BOȚOC Indi

NOTIȚE DE CURS

CTI-RO-1

AM

Analiză matematică

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3

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SERII NUMERICE

ex:

• Seria geometrica $\sum_{n=0}^{\infty} \underbrace{r^n}_{x_n}$, $r \in \mathbb{R}$ convergenta pt $r \in (-1;1)$

$$S_n = x_0 + x_1 + \ldots + x_n = 1 + r + \ldots + r^n = \frac{1(r^{n+1}-1)}{r-1} =$$

$$= \frac{r^{n+1}-1}{r-1} \xrightarrow[n\to\infty]{r\in(-1,1)} \frac{1}{1-r} \underset{not}{=} S$$

• $r \in (-\infty; -1] \Rightarrow$ seria $\sum_{n=0}^{\infty} r^n$ = DIV. (nu are limita)

• $r \in [1, \infty) \Rightarrow$ seria $\sum_{n=0}^{\infty} r^n$ = DIV. ($S_n \to \infty$)

obs: dacă $x_n \not\to 0 \Rightarrow$ seria este divergenta

$\boxed{\text{Propoziție 1}}$ Dacă $\sum_{n=0}^{\infty} x_n$ = conv atunci $x_n \xrightarrow[n\to\infty]{} 0$.

$\boxed{\text{Propoziție 2}}$ Dacă $x_n \underset{n\to\infty}{\not\to} 0$ atunci $\sum x_n$ e divergentă.

Dem P1:

Fie $\sum x_n$ = conv. $\qquad S_{n+1} - S_n = x_{n+1}$

$\Rightarrow (S_n)_{n\geq 0}$ - conv

$\Rightarrow S_n \xrightarrow[n\to\infty]{} S$

$$\underbrace{\lim_{n\to\infty} (S_{n+1} - S_n)}_{0} = \lim_{n\to\infty} x_{n+1} = 0. \text{ adev.}$$

ex: Seria armonică generalizată $\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{conv} & \text{pt } \forall\, p > 1 \\ \text{div} & \text{pt } \forall\, p \leq 1 \end{cases}$

$p \in \mathbb{R}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ = DIV ; $\sum_{n=1}^{\infty} \frac{1}{n^2}$ = conv.

Criterii de convergență pentru serii cu termeni pozitivi

Fie $\sum_{n=0}^{\infty} x_n$, $x_n \geq 0$

① CRITERIUL COMPARAȚIEI

Fie $\sum_{n=0}^{\infty} x_n$ și $\sum_{n=0}^{\infty} y_n$, $x_n \geq 0$, $y_n \geq 0$

și $x_n \leq y_n$, $\forall n$ $\quad$ $\boxed{x_n \leq C \cdot y_n}$, $C>0$.

i) Dacă $\sum_{n=0}^{\infty} y_n = \text{CONV} \rightarrow \sum_{n=0}^{\infty} x_n = \text{CONV}$

ii) Dacă $\sum_{n=0}^{\infty} x_n = \text{DIV} \rightarrow \sum_{n=0}^{\infty} y_n = \text{DIV}$

ex: $\sum_{n=0}^{\infty} \frac{1}{n^2+2n+1}$ $\overset{\text{not}}{=} x_n$

Fie $\sum_{n=1}^{\infty} \frac{1}{n^3} = \text{CONV}$, $\frac{1}{n^3} \overset{\text{not}}{=} y_n$

$$x_n = \frac{1}{n^3+2n+1} < \frac{1}{n^3} = y_n \xRightarrow[\text{COMP (i)}]{\text{crit.}} \sum x_n = \text{CONV}.$$

② CRITERIUL COMPARAȚIEI (forma la limită)

Fie $\sum_{n=0}^{\infty} x_n$ și $\sum_{n=0}^{\infty} y_n$, $x_n, y_n > 0$

Calc. $l = \lim\limits_{n\to\infty} \frac{x_n}{y_n} \in [0, \infty]$

(i) dacă $l \in (0,\infty)$ atunci cele două serii au aceeași natură

$$\sum x_n \sim \sum y_n$$

ii) dacă $l=0$ și $\sum_{n=0}^{\infty} y_n = \text{conv} \Rightarrow \sum_{n=0}^{\infty} x_n = \text{conv.}$

iii) dacă $l=\infty$ și $\sum_{n=0}^{\infty} x_n = \text{Div} \Rightarrow \sum_{n=0}^{\infty} y_n = \text{Div.}$

ex: Stud. conv:

$$\sum_{n=1}^{\infty} \frac{1}{n^3-2n-1}$$

$$\cdot\, l = \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\frac{1}{n^3-2n-1}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{n^3-2n-1} = 1 \in (0,\infty)$$

$$\xrightarrow[\text{LIMITĂ}]{\text{Crit comp}} \sum_{n=1}^{\infty} \frac{1}{n^3-2n-1} \sim \sum_{n=1}^{\infty} \frac{1}{n^3} = \text{CONV} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^3-2n-1} = \text{CONV.}$$

3 CRITERIUL RAPORTULUI

Fie $\sum_{n=0}^{\infty} x_n$, $x_n > 0$ și calc. $l = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} \in [0,\infty]$

i) dacă $l<1 \Rightarrow \sum x_n = \text{CONV.}$

ii) dacă $l>1 \Rightarrow \sum x_n = \text{DIV.}$

iii) dacă $l=1 \Rightarrow$ crit rap. nu decide natura seriei

ex: $\sum_{n=1}^{\infty} \underbrace{\frac{a^n}{n!}}_{=x_n}$, $a>0$.

$$l = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{a^{n+1}}}{(n+1)!} \cdot \frac{\cancel{n!}}{\cancel{a^n}} =$$

$$= \lim_{n\to\infty} \frac{a}{n+1} = 0,\ \forall a \xrightarrow[\text{RAP(i)}]{\text{Crit}} \sum_{n=1}^{\infty} \frac{a^n}{n!} \text{ conv}$$

ex: $$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} = \sum_{n=1}^{\infty} \underbrace{\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2\cdot4\cdot6\cdot\ldots\cdot(2n)}}_{=x_n}$$

$$\cdot \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{(2n+1)\cancel{!!}}{(2n+2)\cancel{!!}} \cdot \frac{\cancel{(2n)!!}}{\cancel{(2n-1)!!}} = \lim_{n\to\infty} \frac{2n+1}{2n+2} = 1.$$

4) CRITERIUL lui RAABE - DUHAMEL

Fie $\sum_{n=0}^{\infty} x_n$ şi $x_n > 0$

Calc. $L = \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right)$.

i) dacă $L < 1 \Rightarrow \sum_{n=0}^{\infty} x_n = \text{DIV}$.

ii) dacă $L > 1 \Rightarrow \sum_{n=0}^{\infty} x_n = \text{CONV}$.

iii) dacă $L = 1 \Rightarrow$ nu putem stabili natura seriei

Revenim la ex de mai sus:

$$L = \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{2n+2}{2n+1} - 1\right) = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}.$$

$$\xRightarrow{\text{Crit RD}} \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} = \text{DIV}.$$

5) CRITERIUL RĂDĂCINII (radicalului)

Fie $\sum_{n=0}^{\infty} x_n$, $x_n > 0$ şi calc $l = \lim_{n\to\infty} \sqrt[n]{x_n} \in [0, \infty]$

i) dacă $l < 1 \Rightarrow \sum_{n=0}^{\infty} x_n = \text{CONV}$.

ii) dacă $l > 1 \Rightarrow \sum_{n=0}^{\infty} x_n = \text{DIV}$

iii) dacă $l = 1 \Rightarrow$ nu putem stabili natura seriei

ex: $\sum_{n=1}^{\infty} \underbrace{\left(1 - \frac{a}{n}\right)^{n^2}}_{= x_n}$, $a > 0$.

$$l = \lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \left(1 - \frac{a}{n}\right)^n = \lim_{n\to\infty} \boxed{\left(1 - \frac{a}{n}\right)^{-\frac{n}{a}}}^{-\frac{a}{n}\cdot n} = e^{-a} = \frac{1}{e^a} < 1$$

$\Rightarrow \sum x_n = \text{CONV}$.

6) CRITERIUL DE CONDENSARE a lui CAUCHY

Fie $\sum_{n=0}^{\infty} x_n$, $x_n > 0$ și $x_n \searrow 0$.

At. $\sum_{n=0}^{\infty} x_n \sim \sum_{n=0}^{\infty} 2^n x_{2^n}$

Obs: nu intră toți termenii lui x_n, ci $x_2, x_4, x_8, \dots$

Ex) Dem. că $\sum \frac{1}{n^p} = \begin{cases} \text{conv}, & p>1 \\ \text{div}, & p \le 1 \end{cases}$

• $p<0 \Rightarrow x_n = \frac{1}{n^p} \xrightarrow[n\to\infty]{\not} 0 \xRightarrow[\text{Div}]{\text{Crit.}} \sum_{n=1}^{\infty} \frac{1}{n^p} = \text{Div}.$

• $p=0 \Rightarrow \sum_{n=1}^{\infty} 1 \xrightarrow[n\to\infty]{\not} 0 \Rightarrow \text{Div}$

• $p>0 \Rightarrow x_n = \frac{1}{n^p} \searrow 0.$

$\Rightarrow \sum_{n=1}^{\infty} x_n \sim \sum_{n=1}^{\infty} 2^n \cdot x_{2^n} = \sum_{n=1}^{\infty} 2^n \frac{1}{2^{np}} = \underbrace{\sum_{n=1}^{\infty} \left(\overset{r}{\frac{1}{2^{p-1}}}\right)^n}_{\text{SERIA GEOMETRICĂ}} = \text{CONV}$ pt $r<1$

$\Updownarrow$

$\frac{1}{2^{p-1}} < 1$

$2^{p-1} > 2^0$

$\Leftrightarrow p > 1.$ q.e.d.

Ex) Stud conv. seriei:

$\sum_{n\ge2}^{\infty} \underset{x_n}{\underline{\frac{1}{\ln n}}} \to$ crit condensare

$\Rightarrow$ stud conv. seriei $\sum_{n\ge2}^{\infty} 2^n \cdot \frac{1}{\ln 2^n} = \sum_{n\ge2}^{\infty} \frac{2^n}{n} \cdot \frac{1}{\ln 2} =$

$= \frac{1}{\ln 2} \sum_{n\ge2}^{\infty} \frac{2^n}{n}$

• calc. $L = \lim\limits_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim\limits_{n\to\infty} n\left(\frac{\cancel{2^n}}{n}\cdot\frac{n+1}{2^{\cancel{n+1}}} - 1\right) =$

$= \lim\limits_{n\to\infty} n\left(\frac{n+1}{2n} - \frac{2n}{2n}\right) = \lim\limits_{n\to\infty} \cancel{n}\cdot\frac{-n+1}{2\cancel{n}} = -\infty \Rightarrow$

$$\Rightarrow \sum_{n\geq 2}^{\infty} \frac{1}{\ln n} \text{ div.}$$