(2)

PAGE NO. :
DATE : / /

15/10/16

ASSIGNMENT-3

Ans-1 In Template Matching, we define

$$MSE = \sum_i \sum_j (f(i,j) - g(i,j))^2 = \sum_i \sum_j [f^2(i,j) + g^2(i-m, j-n) - 2f(i,j)g(i-m, j-n)] \quad (1)$$

Here f is a template & g is the image function s.t (i-m, j-n) are in the domain of definition of the template.
The image energy in the template window $\sum\sum g^2$ in general varies with (m,n).
This effect can be normalized efficiently by subtracting the mean of the patch intensity values and dividing them by their respective standard deviations before comparison.

i.e $f' = \frac{f - \bar{f}}{\sqrt{\sum(f-\bar{f})^2}}$ & for $g' = \frac{g - \bar{g}}{\sqrt{\sum(g-\bar{g})^2}}$ (2)

(Intensity values normalized)

Then values from eq^n (2) can be substituted in eq^n (1) in place of f(i,j) & g(i,j) & {g(i-m, j-n)} respectively with f'(i,j) & g'(i,j) & g'(i-m, j-n)
It can also be substituted in the cross-correlation eq^n for template matching to obtain normalized values ranging from 1 to -1. This is because they become dot product of unit vectors. This is done because:-
By itself the dot product is a poor way to find matching features because the values may be simply large enough due to the brightness of the image region.

PAGE NO.:
DATE: / /

Ans-4 $\mathcal{E}(S) = \int \{\lambda (S'(x))^2 + (f(x) - S(x))^2 \sum_k \delta(x - x_k)\} dx$ — (A)

The E-L eqn: $\int_{x_1}^{x_2} F(S, S', S'', x)\, dx$ ~~b/w~~ ~~some constraints~~ ~~x_1 & x_2~~

$S \to \alpha$; $S' \to \beta$; $S'' \to \gamma$

$$\frac{\delta F}{\delta S} - \frac{d}{dx}\left(\frac{\delta F}{\delta S'}\right) + \frac{d^2}{dx^2}\left(\frac{\delta F}{\delta S''}\right) = 0$$

By substitution we get

$$\frac{\delta F}{\delta \alpha} - \frac{d}{dx}\left(\frac{\delta F}{\delta \beta}\right) + \frac{d^2}{dx^2}\left(\frac{\delta F}{\delta \gamma}\right) = 0 \quad \text{— (1)}$$

Here $\frac{dF}{\delta \alpha} = \frac{d}{dS}\left[(f(x) - S(x))^2 \sum_k \delta(x - x_k)\right] + 0$

$$= \sum_k \delta(x - x_k) \frac{d}{dS}\left[(f(x) - S(x))^2\right]$$

$$= \sum_k \delta(x - x_k)\, -2\,(f(x) - S(x))^{2-1}$$

$$= -2(f(x) - S(x)) \sum_k \delta(x - x_k) \quad \text{— (2)}$$

$$\frac{\delta F}{\delta \beta} = \frac{\delta F}{\delta S'} = \frac{\delta}{\delta S'}\left(\lambda (S'(x))^2\right) + 0$$

$$= 2\lambda S'(x)^{2-1} \Rightarrow \underline{\underline{2\lambda S'(x)}} \quad \text{— (3)}$$

Now,

$$\frac{\delta F}{\delta \gamma} = \frac{\delta F}{\delta S''} = \frac{\delta}{\delta S''}\left(\text{eqn of } \mathcal{E}(S)\right)$$ — (A)

$$= 0 \quad \text{— (4)}$$

Now substituting values of (2), (3) & (4) in eqn no. (1)

(Euler-Lagrange)

we get the resultant E-L eq^n as:-

$$-2(f(x)-S(x))\sum_k \delta(x-x_k) - \frac{d}{dx}(2\lambda S'(x)) = 0$$

$$\Rightarrow \boxed{[\underbrace{f(x)-S(x)}_{\text{Smoothing function}}]\sum_k \underbrace{\delta(x-x_k)}_{\text{Variation b/w each corresponding data points in } f \,\&\, S} + \lambda S''(x) = 0}$$

Assumption: λ is a constant regularization parameter.

The purpose of minimization of the above functional was to find a curve ~~(spline)~~ $S(x)$ (that is why we differentiate the function wrt S, S' & S'' which are S, first & second order derivatives of S respectively) characteristic of a minimum length and optimally fits the function $f(x)$ which is why we differentiate f wrt various forms of S.

Ans-2 To show that curve $f(x)$ that minimizes the integral $\int_1^2 \frac{\sqrt{1+f'^2}}{x}\,dx$ with $f(1)=0$ & $f(2)=1$ is a circle. Radius = ?

Coordinates of its centre = ?

We know that Euler's-Lagrange's eq^n is given by:-

$$\frac{\delta F}{\delta y} - \frac{\delta}{\delta x}\left(\frac{\delta F}{\delta y'}\right) = 0 \quad \text{—— (A)}$$

Now, here in this case we know that

$$\frac{\delta F}{\delta y} = 0$$

Now, $\frac{\delta F}{\delta y'} = \frac{1}{x} \times \frac{1}{2} \times \frac{2f'}{\sqrt{1+f'^2}}$

$$= \frac{f'}{x\sqrt{1+f'^2}} \quad \text{—— (1)}$$

Now, when we substitute (1) in (A) we can deduce that $\frac{f'}{x\sqrt{1+f'^2}} = c$ (constant)

$$\Rightarrow \frac{f'^2}{x^2(1+f'^2)} = c^2 \Rightarrow \frac{f'^2}{x^2 + x^2 f'^2} = c^2$$

$$\Rightarrow f'^2 = c^2x^2 + c^2x^2f'^2 \Rightarrow (1 - c^2x^2)f'^2 = c^2x^2$$

$$\Rightarrow f'^2 = \frac{c^2x^2}{1-c^2x^2} \Rightarrow f' = \frac{cx}{\sqrt{1-c^2x^2}}$$

Now to find f we need to integrate the above equation wrt x

$$\int f' dx = f = c \int \frac{x}{\sqrt{1-c^2x^2}} dx$$

Put $1 - c^2x^2 = u$

$$\Rightarrow -2xc^2 dx = du$$

$$x\,dx = \frac{-du}{2c^2}$$

$$= c \int \frac{-du}{\sqrt{u}\, 2c^2} \Rightarrow \frac{-1}{2c} \times \frac{\sqrt{u}}{1/2} + A$$

$$\Rightarrow -\frac{\sqrt{u}}{c} + A$$

Substituting value of u again we get

$$f = -\frac{1}{c}\sqrt{1-c^2x^2} + A$$

Imposing boundary conditions $x=1$; $f(1)=0$

$$0 = -\frac{1}{c}\sqrt{1-c^2} + A \Rightarrow -Ac = -\sqrt{1-c^2}$$

$$1-c^2-A^2c^2 = 0 \quad — ②$$

$\Rightarrow$ Now we impose conditions $x=2$; $f(2)=1$ we get,

$$c = -1(\sqrt{1-4c^2}) + Ac$$

$$\Rightarrow c - Ac = -(\sqrt{1-4c^2})$$

$$\Rightarrow c^2 + A^2c^2 - 2Ac^2 = 1-4c^2$$

Now from eq^n ② we know $c^2 + A^2c^2 = 1$

$$1 - 2Ac^2 = 1-4c^2$$

$$2Ac^2 = 4c^2$$

$$2A = 4 \Rightarrow A = 2 \quad — ③$$

Now from eq^n ② we also know after rearranging that

$$1 = c^2(1+A^2)$$

$$\Rightarrow c^2 = \frac{1}{1+A^2}$$

$$\Rightarrow c^2 = \frac{1}{1+4} = \frac{1}{5} \text{ from ③} \quad — ④$$

$$\Rightarrow c = \frac{1}{\sqrt{5}}, \ A = 2$$

Hence $f = -\frac{1}{\sqrt{5}}\sqrt{1-\frac{x^2}{5}} + 2$

DATE: / /

Now $(f-2)^2 + x^2 = 5$ (simplified form)
As a result,
The solution to this minimization problem is a circle with radius of magnitude $\sqrt{5}$ & centre $(0,2)$.

Ans-3 Given n data points $\{x_i, y_i\}_{i=1..n}$ s.t $f(x_i) = y_i$

The number of unknowns to fit a quadratic spline to the data mentioned above.
Firstly, this type of ~~splies~~ splines is not used much. and cubic splines are favoured for its minimum curvature property.

Given a set of knots $t_0, t_1, \dots, t_n$ and the data $y_0, y_1, \dots, y_n$, we seek piecewise polynomial represent-ation

$$Q(x) = \begin{cases} Q_0(x) & t_0 \le x \le t_1 \\ Q_1(x) & t_1 \le x \le t_2 \\ \vdots & \\ Q_{n-1}(x) & t_{n-1} \le x \le t_n \end{cases}$$

where $Q_i(x)$ $(i = 0, 1, \dots, n-1)$ are quadratic polynomials. In general $Q_i(x) = a_i x^2 + b_i x + c_i$ and the total number of unknowns: $\boxed{3n}$

Now the conditions we impose on Q_i are:-
$Q_i(t_i) = y_i$, $Q_i(t_{i+1}) = y_{i+1}$; $i = 0, 1, \dots, n-1$: $2n$ conditions

Also $Q_i'(t_i) = Q_{i+1}'(t_i)$; $i = 1, 2, \dots, n-1$: $n-1$ conditions

Hence, the total number of conditions become:

$$2n + (n-1) = 3n-1$$

An extra condition that could be imposed will be $Q_0'(t_0) = 0$ or $Q_0''(t_0) = 0$ depending on the specific problem we encounter

Construction of $Q_i(t)$: Due to the continuous property of Q' we will set

$$z_i = Q'(t_i)$$

We don't know these z_i's which are the unknowns and will be computed later.
Hence, each Q_i must satisfy the conditions

$$Q_i(t_i) = y_i \quad Q_i'(t_i) = z_i \quad Q_i'(t_{i+1}) = z_{i+1} \quad Q_i(t_{i+1}) = y_{i+1} \qquad (1)$$

Using the first 3 conditions we obtain the polynomials

$$Q_i(x) = \frac{z_{i+1} - z_i}{2(t_{i+1} - t_i)}(x-t_i)^2 + z_i(x-t_i) + y_i \qquad (2)$$

for $0 \leq i \leq n-1$

It is convenient to verify the first 3 conditions in eq^n (1)
To compute z_i's we now use the 4^{th} condition in eq^n (1) which gives us the result:

$$z_{i+1} = z_i + 2\left(\frac{y_{i+1} - y_i}{t_{i+1} - t_i}\right) \quad 0 \leq i \leq n-1 \qquad (3)$$

Hence given a z_0 all the z_i's can now be constructed recursively.

Hence given a Z_0 compute Z_i using eqn ③
& compute Q_i by using eqn ②

Ans-5 We know length element of a curve $y(x)$ is
$ds = \sqrt{1+y'^2}\, dx$ — ①
To show area element of image surface $I(x,y)$
is given by $dA = \sqrt{1+I_x^2+I_y^2}\, dx\, dy$
Now say we have some scalar function
$z = I(x,y)$ we have

$$A = \iint_S dA = \iint_T \left\| \frac{\delta r}{\delta x} \times \frac{\delta r}{\delta y} \right\| dx\, dy \quad — ②$$

where $r = (x, y, z) = (x, y, f(x,y))$ so that

$$\frac{\delta r}{\delta x} = (1, 0, f_x(x,y)) \quad — ③$$

& IIIlarly $\frac{\delta r}{\delta y} = (0, 1, f_y(x,y))$ — ④

③&④ $\Rightarrow$ Hence $A = \iint_T \| (1, 0, f_x(x,y)) \times (0, 1, f_y(x,y)) \|$ dx dy

$$= \iint_T \left\| \left(1, 0, \frac{\delta f}{\delta x}\right) \times \left(0, 1, \frac{\delta f}{\delta y}\right) \right\| dx\, dy$$

$$\Rightarrow \iint_T \left\| \left(-\frac{\delta f}{\delta x}, -\frac{\delta f}{\delta y}, 1\right) \right\| dx\, dy$$

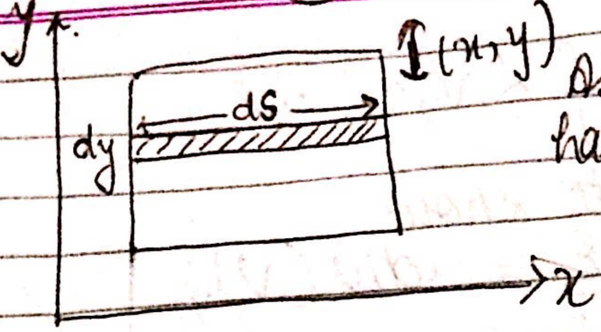
$$\Rightarrow \iint_T \sqrt{\left(\frac{\delta f}{\delta x}\right)^2 + \left(\frac{\delta f}{\delta y}\right)^2 + 1}\; dx\, dy$$

$$\Rightarrow \iint_T \sqrt{I_x^2 + I_y^2 + 1}\; dx\, dy$$

from ②

Hence, we have $dA = \sqrt{I_x^2 + I_y^2 + 1}\, dx\, dy$

An alternate approach by using eqn ① would be



Assume $I(x,y)$ has ~~surface~~ length element dS

Here we have the small area element dA deduced from the figure given by

$$dA = dS\,dy$$

$$\Rightarrow dA = \sqrt{1+y'^2}\,dx\,dy \quad \because \text{from eq}^n \text{ (1)}$$

But here the function $y(x)$ would be $I(x,y)$ since we are talking about the image here.

$$\Rightarrow dA = \sqrt{1+I'^2}\,dx\,dy$$

Now we know if we differentiate I wrt ~~x~~ x & y we need to do partial differentiation

$$\Rightarrow dA = \sqrt{1+\left(\frac{\delta I}{\delta x}\right)^2+\left(\frac{\delta I}{\delta y}\right)^2}\,dx\,dy$$

$$\Rightarrow dA = \sqrt{1+I_x^2+I_y^2}\,dx\,dy \quad \text{— (5)}$$

proved

Now to find E-L eq^n for the functional $\int dA$ we have

$$\int dA \Rightarrow \int \overbrace{\sqrt{1+I_x^2+I_y^2}}^{F}\,dx\,dy$$

The E-L eq^n will be given by

$$\cancel{F_f}^{\,0} - \frac{\delta}{\delta x}F_{f_x} - \frac{\delta}{\delta y}F_{f_y} = 0$$

Cited from B.K.P Horn A-6 Pg 472

Multiply by -1 on both sides

$$\Rightarrow \frac{\delta}{\delta x}\left(F_{f_x}\right) + \frac{\delta}{\delta y}\left(F_{f_y}\right) = 0 \text{ —— } (6)$$

Now for the first term in the above eqn we have:-

$$F_{f_x} = \frac{I_x}{\sqrt{1+I_x^2+I_y^2}} \qquad |||^{larly} \quad F_{f_y} = \frac{I_y}{\sqrt{1+I_x^2+I_y^2}}$$

Now, $\frac{\delta}{\delta x}\left(F_{f_x}\right) = \frac{\delta}{\delta x}\left(\frac{I_x}{\sqrt{1+I_x^2+I_y^2}}\right)$ Apply $\frac{d}{dx}\left(\frac{u}{v}\right)$ rule & chain rule

$$\Rightarrow \frac{\left(\sqrt{1+I_x^2+I_y^2}\right) I_{xx} - I_x \cdot \left(\frac{\cancel{2} I_{xx} I_x + \cancel{2} I_y I_{yx}}{\cancel{2}\sqrt{1+I_x^2+I_y^2}}\right)}{\left(1+I_x^2+I_y^2\right)}$$

$$\Rightarrow \frac{I_{xx}\left(1+I_x^2+I_y^2\right) - \left[\left(I_x\right)^2 I_{xx} + I_x I_y I_{yx}\right]}{\left(1+I_x^2+I_y^2\right)^{3/2}}$$

$$\Rightarrow \frac{I_{xx} + \cancel{\left(I_x\right)^2 I_{xx}} + \left(I_y\right)^2 I_{xx} - \cancel{\left(I_x\right)^2 I_{xx}} - I_x I_y I_{yx}}{\left(1+I_x^2+I_y^2\right)^{3/2}}$$

$$\Rightarrow \frac{I_{xx} + \left(I_y\right)^2 I_{xx} - I_x I_y I_{yx}}{\left(1+I_x^2+I_y^2\right)^{3/2}} \text{ —— } (7)$$

Now, $\frac{\delta}{\delta y}\left(F_{f_y}\right) = \frac{\delta}{\delta y}\left(\frac{I_y}{\sqrt{1+I_x^2+I_y^2}}\right)$

$$\Rightarrow \frac{\left(\sqrt{1+I_x^2+I_y^2}\right) I_{yy} - I_y \cdot \left(\frac{\cancel{2} I_{xy} I_x + \cancel{2} I_y I_{yy}}{\cancel{2}\sqrt{1+I_x^2+I_y^2}}\right)}{\left(1+I_x^2+I_y^2\right)}$$

$$\Rightarrow \frac{I_{yy}\left(1+I_x^2+I_y^2\right) - \left(I_y\right)^2 I_{yy} - I_x I_y I_{xy}}{\left(1+I_x^2+I_y^2\right)^{3/2}}$$

$$\Rightarrow \left(I_{yy} + I_x^2 I_{yy} + \cancel{\left(I_y\right)^2 I_{yy}} - \cancel{\left(I_y\right)^2 I_{yy}} - I_x I_y I_{xy}\right) \Big/ \left(1+I_x^2+I_y^2\right)^{3/2},$$

PAGE NO. :
DATE : / /

$$\Rightarrow \frac{I_{yy} + I_x^2 I_{yy} - I_x I_y I_{xy}}{(1 + I_x^2 + I_y^2)^{3/2}} \quad \text{—} (8)$$

Substitute (7) & (8) in eq^n (6)

$$\frac{I_{xx} + I_y^2 I_{xx} - I_x I_y I_{yx}}{(1 + I_x^2 + I_y^2)^{3/2}} + \frac{I_{yy} + I_x^2 I_{yy} - I_x I_y I_{xy}}{(1 + I_x^2 + I_y^2)^{3/2}} = 0$$

$$\Rightarrow I_{xx} + I_{yy} + I_x^2 I_{yy} + I_y^2 I_{xx} - I_x I_y (I_{yx} + I_{xy}) = 0$$

$$\Rightarrow I_{xx}(1 + I_y^2) + I_{yy}(1 + I_x^2) - I_x I_y (I_{yx} + I_{xy}) = 0$$

where $I_{yx} \longrightarrow \frac{\delta}{\delta x}(I_y)$

$I_{xy} \rightarrow \frac{\delta}{\delta y}(I_x)$

$\because$ since I is a function over both x & y.

also $I_{yx} = I_{xy}$

hence we can write it as

$$I_{xx}(1 + I_y^2) + I_{yy}(1 + I_x^2) - 2 I_x I_y I_{yx} = 0$$

also I_{xx} – 2nd order derivative of I w.r.t 'x'.

I_{yy} – 2nd order derivative of I w.r.t 'y'