Reinforcement Learning KU (DAT.C307UF), WS24
Assignment 1
Monte Carlo Simulation, Bayesian Networks

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Teaching Assistant:	Erwin Pfeiler
Points to achieve:	15 pts
Deadline:	08.11.2024 23:59
Hand-in procedure:	This is a **solo assignment**. No teams allowed. Submit **your report (PDF) and Jupyter Notebook (.ipynb)** to TeachCenter. Upload just these 2 files. Do not upload a folder. Do not zip them. You do not have to add the cover letter since there are no teams allowed.
Plagiarism:	If detected, 0 points for all parties involved. If this happens twice, we will grade the group with "Ungültig aufgrund von Täuschung"

General Remarks

- All results (i.e., plots, results of computations) should be included in your PDF report – the Jupyter Notebook file should only serve as a way to reproduce your results.
- For all pen and paper exercies, report all intermediate steps – only presenting the final solution is insufficient.

Task 1 – Monte Carlo Simulation [7.5 Points]

1. A random variable X with state space $\mathcal{X} = \{0, 1, 2, 3, 4\}$ has the following probability mass function[1]

$$p(X = k) = \begin{cases} M \cdot \frac{\lambda^k e^{-\lambda}}{k!} & \text{for } k = 0, 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

where $k!$ is the *factorial* of k and M is a normalizing constant. Let $\lambda = 4$.

   (a) Find the normalizing constant M such that $\sum_{k \in \mathcal{X}} p(X = k) = 1$.

   (b) Find $\mathbb{E}_X[X]$.

   (c) Use Monte Carlo simulation to approximate the expected value of X and show a convergence plot with the y-axis representing the estimated expected value and the x-axis the number of simulations utilized to calculate the estimated expected value. Use $100, 200, 300, \ldots, 10000$ simulations and obtain an estimate for each value. Draw a horizontal line representing the exact expectation. Include this plot in your report.

[1] Note that this PMF is essentially a *truncated Poisson* distribution.

2. The probability density function of a continuous random variable X is given by:

$$p(x) = \begin{cases} \frac{1}{2}x & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Analytically compute $\mathbb{E}_X[X]$.

   (b) The corresponding *cumulative distribution function* (CDF) is defined as

$$F(x) = \int_{-\infty}^{x} p(z)\; dz.$$

   Write down $F(x)$ as a simple function of x that does not involve an integral (i.e., solve the definite integral).

   (c) We can sample from $p(x)$ using the *inverse transform sampling* trick: First, sample $u \sim \text{Unif}([0,1])$ and then, compute $x = F^{-1}(u)$ where F^{-1} denotes the inverse of F. The result x is a proper sample from p. Write down $F^{-1}(u)$ and implement this sampling procedure.

   (d) Use this sampling procedure to estimate $\mathbb{E}[X]$ via Monte Carlo for $100, 200, 300, \ldots, 10000$ simulations. Similar to Task 1.1., show a convergence plot where you draw a horizontal line at the true expectation. Include this plot in your report.

3. In the previous tasks we found the expected value of random variables by solving the corresponding integral (or sum). It is possible to work the other way around, that is, solve an integral by expressing it as an expectation and then using simulation to get the approximated value. Consider the integral

$$\int_0^{2\pi} \int_0^{2\pi} \sin(\sqrt{x^2 + y^2} + x + y)\; dxdy$$

   This integral is analytically challenging. While it is not an expectation, it can be formulated as the product of a constant, K, times an expectation using a suitable choice of joint density $p(x, y)$.

$$\int_0^{2\pi} \int_0^{2\pi} \sin(\sqrt{x^2 + y^2} + x + y)\; dxdy = K \cdot \mathbb{E}_{(x,y)\sim p(x,y)}[\sin(\sqrt{x^2 + y^2} + x + y)]$$

   (a) Find $p(x, y)$ and K such that the above identity is true.

   (b) Approximate this integral using Monte Carlo simulation (use at least $5,000,000$ samples). Report the final value in your report, along with the number of samples you used.

   (c) Use `scipy.integrate.dblquad`[2] as a different way to compute this integral numerically and compare the output to the Monte Carlo estimate.

[2] `https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.dblquad.html`

Task 2 – Bayesian Networks [7.5 Points]

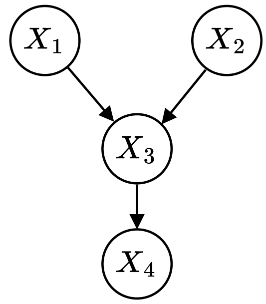

Figure 1: "Extended collider:" A collider $X_1 \to X_3 \leftarrow X_2$ with a descendant X_4.

1. Consider the Bayesian network structure given in Figure 1 (let's call it an "extended collider") and assume that random variables X_1, X_2, X_3 and X_4 are binary, i.e. taking values in $\{0, 1\}$. Provide conditional probability distributions (CPDs) such that in the resulting Bayesian network we have $X_1 \not\perp\!\!\!\perp X_2 \mid X_4$. Demonstrate this by showing that the conditional $p(X_1, X_2 \mid X_4)$ does *not* factorize as $p(X_1, X_2 \mid X_4) = p(X_1 \mid X_4)\, p(X_2 \mid X_4)$ for your choice of CPDs.

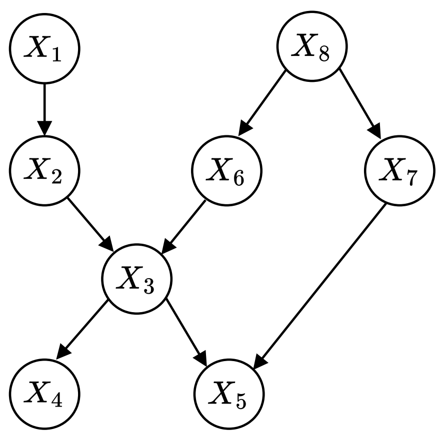

Figure 2: A Bayesian Network over random variables $X_1, \ldots, X_8$.

2. Consider the Bayesian network structure in Figure 2. For each of the following statements, state if they are true for all probabilistic models that follow this DAG. For each answer, give a brief explanation why you think that the statement holds in general (or does not hold in general).
   (a) $X_2 \perp\!\!\!\perp X_8 \mid X_5$
   (b) $X_6 \perp\!\!\!\perp X_7 \mid X_8$
   (c) $X_1 \perp\!\!\!\perp X_8 \mid X_6$
   (d) Can we always write $p(X_6 \mid X_7, X_8) = p(X_6 \mid X_8)$?
   (e) Can we always write $p(X_1, X_8 \mid X_6) = p(X_1 \mid X_6) p(X_8 \mid X_6)$?
   (f) $X_1 \perp\!\!\!\perp X_8 \mid X_6, X_5$

3. Again, consider the Bayesian network structure in Figure 2. For each statement, find a *minimal* set of random variables $\mathcal{X} \subseteq \{X_1, \ldots, X_8\}$ such that the statement is true in all probabilistic models that follow this DAG.

   (a) $X_7 \not\perp\!\!\!\perp X_1 \,|\, \mathcal{X}$

   (b) $X_7 \perp\!\!\!\perp X_6 \,|\, X_5, \mathcal{X}$

   (c) $X_1 \perp\!\!\!\perp X_8 \,|\, X_5, \mathcal{X}$