Homework# 1

Q1.
We would like to find out a, b, θ such that

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

Solving this we get $a = \sqrt{2}, b = 2, \theta = 45°$.

Q2. Show that a line at infinity $\mathbf{l}_\infty$ remains a line at infinity under the projectivity H if and only if H is an affinity.

Proof. Let H is an affinity. Let $\mathbf{l}_\infty$ be a line at infinity and $\boldsymbol{l}'_\infty$ be the transformed line. Then, using line transformation,

$$\begin{aligned} \mathbf{l}'_\infty &= H_A^{-T}\mathbf{l}_\infty \\ &= \begin{bmatrix} A^{-T} & 0 \\ -\mathbf{t^T}A^{-T} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \mathbf{l}_\infty \end{aligned}$$

Thus, line at infinity remains line at infinity under the affine transformation.
Now, in order to prove the converse, we require that a point at infinity, say $\mathbf{x} = (1, 0, 0)^T$, be mapped to a point at infinity. This requires that $h_{31} = 0$ and $h_{32} = 0$, so the transformation is an affinity. ∎

Q4. Given a projective transformation that takes points $\mathbf{x} \to \mathbf{x}'$ via $\mathbf{x}' = H\mathbf{x}$, prove that $Cross(\mathbf{x_1}', \mathbf{x_2}', \mathbf{x_3}', \mathbf{x_4}') = Cross(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4})$ i.e., the cross ratio is a projective invariant.

Proof.

$$\begin{aligned} Cross(x'_1, x'_2, x'_3, x'_4) &= \frac{|x'_1 x'_3|.|x'_2 x'_4|}{|x'_1 x'_4|.|x'_2 x'_3|} \\ &= \frac{|Hx_1 Hx_3|.|Hx_2 Hx_4|}{|Hx_1 Hx_3|.|Hx_2 Hx_4|} \\ &= \frac{det(H)|x_1 x_3|.det(H)|x_2 x_4|}{det(H)|x_1 x_4|.det(H)|x_2 x_3|} \\ &= cross(x_1, x_2, x_3, x_4) \end{aligned}$$

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Q5. Prove that an image line $\mathbf{l}$ defines a plane through the camera center with normal direction $\mathbf{n} = K^t\mathbf{l}$ measured in the camera's Euclidean coordinate frame.

Proof. Points $\mathbf{x}$ on the line $\mathbf{l}$ back-project to directions $\mathbf{d} = K^{-1}\mathbf{x}$ which are orthogonal to the plane normal $\mathbf{n}$, and thus satisfy $\mathbf{d}^T\mathbf{n} = \mathbf{x}^T K^{-T}\mathbf{n} = 0$. Since points on $\mathbf{l}$ satisfy $x^T\mathbf{l} = 0$, it follows that $\mathbf{l} = K^{-T}\mathbf{n}$, and hence $\mathbf{n} = K^T\mathbf{l}$. ■

Q6. In DLT, if $\|Ah\|$ is minimized subject to the constraint $h_9 = H_{33} = 1$, then show that the result is invariant to scaling but not translation of coordinates.

Proof. Recall in 2D homography, in the system of linear equations only two are linearly independent ([1], pp. 89). Hence we get the following

$$\begin{bmatrix} 0 & 0 & 0 & -x_iw'_i & -y_iw'_i & -w_iw'_i & x_iy'_i & y_iy'_i \\ x_iw'_i & y_iw'_i & w_iw'_i & 0 & 0 & 0 & x_ix'_i & y_ix'_i \end{bmatrix} \tilde{\mathbf{h}} = \begin{pmatrix} -w_iy'_i \\ w_ix'_i \end{pmatrix} \tag{1}$$

Where, $\tilde{\mathbf{h}}$ is an $8-$vector consisting of the first 8 components of $\mathbf{h}$. Now, let after scaling by s the new coordinates be $(\bar{x_i},\bar{y_i},\bar{w}_i) = s(x_i, y_i, w_i)$. So, the above equation becomes:

$$s\begin{bmatrix} 0 & 0 & 0 & -x_iw'_i & -y_iw'_i & -w_iw'_i & x_iy'_i & y_iy'_i \\ x_iw'_i & y_iw'_i & w_iw'_i & 0 & 0 & 0 & x_ix'_i & y_ix'_i \end{bmatrix} \tilde{\mathbf{h}} = s\begin{pmatrix} -w_iy'_i \\ w_ix'_i \end{pmatrix}$$

which is same as Eq.1.

Now, let after translation, let the new coordinates be $(\bar{x_i},\bar{y_i},\bar{w}_i) = (x_i + t_i, y_i + u_i, w_i + v_i)$. By substituting in Eq. 1, it's easy to see that

$$\begin{pmatrix} -w_iy'_i \\ w_ix'_i \end{pmatrix} \neq \begin{pmatrix} -\bar{w}_iy'_i \\ \bar{w}_ix'_i \end{pmatrix}$$

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References

[1] Richard Hartley and Andrew Zisserman. *Multiple view geometry in computer vision*. Cambridge university press, 2003.