BOȚOC Indi

NOTIȚE DE CURS

CTI-RO-1

AM

Analiză matematică

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Curs predat de:
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11

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Derivabilitate și diferențiabilitate de ordin superior

Fie $f: D \subset \mathbb{R}^2 \to \mathbb{R}$, $(x_0, y_0) \in \mathring{D}$

$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{x \to x_0} \frac{f(x,y_0) - f(x_0,y_0)}{x - x_0}$$

- $$\frac{\partial^2 f}{\partial x^2}(x_0,y_0) = \left[\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial x}\right)\right]\Bigg|_{(x_0,y_0)} \overset{\text{def}}{=\!=} \lim_{x \to x_0} \frac{\frac{\partial f}{\partial x}(x,y_0) - \frac{\partial f}{\partial x}(x_0,y_0)}{x - x_0}$$

$\frac{\partial^2 f}{\partial x^2}(x_0,y_0) = f''(x_0,y_0)$

$$\frac{\partial \square}{\partial x}$$

= OPERATORUL de DERIVARE în raport cu x

- $$\frac{\partial^2 f}{\partial y^2}(x_0,y_0) = \lim_{y \to y_0} \frac{\frac{\partial f}{\partial y}(x_0,y) - \frac{\partial f}{\partial y}(x_0,y_0)}{y - y_0}$$

- $$\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) = \left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)\right]\Bigg|_{(x_0,y_0)} \overset{\text{def}}{=\!=} \lim_{x \to x_0} \frac{\frac{\partial f}{\partial y}(x,y_0) - \frac{\partial f}{\partial y}(x_0,y_0)}{x - x_0}$$

- $$\frac{\partial^2 f}{\partial y \partial x}(x_0,y_0) = \left[\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right)\right]\Bigg|_{(x_0,y_0)}$$

Obs: de obicei: funcțiile „cuminți", au $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

#LA EXAMEN: $\frac{\partial^{100} f}{\partial x^{99} \partial y}$

Teorema lui SCHWARZ:

Dacă $f: D \subset \mathbb{R} \to \mathbb{R}$ admite derivate parțiale mixte de ordinul 2 în vecinătatea $(x_0,y_0) \in D$ și una dintre derivatele parțiale (de ex: f''_{xy}) este CONTINUĂ în (x_0,y_0) atunci $\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0,y_0)$

- $$\frac{\partial^{m+n} f}{\partial x^m \partial y^n} = \left[\frac{\partial^m f}{\partial x^m}\left(\frac{\partial^n f}{\partial y^n}\right)\right]\Bigg|_{(x_0,y_0)}, \quad m,n \in \mathbb{N}$$

(ex) Calc $\frac{\partial^{13} f}{\partial x^6 \partial y^7} = ?$ pt. $f(x,y) = (x^2+y^2)\sin(x+y)$

$$\frac{\partial^{13} f}{\partial x^6 \partial y^7} = \frac{\partial^6 f}{\partial x^6}\left(\frac{\partial^7 f}{\partial y^7}\right)$$

obs: $$\boxed{(g\cdot h)^{(n)} = C_n^0 g^{(n)}\cdot h + C_n^1 g^{(n-1)} h^{(1)} + \dots + C_n^n g\cdot h^{(n)}}$$ formula lui Leibniz

$$\frac{\partial^7 f}{\partial y^7} = \Big[\underbrace{(x^2+y^2)}_{g(x,y)}\underbrace{\sin(x+y)}_{h(x,y)}\Big]^{(7)}_y = \boxed{C_7^0 \underbrace{(x^2+y^2)^{(7)}_y}_{=0}\sin(x+y)} + \boxed{C_7^1 \underbrace{(x^2+y^2)^{(6)}_y}_{=0}(\sin(x+y))^{(1)}_y} + \underbrace{\dots}_{=0}$$

$$+ C_7^5 \underbrace{(x^2+y^2)^{(2)}}_{2}\underbrace{(\sin(x+y))^{(5)}_y}_{\cos(x+y)} + C_7^6\underbrace{(x^2+y^2)^{(1)}}_{2y}\underbrace{(\sin(x+y))^{(6)}_y}_{-\sin(x+y)} + C_7^7 (x^2+y^2)\underbrace{(\sin(x^2+y^2))^{(7)}}_{-\cos(x+y)}$$

obs: $(\sin(x+y))^{(5)}_y = (\sin(x+y))^{(1)}_y = \cos(x+y)$

$$\Rightarrow \frac{\partial^{13} f}{\partial x^6\partial y^7} = \frac{\partial^6 f}{\partial x^6}\left(\frac{\partial^7 f}{\partial y^7}\right) = \frac{\partial^6 f}{\partial x^6}\Big(\underbrace{42\cos(x+y)}_{\text{simplu}} - \underbrace{14y\sin(x+y)}_{14y - \text{const}} - \underbrace{(x^2+y^2)\cos(x+y)}_{\text{la fel ca sus}}\Big)$$

(ex) Calc $\frac{\partial^{m+n}}{\partial x^m \partial y^n}\left(\frac{1}{y(x+1)}\right) = \frac{(-1)^{m+n}\cdot m!\cdot n!}{y^n\cdot (x+1)^{m+1}}$

aplic: $$\boxed{\left(\frac{1}{x+a}\right)^{(n)} = \frac{(-1)^n n!}{(x+a)^{n+1}}}$$

Diferențiabilitate de ordin superior

$f: D \subset \mathbb{R}^2 \to \mathbb{R}$, $(x_0, y_0) \in \mathring{D}$

$$d^1_{(x_0,y_0)} f = \frac{\partial f}{\partial x}(x_0,y_0)\,dx + \frac{\partial f}{\partial y}(x_0,y_0)\,dy$$

$$d^2_{(x_0,y_0)} f \overset{def}{=} \frac{\partial^2 f}{\partial x^2}(x_0,y_0)\cdot dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\,dx\,dy + \frac{\partial^2 f}{\partial y^2}(x_0,y_0)\,dy^2$$

$$d^3_{(x_0,y_0)} f \overset{def}{=} \frac{\partial^3 f}{\partial x^3}(x_0,y_0)\,dx^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}\,dx^2 dy + 3\frac{\partial^3 f}{\partial x \partial y^2}(x_0,y_0)\,dx\,dy^2 + \frac{\partial^3 f}{\partial y^3}(x_0,y_0)\,dy^3$$

$$d^2_{(x_0,y_0,z_0)} f \overset{def}{=} \frac{\partial^2 f}{\partial x^2}(x_0,y_0,z_0)\cdot dx^2 + \frac{\partial^2 f}{\partial y^2}(x_0,y_0,z_0)\cdot dy^2 + \frac{\partial^2 f}{\partial z^2}(x_0,y_0,z_0)\cdot dz^2 +$$

$$+ 2\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0,z_0)\,dx\,dy + 2\frac{\partial^2 f}{\partial y \partial z}(x_0,y_0,z_0)\,dy\,dz + 2\frac{\partial^2 f}{\partial x \partial z}(x_0,y_0,z_0)\,dx\,dz.$$

ex) $d^{10}_{(x,y)} f = ?$ $\quad f(x,y) = \ln(x+y)$

$$\frac{\partial^{m+n} f}{\partial x^m \partial y^n} = \boxed{}$$

$$\Rightarrow d^{10}_{(x,y)} = \boxed{}\,(dx+dy)^{10}$$

#E-LĂUGIC

Funcții omogene

Def: Fie $f: \mathbb{R}^2 \to \mathbb{R}$ se numește FUNCȚIE OMOGENĂ de GRAD $p \in \mathbb{N}$ dacă $f(tx, ty) = t^p f(x,y)$, $\forall x,y \in \mathbb{R}^2$, $\forall t > 0$

ex: $f(x,y) = (x^2+y^2)\sin\left(\frac{x}{y}\right)$, $y \neq 0$ $\quad \Rightarrow p = 2.$

$$f(tx, ty) = t^2(x^2+y^2)\sin\left(\frac{tx}{ty}\right)$$
$$= t^2 f(x,y)$$

Teorema lui EULER:

Fie $f:\mathbb{R}^2 \to \mathbb{R}$ o funcție OMOGENĂ de ordin $p \in \mathbb{R}_+$. Atunci au loc relațiile:

$\boxed{1}$ $$x \cdot \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = p \cdot f$$ relația lui Euler de ord. 1

$\boxed{2}$ $$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = p(p-1)\, f$$

...

$\boxed{n}$ $$\left(x \cdot \frac{\partial \square}{\partial x} + y \cdot \frac{\partial \square}{\partial y} \right)^{(n)} (f) = p(p-1)(p-2) \ldots (p-n+1)\, f$$ rel. lui Euler de ord n

Obs: $$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \ldots + x_q \frac{\partial f}{\partial x_q} = p \cdot f$$ rel. lui Euler de ord 1 pentru funcție cu q variabile

• ne poate da la examen + mai găsim pe campus