CATEGORIES for the WORKING PROGRAMMER

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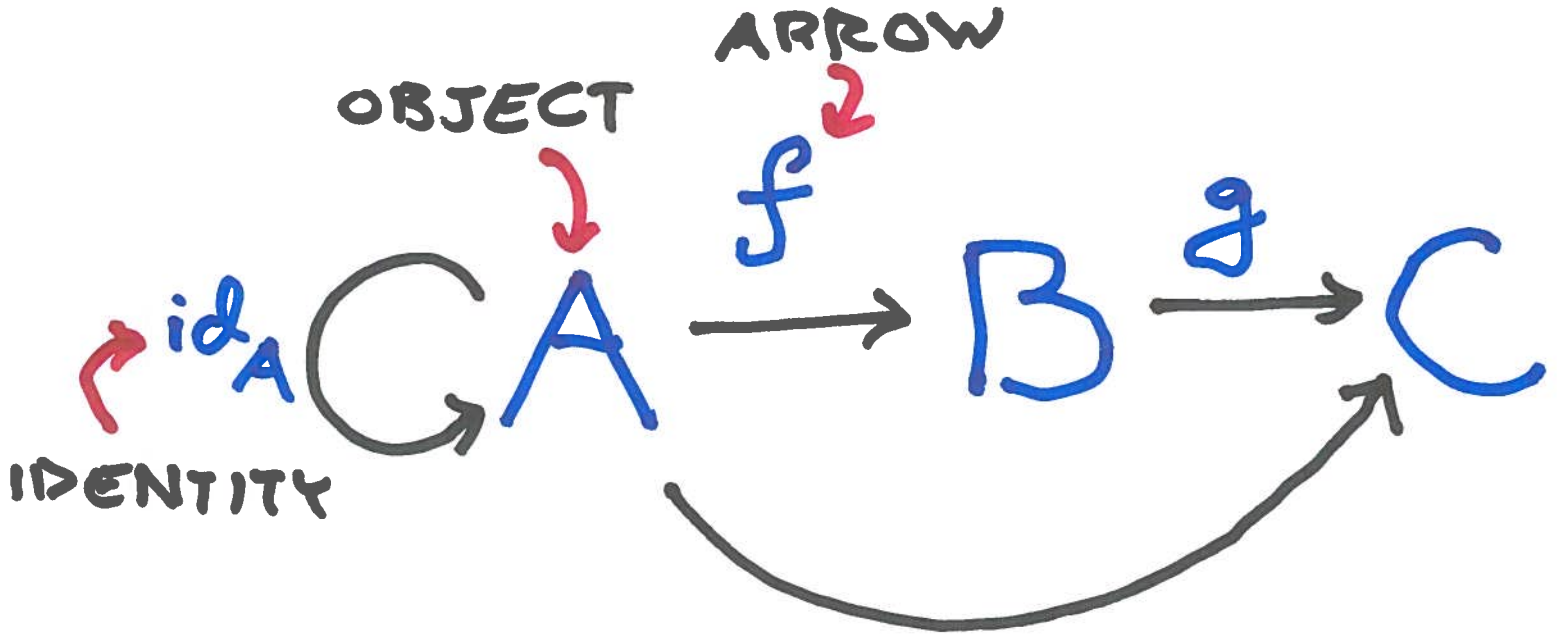
CATEGORY $\mathcal{C}$



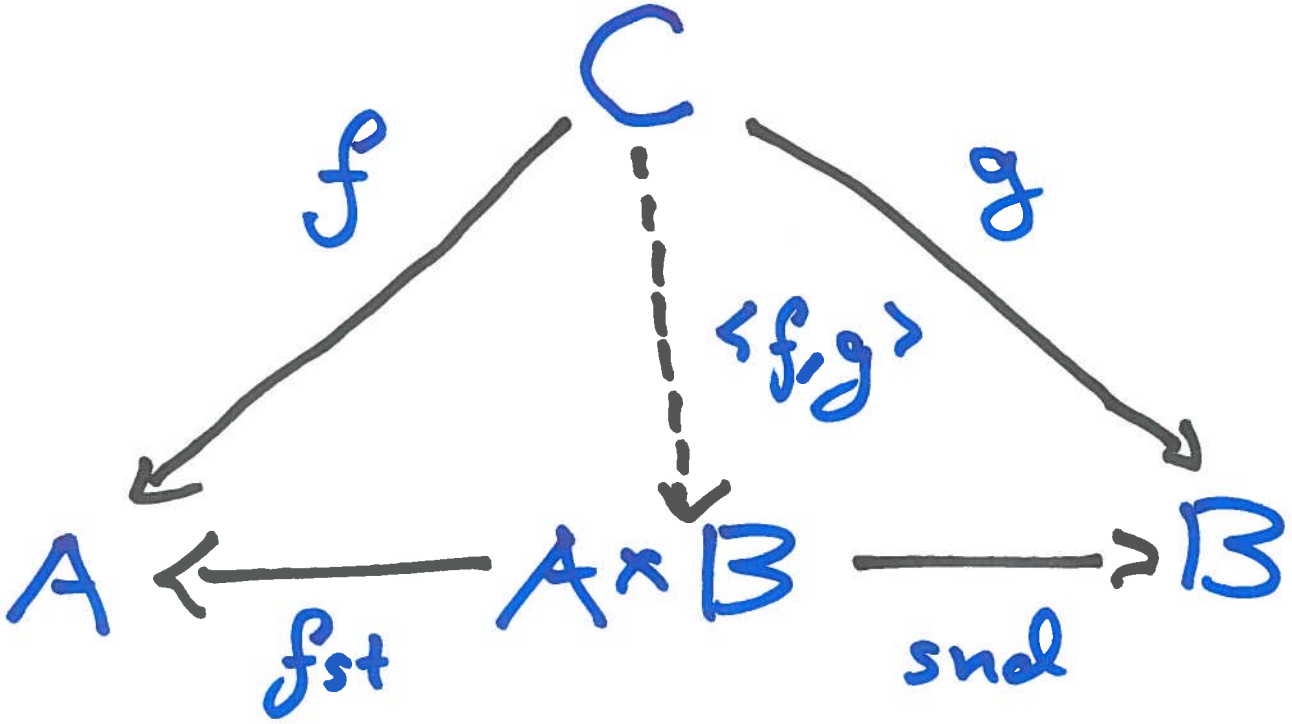
$\mathcal{C}(A,B)$

$= \{ A \xrightarrow{f} B \in \mathcal{C} \}$

$id_A ; f = f ; id_B$

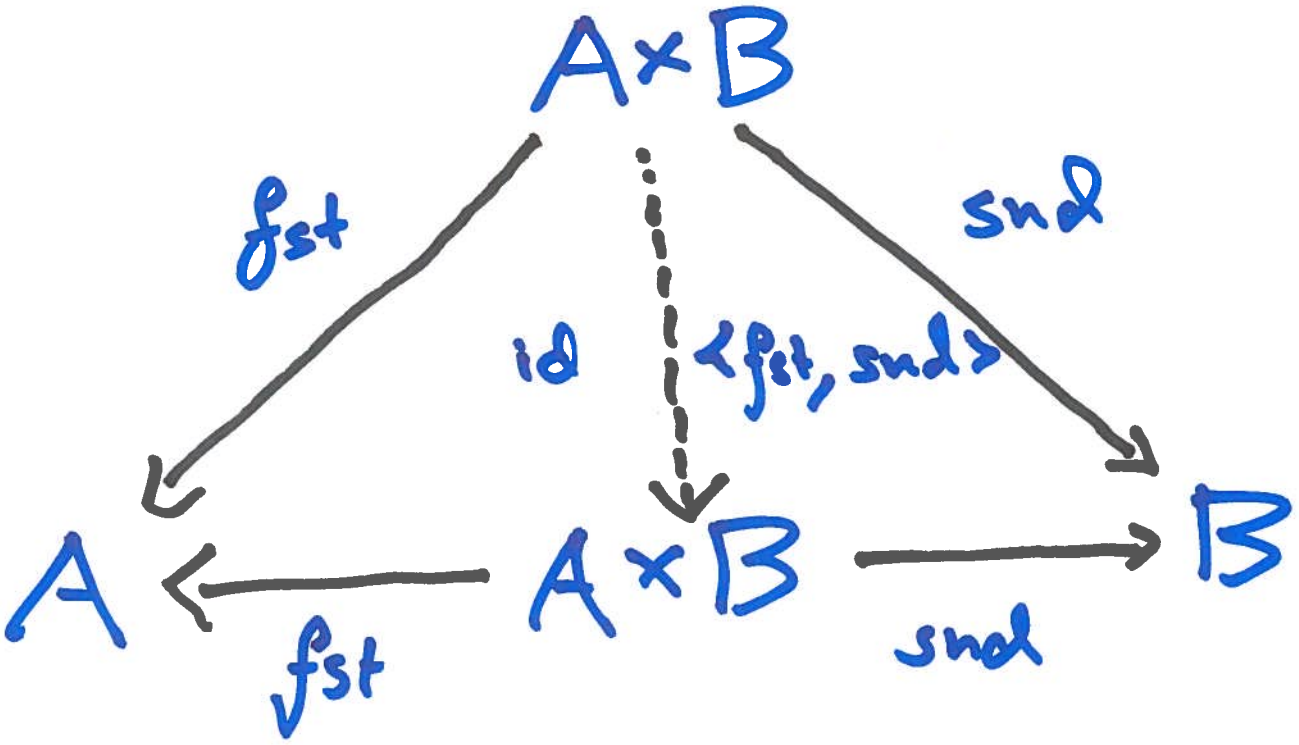
$(f ; g) ; h = f ; (g ; h)$

PRODUCT



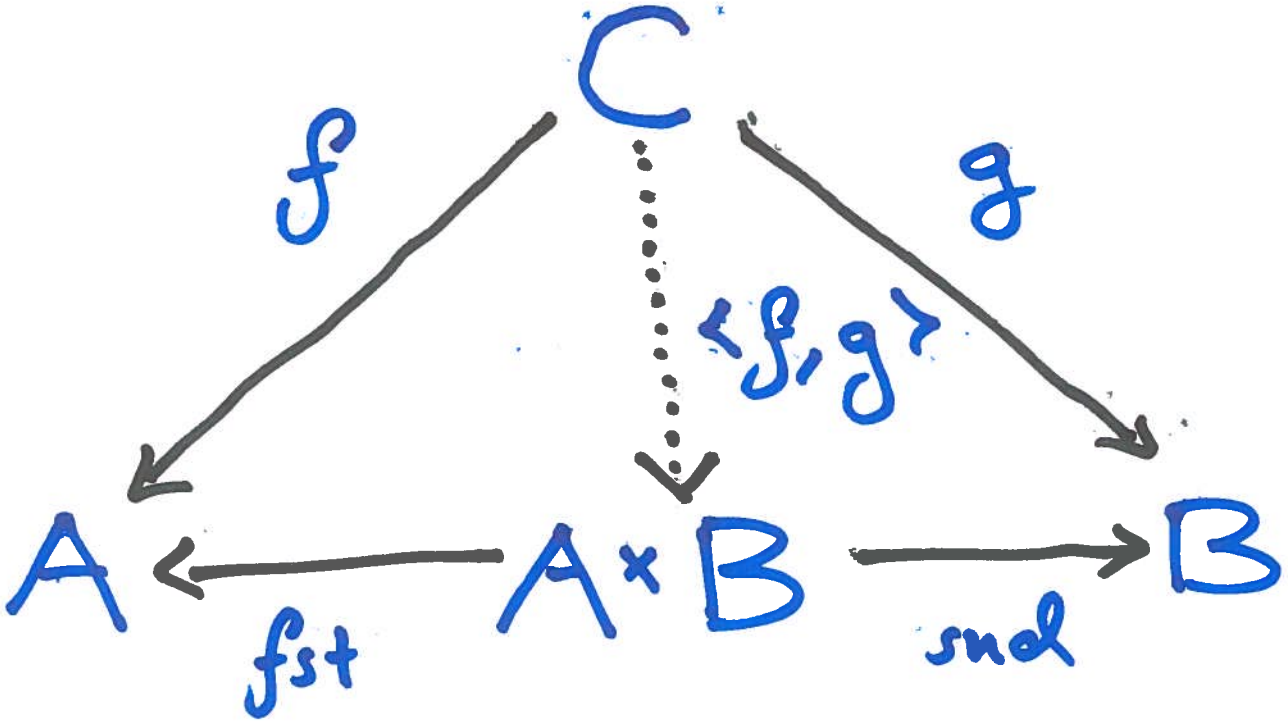
$$\langle -,- \rangle : \mathcal{C}(C,A) \times \mathcal{C}(C,B) \cong \mathcal{C}(C, A \times B)$$

PRODUCT



$$\langle fst, snd \rangle = id$$

PRODUCT



$$\langle f,g\rangle ; fst = f$$
$$\langle f,g\rangle ; snd = g$$
$$h ; fst = f \;\&\; h ; snd = g \Rightarrow h = \langle f,g\rangle$$

PRODUCT

$$\times\text{-I}\ \frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash (M, N) : A \times B}$$

$$\frac{\Gamma \xrightarrow{f} A \qquad \Gamma \xrightarrow{g} B}{\Gamma \xrightarrow{\langle f, g \rangle} A \times B}$$

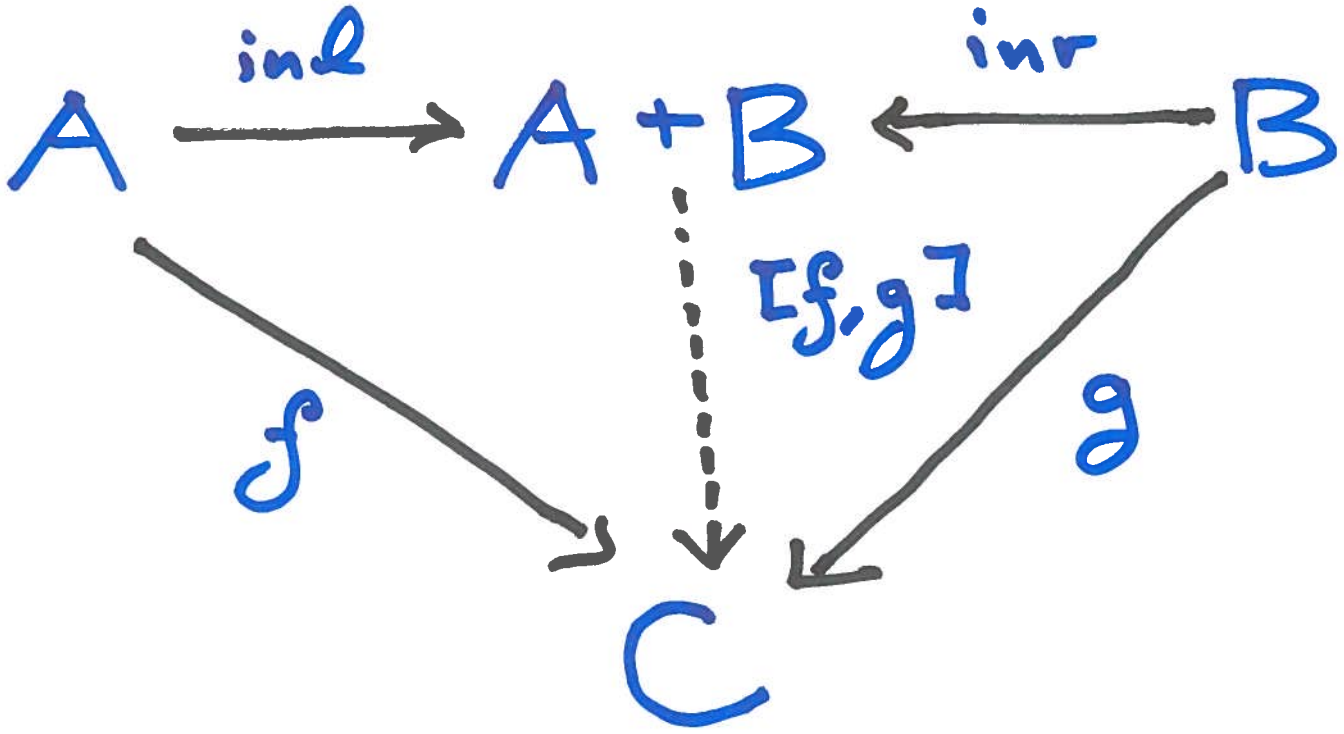
$$\times\text{-E}\ \frac{\Gamma \vdash L : A \times B}{\Gamma \vdash \mathit{fst}\ L : A}$$

$$\frac{\Gamma \xrightarrow{h} A \times B}{\Gamma \xrightarrow{h} A \times B \xrightarrow{\mathit{fst}} A}$$

$$\times\text{-E}\ \frac{\Gamma \vdash L : A \times B}{\Gamma \vdash \mathit{snd}\ L : B}$$

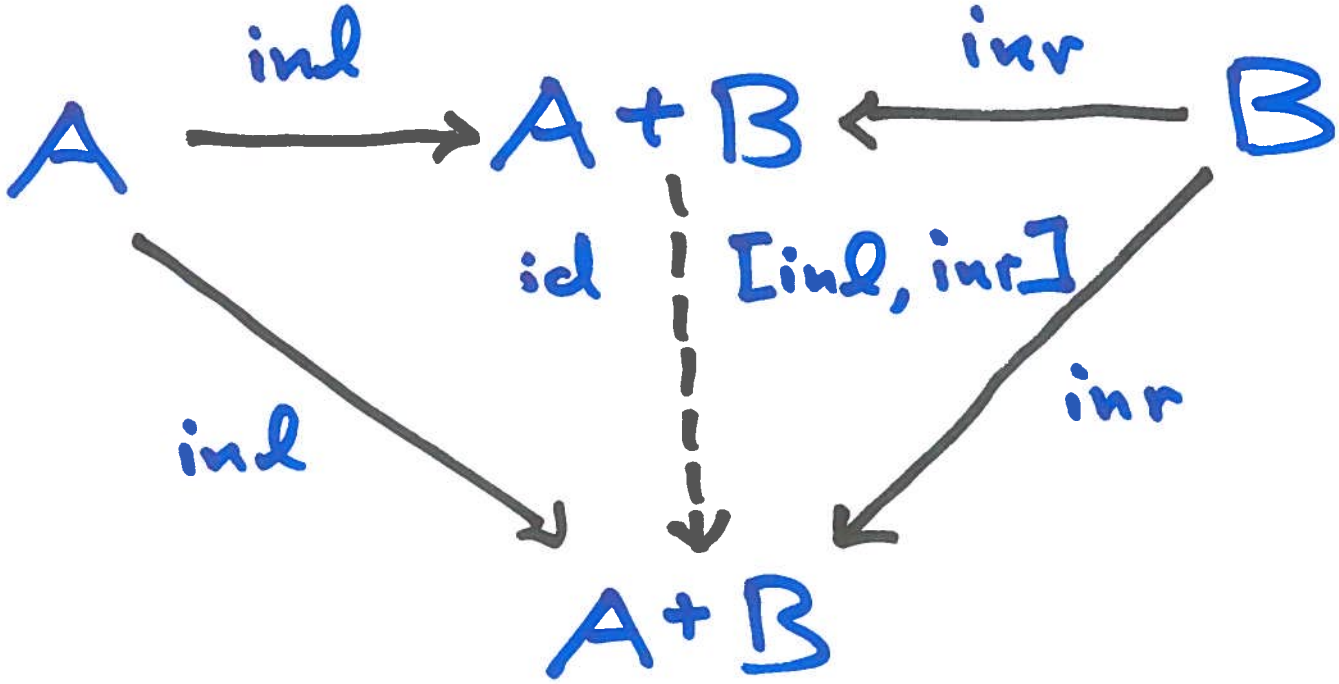
$$\frac{\Gamma \xrightarrow{h} A \times B}{\Gamma \xrightarrow{h} A \times B \xrightarrow{\mathit{snd}} B}$$

SUM



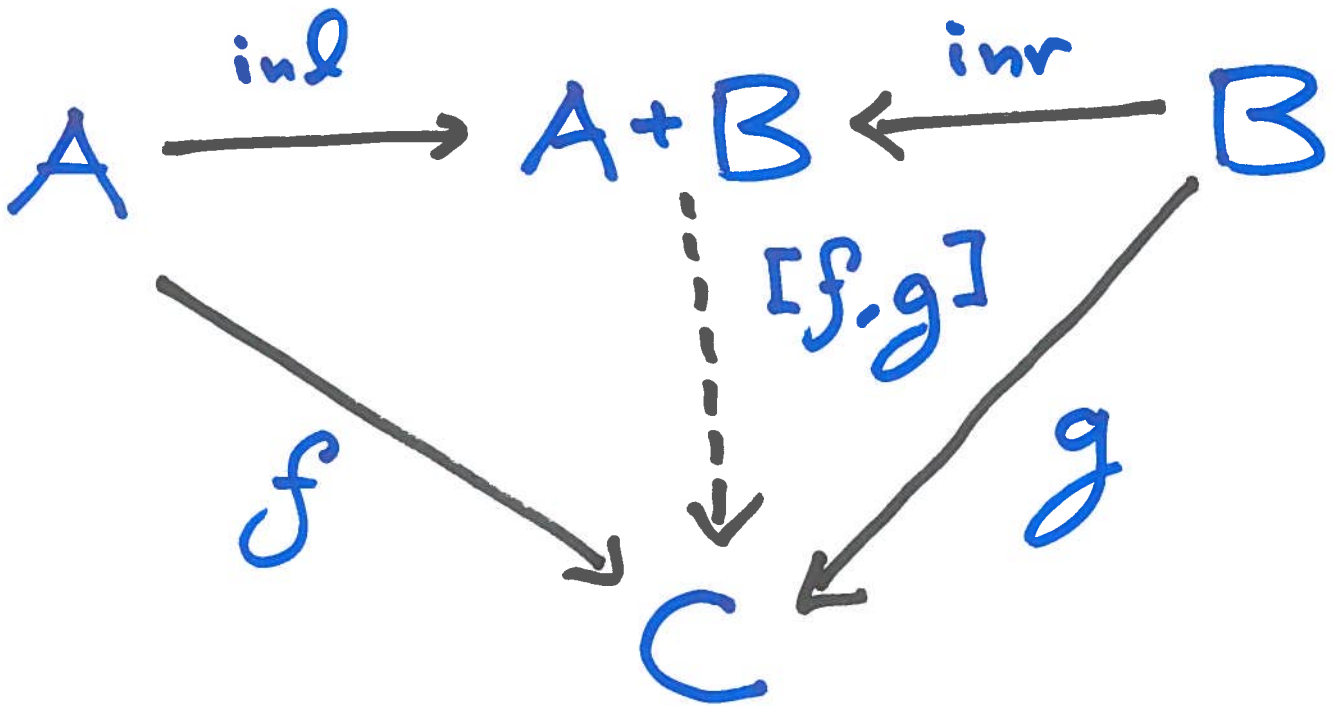
$$inl ; [f, g] = f$$
$$inr ; [f, g] = g$$
$$inl ; h = f \;\&\; inr ; h = g \Rightarrow h = [f, g]$$

SUM



$$[\mathrm{inl}, \mathrm{inr}] = \mathrm{id}$$

SUM



$$[-,-] : \mathcal{C}(A,C) \times \mathcal{C}(B,C) \cong \mathcal{C}(A+B,C)$$

SUM

$$+I\ \frac{\Gamma \vdash M : A}{\Gamma \vdash \mathsf{inl}\ M : A+B} \qquad \frac{\Gamma \xrightarrow{f} A}{\Gamma \xrightarrow{f} A \xrightarrow{\mathsf{inl}} A+B}$$

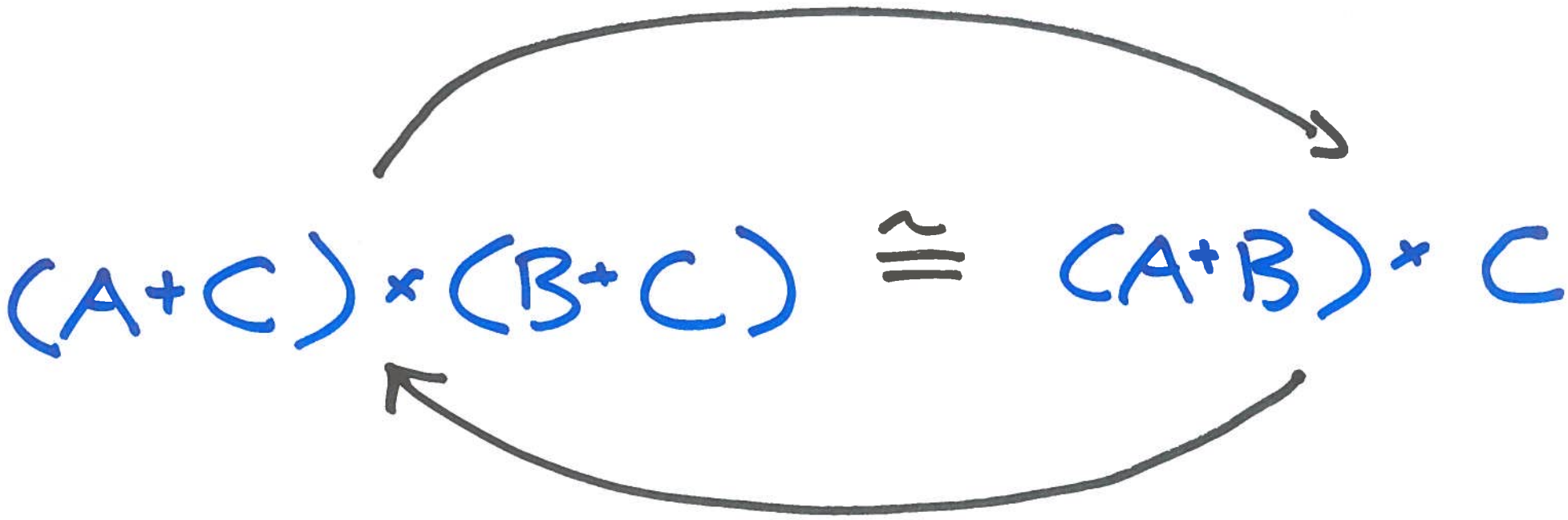
$$+I\ \frac{\Gamma \vdash N : B}{\Gamma \vdash \mathsf{inr}\ N : A+B} \qquad \frac{\Gamma \xrightarrow{g} B}{\Gamma \xrightarrow{g} B \xrightarrow{\mathsf{inr}} A+B}$$

$$+E\ \frac{\begin{array}{l}\Gamma \vdash L : A+B \\ \Gamma, x:A \vdash P : C \\ \Gamma, y:B \vdash Q : C\end{array}}{\Gamma \vdash \left(\begin{array}{l}\mathsf{case}\ L\ \mathsf{of} \\ \quad \mathsf{inl}\ x \to P \\ \quad \mathsf{inr}\ y \to Q\end{array}\right) : C} \qquad \frac{\begin{array}{l}\Gamma \xrightarrow{h} A+B \\ \Gamma \times A \xrightarrow{f} C \\ \Gamma \times B \xrightarrow{g} C\end{array}}{\Gamma \xrightarrow{\langle id, h\rangle} \Gamma \times (A+B) \xrightarrow{[f,g]} C}$$

$$(\Gamma \times A) + (\Gamma \times B)$$

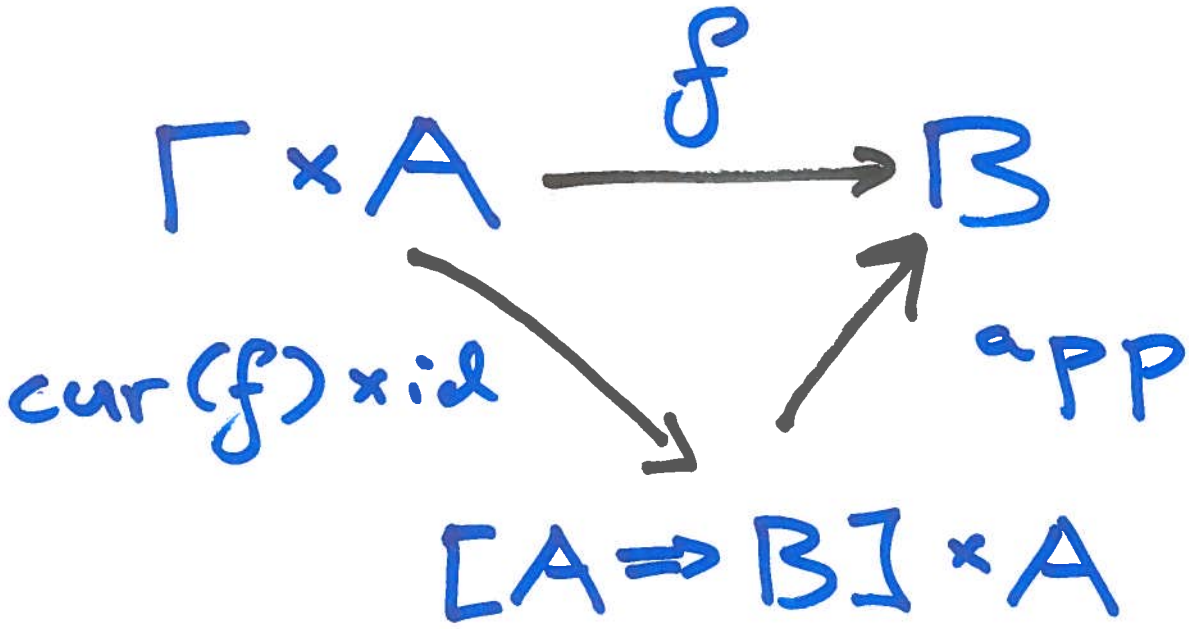
DISTRIBUTIVE

$\langle [fst; inl, fst; inr], [snd, snd] \rangle$



$\langle fst; [cur(inl), cur(inr)], snd \rangle; app$

EXPONENTIAL



$$\mathcal{C}(\Gamma \times A, B) \cong \mathcal{C}(\Gamma, [A \Rightarrow B])$$

VARIABLES

$$\frac{}{x:A \vdash x:A}$$

$$\frac{\Gamma \vdash N:B}{\Gamma, x:A \vdash N:B}$$

$$\frac{\Gamma \vdash N:B}{x:A, \Gamma \vdash N:B}$$

$$\frac{}{A \xrightarrow{id} A}$$

$$\frac{\Gamma \xrightarrow{g} N}{\Gamma \times A \xrightarrow{fst} \Gamma \xrightarrow{g} N}$$

$$\frac{\Gamma \xrightarrow{g} N}{A \times \Gamma \xrightarrow{snd} \Gamma \xrightarrow{g} N}$$

EXPONENTIAL

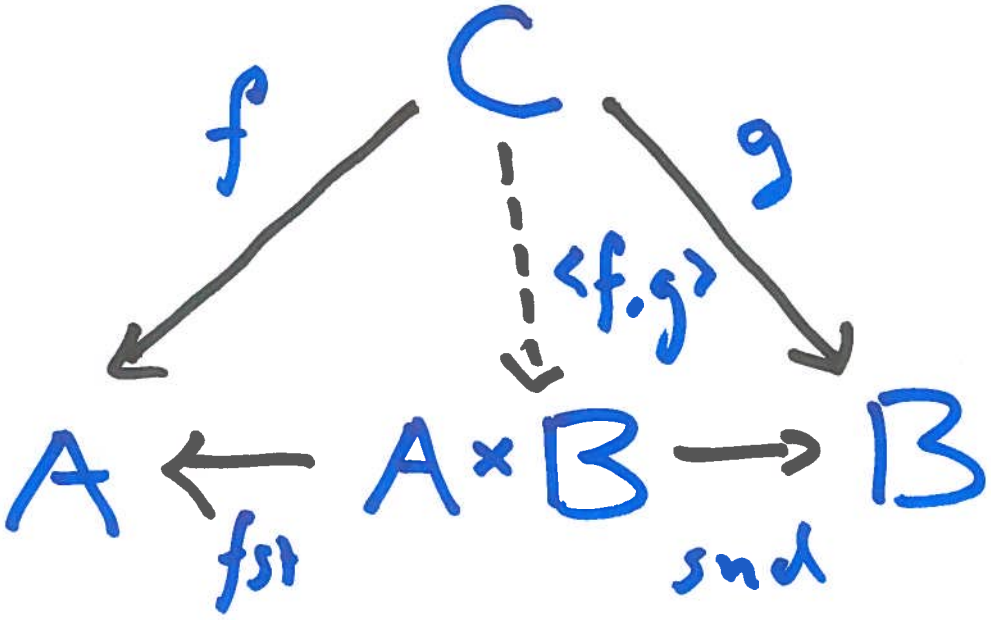
$$\to I\ \frac{\Gamma, x:A \vdash N : B}{\Gamma \vdash (\lambda x{:}A.N) : [A \Rightarrow B]}$$

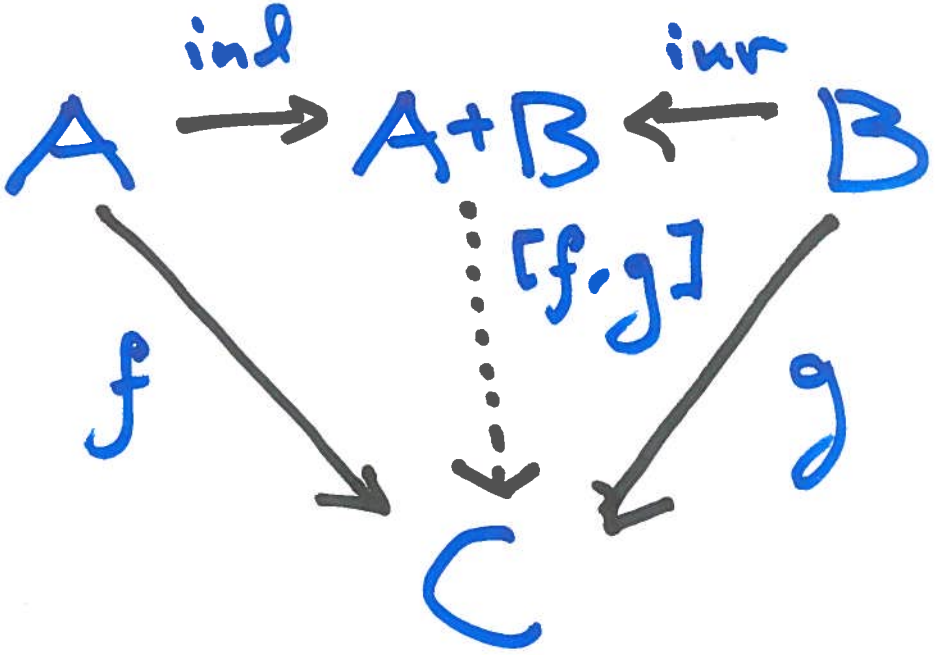
$$\frac{\Gamma \times A \xrightarrow{f} B}{\Gamma \xrightarrow{cur(f)} [A \Rightarrow B]}$$

$$\to E\ \frac{\Gamma \vdash L : [A \Rightarrow B] \quad \Gamma \vdash M : A}{\Gamma \vdash L\ M : B}$$

$$\frac{\Gamma \xrightarrow{f} [A \Rightarrow B] \quad \Gamma \xrightarrow{g} A}{\Gamma \xrightarrow{\langle f, g \rangle} [A \Rightarrow B] \times A \xrightarrow{app} B}$$

PRODUCTS & SUMS





ISOMORPHISMS

$$\mathcal{C}(C,A) \times \mathcal{C}(C,B) \cong \mathcal{C}(C, A * B)$$

$$\mathcal{C}(A,C) \times \mathcal{C}(B,C) \cong \mathcal{C}(A + B, C)$$

$$\mathcal{C}(C * A, B) \cong \mathcal{C}(C, [A \Rightarrow B])$$

HIGH SCHOOL

$$A^C \times B^C = (A \times B)^C$$

$$C^A \times C^B = C^{(A+B)}$$

$$B^{(C \times A)} = (B^A)^C$$