Real Analysis + Differentiation Prelude

Lecture Notes 1:

I assume the natural numbers, with properties we are aware of.

$\mathbb{N} = \{1, 2, 3, \dots\}$

Consider $S \subset \mathbb{N}$ such that:

1. $1 \in S$
2. if $k \in S$ then $k+1 \in S$.

what is S then? S must then be $\mathbb{N}$. ... (1)

So, we arrive at the principle of induction, and can prove by induction.

$P(x)$ be a statements, $x \in \mathbb{N}$. To show $P(x)$ holds $\forall x \in \mathbb{N}$:

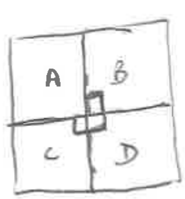
1. Show $P(1)$ is true (base case)
2. $\underbrace{\text{Assume } P(k) \text{ is true}}_{\text{Induction Hypothesis}}$. Show $P(k) \Rightarrow P(k+1)$.

Then, $k \in \mathbb{N}$ (From (1)).

Example 1: Every $2^n \times 2^n$ board can be tiled by  if one tile is removed.

Proof by induction:

(1) For base case: 2×2 . Remove any tile. By symmetry we get the L shaped tile 

(2) 

Now consider $2^{n+1} \times 2^{n+1}$: Any one of A, B, C, D can be tiled as they are $2^n \times 2^n$. The remaining 3 $2^n \times 2^n$ blocks can have a tile removed that as show in the figure. This is L-shaped and the rest can be tiled as well.

Counting a set A means putting its elements in one-one correspondence (bijection correspondence) with some subset S of N.

$$f : S \to A.$$

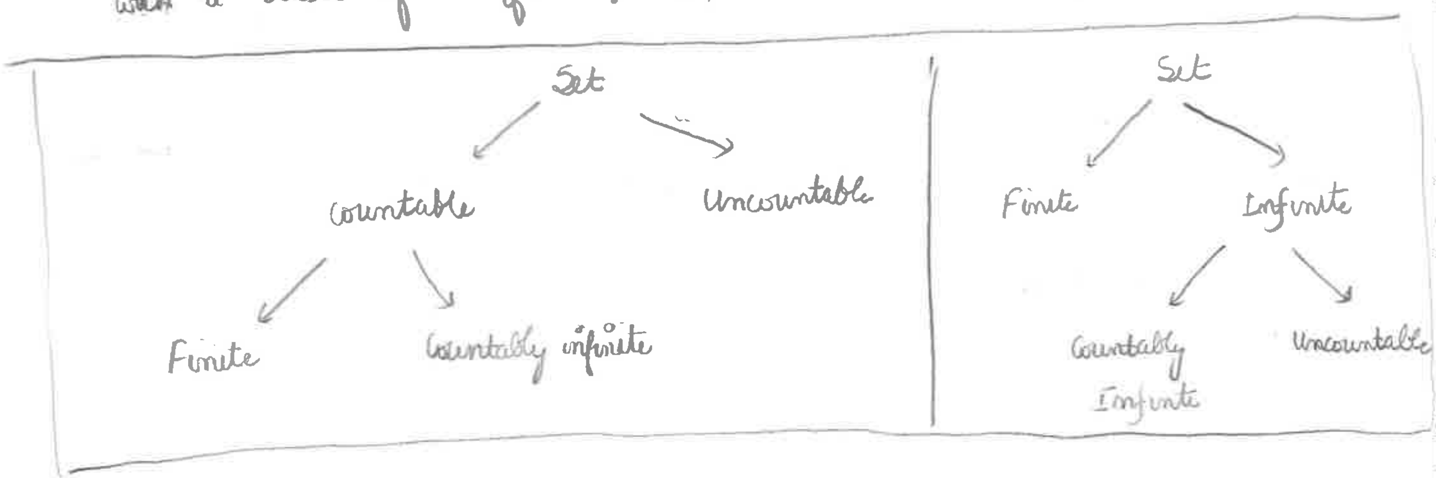
① A set A is finite if $A \sim J_n$, $J_n = \{1, 2, \dots n\}$ for some n. Else, infinite

② A set A is countably infinite if $A \sim N$.

A sequence $x_1, x_2, \dots$ is countably infinite.

Any enumerable set is countable.

A key point here is that a countably infinite can be put in 1-1 correspondence with a subset of itself. Eg: $\{1, 2, \dots\} \sim \{2, 3, \dots\}$ $f(n) = n+1$



Are there sets that are uncountable?

Rational Numbers : $B = \{ \frac{m}{n} \mid \frac{m}{n} \in \mathbb{Z}$ (integers), $n \neq 0 \}$

But $2/1$ can also be written as $4/2, \dots$ and so on. We want to consider all these elements of B as equivalent.

We think of B as a set of ordered pairs $\langle m, n \rangle$, $(m, n \in \mathbb{Z}, n \neq 0)$

We identify $\langle m, n \rangle$ and $\langle p, q \rangle$ together if $m \cdot q = p \cdot n$

Now, consider this relation, what properties does it have??

1. $\langle m,n\rangle \; R \; \langle m,n\rangle$ (Reflexive)
2. $\langle m,n\rangle \; R \; \langle p,q\rangle \iff \langle p,q\rangle \; R \; \langle m,n\rangle$. (Symmetric)
3. $\langle m,n\rangle \; R \; \langle p,q\rangle$ and $\langle p,q\rangle \; R \langle u,v\rangle \Rightarrow \langle m,n\rangle \; R \; \langle u,v\rangle$ (Transitive)

If $mq = pn$ & $pv = qu$. then $mv = nu$

$p = \frac{mq}{n} \Rightarrow \frac{mq}{n} v = qu \Rightarrow mv = nu$.

A relation following these 3 properties is called an Equivalence relation.

What does an equivalence relation do?

$S = \{1, 2, 3\}$ with relation $R = \{(1,1), (2,2), (2,3), (3,3), (3,2)\}$.

Set of element 1 is related to $\{1\}$

Set of elements 2 is related to $\{2, 3\}$

Set of element 3 is related to $\{2, 3\}$

It gives rise to a natural grouping of elements related to each other.

It partitions a set into a set of equivalence classes, each containing element related to one another.

Equivalente classes of S under R: $\{1\}$ $\{2,3\}$

Aside:

A slightly different set of conditions.

1. $a \; R \; a$
2. $a \; R \; b$ and $b \; R \; a \Rightarrow a = b$
3. $a \; R \; b$ and $b \; R \; c \Rightarrow a \; R \; c$.

Gives rise ~~to~~ a partial order on the sets elements.

The set of rational numbers $\mathbb{Q}$ are* the equivalence class of ordered pairs $\langle m,n\rangle$ $m, n \in \mathbb{Z}$; $n \neq 0$ under the equivalence relation $\langle m,n\rangle \sim \langle p,q\rangle$ if $mq = pn$.

* when I say are, I mean can be put in 1-1 correspondence.

We say that the rationals are the quotient set of the ordered pairs of natural numbers under the equivalence relation.

Aside: This concept of equivalence is very important even in geometry for creating new objects. For example, consider a straight line and identify end points.

Relation: $\forall p \in$ line $\quad pRp$.

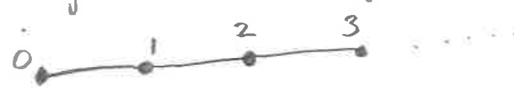
For the end points p_1 and p_2 $\quad p_1 R p_2$ and $p_2 R p_1$.

Equivalence class: circle (topologically)

Real Numbers: $3x = 5$ has no solution in the integers. So, we get $\mathbb{Q}$.

$x^2 = 2$ has no solution in $\mathbb{Q}$. (Prove for exercise).

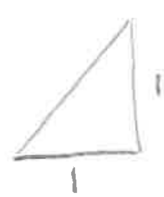
Number line: Take a measuring stick. This is of unit 1. Draw a line and extend this:



This gives us a way of mapping integers and rationals to a line.

Thus, we can "measure" the length of an object, compared to the stick.

Measure the hypotenuse of:



It will not coincide with any of the markings. Thus, we are missing some markings.

$\mathbb{Q}$ actually does not have the least upper bound property. (Not every bounded set has a supremum (defined shortly)).

Upper bound: x is called an upper bound of set A that is ordered, if $x > a_\alpha \ \forall \ a_\alpha \in A$.

Least upper bound: x is called least upper bount of set A that is ordered if

x is an upper bound and if $y < x$ then y is not an upper bound of A.
This is also called the supremum.

Examples

① $A = \{1, 2, 3, 4\}$ $\quad$ $\sup(A) = ?$

② $A = \mathbb{Q}^-$ $\quad$ $\sup(A) = ?$ $\quad$ why?

$\mathbb{Q}$. so, if $-a$ is $\sup(\mathbb{Q}^-)$ then $\frac{-a}{2}$ is a rational $> -a$

$\therefore \frac{-a}{2} \in \mathbb{Q}^-$. But, then $-a$ is not an upper bound at all.

③ $A = \mathbb{Q}$. $\quad$ $\sup(A) = \infty$ (unbounded symbol).

④ $A = \{x \mid x^2 < 2\}$ $\quad$ $\sup(A) = ?$

Not defined in $\mathbb{Q}$. Similar to Example ②, we can construct a rational $y = f(x)$ such that $y^2 < 2$ but $y > x$ for any x.

We construct a larger set $\mathbb{R}$ that contains $\mathbb{Q}$, and that has least upper bound property. (If there is an upper bound x for A, then there is a least upper bound y of A).

A Dedekind cut α is a subset of $\mathbb{Q}$ s.t.

① If $p \in \alpha$, $q \in \mathbb{Q}$ and $q < p$ then $q \in \alpha$.

② If $p \in \alpha$ then $p < r$ for some $r \in \alpha$.

$-x < p < a$

Examples

① $\mathbb{Q}^-$ $\Rightarrow$ cut or not? Cut.

② $\{x \mid x \leq 2\}$ $\Rightarrow$ cut or not? Not cut.

Define $\mathbb{R}$: $\{\alpha : \alpha \text{ is a cut}\}$

We ~~associate~~ identify a real number with the cut that consists of all rationals less than the number.

But right now, we just have a set. Now, we endow structure:

① Order on the cuts: $\alpha, \beta \in$ cuts $\quad \alpha < \beta$ if $\alpha \subset \beta$.

↳ Satisfies properties of the ordering relation

② Addition on the cuts:

$\alpha + \beta = \{ r + s \mid r \in \alpha \text{ and } s \in \beta \}$; $0^* = 0_*$ (addition identity)

↳ Satisfies the addition axioms of a group

(Prove for exercise)

Quick Review:

Monoid: Set together with '+' with:

① Associativity: $(a+b)+c = a+(b+c)$

② Identity: $a + 0 = a = 0 + a$

Group: Monoid with inverse element:

① Inverse: $a + (-a) = 0 = (-a) + a$.

Commutation Group: Group with commutativity: Also called Abelian Group

① Commutation: $(a+b) = (b+a)$

Ring: Commutative addition group with a monoid • operator with • distributing over addition

Field: Ring with non-zero elements forming an abelian group under multiplication.

Thus, R is an ordered field.

<u>R contains $\mathbb{Q}$ as a subfield.</u> Associate to $q \in \mathbb{Q}$ the cut $q^* = \{ r \in \mathbb{Q} : r < q \}$

(1) $f: \mathbb{Q} \longrightarrow \mathbb{R}$ is injection ;

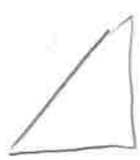
(2) And it preserves structure, by which we mean :

$p, q \in \mathbb{Q}$ and $p < q \Rightarrow f(p) < f(q)$

(defined as rational order) (defined as set containment)

So, what have we done? We started with a set of elements (rationals), and defined a set of objects that have the properties of real numbers.

Length :



1 1

$=$ cut $\alpha = \{ q : q^2 < 2 \text{ or } q < 0, q \in \mathbb{Q} \}$

is a cut.

α^2 creeps up on 2. α^2 is associated with rational 2.

Also, R has least upper bound property. (It is the only ordered field with this property).

So, ~~then~~ are no "gaps" in the real line now

Real number x such that $x^5 = 20$?

corresponding to cut $\alpha = \{ q : q^5 < 20, q \in \mathbb{Q} \}$

Also, R is uncountable. Assume it is countable. Enumerate them as set S.

$s_1 = 0.1234\ldots$

$s_2 = 0.234\ldots$

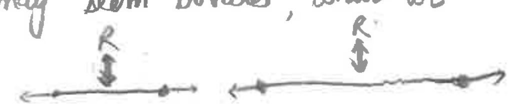
$\vdots$

Create new number x as follows: $x[i] = 7$ if $s_i[i] \neq 7$ else $x[i] = 9$.

$x[i]$ differs from every element of S, and, thus, $x \notin S$. But,

$x \in R$. $\therefore R \neq S$.

Till now we have sets and endow them with additional structure. We mention that will take a measuring stick and use it to identify integers, rationals and reals on a line. While this may seem obvious, what we are subtly doing is creating a metric space



A set X is called a metric space if $\exists\, d : X \times X \longrightarrow R$ such that

$\forall\, p, q \in R$:

1. $d(p,q) \geq 0$, $d(p,q) = 0 \iff p = q$
2. $d(p,q) = d(q,p)$
3. $d(p,q) \leq d(p,r) + d(r,q)$

Examples of d :

1. $[R,\ d(x,y) = |x-y|]$ — Real line
2. $\left[R,\ d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}\right]$ — Cloud of points

The metric induces "Geometry" in a set, converting it to a space where distances can be measured.

Now that we have a set endowed with a metric, we can talk of concepts of open and closed balls

Open Ball around x $= \{y \mid d(x,y) < r\}$. (also called neighborhood)

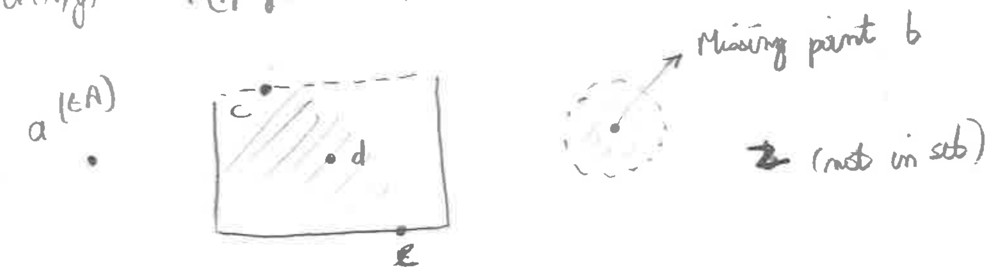
Closed ball around x : $\{y \mid d(x,y) \leq r\}$

We may also define the important concept of a limit point. A point x is called a limit point of a set A if <u>every</u> open ball around x contains a point of A different from x.

<u>Examples</u>:

① $G = \left\{ \frac{1}{n} : n \in N \right\}$. What is a limit point? Use $(p-\varepsilon, p+\varepsilon)$ as open balls. 0 is a limit point.

② $\left(R^2, d(x,y) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}\right)$



Which points are limit points?

e, d, c (don't requires x to be in set A), b (same as prev)

a is not a limit point, as, though it contains a point of A but the point is itself.

Exercise <u>Contraposition</u>: A point x is not a limit point of A if there exists an open ball around x that does not contain any point of A other than x itself.

a is in the set but not a limit point $\Rightarrow$ isolated points.

③ What are isolated points of $G = \{\frac{1}{n} : n \in \mathbb{N}\}$? (All of them. 0 is not in G)

d $\Rightarrow$ interior point. A point is an interior point of the set A if there exists an open ball around it completely contained in A (this includes the fact that the point must itself be in the set)

④ What are limit points of $\mathbb{R}$ in the discrete metric?

No limit points. Take a ball of radius $\frac{1}{2}$.

⑤ What are the interior points of $\mathbb{R}$ in discrete metric? All points are interior points.

⑥ What are the limit points of $\mathbb{Q}$ (seen embedded in $\mathbb{R}$)? All points.

All balls around $\mathbb{Q}$ has another rational in it.

⑨ If p is a limit point of E, does every neighborhood contain infinitely many points of E?

Assume $\exists$ nbd that doesn't

Select $r = \min_{q \in E} \{d(p,q)\}$. The minimum exists and > 0, as the set is finite

A set E in metric space X is open, if every point p is an interior point of E

Ex: Is the open ball open? Or the open interval (a,b) open?

A set is closed if its complement is open. Or a set is closed if it contains all its limit points.

Ex: In R, is $\{p\}$ closed? Limit points $\{p\} = \phi$ ∴ Contained vacuously

$(a,b]$ ⇒ half open : Neither closed nor open.

Is $(0,1)$ open in R? Yes.

$(0,1)$ open in R^2? No

An open set can be closed by including all its limit points. $A \cup$ limit points of A is called its closure. $\overline{A} = A \cup lp(A)$

<u>Sequences</u>: An infinite sequence $\{p_n\}$ in X is a function $f: N \longrightarrow X$ maps $n \longrightarrow p_n$ (a point in X. X here is a metric space)

Such a sequence in a metric space may have the useful property of convergence.

$\{p_n\}$ converges if $\exists\, p \in X$ such that $\forall\, \varepsilon > 0$, $\exists\, n_0$ s.t. $\forall n \geq n_0 \Rightarrow$ $dist(p_n, p) < \varepsilon$.

We write $p_n \longrightarrow p$ or $\lim\limits_{n \to \infty} p_n = p$.

Example : $p_n = \frac{n+1}{n}$. Does it converge $\in \mathbb{R}$?

Intuitively, if it converges, it will do so for 1.

Given ε, task is to find n_0 such that $\forall\, n \geq n_0$

Processing in your head.

$$\left| \frac{n+1}{n} - 1 \right| < \varepsilon \qquad \left| \cancel{1+} \frac{1}{n} \cancel{-1} \right| < \varepsilon$$

If $n > \frac{1}{\varepsilon}$, $\varepsilon < \frac{1}{n}$.

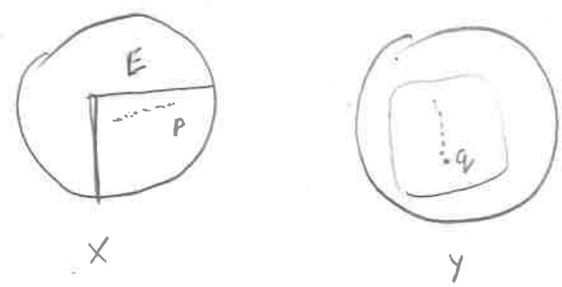
$\therefore$ select n_0 as $\left[\frac{1}{\varepsilon}\right] + 1$. Then, for $n > n_0$, $n > \frac{1}{\varepsilon}$

$$\left| \frac{n+1}{n} - 1 \right| = \left| \frac{1}{n} \right| < \varepsilon.$$

Note : $\frac{n+1}{n}$ never really becomes 1.

$\therefore$ We now know what $\lim_{n \to \infty} x_n = \alpha$ means.

What does $\lim_{x \to p} f(x) = q$ mean? Does it make sense?



$f : E \to Y$

Intuitively this means that, if we consider a sequence $\{p_n\}$ that converges to p, the sequence $\{f(p_n)\}$ converges to q in (just a limit point is OK)

Note : It is not required for p to be in E, nor for q to be $f(p)$. This is coming from the definition of sequence convergence when we only "approach" a value as closely as we want.

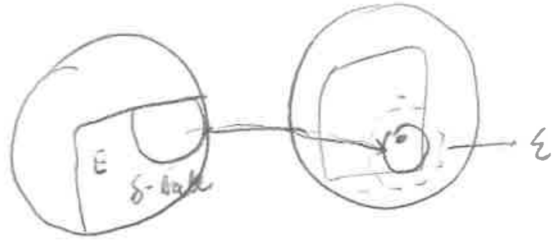
We are not concerned with what is happening at the point itself.

$f(x) \to q$ as $x \to p$ or $\lim_{x \to p} f(x) = q$.

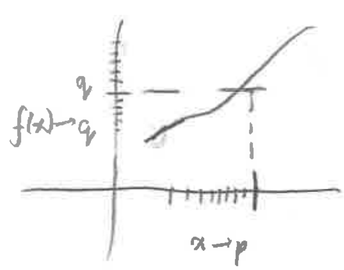
<u>Means:</u> $\exists\, q \in Y$ such that $\forall \varepsilon > 0$ $\exists\, \delta > 0$ s.t.

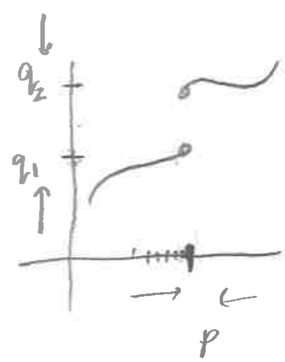
$\forall x \in E$, $0 < d(x,p) < \delta \Rightarrow d(f(x), q) < \varepsilon$.

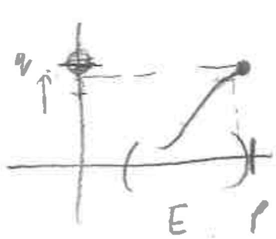
Both limits of functions and limits of sequence have this flavor of probing from the user. If a sequence has a limit p, no matter the closeness I want to attain from p, I can do it. Similarly, for limit of a function at a point, if I give an ε that I want to land in, of q, I know that selecting <u>any</u> point in E in δ- open ball of q will allow me to do so.



<u>Examples:</u>







Continuous functions : A function is continuous at $x = a$, if $\lim_{x \to a} f(x)$ exists and $= f(a)$.

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in E$, $d'(x,p) < \delta \Rightarrow d(f(x), q) < \varepsilon$

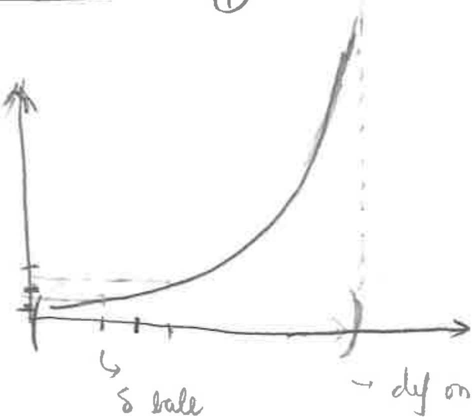
Example

continuous ??

Uniformly continuous :

①

ε- ball



②

Continuity is not that strong a condition

The 2nd function is much more well behaved. 1 δ- ball will work for the entire set of values of X.