Machine Learning on Graphs - Homework 1

Mohammad Bahrami - 9724133

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1 Question 1

1.1 A

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

1.2 B

In we solve this equation: $(A - \lambda I).v = 0$ we put $det(A - \lambda I) = 0$ which results in the equation:

$$-\lambda^7 + 10\lambda^5 + 4\lambda^4 - 21\lambda^3 - 6\lambda^2 + 11\lambda = 0$$

by solving the above equation, we get 7 eigen values which are:

$$\lambda_1 \approx 2.982, \lambda_2 \approx -2.227, \lambda_3 \approx -1.678, \lambda_4 \approx -1.077, \lambda_5 \approx 1.290, \lambda_6 \approx 0.709, \lambda_7 \approx 0$$

now to find the eigen vector of each value, we put the value in the $Av = \lambda v$ equation and solve for vector v, which results in eigen vectors:

$$\begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{matrix} \\ \begin{bmatrix} -0.385 & -0.391 & -0.256 & 0.553 & 0.429 & 0.376 & 0 \\ -0.352 & 0.566 & 0.004 & -0.095 & -0.230 & 0.633 & -0.301 \\ -0.249 & -0.382 & 0.325 & -0.138 & -0.461 & 0.298 & 0.603 \\ -0.296 & 0.070 & -0.249 & -0.687 & 0.529 & 0.006 & 0.301 \\ -0.417 & -0.487 & -0.076 & -0.312 & -0.264 & -0.225 & -0.603 \\ -0.391 & 0.285 & -0.550 & 0.244 & -0.365 & -0.421 & 0.301 \\ -0.499 & 0.235 & 0.675 & 0.187 & 0.254 & -0.372 & 0 \end{bmatrix} \end{array}$$

1.3 C

The right eigen value for eigen node centrality is $\lambda_1 = 2.982$ because it is the largest eigen value and it will make the algorithm converge.

Yes, it is the only value that makes the algorithm converge. Uniqueness Proof:

$$1.\quad \lambda < \lambda_1 : \lim_{t\to\infty} (\frac{\lambda_1}{\lambda})^{t+1} = \infty$$

$$2.\quad \lambda > \lambda_1 : \lim_{t\to\infty} (\frac{\lambda_1}{\lambda})^{t+1} = 0$$

We see that if the λ is different, the main summation equation of the algorithm will diverge to infinity or converge to 0 which is not we want.

1.4 D

We put $\lambda = \lambda_1 = 2.982$

Stage 1:

$$X_1 = \frac{1}{\lambda} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.006 \\ 1.006 \\ 0.670 \\ 0.670 \\ 1.006 \\ 1.006 \\ 1.341 \end{bmatrix}$$

Stage 2:

$$X_2 = \frac{1}{\lambda} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1.006 \\ 1.006 \\ 0.670 \\ 0.670 \\ 1.006 \\ 1.006 \\ 1.341 \end{bmatrix} = \begin{bmatrix} 1.012 \\ 0.899 \\ 0.674 \\ 0.787 \\ 1.124 \\ 1.012 \\ 1.237 \end{bmatrix}$$

Stage 3:

$$X_3 = \frac{1}{\lambda} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1.012 \\ 0.899 \\ 0.674 \\ 0.787 \\ 1.124 \\ 1.012 \\ 1.237 \end{bmatrix} = \begin{bmatrix} 0.980 \\ 0.942 \\ 0.641 \\ 0.754 \\ 1.055 \\ 1.018 \\ 1.319 \end{bmatrix}$$

Yes, as we see the numbers are getting smaller and the vector is converging. If we repeat this process many times, X will converge to a coefficient of $\lambda_1 v_1$.

2 Question 2

The Answer to this Question is in: `./HW1-MohammadBahrami-9724133.ipynb > Question 2`

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3 Question 3

We want to choose 3 nodes to form a triangle with. Let $q = \binom{n}{3}$ be the ways we can select these 3 nodes. Now consider $X_1, X_2, \ldots, X_q$ as a set of random variables where X_i is set to 1 if and only if $i'th$ set defines a triangle. We need to calculate $\mathbb{E}(\sum_{i=1}^{q} X_i)$. By linearity of Expectation:

$$\mathbb{E}(\sum_{i=1}^{q} X_i) = \sum_{i=1}^{q} \mathbb{E}(X_i)$$

and because all X_is are identically distributed:

$$\sum_{i=1}^{q} \mathbb{E}(X_i) = q\mathbb{E}(X_1) = qp^3 = \binom{n}{3} p^3$$

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4 Question 4

The Answer to this Question is in: `./HW1-MohammadBahrami-9724133.ipynb > Question 4`

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5 Question 5

The Answer to this Question is in: `./HW1-MohammadBahrami-9724133.ipynb > Question 5`
