Sorbonne Université

Calcul haute performance : programmation et algorithmique avancées

Projects: Merge Sort, Monte Carlo and PDE simulations

The presentation of the results on December 12th, 2024

/!\General introduction common to all projects

Students are required to submit the slides and the source code for their project to their instructor (by an email to L. ABBAS TURKI) by December 11th. The source code must be readable and, therefore, adequately commented to ensure clarity. In addition to evaluating the source code, grading will also be based on a presentation delivered during the session on December 12th. The entire presentation, including questions, should not exceed 15 minutes and should be structured as follows: 6 to 8 minutes for presenting results, 2 to 4 minutes for explaining the code, and 4 to 5 minutes for questions. Accordingly, the number of slides dedicated to presenting results should be limited to 8 maximum, and the duration of the presentation (excluding the question period) should fall between 10 and 11 minutes. During the 10 to 11 minutes of the presentation, speaking time must be evenly divided among students within the same group. If this balance is not maintained, all questions will be directed solely to the student who has spoken the least.

1 Monte Carlo simulation of Heston model

The Heston model for asset pricing has been widely examined in the literature. Under this model, the dynamics of the asset price S_t and the variance v_t are governed by the following system of stochastic differential equations:

$$dS_t = rS_t dt + \sqrt{v_t} S_t d\hat{Z}_t \tag{1}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t \tag{2}$$

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2} Z_t \tag{3}$$

where:

- the spot values $S_0 = 1$ and $v_0 = 0.1$,
- r is the risk-free interest rate, we assume $r = 0$,
- κ is the mean reversion rate of the volatility,
- θ is the long-term volatility,
- σ is the volatility of volatility,
- W_t and Z_t are independent Brownian motions

In this project, we aim to compare two distinct methods for simulating an at-the-money call option (where "at-the-money" here means $K = S_0 = 1$) at maturity $T = 1$ under the Heston model. The option has a payoff given by $f(x) = (x - K)_+$, and, thus, we want to simulate with Monte Carlo the expectation $E[f(S_T)] = E[(S_1 - 1)_+]$. This comparison will focus on the efficiency and accuracy of each simulation method in pricing the call option within the stochastic volatility framework of the Heston model.

We begin with the Euler discretization scheme, which updates the asset price S_t and the volatility v_t at each time step as follows:

$$S_{t+\Delta t} = S_t + rS_t\Delta t + \sqrt{v_t}S_t\sqrt{\Delta t}(\rho G_1 + \sqrt{1-\rho^2}G_2) \tag{4}$$

$$v_{t+\Delta t} = g\left(v_t + \kappa(\theta - v_t)\Delta t + \sigma\sqrt{v_t}\sqrt{\Delta t}G_1\right) \tag{5}$$

where G_1 and G_2 are independent standard normal random variables, and the function g is either taken to be equal to $(\cdot)_+$ or to $|\cdot|$.

1. Assuming $\kappa = 0.5$, $\theta = 0.1$, $\sigma = 0.3$ and using a discretization $\Delta t = 1/1000$, write down a Monte Carlo simulation code using Euler discretization to approximate $E[(S_1 - 1)_+]$. (/8.5 MAIN5, /7 other M2)

As an alternative to the Euler scheme, we are going to implement a version of the exact simulation procedure presented in the paper [1]. The steps that we choose differ a little bit as we take the discretization $\Delta t = 1/1000$ and we do the following:

step 1. Within a for loop on time steps, we define

$$d = 2\kappa\theta/\sigma^2, \quad \lambda = \frac{2\kappa e^{-\kappa\Delta t}v_t}{\sigma^2(1-e^{-\kappa\Delta t})}, \quad N = \mathcal{P}(\lambda), \quad \text{with } \mathcal{P} \text{ simulated by } \texttt{curand\_poisson}$$

and denoting $\mathcal{G}(\alpha)$ the standard gamma distribution whose simulation is presented in [4]

$$v_{t+\Delta t} = \frac{\sigma^2(1-e^{-\kappa\Delta t})}{2\kappa}\mathcal{G}(d+N).$$

step 2. The integral $\int_0^1 v_s ds$ is stored in a variable `vI` set to zero before the for loop on time steps. Then, in the for loop we update `vI+=`$0.5 * (v_t + v_{t+\Delta t})$.

step 3. Once we finish the for loop, we compute $\int_0^1 \sqrt{v_s}dW_s$, using the expression

$$\int_0^1 \sqrt{v_s}dW_s = \frac{1}{\sigma}\left(v_1 - v_0 - \kappa\theta + \kappa\texttt{vI}\right)$$

Then we compute

$$m = -0.5\texttt{vI} + \rho\int_0^1 \sqrt{v_s}dW_s, \quad \Sigma^2 = (1-\rho^2)\texttt{vI}$$

and we set $S_1 = \exp(m + \Sigma G)$ where G is a standard normal random variable.

2. Define a device function that simulates the standard gamma distribution $\mathcal{G}(\alpha)$ presented in [4]. Make sure to deal with both situations $\alpha \geq 1$ and $\alpha < 1$. (/6.5 MAIN5, /5 other M2)

3. For many values of $\kappa \in [0.1, 10]$, $\theta \in [0.01, 0.5]$ and $\sigma \in [0.1, 1]$ such that $20\kappa\theta > \sigma^2$, compare the execution time of the Euler discretization scheme to the exact simulation procedure. (/5 MAIN5, /4 other M2)

2 Nested Monte Carlo for exponential Ornstein-Uhlenbeck model

Numerous mean-reverting stochastic volatility models are utilized in financial mathematics to describe asset price dynamics. Here, we focus on the exponential Ornstein-Uhlenbeck (OU) volatility model, in which the volatility process follows an Ornstein-Uhlenbeck-type dynamics. Specifically, this model is characterized by the following equations:

$$dS_t = S_t\left(rdt + \exp(Y_t)dW_t\right)$$

$$dY_t = \alpha(m - Y_t)dt + \beta d\hat{Z}_t$$

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2}Z_t$$

where:

- the spot values $S_0 = 1$ and $Y_0 = \log(0.1)$,
- r is the risk-free interest rate, we assume $r = 0$,
- α is the mean reversion rate of the volatility,
- m is the long-term log-volatility,
- β is the volatility of the log-volatility,
- W_t and Z_t are independent Brownian motions.

For further details on this model, we refer the reader to [2]. In this project, our goal is to train a neural network that can accurately predict the price of a call option for any maturity, strike, and a range of model parameters α, β, m, ρ, and Y_0.

To achieve this, starting from question 2, we will generate a large number (nested) of Monte Carlo simulated prices on GPUs, storing the results in .csv files to create a comprehensive dataset of call option prices. This dataset will then serve as the training set for the neural network, enabling it to learn and generalize the relationship between option prices and the various model parameters across different maturities and strikes.

Before nested Monte Carlo of question 2, we start in question 1 performing a regular Monte Carlo simulation. This Monte Carlo simulation requires an Euler discretization scheme, which updates the asset price S_t and the log-volatility Y_t at each time step as follows:

$$S_{t+\Delta t} = S_t + rS_t\Delta t + \exp(Y_t)S_t\sqrt{\Delta t}G_1 \tag{6}$$

$$Y_{t+\Delta t} = Y_t + \alpha(m - Y_t)\Delta t + \beta\sqrt{\Delta t}(\rho G_1 + \sqrt{1-\rho^2}G_2) \tag{7}$$

where G_1 and G_2 are independent standard normal random variables.

1. Assuming $\alpha = 1$, $m = 0.1$, $\beta = 0.1$ and using a discretization $\Delta t = 1/1000$, write down a Monte Carlo simulation code using Euler discretization to approximate $E[(S_1 - 1)_+]$ which is the price of a call option of maturity $T = 1$ and strike $K = S_0 = 1$. (/8.5 MAIN5, /7 other M2)

2. Given that β is obtained by the the expression

$$\beta = \sqrt{2\alpha\nu^2}(1 - e^m),$$

in the file `MC.cu`, we want to simulate $E((X_T - K)_+)$ for many combinations of values of α, m, ν^2, Y_0, ρ, T, K. Complete the file `MC.cu` and parallelize with respect to parameter combinations. (/6.5 MAIN5, /5 other M2)

3. (/5 MAIN5, /4 other M2)

   a) Generate both a training dataset and a (test/validation) dataset

   b) Using the jupyter notebook given to you train (on CPU) a neural network and compute its execution time (on CPU) during the inference.

3 Nested Monte Carlo for bullet options

In mathematical finance, the Black & Scholes (B&S) model is commonly used to describe the dynamics of asset prices over a time interval $[0, T]$. For time increments s and t with $0 \leq s < t \leq T$, the asset price process under the B&S model can be simulated using the following time induction formula:

$$S_t = S_s \exp\left((r - \sigma^2/2)(t-s) + \sigma\sqrt{t-s}G\right) \text{ and } S_0 = x_0 \text{ where}$$

- σ is the volatility assumed here equal to 0.2,
- r is the risk-free rate assumed here equal to 0.1,

- x_0 is the initial spot price of S at time 0, assumed here equal to 100
- G is independent from S_s and has a standard Normal distribution $\mathcal{N}(0,1)$.

This formula allows for simulating the path of the asset price from time s to t. This approach can be iteratively applied to generate an entire price path S for any time schedule $T_0 = 0 < T_1 < ... < T_M < T_{M+1} = T$.

Assuming the B&S model, we will begin by using Monte Carlo simulation, in question 1, to approximate the price of a bullet option at the initial time $t = 0$. In questions 2 and 3, we will employ two different nested Monte Carlo simulations to approximate the price of the bullet option across multiple increments of time and space.

The price of a bullet option $F(t,x,j) = e^{-r(T-t)}E(X\big|S_t = x, I_t = j)$, $X = (S_T - K)_+ 1_{\{I_T \in [P_1,P_2]\}}$ with $I_t = \sum_{T_i \leq t} 1_{\{S_{T_i} < B\}}$ and

- K is the contract's strike assumed here equal to $x_0 = 100$,
- T is the contract's maturity assumed here equal to 1
- barrier B should be bigger than S I_T times $\in \{P_1, ..., P_2\} \subset \{0, ..., M\}$ where P_1 and P_2 are two integers.

First, we want to use MC to approximate $F(0, x_0, 0) = e^{-rT}E(X)$ with $\{X_i\}_{i \leq n}$ being independent random variables that have the same distribution as X.

When $P_1 = P_2 = B = 0$, convince yourself that $E(X) = E((S_T - K)_+)$. In this case, use MC to approximate $F(0, x_0, 0)$ and check that you get a value that is close enough to

$$N(d_1)x_0 - N(d_2)Ke^{-rT} \text{ with } N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy,\ d_1 = \frac{\ln(x_0/K) + (r + \sigma^2/2)(T)}{\sigma\sqrt{T}}, \tag{8}$$

and $d_2 = d_1 - \sigma\sqrt{T}$. N is `NP` function already given in this course.

1. Assuming $B = 120$, $M = 100$, $P1 = 10$, $P2 = 50$ and $T_i = i/M$ for $i = 0, ..., M$, write down a Monte Carlo simulation code to approximate $F(0, x_0, 0)$. (/8.5 MAIN5, /7 other M2)

2. Define an array that contains many possible values of the triplet $(k, x, j) \in \{0, 1, ..., M\} \times [70, 130] \times \{0, 1, ..., \min(k, P_2)\}$. For each value from this array we want to have a Monte Carlo simulation that approximates the value of $F(T_k, x, j)$. (/6.5 MAIN5, /5 other M2)

   a). Define a parallelization strategy that allows to compute these many Monte Carlo simulations.

   b) Write down a CUDA/C code that allows to do it.

3. Another strategy would be first to simulate trajectories of $(S_t, I_t)_{t=0,T_1,...,T_M}$. Then, on the top of these outer trajectories and starting at $(k, x, j) \in \{0, 1, ..., M\} \times \mathbb{R}_+ \times \{0, 1, ..., \min(k, P_2)\}$, we simulate realizations of $F(T_k, x, j)$ using Monte Carlo. (/5 MAIN5, /4 other M2)

   a) Write down a CUDA/C code that allows to do it.

   b) Compare the efficiency of this solution to the one associated to question 2.

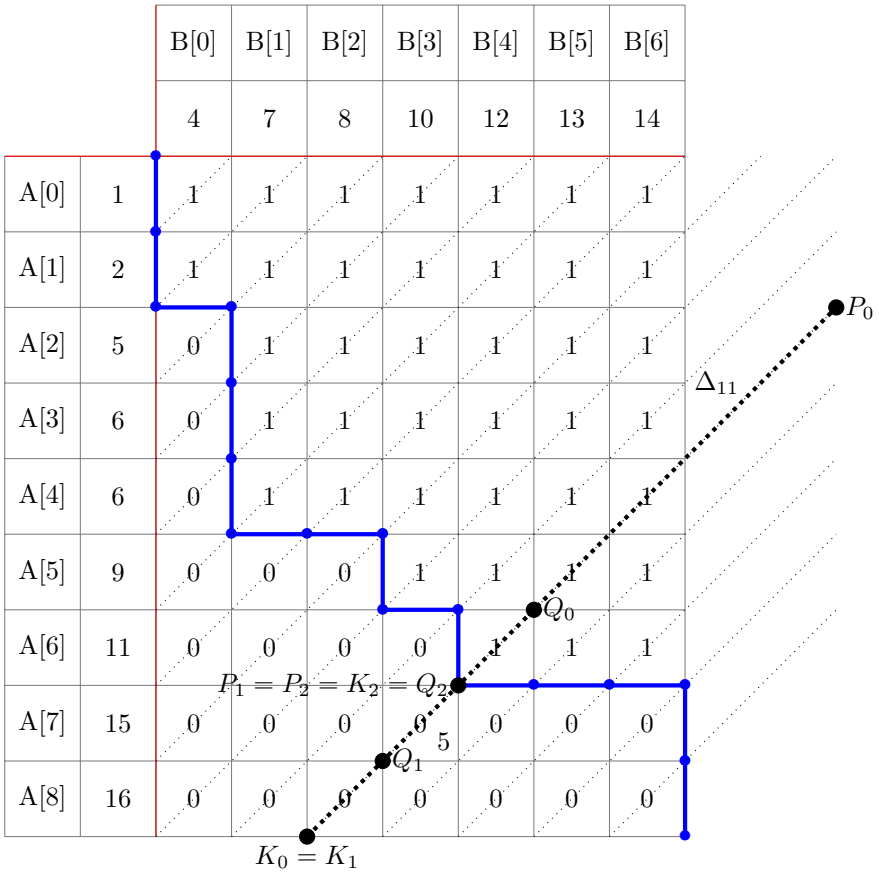
4 Merge large

This subject starts with the same algorithm of merge path, presented in [3], implemented on only one block. The students are then asked to translate it into the merge of large arrays.

We start then with the merge path algorithm. Let A and B be two ordered arrays (increasing order), we want to merge them in an M sorted array. The merge of A and B is based on a path that starts at the top-left corner of the $|A| \times |B|$ grid and arrives at the down-right corner. The Sequential Merge Path is given by Algorithm 1 and an example is provided in Figure 1.

Algorithm 2 Merge Path (Indices of n threads are integers from 0 to $n-1$)

Require: A and B are two sorted arrays
Ensure: M is the merged array of A and B with $|M| = |A| + |B|$
  for each thread **in parallel do**
    i=index of the thread
    if $i > |A|$ **then**
      $K = (i - |A|, |A|)$ ▷ Low point of diagonal
      $P = (|A|, i - |A|)$ ▷ High point of diagonal
    else
      $K = (0, i)$
      $P = (i, 0)$
    end if
    while True **do**
      $offset = abs(K_y - P_y)/2$
      $Q = (K_x + offset, K_y - offset)$
      if $Q_y \geq 0$ and $Q_x \leq B$ and
        $(Q_y = |A|$ or $Q_x = 0$ or $A[Q_y] > B[Q_x - 1])$ **then**
        if $Q_x = |B|$ or $Q_y = 0$ or $A[Q_y - 1] \leq B[Q_x]$ **then**
          if $Q_y < |A|$ and $(Q_x = |B|$ or $A[Q_y] \leq B[Q_x])$ **then**
            $M[i] = A[Q_y]$ ▷ Merge in M
          else
            $M[i] = B[Q_x]$
          end if
          Break
        else
          $K = (Q_x + 1, Q_y - 1)$
        end if
      else
        $P = (Q_x - 1, Q_y + 1)$
      end if
    end while
  end for



Algorithm 1 Sequential Merge Path

Require: A and B are two sorted arrays
Ensure: M is the merged array of A and B with $|M| = |A| + |B|$
  procedure MERGEPATH (A, B, M)
    $j = 0$ and $i = 0$
    while $i + j < |M|$ **do**
      if $i \geq |A|$ **then**
        M[i+j]=B[j]
        $j = j + 1$ ▷ The path goes right
      else if $j \geq |B|$ or $A[i] < B[j]$ **then**
        M[i+j]=A[i] ▷ The path goes down
        $i = i + 1$
      else
        M[i+j]=B[j]
        $j = j + 1$ ▷ The path goes right
      end if
    end while
  end procedure

Each point of the grid has a coordinate $(i, j) \in [\![0, |A|]\!] \times [\![0, |B|]\!]$. The merge path starts from the point $(i, j) = (0, 0)$ on the left top corner of the grid. If $A[i] < B[j]$ the path goes down else it goes right. The array $[\![0, |A|-1]\!] \times [\![0, |B|-1]\!]$ of boolean values $A[i] < B[j]$ is not important in the algorithm. However, it shows clearly that the merge path is a frontier between ones and zeros.

To parallelize the algorithm, the grid has to be extended to the maximum size equal to $\max(|A|, |B|) \times \max(|A|, |B|)$. We denote K_0 and P_0 respectively the low point and the high point of the ascending diagonals Δ_k. On GPU, each thread $k \in [\![0, |A| + |B| - 1]\!]$ is responsible of one diagonal. It finds the intersection of the merge path and the diagonal Δ_k with a binary search described in Algorithm 2.

1. For $|A| + |B| \leq 1024$, write a kernel `mergeSmall_k` that merges A and B using only one block of threads. (/8.5 MAIN5, /7 other M2)

Actually, by using the index `idx = threadIdx.x + blockIdx.x * blockDim.x`, it is possible to replace the per-block thread index `threadIdx.x` used in question 1 to enable merging of large arrays across multiple blocks. However, this approach is suboptimal. Indeed, as mentioned in [3], merge path algorithm should be divided into 2 stages: partitioning stage and merging stage. The partitioning stage is important to propose an algorithm that involves various blocks.

2. For an input size $|A| + |B| = d$ large enough to fully engage the GPU's computational resources, design a parallel solution that merges arrays A and B by distributing the workload across multiple blocks through partitioning. (/6.5 MAIN5, /5 other M2)

3. Study the execution time with respect to d and compare the answer to question 2 to the answer to question 1 when we simply use `idx` to merge large arrays. (/5 MAIN5, /4 other M2)

5 Batch merge small and batch sort small

This subject starts with the same algorithm of merge path, presented in [3], implemented on only one block. The students are then asked to translate it into a batch merge and a batch sort. **Consequently, question 1 is common with question 1 of subject 4**

1. For $|A| + |B| \leq 1024$, write a kernel `mergeSmall_k` that merges A and B using only one block of threads. (/8.5 MAIN5, /7 other M2)

In this part, we assume that we have a large number $N(\geq 1e3)$ of arrays $\{A_i\}_{1 \leq i \leq N}$ and $\{B_i\}_{1 \leq i \leq N}$ with $|A_i| + |B_i| = d \leq 1024$ for each i. Using some changes on `mergeSmall_k`, we would like to write `mergeSmallBatch_k` that merges two by two, for each i, A_i and B_i.

Given a fixed common size $d \leq 1024$, `mergeSmallBatch_k` is launched using the syntax

```
mergeSmallBatch_k<<<numBlocks, threadsPerBlock>>>(...);
```

with `threadsPerBlock` is multiple of d but smaller than 1024 and `numBlocks` is an arbitrary sufficiently big number.

2. (/5 MAIN5, /4 other M2)
   a) After figuring out why the indices
   ```
   int Qt = threadIdx.x/d;
   int tidx = threadIdx.x - (Qt*d);
   int gbx = Qt + blockIdx.x*(blockDim.x/d);
   ```
   are important in the definition of `mergeSmallBatch_k`, write the kernel `mergeSmallBatch_k` that batch merges two by two $\{A_i\}_{1\leq i\leq N}$ and $\{B_i\}_{1\leq i\leq N}$. Study the execution time with respect to d.
   b) How the code changes if we decide to dedicate at most 32 threads (instead of d) for each merge.
3. Write the kernel `sortSmallBatch_k` that sorts many arrays $\{M_i\}_{1\leq i\leq N}$ of size $d \leq 1024$. Study the execution time with respect to d. (/5 MAIN5, /4 other M2)

6 PDE simulation of bullet options

The students have to implement and compare Thomas algorithm to PCR for tridiagonal systems in question 1. Then, they have to simulate a PDE of a bullet option using Crank-Nicolson scheme based on either Thomas or PCR in questions 2 and 3

We refer to [6] for a fair description of PCR. Regarding Thomas algorithm, as described in [5], it allows to solve tridiagonal systems:

$$\begin{pmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & 0 & \\ & a_3 & b_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & c_{n-1} \\ & & & & a_n & b_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \tag{9}$$

using a forward phase

$$c'_1 = \frac{c_1}{b_1},\ y'_1 = \frac{y_1}{b_1},\ c'_i = \frac{c_i}{b_i - a_i c'_{i-1}},\ y'_i = \frac{y_i - a_i y'_{i-1}}{b_i - a_i c'_{i-1}} \text{ when } i = 2, ..., n \tag{10}$$

then a backward one

$$z_n = y'_n,\ z_i = y'_i - c'_i z_{i+1} \text{ when } i = n-1, ..., 1. \tag{11}$$

1. Using Thomas method, write a kernel that solves various tridiagonal systems ($d \leq 1024$) at the same time, one system per block. Do the same thing for PCR then compare both methods for $d \leq 1024$. To compare these two methods, generate strictly diagonally dominant tridiagonal matrices i.e. $|b_i| > |a_i| + |c_i|$ (/8.5 MAIN5, /7 other M2)

Let $u(t, x, j) = e^{r(T-t)} F(t, e^x, j)$ where F is the price of a bullet option as in Section 3. One can then show that, on any interval $t \in [T_{M-k}, T_{M-k+1})$, $k = M, ..., 0$, $u(t, x, j)$ is the solution of the PDE

$$\frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2}(t, x, j) + \mu \frac{\partial u}{\partial x}(t, x, j) = -\frac{\partial u}{\partial t}(t, x, j)$$

$$\text{with: } \mu = r - \frac{\sigma^2}{2}$$

. The final and boundary conditions are:

$$\begin{aligned} \bullet\ u(T,x,j) &= \max(e^x - K, 0)\mathbb{1}_{\{j\in[P_1,P_2]\}} \text{ for any (x, j)} \\ \bullet\ u(t,\log[K/3],j) &= \texttt{pmin} = 0 \\ \bullet\ u(t,\log[3K],j) &= \texttt{pmax} = 2K \end{aligned}$$

In addition, because of the discontinuity of the process I at times T_{M-k}, we need to set

$$u_{T_{M-k}}(S_{T_{M-k}},j) = \begin{cases} u_{T_{M-k}}(S_{T_{M-k}},P_2)\mathbb{1}_{\{S_{T_{M-k}}\geq B\}} & \text{if } j = P_2 \\ u_{T_{M-k}}(S_{T_{M-k}},P_k^1)\mathbb{1}_{\{S_{T_{M-k}}<B\}} & \text{if } j = P_k^1 - 1 \\ \left[\begin{array}{c} u_{T_{M-k}}(S_{T_{M-k}},j)\mathbb{1}_{\{S_{T_{M-k}}\geq B\}} \\ +u_{T_{M-k}}(S_{T_{M-k}},j+1)\mathbb{1}_{\{S_{T_{M-k}}<B\}} \end{array}\right] & \text{if } j \in [P_k^1, P_2 - 1] \end{cases} \tag{12}$$

with $P_k^1 = max(P_1 - k, 0)$.

From now on we use notations $u_t(x,j) = u(t,x,j)$ and $u_{k,i} = u(t_k, x_i, j)$. Following Crank Nicolson scheme, we get

$$q_u u_{k,i+1} + q_m u_{k,i} + q_d u_{k,i-1} = p_u u_{k+1,i+1} + p_m u_{k+1,i} + p_d u_{k+1,i-1}$$

$$q_u = -\frac{\sigma^2\Delta t}{4\Delta x^2} - \frac{\mu\Delta t}{4\Delta x}, \quad q_m = 1 + \frac{\sigma^2\Delta t}{2\Delta x^2}, \quad q_d = -\frac{\sigma^2\Delta t}{4\Delta x^2} + \frac{\mu\Delta t}{4\Delta x}$$

$$p_u = \frac{\sigma^2\Delta t}{4\Delta x^2} + \frac{\mu\Delta t}{4\Delta x}, \quad p_m = 1 - \frac{\sigma^2\Delta t}{2\Delta x^2}, \quad p_d = \frac{\sigma^2\Delta t}{4\Delta x^2} - \frac{\mu\Delta t}{4\Delta x}$$

The figure below shows an example of how PDE's backward resolution algorithm (with M = 10, $P_1 = 3, P_2 = 8$) is deployed with time on the x-axis and the set of values of I_t in the ordinate.

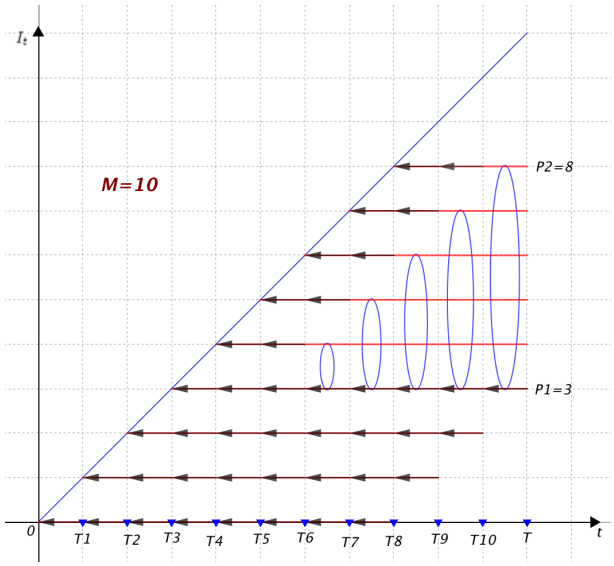


Figure 2: Backward induction scheme

2. Write the code that solves the PDE on $[T_M, T)$. (/6.5 MAIN5, /5 other M2)

3. After defining the kernel that performs (12), finish the code that solves all the PDE. (/5 MAIN5, /4 other M2)

References

[1] Mark Broadie and Özgür Kaya. Exact simulation of stochastic volatility and other affine jump diffusion processes. *Operations research*, 54(2):217–231, 2006.

[2] Jean-Pierre Fouque, George Papanicolaou, and K Ronnie Sircar. Mean-reverting stochastic volatility. *International Journal of theoretical and applied finance*, 3(01):101–142, 2000.

[3] O. Green, R. McColl and D. A. Bader GPU Merge Path - A GPU Merging Algorithm. *26th ACM International Conference on Supercomputing (ICS),* San Servolo Island, Venice, Italy, June 25-29, 2012.

[4] G. Marsaglia and T. Wai-Wan. A simple method for generating gamma variables. *ACM Transactions on Mathematical Software*, 26(3):363–372, 2000.

[5] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery (2002): *Numerical Recipes in C++: The Art of Scientific Computing.* Cambridge University Press.

[6] Y. Zhang, J. Cohen and J. D. Owens (2010): Fast Tridiagonal Solvers on the GPU. *Proceedings of the 15th ACM SIGPLAN Symposium on Principles and Practice of Parallel Programming*, 127–136.