BOȚOC Indi

NOTIȚE DE CURS

CTI-RO-1

AM

Analiză matematică

an univ. 2022-2023

15

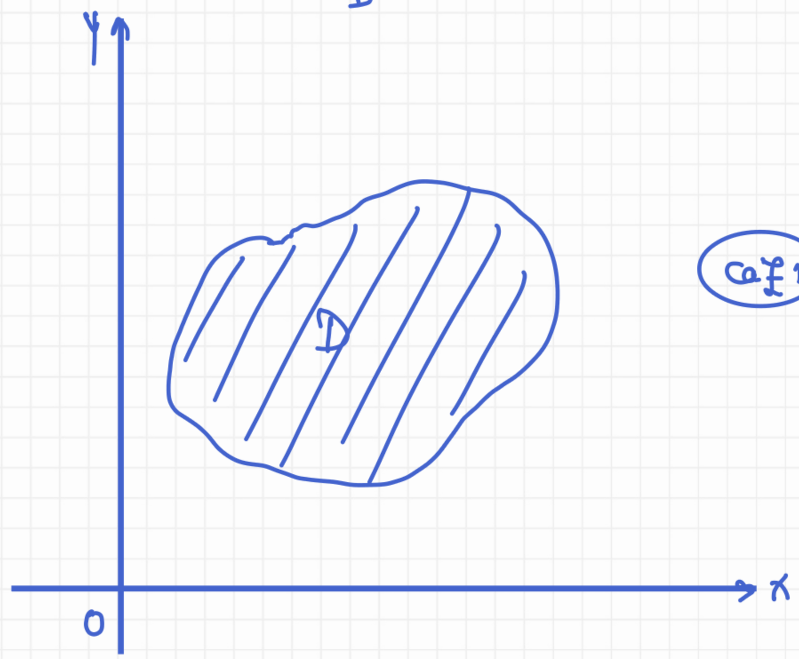
Curs predat de:
Conf. univ. dr.
CĂDARIU
Liviu

13.01.2022

Integrale duble

$\mathbb{R}^2 \supset D$, $I = \iint\limits_D f(x,y)\,dx\,dy$, $f: D \subset \mathbb{R}^2 \to \mathbb{R}$

integrabilă pe D



caz 1 $f(x,y) \equiv 1$

$$I = \iint\limits_D 1\,dx\,dy = \text{Aria}(D)$$

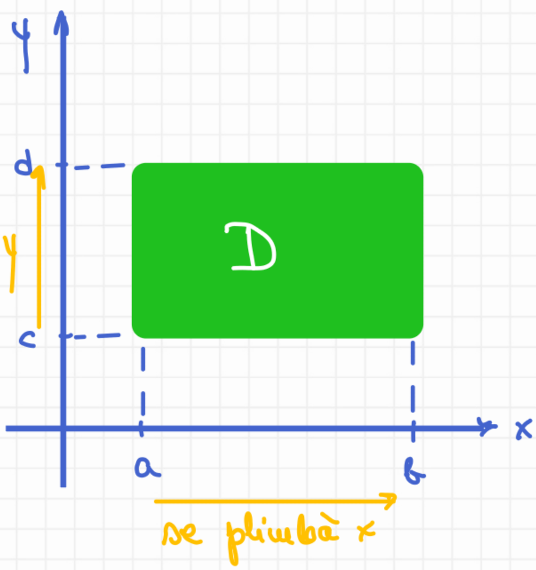
caz 2 $f(x,y) =$ densitatea de masă

$$I = \iint\limits_D f(x,y)\,dx\,dy = \text{Masa}(D)$$

Mod de calcul

1. $D = [a,b] \times [c,d]$ ~ dreptunghi

Ⓐ Funcție de 2 variabile



$$\boxed{I = \iint\limits_D f(x,y)\,dx\,dy = \underbrace{\int_a^b \Big(\underbrace{\int_c^d f(x,y)\,dy}_{\text{o rezolvi și dă în rap. cu } x}\Big)\,dx}_{\text{asta depinde numai de } x}}$$

ex: Calc. $I=\iint_D e^{x+2y}\,dx\,dy$, unde $D=[0,1]\times[-1,2]$

$$\Rightarrow I=\int_0^1\left(\int_{-1}^2 e^{x+2y}\,dy\right)dx=\int_0^1\left(\frac{e^{x+2y}}{2}\Big|_{-1}^{2}\right)dx=\frac{1}{2}\int_0^1\left(e^{x+4}-e^{x-2}\right)dx=$$

$$=\frac{1}{2}\left[e^{x+4}\Big|_0^1-e^{x-2}\Big|_0^1\right]=\frac{1}{2}\left[e^5-e^4-e^{-1}+e^{-2}\right]=\dots$$

B) Produsul a 2 funcții de o singură variabilă

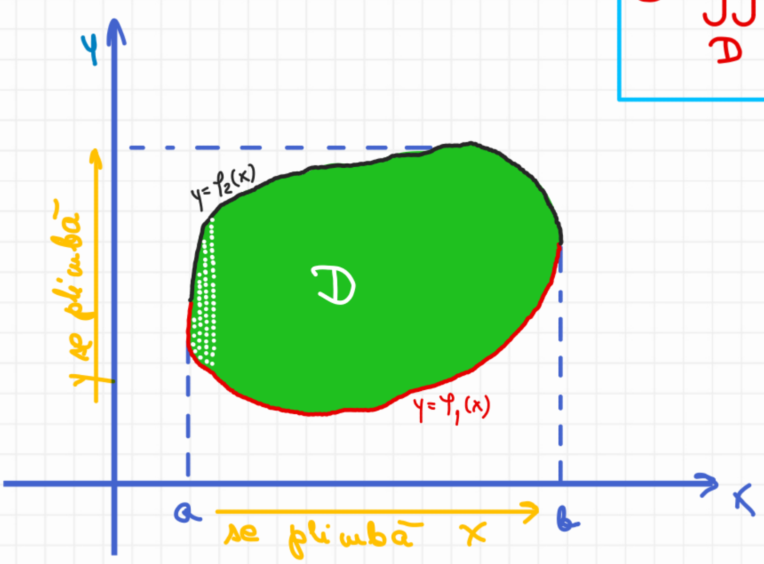
$$I=\iint_D [\underbrace{f(x)}_{}\cdot g(y)]\,dx\,dy=\left(\int_a^b f(x)\,dx\right)\cdot\left(\int_c^d g(y)\,dy\right)$$

constant în raport cu y ⇒ iese afară

② D - neregulat

A) Cu proiecția pe Ox

$$I=\iint_D f(x,y)\,dx\,dy=\int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\right)dx$$

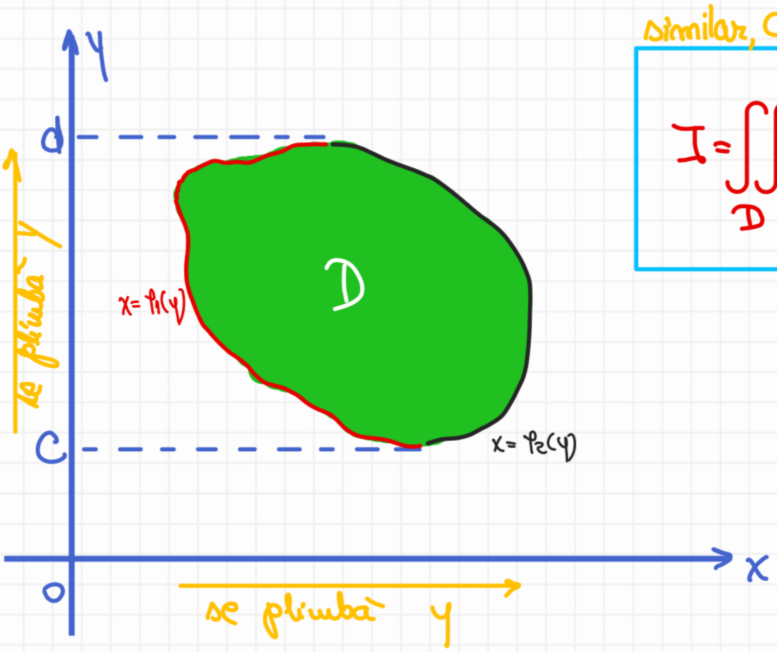


• proiectez D pe Ox ⇒ $a\le x\le b$

• ca o imprimantă cu ace, începem să „desenăm" de la curba inferioară

$$D:\begin{cases} a\le x\le b\\ \varphi_1(x)\le y\le \varphi_2(x)\end{cases}$$

(B) Cu proiecția pe Oy



similar, ca și mai sus:

$$\boxed{I=\iint_D f(x,y)\,dx\,dy=\int_c^d\left(\int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)\,dx\right)dy}$$

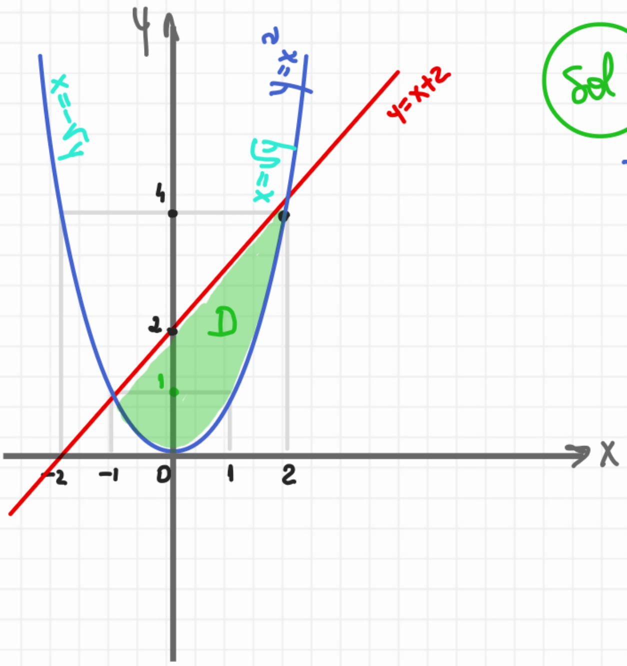
$D: c \le y \le d$

$\quad \varphi_1(x) \le x \le \varphi_2(y)$

(ex) $I=\iint_D y\,dx\,dy \quad , D: \begin{cases} y=x^2 \\ y=x+2 \end{cases} \Rightarrow x^2=x+2 \quad : \quad x_0=2,\ y_0=4 \Rightarrow A(2,4)$

SAU

$x_1=-1 \quad , y_1=1 \Rightarrow B(-1,1)$



(Sol 1) Cu proiecția pe Ox

$D: \begin{cases} -1 \le x \le 2 \\ x^2 \le y \le x+2 \end{cases}$

$\Rightarrow I=\int_{-1}^{2}\left(\int_{x^2}^{x+2} y\,dy\right)dx =$

$=\int_{-1}^{2} \frac{y^2}{2}\Big|_{x^2}^{x+2} dx = \frac{1}{2}\int_{-1}^{2}\left[(x+2)^2-x^4\right]dx =$

$=\frac{1}{2}\int_{-1}^{2}\left(x^2+4x-4-x^4\right)dx = \ldots$

Sol 2 cu proiecția pe Oy

$$D:\begin{cases} 0 \le y \le 4 \\ \underset{\text{FRONTIERE}}{2} \le x \le \sqrt{y} \end{cases}$$

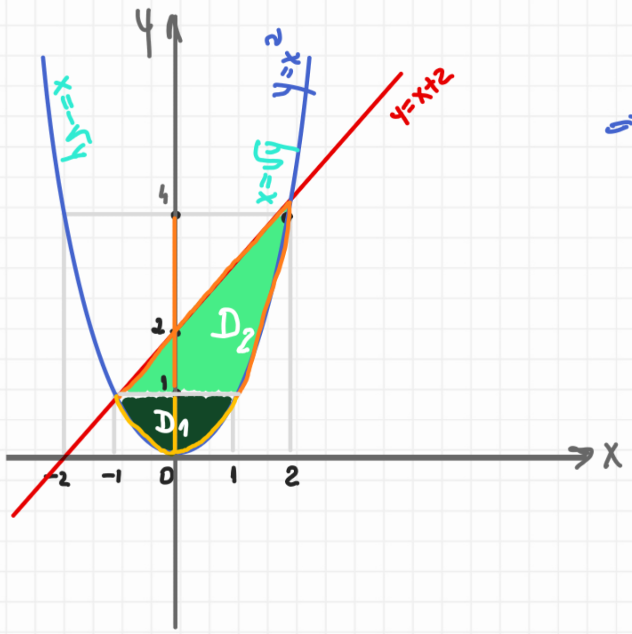
$$D_1:\begin{cases} 0 \le y \le 1 \\ -\sqrt{y} \le x \le \sqrt{y} \end{cases}$$

pentru $y \in [0,1]$ front. st e $-\sqrt{y}$, front. dr. e $\sqrt{y}$

am desfăcut pentru că frontiera din stânga a lui x DIFERĂ

$$D_2:\begin{cases} 1 \le y \le 4 \\ y-2 \le x \le \sqrt{y} \end{cases}$$

pentru $y \in [1,4]$ front st. e $y-2$, front. dr. e $\sqrt{y}$

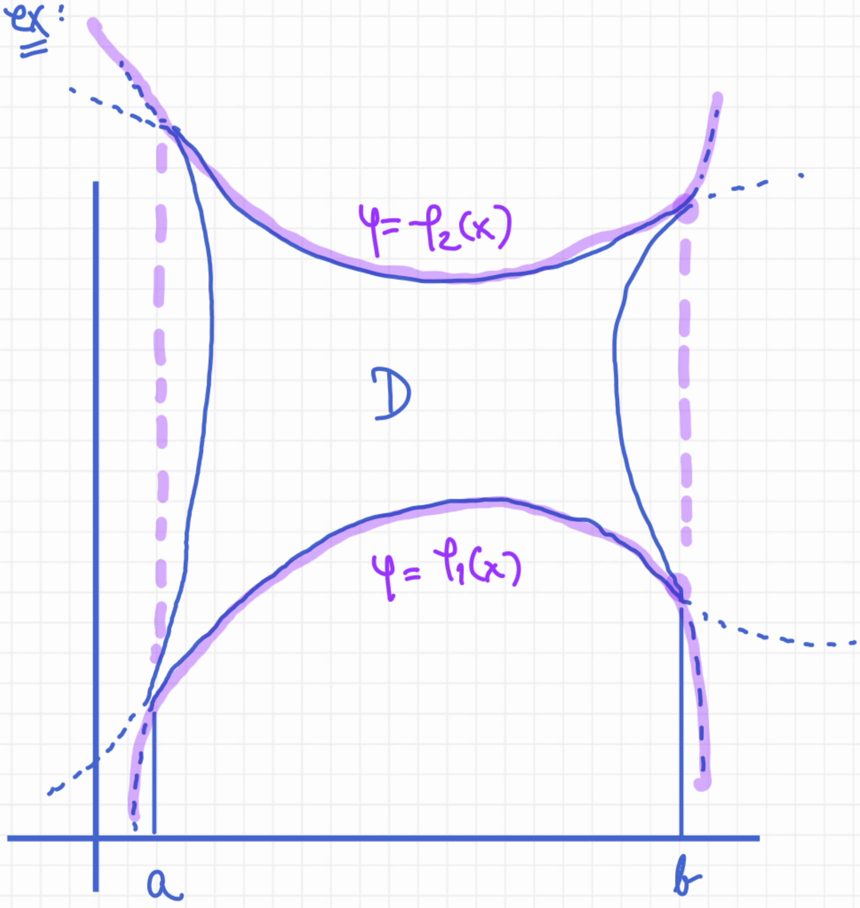


$$\Rightarrow I = I_1 + I_2 \underline{\underline{\text{aplic 2B}}}$$

$$= \int_0^1 \left(\int_{-\sqrt{y}}^{\sqrt{y}} y\, dx \right) dy + \int_1^4 \left(\int_{y-2}^{\sqrt{x}} y\, dx \right) dy$$

---

ex:

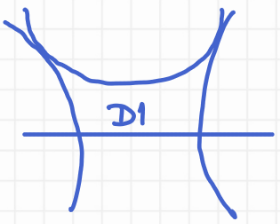


$$I = \iint_D f(x,y)\, dx dy$$

$$D:\begin{cases} a \le x \le b \\ \varphi_1(x) \le y \le \varphi_2(x) \end{cases}$$

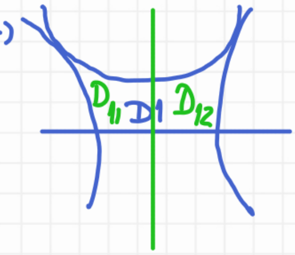
- nu-i bun, că dă pe afară, „imprimanta" iese din D

• obs: avem nevoie de frontiere drepte. Deci, împărțim după, de ex, Ox

$\Rightarrow$ 

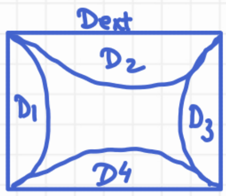
$D = D_1 \cup D_2$

dacă nici așa nu e bine, împărțim și după Oy

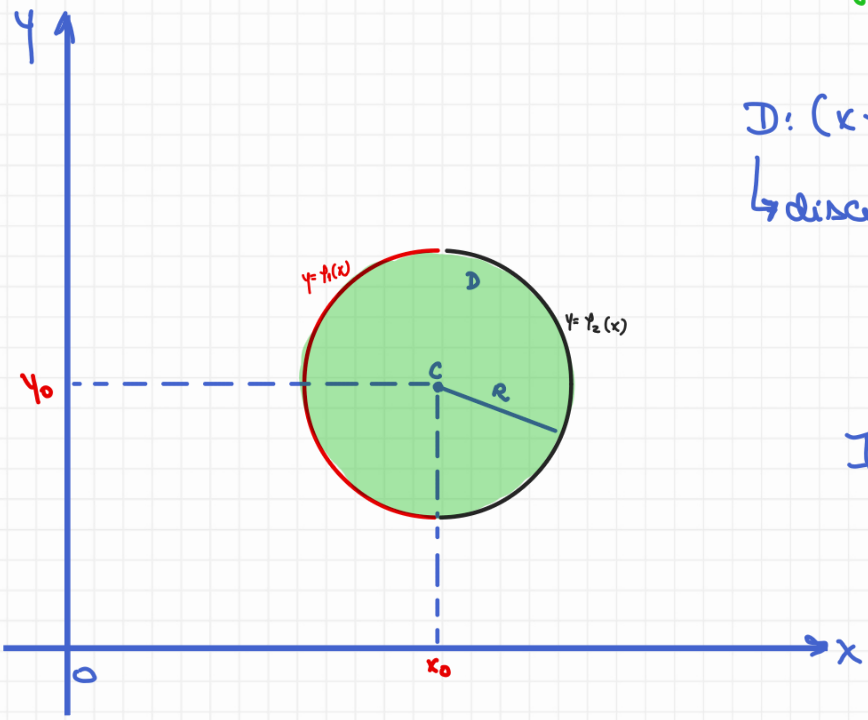
$\Rightarrow$ 

• obs: putem împărți cât avem nevoie

• obs: putem scădea mulțimi: $D = D_{ext} \setminus D_1 \setminus D_2 \setminus D_3 \setminus D_4$



COORDONATE POLARE la integrale duble



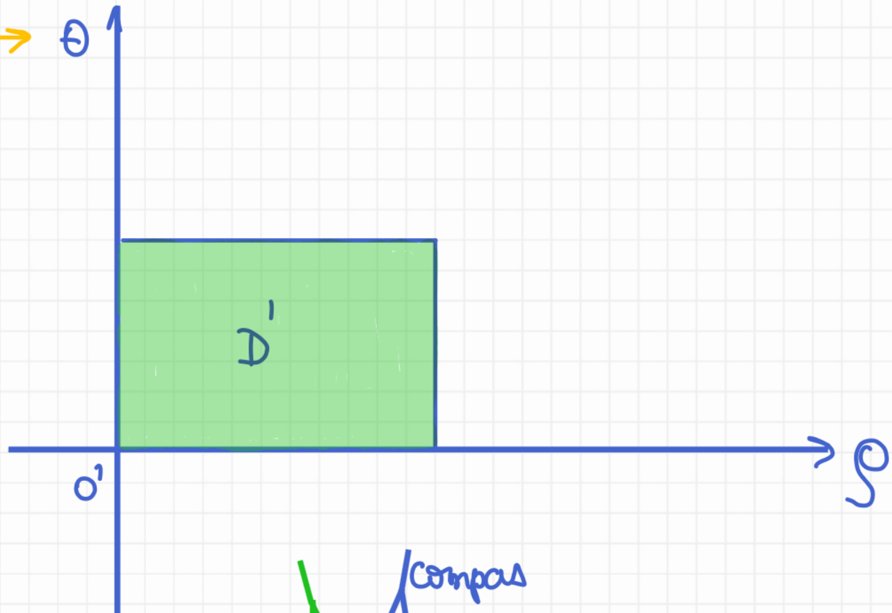
$D: (x-x_0)^2+(y-y_0)^2 \le R^2$

↳ discul de cerc de rază R cu centrul în $C(x_0,y_0)$

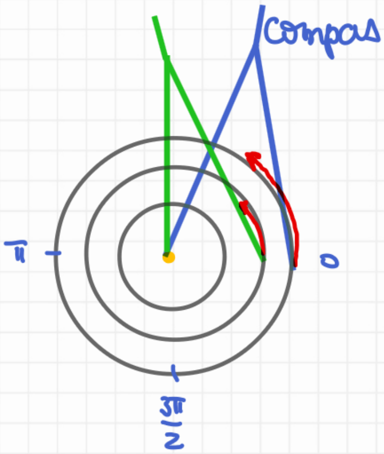
$$I=\iint_D f(x,y)\,dx\,dy$$

$$\begin{cases} x = x_0+\rho\cos\theta \\ y = y_0+\rho\sin\theta \end{cases} \xrightarrow[\text{devine}]{\text{graficul}}$$

$(x,y) \longrightarrow (\rho,\theta)$



$$D': \begin{cases} 0 \le \rho \le R \\ 0 \le \theta \le 2\pi \end{cases}$$ de ce?



• compasul, „imprimanta" mea, merge punct cu punct

$\Rightarrow$ I devine: $I=\iint_{D'} f(x_0+\rho\cos\theta,\, y_0+\rho\sin\theta)\,d\rho\,d\theta$

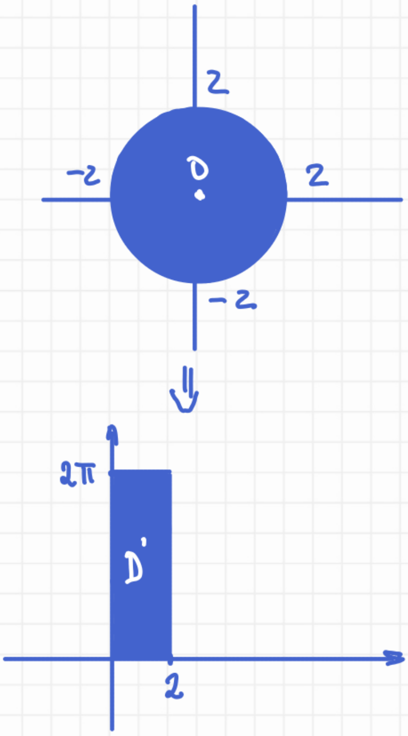
• dar $dx\,dy = \left|\dfrac{D(x,y)}{D(\rho,\theta)}\right| = \begin{vmatrix} x'_\rho & y'_\rho \\ x'_\theta & y'_\theta \end{vmatrix} d\rho\,d\theta = \begin{vmatrix} \cos\theta & \sin\theta \\ -\rho\sin\theta & \rho\cos\theta \end{vmatrix} d\rho\,d\theta =$

$$= \rho(\cos^2\theta + \sin^2\theta)\, d\rho\, d\theta = \rho\, d\rho\, d\theta$$

(ex) D: $x^2+y^2 \le 4$, calc $I = \iint_D 1\, dx\, dy$ = aria lui D = ?

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ aria discului



Folosim coord polare: $\begin{cases} x = 0 + \rho\cos\theta \\ y = 0 + \rho\sin\theta \end{cases}$, $D': \begin{cases} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \end{cases}$

$$dx\,dy = \rho\, d\rho\, d\theta$$

$$\Rightarrow I = \iint_{D} \rho\, d\rho\, d\theta = \left(\int_0^2 \rho\, d\rho\right) \cdot \left(\int_0^{2\pi} 1\, d\theta\right) =$$

$$= \frac{\rho^2}{2}\Big|_0^2 \cdot \theta\Big|_0^{2\pi} = \frac{4-0}{2} \cdot 2\pi = 4\pi$$