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Assignment-2

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Ans-1 A funct^n of 2 independent variables is called seperable wrt to a specific coordinate system if it can be written as a product of two functions, each of which depends on only one of the independent variables.
Hence f is seperable such that

$$f(x,y) = f_x(x) f_y(y) \quad \text{——— ①}$$

It is also seperable in polar coordinates i.e

$$~~It~~ f(r,\theta) = f_R(r) f_\theta(\theta) \quad \text{——— ②}$$

Seperability allows 2-D manipulations to be reduced to simplistic 1-D manipulations.

Now Take F.T on both sides of eqn ①

$$\mathcal{F}[f(x,y)] = \iint_{-\infty}^{\infty} f(x,y) e^{-j2\pi(f_x x + f_y y)} dx\, dy$$

no brackets here

$$\Rightarrow \int_{-\infty}^{\infty} f_x(x) e^{-j2\pi f_x x} dx \int_{-\infty}^{\infty} f_y(y) e^{-j2\pi f_y y} dy$$

$$\Rightarrow \mathcal{F}_x(f_x)\, \mathcal{F}_y(f_y) \quad \text{——— ③}$$

Hence eqn ③ is evidence that if f(x,y) is seperable so is its Fourier Transform.

Ans-2 Similarity ~~Shift~~ Theorem:- If $\mathcal{F}(g(x,y)) = G(f_x, f_y)$ then

$$\mathcal{F}[g(ax, by)] = \frac{1}{|ab|} G\left(\frac{f_x}{a}, \frac{f_y}{b}\right) \quad \text{——— ①}$$

Shift Theorem:- If $\mathcal{F}(g(x,y)) = G(f_x, f_y)$ then

$$\mathcal{F}[g(x-a, y-b)] = G(f_x, f_y)\, e^{-j2\pi(f_x a + f_y b)} \quad \text{——— ②}$$

For a rectangular funct. it's F.T is the sinc funct^n.

Now, we have

(i) $rect\left(\frac{x}{3}, \frac{y}{2}\right)$

Since it's a separable f^n it's F.T is also seperable.

Hence $\mathcal{F}\left[rect\left(\frac{x}{3}, \frac{y}{2}\right)\right] = \mathcal{F}\left[rect\left(\frac{x}{3}\right)\right] \cdot \mathcal{F}\left[rect\left(\frac{y}{2}\right)\right]$

and from eq^n 1

$$= 6\, sinc(3f_x)\, sinc(2f_y)$$

Also, the proof that the Fourier transform of rect() is sinc() function is given below; assume g(t) is rect(t)

$$\mathcal{F}[g(t)] = G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-T/2}^{T/2} A e^{-j2\pi f t}\, dt \quad \because \text{By def}^n \text{ of a rectangular function}$$

$$\Rightarrow \frac{A}{-2\pi j f}\left[e^{-j2\pi f t}\right]_{-T/2}^{T/2} \Rightarrow \frac{A}{-2\pi j f}\left[e^{-j\pi f T} - e^{j\pi f T}\right]$$

$$\Rightarrow \frac{AT}{\pi f T}\left(\frac{e^{j\pi f T} - e^{-j\pi f T}}{2j}\right) \Rightarrow \frac{AT}{\pi f T} \sin(\pi f T) \Rightarrow AT\, sinc(fT)$$

This proof will be valid for subsequent parts too.

ii) $rect(x-4, y-5) \Rightarrow$ Again since its seperable

$$F.T(rect(x-4, y-5)) = \mathcal{F}[rect(x-4)] \cdot \mathcal{F}[rect(y-5)]$$

from eq^n 2

$$= sinc(f_x)\, sinc(f_y)\, e^{-j2\pi[4f_x + 5f_y]}$$

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iv) $3\,rect(x-8, 3y)$ Again since its seperable

$$F.T[3\,rect(x-8,3y)] = F.T[rect(x-8)] \cdot F[rect(3y)] \cdot 3$$

$$= 3\,sinc(f_x)\, e^{-j2\pi \times 8 f_x} \cdot \frac{1}{3}\, sinc\left(\frac{f_y}{3}\right)$$

$$= sinc(f_x)\, sinc\left(\frac{f_y}{3}\right) e^{-j16\pi f_x} \quad \text{—— (4)}$$

The eq^n (3) can be further simplified i.e Ans-iii)

$$sinc(f_x)\, sinc(f_y)\, e^{-j\pi[8f_x + 10f_y]} \quad \text{—— (5)}$$

iii) $sinc(x-5, 2y-7)$

We know F.T of sinc function is rect function since both of them belong to a category known as Fourier pairs.

$\Rightarrow F.T[sinc(x-5)] \cdot F[sinc(2y-7)]$ since they are separable

$$\Rightarrow rect(f_x)\, e^{-j\pi 10 f_x} \cdot F\left[sinc\left(2\left(y-\frac{7}{2}\right)\right)\right]$$

$$\Rightarrow \frac{1}{2}\, rect(f_x)\, rect\left(\frac{f_y}{2}\right) e^{-j\pi[10f_x + 7f_y/2]}$$

$$\Rightarrow \frac{1}{2}\, rect(f_x)\, rect\left(\frac{f_y}{2}\right) e^{-j\pi(10f_x + 3.5f_y)}$$

v) $e^{j16\pi x}\, sinc\left(x, \frac{y}{3}\right)$ again since its separable

$$F.T\left[sinc\left(x, \frac{y}{3}\right) e^{j16\pi x}\right] = F.T[sinc\,x \cdot e^{j16\pi x}]\; F\left[sinc\left(\frac{y}{3}\right)\right]$$

$$= 3\,rect(3f_y)\,[rect(f_x - 8)]$$

Ans-4 To compute the convolutions for the following sets of matrices :-

$$\overset{I_1}{\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}} * \overset{I_2}{\begin{bmatrix} 1 & -1 \end{bmatrix}}$$

n Rows of Convolution Matrix = $5+1-1=\underline{5}$ n Cols = $\underline{6}$

1) Invert the 2^{nd} input matrix by 180 degrees
2) Slide the matrix I_2 such that its second element lies on top of the (0,0) element of I_1
3) Multiply each element of inverted I_2 matrix by the element of I_1 underneath & summate the individual products.

i.e

$\boxed{-1}\boxed{1}$

I_2

Convolution mask

$$\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

Now $C_{11} = -1 \cdot 0 + 1 \cdot 1 = \underline{1}$ $\quad C_{14} = -1 \cdot 6 + 1 \cdot 4 = -2$

$C_{12} = -1 \cdot 1 + 1 \cdot 4 = \underline{3}$ $\quad C_{15} = -1 \cdot 4 + 1 \cdot 1 = -3$

$C_{13} = -1 \cdot 4 + 1 \cdot 6 = \underline{2}$ $\quad C_{16} = -1 \cdot 1 + 1 \cdot 0 = -1$

Repeating the same procedure for the 2nd row

$C_{21} = -1\cdot 0 + 1\cdot 4 = 4$ Similarly $C_{24} = -8$

$C_{22} = -1\cdot 4 + 1\cdot 16 = 12$ $C_{25} = -12$

$C_{23} = -1\cdot 16 + 1\cdot 24 = 8$ $C_{26} = -4$

For the 3rd row

$C_{31} = -1\cdot 0 + 1\cdot 6 = 6$ $C_{34} = -12$

$C_{32} = -1\cdot 6 + 1\cdot 24 = 18$ $C_{35} = -18$

$C_{33} = -1\cdot 24 + 1\cdot 36 = 12$ $C_{36} = -6$

Now since 4th & 5th rows of I_1 are same as 1st & 2nd the convolution sums will be the same for this case.

Hence,

$$\text{Ouput} = \text{Convolution Sum} = \begin{bmatrix} 1 & 3 & 2 & -2 & -3 & -1 \\ 4 & 12 & 8 & -8 & -12 & -4 \\ 6 & 18 & 12 & -12 & -18 & -6 \\ 4 & 12 & 8 & -8 & -12 & -4 \\ 1 & 3 & 2 & -2 & -3 & -1 \end{bmatrix}$$

$$\overset{I_1}{\begin{bmatrix} 3 & 2 & 3 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}} * \overset{I_2}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

1) To rotate I_2 by 180 we must flip it horizontally & vertically the elements of I_2.

Rest of the steps are same.

$n\,\text{Rows} = 3+2-1 = \underline{\underline{4}}$ $n\,\text{Cols} = 3+2-1 = \underline{\underline{4}}$

~~0 −1~~ $\overset{I_2}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$ 2) Here slide the new matrix I_2 such that its bottom right corner lies on the top-left corner of I_1.

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Now, $C_{11} = -1\cdot 0 + 0 + 0 + 1\cdot 3 = 3$

$C_{12} = -1\cdot 0 + 0 + 0 + 1\cdot 2 = 2$

$C_{13} = 0 + 0 + 0 + 1\cdot 3 = 3$

$C_{14} = 0 + 0 + 0\cdot 3 + 1\cdot 0 = 0$

Again for the 2nd row, here also note that the first row of I_2 would now slide over 1st row of I_1. we have,

$C_{21} = 0 + 0 + 0 + 1\cdot 2 = 2$

$C_{22} = -3 + 0 + 0 + 3 = 0$

$C_{23} = -2 + 0 + 0 + 2 = 0$

$C_{24} = -3 + 0 + 0 + 0 = -3$

The values for 3rd & 4th rows are obtained as follows:

$C_{31} = 0 + 0 + 0 + 1 = 1$ $\qquad$ $C_{41} = 0 + 0 + 0 + 0 = 0$

$C_{32} = -2 + 0 + 0 + 2 = 0$ $\qquad$ $C_{42} = -1$

$C_{33} = -3 + 0 + 0 + 3 = 0$ $\qquad$ $C_{43} = -2$

$C_{34} = -2 + 0 + 0 + 0 = -2$ $\qquad$ $C_{44} = -3$

Ouput Convolution Sum Matrix is given by:-

$$\begin{bmatrix} 3 & 2 & 3 & 0 \\ 2 & 0 & 0 & -3 \\ 1 & 0 & 0 & -2 \\ 0 & -1 & -2 & -3 \end{bmatrix}$$

Ans-5 The squared gradient of an image $I(x,y)$ is given by:-

$\nabla\cdot\nabla$ where ∇ is the column vector $\begin{bmatrix} \frac{\delta}{\delta x} \\ \frac{\delta}{\delta y} \end{bmatrix}$

A funct^n is said to be rotationally symmetric iff it yields the same value under an arbitrary rotat^n of coordinates

i.e $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = R \begin{bmatrix} x \\ y \end{bmatrix}$

Here it can be applied for an operator that acts on such functions

i.e $\nabla^2 = \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}$ (2-D Laplace operator)

Now, the squared gradient of $I(x,y)$

squared gradient

$$s(x,y) = \left(\frac{\delta I}{\delta x}\right)^2 + \left(\frac{\delta I}{\delta y}\right)^2 = [(B_2 - B_1)\delta(x\sin\theta - y\cos\theta + \rho)]^2 \quad \text{--- (1)}$$

Now we apply a rotation of 'ϕ' to the squared gradient and get a modified eq^n (1) as:-

$$R[s(x,y)] = [(B_2 - B_1)\delta[(x\cos\phi - y\sin\phi)\sin\theta - (x\sin\phi + y\cos\phi)\cos\theta + \rho]]^2$$

$$= [(B_2 - B_1)\delta((x\cos\phi\sin\theta - y\sin\phi\sin\theta) - (x\sin\phi\cos\theta + y\cos\phi\cos\theta) + (\rho)))^2$$

$$= [(B_2 - B_1)]\delta((x(\cos\phi\sin\theta - \sin\phi\cos\theta) \mp y(\cos\phi\cos\theta + \sin\phi\sin\theta) + (\rho)))^2$$

$$= [[B_2 - B_1)]\delta(x\sin(\theta - \phi) - y\cos(\theta - \phi) + \rho)]^2 \quad \text{--- (2)}$$

Here eq^n (2) shows that even when we rotated the x- & y- coordinates by an angle ϕ we were able to observe the similar effect on $s(x,y)$ from eq^n (2).

Hence, Squared gradient of an image is rotationally symmetric.

Ans-6 We know that

a) $I(x,y) = B_1 + (B_2 - B_1)\, u(x\sin\theta - y\cos\theta + p)$
cited from B.K.P.

Now the partial derivatives are:-

$$\frac{\delta I}{\delta x} = +\sin\theta(B_2 - B_1)\,\delta(x\sin\theta - y\cos\theta + p)$$

$$\frac{\delta I}{\delta y} = -\cos\theta(B_2 - B_1)\,\delta(x\sin\theta - y\cos\theta + p)$$

Here θ is the angle the eq^n of edge makes with the x-axis assuming the edge is along the line ~~$x\cos\theta - y$~~. $x\sin\theta - y\cos\theta + p = 0$

Also, we should note that integral of one-dimensional unit-impulse funct^n is a unit-step function. hence, the derivative of the latter results in the former.

Now the vector in direct^n of brightness gradient as specified (implicitly) in Ans-5 will be given by:-

$$\begin{bmatrix} \frac{\delta I}{\delta x} \\ \frac{\delta I}{\delta y} \end{bmatrix} \quad \text{——— (1)}$$

So the unit vector will be formed by dividing the vector with its total magnitude

where magnitude $= \dfrac{1}{\sqrt{\left(\frac{\delta I}{\delta x}\right)^2 + \left(\frac{\delta I}{\delta y}\right)^2}} = \dfrac{1}{\delta(x,y)}$ — (2)

$\Rightarrow$ From (1) & (2)

$$\frac{1}{\sqrt{\left(\frac{\delta I}{\delta x}\right)^2 + \left(\frac{\delta I}{\delta y}\right)^2}} \left[\delta I/\delta x \quad \delta I/\delta y\right]^T$$

Now θ is the angle with x-axis hence,

$$\theta = \tan^{-1}\left(\frac{\delta I/\delta y}{\delta I/\delta x}\right) \quad \text{—— (3)}$$

$$\Rightarrow \tan\theta = \frac{\delta I/\delta y}{\delta I/\delta x}$$ By squaring both sides

$$\tan^2\theta = \left(\frac{\delta I/\delta y}{\delta I/\delta x}\right)^2 = \sec^2\theta - 1$$

$$\Rightarrow \sec^2\theta = 1 + \left(\frac{\delta I/\delta y}{\delta I/\delta x}\right)^2$$

$$\Rightarrow \cos^2\theta = \frac{1}{1+\left(\frac{\delta I/\delta y}{\delta I/\delta x}\right)^2} \quad \Rightarrow \cos\theta = \frac{1}{\sqrt{1+\left(\frac{\delta I/\delta y}{\delta I/\delta x}\right)^2}} \quad \text{(4)}$$

$|||^{\text{arly}}$ $\because \sin^2\theta = 1-\cos^2\theta$

$$\sin^2\theta = 1 - \frac{1}{1+\left(\frac{\delta I/\delta y}{\delta I/\delta x}\right)^2}$$

$$\Rightarrow \sin^2\theta = \frac{\left(\frac{\delta I/\delta y}{\delta I/\delta x}\right)^2}{\left(\sqrt{1+\left(\frac{\delta I/\delta y}{\delta I/\delta x}\right)^2}\right)^2} \quad \sin\theta = \frac{\delta I/\delta y}{\sqrt{\left(\delta I/\delta x\right)^2+\left(\frac{\delta I}{\delta y}\right)^2}} \quad \text{(5)}$$

$|||^{\text{arly}}$ eq^n (4) can be modified as

$$\cos\theta = \frac{\delta I/\delta x}{\sqrt{\left(\delta I/\delta x\right)^2+\left(\delta I/\delta y\right)^2}}$$

b) $E' \rightarrow$ the first directional derivative of brightness in the direc. of brightness gradient
The unit vector is written as

$$\frac{\delta E/\delta x}{\sqrt{\left(\delta E/\delta x\right)^2+\left(\delta E/\delta y\right)^2}}\,\hat{i} + \frac{\delta E/\delta y}{\sqrt{\left(\frac{\delta E}{\delta x}\right)^2+\left(\delta E/\delta y\right)^2}}\,\hat{j}$$

Hence if we ignore the unit vectors we know that the magnitude of a unit vector is given! $M = \sqrt{a^2+b^2}$ if vector is $a\hat{i}+b\hat{j}$. Also to be written more simply it becomes

$$\Rightarrow \frac{1}{\sqrt{\left(\frac{\delta E}{\delta x}\right)^2+\left(\frac{\delta E}{\delta y}\right)^2}} \left[\left(\frac{\delta E}{\delta x}\right)^2 \left(\frac{\delta E}{\delta y}\right)^2\right]$$

Hence magnitude (E') = magnitude of brightness gradient

where $E' \rightarrow$ first directional derivative of brightness in the direction of brightness gradient.

Ans-3 Given $H(u,v) = P(u,v) * P(u,v)$

$$P(u,v) = \begin{cases} 1 & \text{for } |u| \leq 1/2 \;\&\; |v| \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$P(u,v)$ is clearly seen to be a rectangular funct!

Now, we know the convolution formula is given by:-

$$H(u,v) = \int\int_{-\infty}^{\infty} P(\xi,\eta)\, P(u-\xi, v-\eta)\, d\xi\, d\eta$$

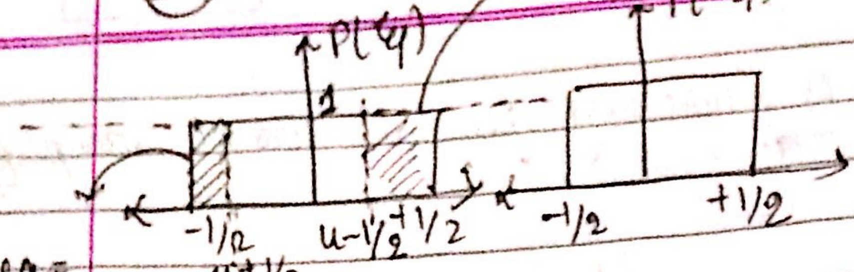
Now we have to observe the convolution for different values of u & v. Also, since the rectangular function is linearly separable.

We have,

$$H(u,v) = \int_{-\infty}^{\infty} P(\xi)\, P(u-\xi)\, d\xi \int_{-\infty}^{\infty} P(\eta)\, P(v-\eta)\, d\eta$$

(11)

Area $= \left(\frac{1}{2} - u + \frac{1}{2}\right)\cdot 1 = 1-u$

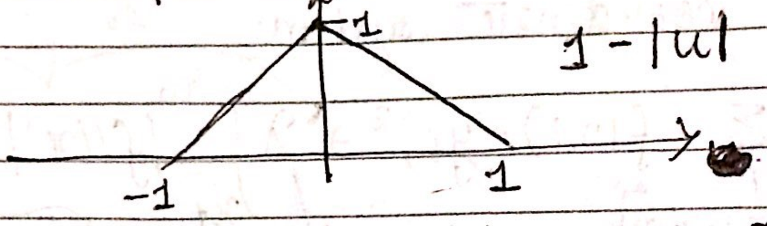


Area $= \left(u+\frac{1}{2}+\frac{1}{2}\right)\cdot 1 = 1+u$

Now, under different limits we have different values for e.g. for $u+\frac{1}{2} < -\frac{1}{2} \Rightarrow u < -1$
& $u - \frac{1}{2} > \frac{1}{2} \Rightarrow u > 1$ — Both cases will yield convolution to be 0.

Now: $-\frac{1}{2} < u + \frac{1}{2} < \frac{1}{2}$
$\Rightarrow -1 < u < 0$: $1+u$ as convolution sum
& for $-\frac{1}{2} < u - \frac{1}{2} < \frac{1}{2} \Rightarrow 0 < u < 1$: $1-u$ as convolution sum.

Hence the representatn would be:-



The function $H(u,v)$ can be represented as

$$H(u,v) = \begin{cases} (1+u)(1+v) & -1 < u,v < 0 \\ (1-u)(1-v) & 0 < u,v < 1 \\ 0 & \text{otherwise} \end{cases} = (1-|u|)\times(1-|v|)$$

Now the above function is obtained because the convolution of the function along with the functions are linearly seperable. We will have similar results for v. i.e
for $v < -1$ & $v > 1$ we have convolution sum : 0.
for $-1 < v < 0$ we have convolution sum: $1+v$
for $0 < v < 1$ we have convolution sum: $1-v$

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Now to find the corresponding point spread function $h(x,y)$ from $H(u,v)$

$$H(u,v) = \begin{cases} (1+u)(1+v) & -1 < u < 0 \ ; -1 < v < 0 \\ (1-u)(1-v) & 0 < u < 1 \ ; \ 0 < v < 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

or
$$\begin{cases} (1-|u|)(1-|v|) & -1 < u,v < 1 \\ 0 & \text{otherwise} \end{cases}$$

Now $h(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(u,v)\, e^{+i(ux+vy)}\, du\, dv$

Substituting values from eqn (1) & plugging in the limits we get $h(x,y)$

But an alternative method could have been chosen using convolution property of fourier transform.

We will adopt the second approach since its simpler in complexity.

We know that $H(u,v)$ is a triangular function

First say for a triangular function $x(t)$ we have, $x(t) = \begin{cases} 1-|t| & |t| < 1 \\ 0 & |t| > 1 \end{cases}$

$$\mathcal{F}[x(t)] = \int_{-\infty}^{\infty} e^{-2\pi j f t}\, x(t)\, dt = 2\int_0^1 (1-t)\cos(\omega t)\, dt$$

$$\Rightarrow 2\left[\int_0^1 \cos(\omega t)\, dt - \int_0^1 t\cos(\omega t)\, dt\right]$$

$$\Rightarrow \frac{2}{\omega}\left[\sin(\omega t) - \int_0^1 t\, d\sin(\omega t)\right]$$

$$\Rightarrow \frac{2}{\omega}\left[\sin(\omega) - t\sin(\omega t)\Big|_0^1 - \int_0^1 \sin(\omega t)\, dt\right]$$

(13)

$$= \frac{2}{\omega^2}\cos(\omega t)\Big|_1^0 - \frac{2}{\omega^2}(1-\cos\omega) - \left(\frac{2}{\omega}\right)^2 \sin^2\left(\frac{\omega}{2}\right)$$

$$= \frac{\sin^2(\pi f)}{(\pi f)^2} = \underline{\underline{\operatorname{sinc}^2(\omega)}}$$

Also since from the first part of the question we know,

$\mathcal{F}[A(t)] = \mathcal{F}[\operatorname{rect}(t) * \operatorname{rect}(t)] = \operatorname{sinc}(\omega)\operatorname{sinc}(\omega)$

$= \underline{\underline{\operatorname{sinc}^2(\omega)}}$ Same will hold for 2-D

From above we know

$$P(u,v) \circledast P(u,v) = \begin{cases} (1-|u|)(1-|v|) & \text{for } 0<|u|,|v|<1 \\ 0 & \text{otherwise} \end{cases}$$

From convolution theorem

$$h(x,y) = p(x,y)\cdot p(x,y)$$
$$= \operatorname{sinc}(x,y)\cdot \operatorname{sinc}(x,y)$$
$$= \underline{\underline{\operatorname{sinc}^2}}(x,y)$$