1: Let $f(x,y)$ be separable, i.e., $f(x,y) = g(x)h(y)$.

Then $\mathcal{F}(f(x,y)) = \iint_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi(ux+vy)}\, dx\,dy$

$$= \int_{-\infty}^{\infty} g(x)\, e^{-i2\pi ux}\, dx \int_{-\infty}^{\infty} h(y)\, e^{-i2\pi vy}\, dy$$

$$= G(u)\, H(v)$$

$\Rightarrow F(u,v) = G(u)\,H(v) \Rightarrow$ Fourier transform is also separable. ▨

3. $H(u,v) = P(u,v) * P(u,v)$

$$P(u,v) = \begin{cases} 1 & |u| \le 1/2,\ |v| \le 1/2 \\ 0 & \text{o.w.} \end{cases}$$

$\Rightarrow P(u,v) = \text{rect}(u,v)$

$$\therefore H(u,v) = \text{triag}(u,v) = \begin{cases} (1-|u|)(1-|v|), & |u| < 1,\ |v| < 1 \\ 0 & \text{o.w.} \end{cases}$$

$h(x,y) = \mathcal{F}^{-1}(H(u,v)) = \text{sinc}^2(x,y)$ ▪

4.

$$\underbrace{\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}}_{A} * \underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{B}$$

Flip B both vertically and horizontally to get $[-1\ 1]$.

Now the output is

$$\begin{bmatrix} 1 & 3 & 2 & -2 & -3 & -1 \\ 4 & 12 & 8 & -8 & -12 & -4 \\ 6 & 18 & 12 & -12 & -18 & -6 \\ 4 & 12 & 8 & -8 & -12 & -4 \\ 1 & 3 & 2 & -2 & -3 & -1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 3 & 2 & 3 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}}_{C} * \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{D}$$

Flip D to get $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

The output is

$$\begin{bmatrix} 3 & 2 & 3 & 0 \\ 2 & 0 & 0 & -3 \\ 1 & 0 & 0 & -2 \\ 0 & -1 & -2 & -3 \end{bmatrix}$$

5. Here we will first prove a more general theorem.

<u>Theorem</u>:- $f(x,y)$ is rotationally symmetric <u>iff</u>

$$\frac{1}{x}\frac{\partial f}{\partial x} = \frac{1}{y}\frac{\partial f}{\partial y}$$

($\Rightarrow$) Let $f(x,y)$ be rotationally symmetric. Hence, in polar coordinate it depends only on the radius $r = \sqrt{x^2+y^2}$ and does not depend on $\theta = \tan^{-1}(y/x)$.

Hence, $f(x,y) = \bar{f}(r)$.

$$\therefore \frac{1}{x}\frac{\partial f}{\partial x} = \frac{1}{x}\frac{\partial \bar{f}}{\partial r}\frac{\partial r}{\partial x} = \frac{1}{x}\frac{\partial \bar{f}}{\partial r}\frac{x}{\sqrt{x^2+y^2}} = \frac{\partial \bar{f}}{\partial r}\frac{1}{\sqrt{x^2+y^2}}$$

$$\frac{1}{y}\frac{\partial f}{\partial y} = \frac{1}{y}\frac{\partial \bar{f}}{\partial r}\frac{y}{\sqrt{x^2+y^2}} = \frac{\partial \bar{f}}{\partial r}\frac{1}{\sqrt{x^2+y^2}}$$

($\Leftarrow$) Let $\frac{1}{x}\frac{\partial f}{\partial x} = \frac{1}{y}\frac{\partial f}{\partial y}$

Let $f(x,y) = g(r,\theta)$.

$$\frac{1}{x}\left[\frac{\partial g}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial x}\right] = \frac{1}{y}\left[\frac{\partial g}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial g}{\partial \theta}\frac{\partial \theta}{\partial y}\right]$$

$$\Rightarrow \frac{\partial g}{\partial r}\frac{1}{\sqrt{x^2+y^2}} + \frac{\partial g}{\partial \theta}\frac{(-1)}{xy} = \frac{\partial g}{\partial r}\frac{1}{\sqrt{x^2+y^2}} + \frac{\partial g}{\partial \theta}\frac{1}{xy}$$

$$\Rightarrow \frac{\partial g}{\partial \theta}\frac{2}{xy} = 0 \Rightarrow \frac{\partial g}{\partial \theta} = 0 \quad \left[\text{as } \frac{1}{xy} \neq 0\right]$$

$\therefore$ f is rotationally symmetric.

Now here

$$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2$$

So, $$\frac{1}{(\partial/\partial x)} \frac{\partial f}{(\partial/\partial x)} = 2 = \frac{1}{(\partial/\partial y)} \frac{\partial f}{(\partial/\partial y)}$$

So, by above Theorem, f (squared gradient) is rotationally symmetric $\boxed{//}$

8-3

a) The unit vector in the direction of the brightness gradient

$$\vec{u} = \frac{\left(\frac{\partial E}{\partial x}, \frac{\partial E}{\partial y}\right)^t}{\sqrt{\left(\frac{\partial E}{\partial x}\right)^2 + \left(\frac{\partial E}{\partial y}\right)^2}}$$

$$\cos\theta = \frac{|\partial E/\partial x|}{\sqrt{(\partial E/\partial x)^2 + \left(\frac{\partial E}{\partial y}\right)^2}}, \quad \sin\theta = \frac{|\partial E/\partial y|}{\sqrt{(\partial E/\partial x)^2 + \left(\frac{\partial E}{\partial y}\right)^2}}$$

b) the first directional derivative of brightness in the direction $\vec{u}$

is $$E' = \left(\frac{\partial E}{\partial x}, \frac{\partial E}{\partial y}\right)^t \vec{u} = \sqrt{\left(\frac{\partial E}{\partial x}\right)^2 + \left(\frac{\partial E}{\partial y}\right)^2}$$

= Magnitude of the brightness gradient.

2.

(i) $\mathcal{F}\left(\text{rect}(x/3, y/2)\right) = 6 \,\text{sinc}(3u)\,\text{sinc}(2v)$

(ii) $\mathcal{F}\left(\text{rect}(x-4, y-5)\right) = e^{-i2\pi(4u+5v)}\,\text{sinc}(u)\,\text{sinc}(v)$

(iii) $\mathcal{F}\left(\text{sinc}(x-5, 2y-7)\right) = \frac{1}{2} e^{-i2\pi(5u+7/2v)}\,\text{rect}(u, v/2)$

(iv) $\mathcal{F}\left(\exp\{i16\pi x\}\,\text{sinc}(x, y/3)\right) = \delta(u-8) \otimes 3\,\text{rect}(u, 3v)$ (*)

(iv) $\mathcal{F}\left(3\,\text{rect}(x-8, 3y)\right) = e^{-i2\pi 8u}\,\text{sinc}(u, v/3)$

(*) as $\mathcal{F}\left(\exp\{i16\pi x\}\,\text{sinc}(x, y/3)\right)$

$$= \mathcal{F}\left(\exp\{i16\pi x\}\right) \otimes \mathcal{F}\left(\text{sinc}(x, y/3)\right)$$

$$= \delta(u-8) \otimes 3\,\text{rect}(u, 3v)$$

For parts (i)-(v), we have used the more general FT theorem:

Let $g(x,y) = f(ax+by+c,\ dx+ey+f)$ then

<u>Theorem:-</u>

$$G(u,v) = \frac{1}{|\Delta|} \exp \frac{i2\pi}{\Delta}\left[(ce-bf)u + (af-cd)v\right] F\left(\frac{eu-dv}{\Delta}, \frac{-bu+av}{\Delta}\right)$$

where $\Delta = \det\left(\begin{bmatrix} a & b \\ d & e \end{bmatrix}\right)$