show that the there is one zero in this contour. This contour along with the Riemann Zeta function can be illustrated Figure 3 below

Figure 3 - Plot of Riemann-Zeta with Contour $C : |s| = 3$

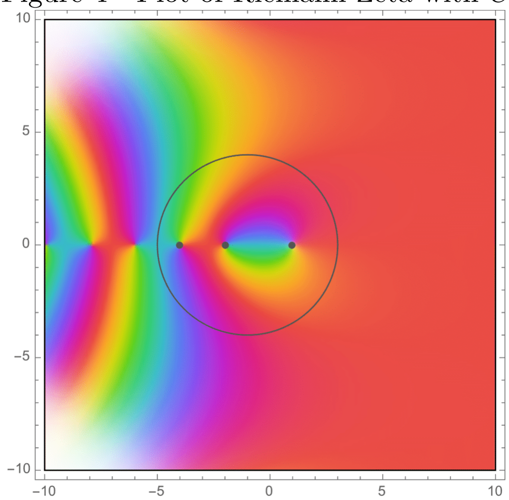


This plot marks the pole of $s = 1$ and the zero at $s = -2$ that are within the contour. Through argument principle, it can be determined that since within this contour $N = 1$ and $P = 1$, $I = 0$.

If the contour of a circle centered at $(-1, 0)$ with radius of 4 used, named C_a, there will be two zeros, $s = -2, -4$, within the contour along with the known pole. This is illustrated below in

Figure 4. By Argument Principle, it would be assumed that $I = 1$ because $N - P = 1$. To verify this, evaluate the integral I using C_a.

Figure 4 - Plot of Riemann-Zeta with C_a



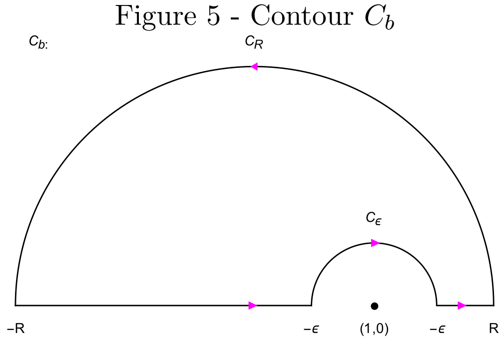
See appendix for verification of these contour integrals through Mathematica

Additionally, it would be curious to consider real integrals for the Riemann Zeta function. For example, consider the integral,

$$I = \int_{-\infty}^{\infty} \zeta(x)\, dx \tag{5}$$

To evaluate this integral, the following contour C_b in Figure 5 and method in Equation 6 could be used,

Figure 5 - Contour C_b



$$\oint_C \zeta(s)\, ds = \lim_{R\to\infty, \epsilon\to 0^+} \left(\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} + \int_{C_R} + \int_{C_\epsilon} \right) \zeta(s)\, ds = 0 \tag{6}$$

As this is a very complicated function and therefore integral, we do not have the skill set to do find the value of I at this time, but it is an interesting problem to think about.

3.2 Singularity of Riemann-Zeta

As mentioned before, the Riemann-Zeta function is most usually defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{7}$$

where s is a complex variable, detailed as $s = \sigma + it$. By definition, the singularities of a function are where it's undefined or behaves in an unusual way. For a series such as this, that would occur when a value of s causes the series to diverge, meaning it grows without bound as more terms are added.

By inspecting the right half of the complex plane (when $\sigma > 0$), it's clear that values of $\sigma > 1$ cause $\zeta(s)$ to converge. This is because the values of n^s grow so rapidly that $\zeta(s)$ converges to zero as the number of terms increase. As well as this, there are no discontinuities of $\zeta(s)$ when $\sigma > 1$.

Within this planar constriction, remaining is the interval $0 < \sigma \leq 1$. When $\sigma = 1$, the Riemann Zeta function becomes analogous with the Harmonic Series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... \tag{8}$$

which **diverges**, suggesting that the Riemann Zeta function at s = 1 will, too.

Lastly, and arguably most interestingly, is $\zeta(s)$ when $\sigma < 1$. The Riemann Zeta function does not converge absolutely here, but it can be analytically continued using techniques from complex analysis. This continuation gives rise to what's known as the Dirichlet Eta function, denoted by

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \tag{9}$$

which contains the first four nonzero terms

$$\eta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + ... \tag{10}$$

This Dirichlet Eta function converges absolutely for $\sigma > 0$, and the Riemann Zeta function can be written in terms of $\eta(s)$ [5] (see **Calculations within Singularities of the Riemann Zeta Function** within the appendix for detailed steps)

$$\zeta(s) = \frac{1}{1 - 2^{1-s}}\eta(s) \tag{11}$$

Creating an analytic continuation of the Riemann Zeta function in the domain $0 < \sigma < 1$.

Now, in order to analyze any potential singularities of this continuation, it's important to re-address the fact that $\eta(s)$ converges absolutely for $\sigma > 0$, implying that these potential singularities would occur from

$$\frac{1}{1 - 2^{1-s}} \tag{12}$$

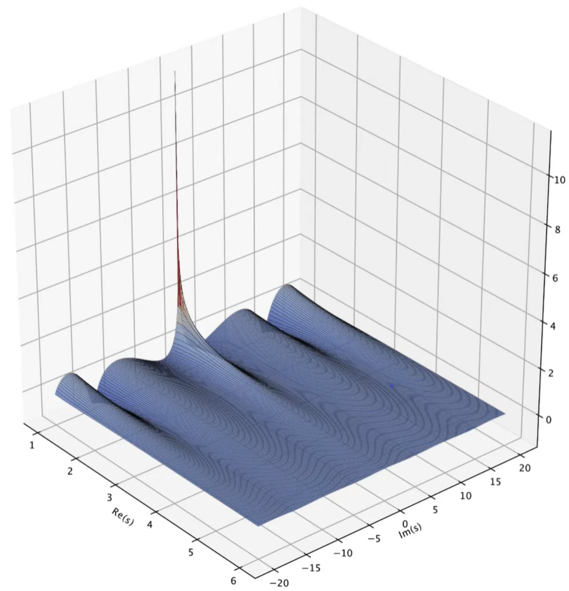
Trivially, this diverges when $1 - 2^{1-s} = 0$, or when $s = 1$. However, since s is a complex variable, there are other possibilities as well. See appendix for details, and what results is a condition for s:

$$s = 1 - \frac{2\pi i n}{\log(2)} \tag{13}$$

This suggests that there may actually be **infinitely many** divergences between $0 < \sigma < 1$ (because there are infinitely many integers that n could be). [5]

However, a visualization of the Riemann Zeta function implies otherwise: [5]

Figure 6 - Riemann Zeta visualization



As shown, the function only has a clear singularity at s = 1 (the potential divergence from when $\zeta(s)$ equals the Harmonic Series).

This incorporates a contradiction with the previous analysis: how can a function have one singularity and also infinitely many? This may, however, be possible if every singularity when $s \neq 1$ is **removable**.

Returning to the analytic continuity of $\zeta(s)$ on $0 < \sigma < 1$,

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \eta(s) \tag{14}$$

removable discontinuities are possible if at each point where $1 - 2^{1-s} = 0$, $\eta(s)$ is also equal to zero. This is because the divergence of $\frac{1}{2^{1-s}}$ could become obsolete if the zero of $\eta(s)$ occurs at that same point, negating it. However, verifying this correlation could prove to be a rather daunting task.

Fortunately, there is another approach that can imply this same theory. Starting with the introduction of another, more specially constructed, Dirichlet series: $X(s)$: [5]

$$X(s) = 1 + \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + ... \tag{15}$$

This series is quite similar to the Dirichlet Eta function, but only has a negative sign on every third term, instead of every other. Using the same technique as before, $\zeta(s)$ can be written in terms of X(s) (see appendix)

$$\zeta(s) = \frac{1}{1-3^{1-s}} X(s) \tag{16}$$

This result is critically similar to the previous relation containing $\eta(s)$ and all the same logic can be applied, including the conviction that X(s) also converges for $Re(s) > 0$. This implies that the singularities of $\zeta(s)$ **must** occur from $\frac{1}{1-3^{1-s}}$, or when $1 - 3^{1-s} = 0$. By applying the same algebra as before, it's apparent that

$$s = 1 - \frac{2\pi i m}{\log(3)}, \quad m \in Z \tag{17}$$

must **also** be the case for $\zeta(s)$ to attain a singularity.

Therefore, by equating these two conditions

$$1 - \frac{2\pi i n}{\log(2)} = 1 - \frac{2\pi i m}{\log(3)}, \quad n, m \in Z \tag{18}$$

See appendix for detailed simplification steps

This results in the requirement that

$$3^n = 2^m \tag{19}$$

Containing only one trivial solution: when $n = m = 0$. By returning to the previously contrived condition of singularity, plugging in $m = n = 0$ will always result in $s = 1$, which happens to be the acclaimed singularity retrieved from the visualization of $\zeta(s)$. This not only proves that $s = 1$ is indeed an isolated singular point of $\zeta(s)$, but it's the **only** singularity of the Riemann Zeta function. [5]

An isolated singularity is classified as a pole if the function approaches either positive or negative infinity as s approaches the singularity point from either direction.

For the Riemann Zeta function, as s approaches 1 from the right ($s > 1$), the function diverges to positive infinity. Similarly, as s approaches 1 from the left ($s < 1$), the function also diverges to positive infinity.

Thus, since the function diverges to positive infinity from both directions as s approaches 1, $s = 1$ is classified as a pole of $\zeta(s)$. Specifically, it's considered a **simple pole** because the divergence is not oscillatory or otherwise complex; it's a straightforward unbounded increase in the function value.

Now, this all proves that when $\sigma > 0$, the only singularity of $\zeta(s)$ is when $s = 1$. Yet it doesn't directly show anything about the behavior of $\zeta(s)$ when $\sigma < 0$. In order to consider $\zeta(s)$ on the left half plane, the previously mentioned **reflection formula** can be used to relate the function's value at s to its value at $1 - s$. This builds an analytic continuation of the Riemann Zeta function for $\sigma < 0$. By definition, this analytic continuation is analytic on the left half complex plane, meaning that it is nowhere singular with respect to all finite points. $s = 1$ then remains as the only singularity of the Riemann Zeta function.

Therefore, with the classification of $\zeta(s)$'s only singularity as a simple pole at $s = 1$, this implies that the Riemann Zeta function is a **meromorphic function**. A meromorphic function is a function whose singularities are all poles, and since $\zeta(s)$ contains only one singularity, which is a pole, it is hence meromorphic.

In summary, the Riemann Zeta function and its analytic continuations are analytic everywhere on the complex plane except for at $s = 1$, where it corresponds to the divergent Harmonic Series.

3.3 Finding Scattering Amplitudes using the Riemann Zeta Function

Properties of the Riemann Zeta function can be applied to solving for scattering amplitudes in quantum physics. These scattering amplitudes represent the probability for particles to scatter or interact with each other. When particles collide or interact, they exchange various forces, such as electromagnetic or weak forces. The scattering amplitude describes the likelihood of different outcomes of these interactions, including how particles might change direction, exchange momentum, or transform into different types of particles.

An example of this application is presented in Doctor Grant N. Remmen's Letter on *Amplitudes and the Riemann Zeta Function* [[3]], where he describes how physical properties of scattering amplitudes are mapped to the Riemann Zeta function.

Without going too much into the quantum mechanics listed in Remmen's article, there are some fascinating parallels he discovers with respect to the Riemann Zeta function and a specific scattering amplitude function. By constructing this specific scattering amplitude function in terms of the Riemann Zeta function, he finds that many of the specific properties of the Zeta function correspond to characteristics of all scattering amplitudes. This allows him to establish that his unique function does indeed behave as an appropriate scattering amplitude.

He uses the singularities of the Riemann Zeta function, a Laurent series expansion, and the Residue theorem to prove the necessary characteristics of his scattering amplitude. As well as this, he takes into account the multiple possibilities of what could occur if the Riemann hypothesis were true, and how that would affect his amplitude function.

Please see **Explanation of Riemann Zeta function and Scattering Amplitudes** within the appendix for a detailed report on how Remmen manipulates properties of the Riemann Zeta function to prove the legitimacy of his scattering amplitude function.

3.4 Approximating the Distribution of Prime Numbers with the Zeta Function

The Riemann Zeta function reveals a surprising amount of information on the distribution of prime numbers. To start, in the 18th century, Euler discovered an important formula relating primes and the Zeta function called the Euler product formula.[4]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{20}$$

Where *pprime* is the set of all prime numbers. This at first seems unnatural; however, a quick proof can help visualize this better and can be seen in Appendix Section 5.3. The Euler's Product formula showcases that every natural number can be uniquely represented by a product of primes (also known as the fundamental theorem of arithmetic). It is also the key method for relating the Zeta function to the very important prime counting function.

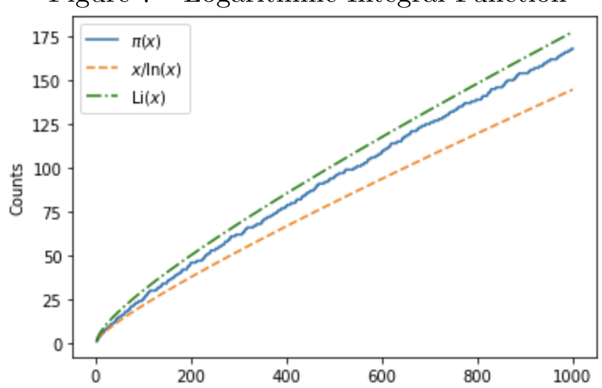
The prime counting function denoted by $\pi(x)$ was introduced by Gauss in 1793.[9] It is an easy way to see how primes are distributed high up on the number line, and it is defined by how many prime numbers are less than or equal to some real number x. For example, $\pi(10) = 4$, (2,3,5,7). This function sparked the curiosity of many mathematicians, which ultimately led to the discovery of the prime number theorem:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln(x)} = 1 \tag{21}$$

This theorem states that as x approaches infinity, $\pi(x)$ will approach $x/logx$.[4] At around the same time, another function called the logarithmic integral function was discovered, which happened to be a better approximation for primes than the prime number theorem and can be seen below.

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\ln(t)}\, dt \tag{22}$$

Figure 7 - Logarithmic Integral Function



To relate this back to the Zeta function, the zeta zeros contain the most interesting insights about prime numbers. As noted earlier in this paper, there are the trivial zeros that occur at values $s = -2n$, and there are the nontrivial zeros that occur in the critical strip $0 < Re(s) < 1$. The Riemann Hypothesis proposes that all of these nontrivial zeros occur on the vertical line Re(s)=$\frac{1}{2}$. This hypothesis is strongly suggested to be true, but it has not been proven. To see the full implications of these nontrivial zeros, a function was created that used the zeros to estimate $\pi(x)$. Riemann came up with his own version of the prime number theorem that went as follows:

$$J(x) = \mathrm{Li}(x) - \sum_p \mathrm{Li}(x^p) - \log 2 + \int_x^{\infty} \frac{1}{t(t^2-1)\log t}\, dt \tag{23}$$

[4]

This function can be easily understood by breaking it down into four parts. The first term, Li(x), is the logarithmic integral function. “The second term, or ‘periodic term’ is the sum of the logarithmic integral of x to the power ρ, summed over ρ, which are the non-trivial zeros of the Riemann zeta function. It is the term that adjusts the overestimate of the principal term.”[4] The third and fourth terms are simply error-correcting terms that Riemann found. This formula was a massive improvement at modeling $\pi(x)$ compared to $x/logx$ or any other prime counting function for that matter. To illustrate the accuracy of the function, there are two graphs below.

Figure 8 - $\pi(x)$ Being Approximated by $J(x)$ Using the First 35 Nontrivial Zeros[4]

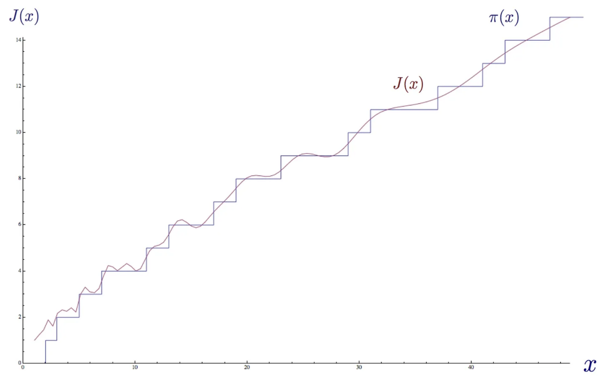
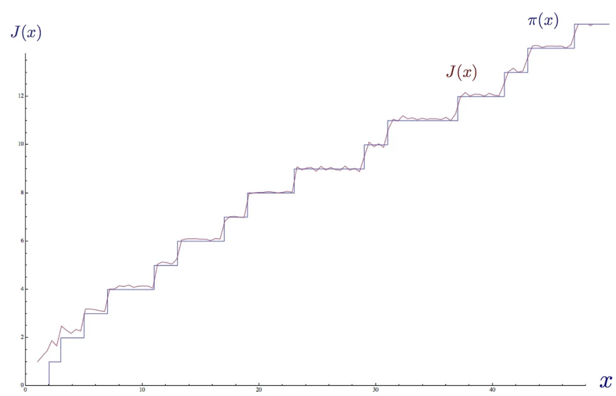


Figure 9 - $\pi(x)$ Being Approximated by $J(x)$ Using the First 100 Nontrivial Zeros[4]



These results are nothing short of extraordinary. As the number of zeta zeros used approaches infinity, Riemann’s counting functions approach the prime counting function. So if Riemann’s hypothesis is true, the zeta function could be used to find a near-perfect match for the distribution of prime numbers.

4 Conclusions and Further Directions

The Zeta function, denoted by Equation 1, was first explored in the 18th century by L. Euler and was only defined with values $s > 1$. It wasn't until 100 years later that B. Riemann extended the function to the entirety of the complex plane and derived the functional equation (Equation 3).[9] This extended equation brought along many interesting properties and applications. First, there are two types of zeros in the function: trivial and nontrivial. The trivial zeros can be found at $s = -2n$, where n is a positive integer. The nontrivial zeros are claimed to be at $\sigma(s) = \frac{1}{2}$ by the Riemann Hypothesis. We used the Argument Principle to show the number of zeros within a given contour. Following the investigation into the zeta zeros, we examined the singularities of the function. With the help of multiple functions such as the Dirichlet Eta function and the Dirichlet series and through multiple calculations, the results concluded that the Zeta function contains only one isolated singular point at $s = 1$. Lastly, we performed an investigation on the distribution of the prime numbers using the Zeta function, highlighting the relationship between prime numbers and the Zeta function through the Euler product formula and also showing how to almost exactly imitate the prime counting function using the nontrivial zeros and Riemann's Prime Counting Function. This investigation provided the fundamentals for understanding the Riemann Zeta function; however, there are still many aspects of the function that were left uncovered.

With more time, it may be interesting to further explore real integrals of the zeta function, as discussed briefly in Section 3.1. Doing further research on this topic could lead to interesting applications of real integrals of the zeta function. Additionally, it would be interesting to investigate further applications of the Zeta Function such as cryptography and probability. Another topic to look into in more detail would be to go into the derivation of the completed Riemann Zeta function. Further, it could be exciting to attempt a proof for the Riemann Hypothesis individually. This would require intense research and dedication but could be fun to see what one would learn in this attempt.

5 Appendix

5.1 Calculations within Singularities of the Riemann Zeta Function

Calculation for Riemann Zeta function in terms of Eta Function:

Write

$$\zeta(s) - \eta(s) = (\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + ...) - (1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + ...) \tag{24}$$

Which simplifies to

$$\frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{8^s} + ... = \frac{2}{2^s}\zeta(s) \tag{25}$$

Then plugging back into $\zeta(s) - \eta(s)$ and solving for $\zeta(s)$

$$\zeta(s) - \eta(s) = \frac{2}{2^s}\zeta(s) \tag{26}$$

$$\zeta(s) - \frac{2}{2^s}\zeta(s) = \eta(s) \tag{27}$$

$$\zeta(s)(1 - \frac{2}{2^s}) = \eta(s) \tag{28}$$

$$\zeta(s) = \frac{1}{1 - 2^{1-s}}\eta(s) \tag{29}$$

See equation (11)

Calculating Condition on s:

$2^{1-s} = 1$ can be written as

$$e^{\log(2)(1-s)} = 1 \tag{30}$$

Which is only true when the argument of $e^{\log(2)(1-s)}$ is equal to $2\pi i$, or any integer multiple thereof,

$$\log(2)(1 - s) = 2\pi i n, \quad n \in Z \tag{31}$$

since this is when the phase will be pointing directly right, where $Re(s) = 1$. Simplifying this expression results in

$$s = 1 - \frac{2\pi i n}{\log(2)} \tag{32}$$

See equation (13)

Calculation for Riemann Zeta Function in terms of $X(s)$ Function:

Write

$$\zeta(s) - X(s) = (\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + ...) - (1 + \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} + ...) \tag{33}$$

Simplifying to (same method as with $\eta(s)$, see above)

$$\zeta(s) = \frac{1}{1 - 3^{1-s}} X(s) \tag{34}$$

See equation (16)

Simplification of Equating Singularity Conditions:

Start with

$$1 - \frac{2\pi i n}{\log(2)} = 1 - \frac{2\pi i m}{\log(3)}, \quad n, m \in Z \tag{35}$$

The constant 1 cancels from both equations, and so does $2\pi i$. Then, cross multiply each log term. What's left is

$$n \log(3) = m \log(2) \tag{36}$$

By evocation of the exponential, e, this can be rewritten as

$$e^{n \log(3)} = e^{m \log(2)} \tag{37}$$

Which finally becomes

$$3^n = 2^m \tag{38}$$

See equation (19)

5.2 Explanation of Riemann Zeta Function and Scattering Amplitudes

To begin with, Dr. Remmen defines a function of the complex variable, s,

$$\alpha(s) = -\frac{i}{4\sqrt{s}} \left[\Psi\left(\frac{1}{4} + \frac{i}{2}\sqrt{s}\right) + \frac{2\zeta'(\frac{1}{2} + i\sqrt{s})}{\zeta(\frac{1}{2} + i\sqrt{s})} \right] + \frac{i \log \pi}{4\sqrt{s}} - \frac{1}{s + \frac{1}{4}} \tag{39}$$

where $(\frac{1}{2} + i\sqrt{s}) = z$, and $\Psi(\frac{z}{2})$ is the Digamma function. He then writes $\alpha(s)$ in terms of the Landau-Riemann (capital) xi function $\Xi(z) = \xi(\frac{1}{2} + iz)$, where $\xi(z)$ is defined as $\frac{1}{2}z(z-1)\pi^{-\frac{z}{2}}\Gamma(\frac{z}{2})\zeta(z)$.

This $\alpha(s)$ can then be compacted into:

$$\alpha(s) = -\frac{d \log \Xi(\sqrt{(}s)}{ds} \tag{40}$$

and then finally, $\alpha(s)$ is used to "define an amplitude describing the four-point scattering of massless particles in terms of the Mandelstam variables, $s = -(p_1 + p_2)^2, t = -(p_1 + p_3)^2$, and

$u = -s - t$"[3]. This constructs a specific scattering amplitude, denoted as $M(s,t)$, that exhibits striking parallels with the properties of the Riemann Zeta function.

$$M(s,t) = \alpha(s) + \alpha(u) \tag{41}$$

Since the Digamma and Zeta functions are both meromorphic, this scattering amplitude is then shown to be meromorphic as well, and its poles correspond to the nontrivial zeros of the Riemann zeta function, $\zeta(\frac{1}{2} + i\mu_n) = 0$, at $s = \mu_n^2$.

This then creates a connection with the Riemann hypothesis: if the Riemann hypothesis is true, then μ_n must be real, and all of $M(s,t)$'s poles lie on the real axis. "For a theory with scattering described by $M(s,t)$, the Riemann hypothesis then becomes the physical requirement of real masses for the on-shell states in the spectrum."[3] In other words, the scattering amplitude $M(s,t)$ describes the behavior of particles undergoing scattering interactions, and the poles of this amplitude correspond to the masses of hypothetical particles exchanged in the scattering process. The Riemann hypothesis particularly concerns the realness of the masses associated with the poles of the amplitude, $M(s,t)$.

This is substantial because if the Riemann hypothesis is valid, then the properties of particles involved in scattering interactions are constrained in a specific way dictated by the distribution of zeros of the Zeta function. This interplay between mathematical conjecture and physical theory highlights the deep and potentially unexpected connections that can emerge between seemingly unrelated fields.

Remmen next states that the amplitude $M(s,t)$ exhibits **locality**. Locality refers to the principle that interactions between distant objects occur only through their immediate surroundings and not instantaneously across space. This principle ensures that the effects of interactions are confined to nearby regions, leading to a predictable and consistent description of physical processes. [6]

The text claims that this assurance "requires $\alpha(s) \sim \frac{1}{-s+\mu_n^2}$, and a failure of locality in $\alpha(s)$ via a pole $\sim \frac{1}{(-s+\mu_n^2)^k}$ for some $k > 1$ would require $\zeta(z) \sim exp[\frac{b}{(z-z_n)^{k-1}}]$ near the corresponding zero z_n, for some b." [3] Since this exponential term would approach either zero or infinity as $z \to z_n$, depending on the direction of approach, this creates an **essential singularity**. However, by the fact that the Riemann zeta function is meromorphic and lacks essential singularities, locality in $\alpha(s)$ is enforced.

Another consequence of employing the Riemann Zeta function to incorporate a specific amplitude $M(s,t)$ is the adoption of **unitarity**. Unitarity is a fundamental property of quantum theories that ensures the conservation of probability, a requirement that imposes constraints on the behavior of scattering amplitudes and is essential for maintaining the consistency and predictability of quantum mechanical systems. [2]

By definition, since $M(s,t)$ is classified as an amplitude,

$$\lim_{s\to 0} \frac{d^{2k}}{ds^{2k}} M(s,0) > 0 \tag{42}$$

must be true for all $k > 0$. This occurs as a result of analyticity and unitarity, and is known as a **positivity property**.

To link this to complex variables, Remmen declares that "forward amplitudes in an infrared effective field theory coming from a well behaved ultraviolet completion are known to possess positivity properties coming from **analytic dispersion relations**." [3] The derivation of these dispersion relations involves computing contour integrals of the scattering amplitude over certain regions in the complex s plane. Analyticity allows the contours to be deformed to new paths, enabling the use of techniques such as the residue theorem to relate the integrals to physical quantities like cross sections associated with the scattering process.

Remmen computes the contour integral

$$c_{2k} = \oint_C \frac{ds}{s^{2k+1}} M(s,0) \tag{43}$$

for C a small, simple closed contour around the origin. The analyticity of $M(s,t)$ can be "deformed to a new contour running just above and below the real s axis, plus a circle at infinity." [3] Using unitarity, Remmen applies the optical theorem and crossing symmetry to imply that

$$c_{2k} = \frac{2}{\pi} \int_0^\infty \left(\frac{ds}{s^{2k}}\right) \sigma(s) + c_\infty^{(2k)} \tag{44}$$

where $\sigma(s)$ is the "(positive) cross section associated with the scattering in the amplitude, and

$$c_\infty^{(2k)} = \left(\frac{1}{2\pi i}\right) \oint_{|s|=\infty} \left(\frac{ds}{s^{2k+1}}\right) M(s,0) \tag{45}$$

is a boundary term." [3]

Remmen then states that "a nonzero boundary term for some $k \geq 0$ would imply that $\Xi(z)$ grows at least as fast as $exp(\frac{d}{z^{4k+2}})$ for some d (i.e., growth order at least $4k+2$), which is inconsistent with the fact that $\Xi(z)$ has a known growth order of unity; thus, all of the $c_\infty^{(2k)}$ must vanish." [3] Since it's already been shown that $M(s,0)$ is analytic everywhere in the complex s plane except for its poles at $s = \pm\mu_n^2$, the residue of each pole can be summed using the Residue Theorem to find that

$$c_0 = c_\infty^{(0)} + \frac{1}{2\pi i} \sum_n \oint_{s=\mu_n^2} \frac{ds}{s} M(s,0) = c_\infty^{(0)} + \sum_n \frac{2}{\mu_n^2} \tag{46}$$

which explicitly predicts the value of c_{2k} as

$$c_{2k} = \sum_{n=1}^{\infty} \frac{2}{\mu_n^{2(2k+1)}} \tag{47}$$

Returning to the Riemann hypothesis, it can be used here to imply the positivity of c_{2k}, required by unitarity and analyticity. Finally, Remmen claims that this "is a nontrivial check of the analytic and asymptotic structure of $M(s,0)$, confirming that it indeed behaves like a forward amplitude." [3]

Therefore, Dr. Grant N. Remmen finally concludes that the function he formed, $M(s,t)$, is indeed an accurate representation of a scattering amplitude. By incorporating the Riemann Zeta function within his amplitude function, Remmen was able to use complex variable techniques to link known properties of the Zeta function with necessary conditions of all scattering amplitudes, explicitly narrowing down his function as substantial. This finding also exposes a link between the distinct topics of complex variables and quantum mechanics, introducing the query of what other parallels this use of complex variables could uncover.

5.3 Proof of the Euler Product Formula

Euler begins with the general zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots \tag{48}$$

First, he multiplies both sides by the second term:

$$\frac{1}{2^s} \times \zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \dots \tag{49}$$

He then subtracts the resulting expression from the zeta function:

$$\left(1 - \frac{1}{2^s}\right) \times \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \dots \tag{50}$$

He repeats this process, next multiplying both sides by the third term

$$\frac{1}{3^s} \times \left(1 - \frac{1}{2^s}\right) \times \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \dots \tag{51}$$

And then subtracting the resulting expression from the zeta function

$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right) \times \zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \dots \tag{52}$$

The zeta function minus $1/2^s$ times the zeta function minus $1/3^s$ times the zeta function.

Repeating this process to infinity, one would in the end be left with the expression:

$$\dots\left(1 - \frac{1}{13^s}\right)\left(1 - \frac{1}{11^s}\right)\left(1 - \frac{1}{7^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 \tag{53}$$

1 minus all the prime reciprocals, multiplied by the zeta function.

Next, divide the expression by all of the prime reciprocal terms, and obtain:

$$\zeta(s) = \left(1 - \frac{1}{2^s}\right)^{-1} \times \left(1 - \frac{1}{3^s}\right)^{-1} \times \left(1 - \frac{1}{5^s}\right)^{-1} \times \left(1 - \frac{1}{7^s}\right)^{-1} \times \left(1 - \frac{1}{11^s}\right)^{-1} \times \dots \tag{54}$$

Shortened, we have shown that:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \tag{55}$$

[4]

5.4 Contour Integration Code

Contour Integral 1 - ContourIntegrate with $C : |z| = 3$

```
h[z_] = Zeta'[z]/Zeta[z]
```

```
ContourIntegrate[h[z], z ∈ Circle[{0,0},3]]
```

```
0
```

See Contour Integral 1

Contour Integrate 2 - ContourIntegrate with C_a

```
h[z_] = Zeta'[z]/Zeta[z]
```

$$\frac{\text{Zeta}'[z]}{\text{Zeta}[z]}$$

```
ContourIntegrate[h[z], z ∈ Circle[{-1,0},4]]
```

$2\,i\,\pi$

See Contour Integral with C_a

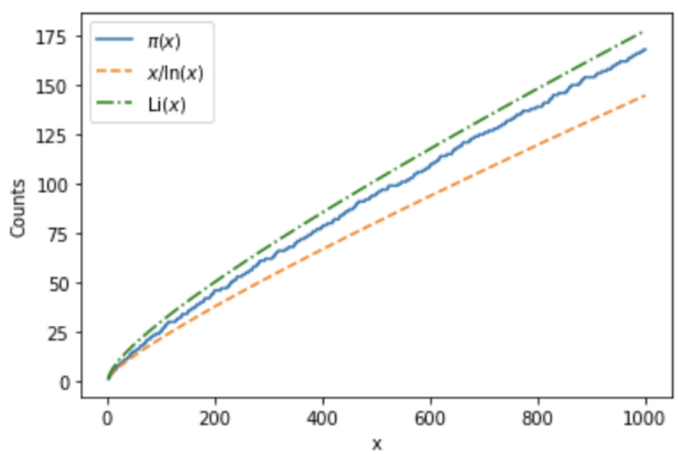
5.5 Logarithmic Integral Function Code

```python
import matplotlib.pyplot as plt
from sympy.ntheory.generate import primepi
from sympy import li
import numpy as np

# Define the range for x values, avoiding division by zero at x=1 for the x/log(x) plot
x_vals = np.linspace(2, 1000, 1000)
# Calculate y values using the primepi function
y_primepi = [primepi(x) for x in x_vals]
# Calculate y values for x / log(x)
y_x_logx = x_vals / np.log(x_vals)
# Calculate y values for the logarithmic integral
y_li = [li(x).evalf() for x in x_vals]

# Plot the results for pi(x), x/log(x), and Li(x)
plt.plot(x_vals, y_primepi, label=r'$\pi(x)$')
plt.plot(x_vals, y_x_logx, label=r'$x / \ln(x)$', linestyle='--')
plt.plot(x_vals, y_li, label=r'$\operatorname{Li}(x)$', linestyle='-.', color='green')

plt.xlabel('x')
plt.ylabel('Counts')
plt.legend()
plt.show()
```



6 Reference

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