Non-Linear Shape Optimization using Local Subspace Projections

Przemyslaw Musialski, Christian Hafner, Florian Rist, Michael Birsak, Michael Wimmer, Leif Kobbelt

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Presenter – Mukul Sati

Motivation

- Digital Fabrication
  - Optimize shape geometry to achieve user desired goal.

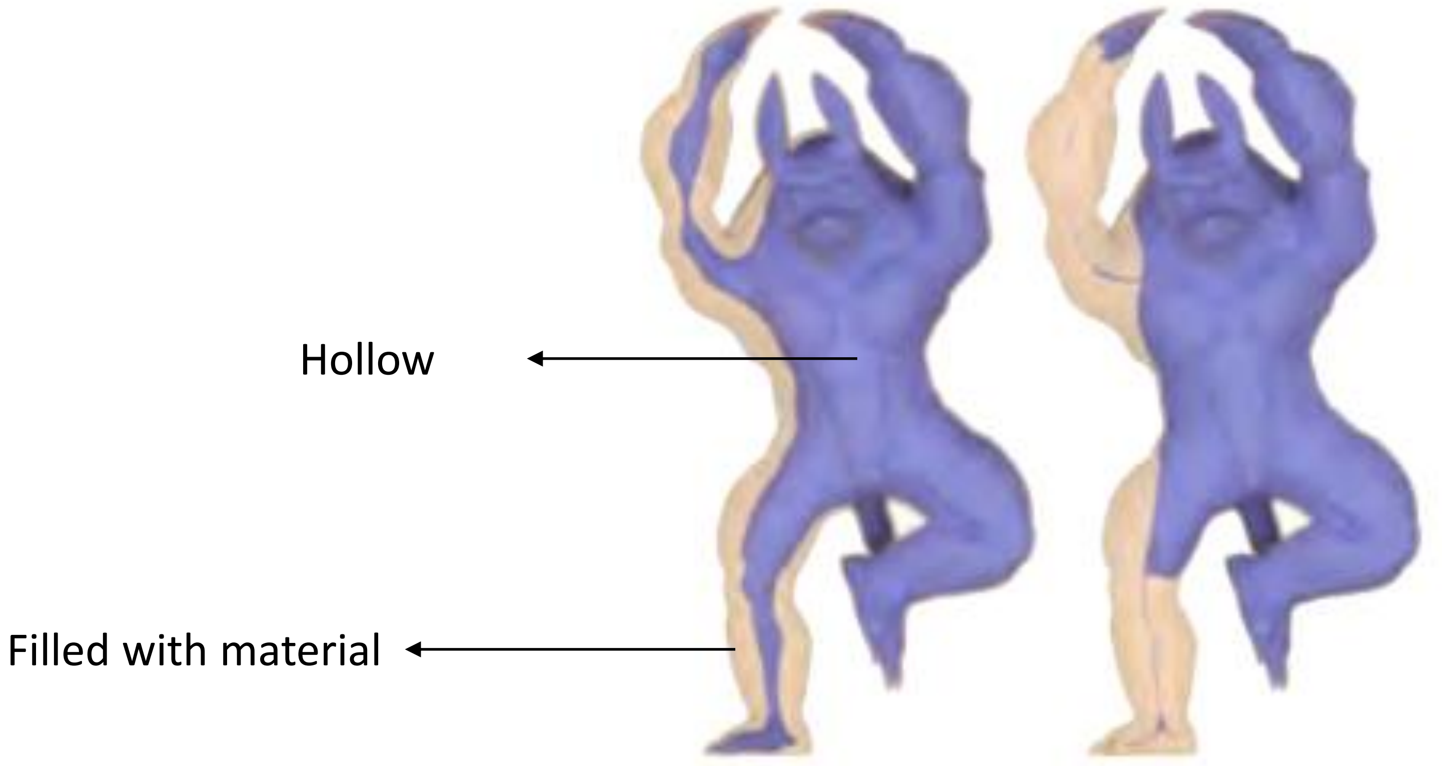
    http://www.youtube.com/watch?v=Kxk63ljYxEY&t=1m20s

Contributions (and agenda for the talk)

- Generic characterization of the shape optimization problem
- Offset surface parameterization using manifold harmonics
- Parameter reduction using local sub-space projection
- Box constraint elimination

- Grounding application – optimize shape for static stability.

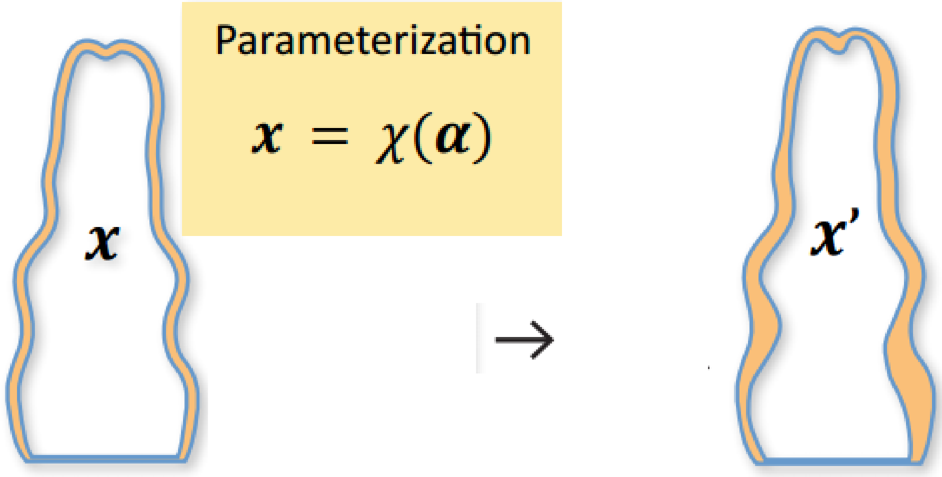
Optimize shape for static stability



Shape Optimization

In this paper:

- Shape is considered to be represented by an interior and exterior surface. The shape is the "gap" (closure of set of points in the volume between the two bounding surfaces)
- The outer surface is represented by a mesh that provides vertex positions and normal. The outer surface is fixed during the optimization.
- The inner surface is parameterized (as a normal offset – details to follow) with respect to the outer surface, and the optimum parameter space point specifies the inner surface, and thus the shape.



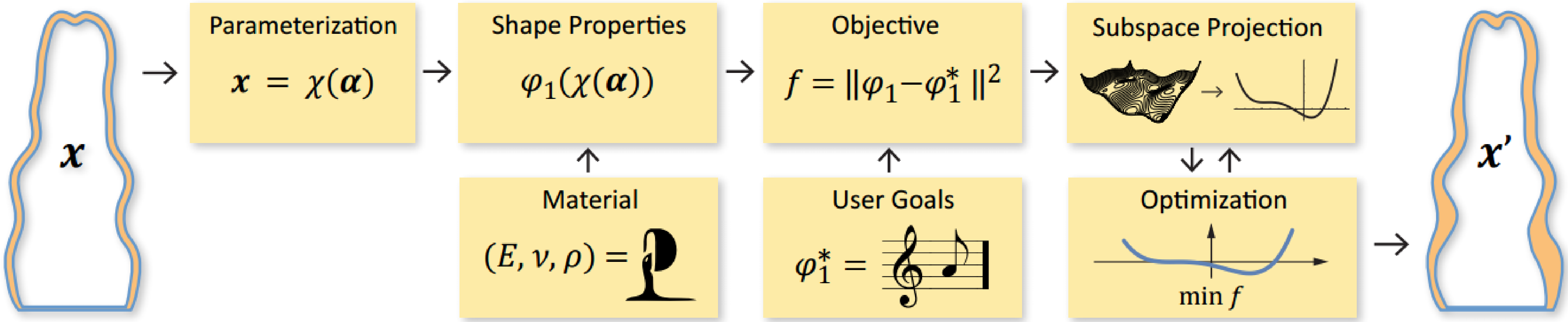
Shape Optimization

In any shape optimization problem:

- The user provides *material properties* of the shape.
  - For the static stability problem, assume the material has homogeneous volumetric mass density.
- The user provides a (set) of values of some *shape properties* (which are abstractly, some function of the shape).
  - For static stability, the user would specify that the center of mass (which, given a shape is calculable), projects onto the base of support (convex hull of ground contact points of outer surface).
  - Additionally, the user would want the center of mass to be as low as possible.
- We drive the optimization so that the optimized shape's properties minimize (summed least-squares) the user desired values.
- There are constraints on the shape geometry (self-intersection free, etc).

Center of mass should project onto the base of support

A generic shape optimization pipeline



$$
\begin{aligned}
&\min_{\alpha} \; f(\varphi(\chi(\alpha))) \\
&\text{s.t.} \quad g_j(\chi(\alpha), \varphi(\chi(\alpha))) \leqslant 0 \,.
\end{aligned}
$$

$$\nabla_\alpha f = \frac{\partial f}{\partial \varphi} \frac{\partial \varphi}{\partial \chi} \frac{\partial \chi}{\partial \alpha} \qquad \nabla_\chi g_i = \frac{\partial g_i}{\partial \chi} \frac{\partial \chi}{\partial \alpha} \qquad \nabla_\varphi g_i = \frac{\partial g_i}{\partial \varphi} \frac{\partial \varphi}{\partial \chi} \frac{\partial \chi}{\partial \alpha}$$

A specialized shape optimization pipeline for static stability

Offset parameterized inner surface

If the coordinate frame is such that the object is assumed to placed in the manner it is desired to stand, and the center of base is at zero, then, user desired center of mass **c*** – $(c^*_x, c^*_y) = 0$. For additional stability, $c^*_z = 0$

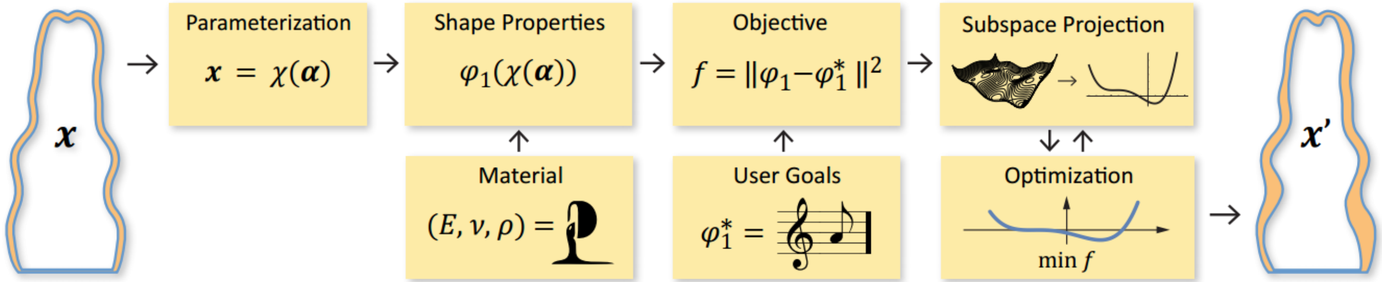
At any state of the optimization, let the parameters be $\boldsymbol{\alpha}$ (vector). The center of mass depends on the shape, and thus on $\boldsymbol{\alpha}$.

Objective f:

$w_1(c_x^2(\boldsymbol{\alpha}) + c_y^2(\boldsymbol{\alpha})) + w_2|c_z(\boldsymbol{\alpha})|$

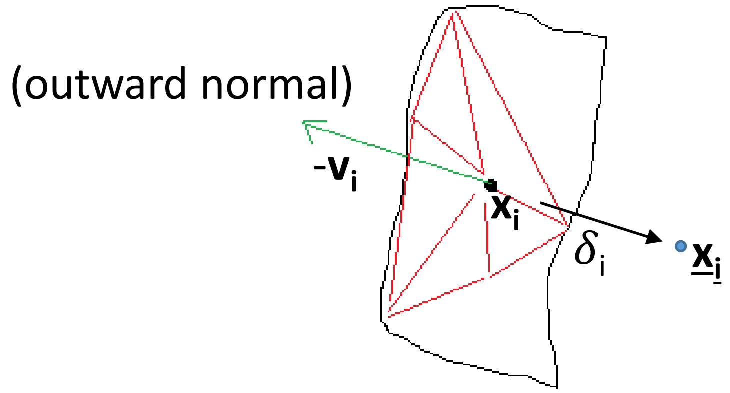
Optimize

$$\min_{\alpha} f(\varphi(\chi(\alpha)))$$
$$\text{s.t.}\ \ g_j(\chi(\alpha), \varphi(\chi(\alpha))) \leqslant 0.$$



Offset surface parameterization

- $\underline{\mathbf{x}}_{\underline{i}} = \mathbf{x}_\mathbf{i} + \delta_i \mathbf{v}_\mathbf{i}$



- Per vertex normal remains constant.
- The inner surface is parameterized by $\boldsymbol{\delta} = (\delta_0, \ldots \delta_n)$.
- Optimize over $\boldsymbol{\delta}$.
- Add constraints: $\delta_i \in (0, b_i)$ (or (a_i, b_i) to enforce minimum wall thickness).

Manifold harmonics parameterization

- The parameter space is equal to the number of vertices.

- We have a discrete digital signal $\boldsymbol{\delta}$ that defined by its value at n vertices.
- Want to represent all such signals $\boldsymbol{\Delta} = \{\boldsymbol{\delta} \mid \boldsymbol{\delta} \in \mathbf{R}^n\}$ in terms of a suitable basis $\mathbf{b}_1$, $\mathbf{b}_2$, …$\mathbf{b}_n$. Such that for any $\boldsymbol{\delta} \in \boldsymbol{\Delta}$:

$$\delta = \sum_{k=0}^{n} < \boldsymbol{\delta}, \boldsymbol{b_k} > \boldsymbol{b_k}$$

Manifold harmonics parameterization

- If the basis has a suitable frequency interpretation, analogous to the Fourier transform, we can just use the first m basis vectors (low frequency) to obtain a transformation R^m->R^n as: $\mathbf{\Gamma} = [\mathbf{b_1}\ \mathbf{b_2}\ \mathbf{b_3} \ldots \mathbf{b_m}]$

- Note $\mathbf{\Gamma}$ is a nXm matrix that encodes the R^m->R^n transformation in a "good" basis.

- Then, we can use a m dimension vector $\boldsymbol{\alpha}$ and obtain parameterization of offset surface using a lower dimensional parameter vector:

$$\delta_i = \mathbf{\Gamma}\ \boldsymbol{\alpha}$$

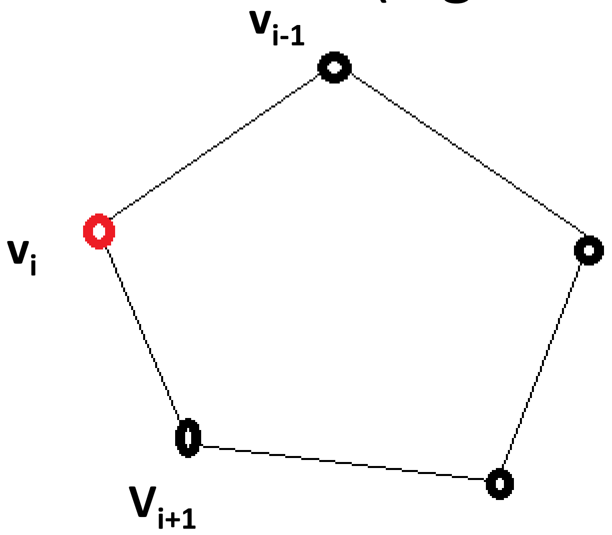
$$\underline{\mathbf{x}}_{\underline{i}} = \mathbf{x_i} + \delta_i \mathbf{v_i}$$

Manifold harmonics

- The cosine transform represents the even extension of a compactly supported real signal as the linear combination of cosine functions with increasing frequencies (in general, expresses a signal as the linear combination of complex exponentials).
- Higher correlation with a particular particular frequency is captured by the integral based dot product.
- The key to extending this to arbitrary domains is to realize that the complex exponentials are the eigen-functions (/eigen-values) of the Laplace operator d^2/dx.
- This definition generalizes to any dimension using the generalization of the Laplace operator – the Laplace-Beltrami operator: $\Delta f = \nabla^2 f = \nabla \cdot \nabla f$

Manifold harmonics

- To extend the concept to arbitrary domains, define an analogous Laplace operator, and extract its eigen-functions (eigen-values).



- Laplacian: Δ: v_{ix} -> $(v_{(i+1)x} - v_{ix})/2 - (v_{(i-1)x} - v_{ix})/2$
- In matrix form: $\Delta v_x = -Kv_x$ where

$$K = \frac{1}{2}\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$$

Manifold harmonics

- Can be extended to triangulated domains (and general graphs) similarly by defining:

$$\Delta v_{ix} = \Sigma\, w_{ij}(v_{jx} - v_{ix})$$

  for some neighborhood of $v_{i.}$

- One easy choice of neighborhood is the 1-ring neighborhood, and one easy choice of weights is to equally weigh all the neighbors.
- Can incorporate geometry into the weights as well, to get an operator that is influenced by both the geometry and connectivity.

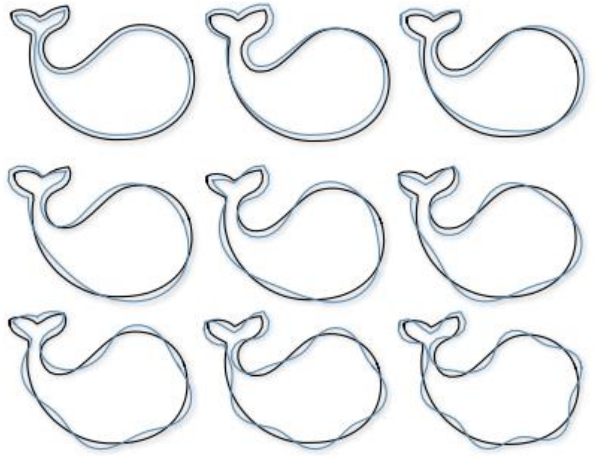
Manifold harmonics

- The above discrete operator is representable as a nXn matrix for a mesh with n vertices. One can extract the eigen-values.

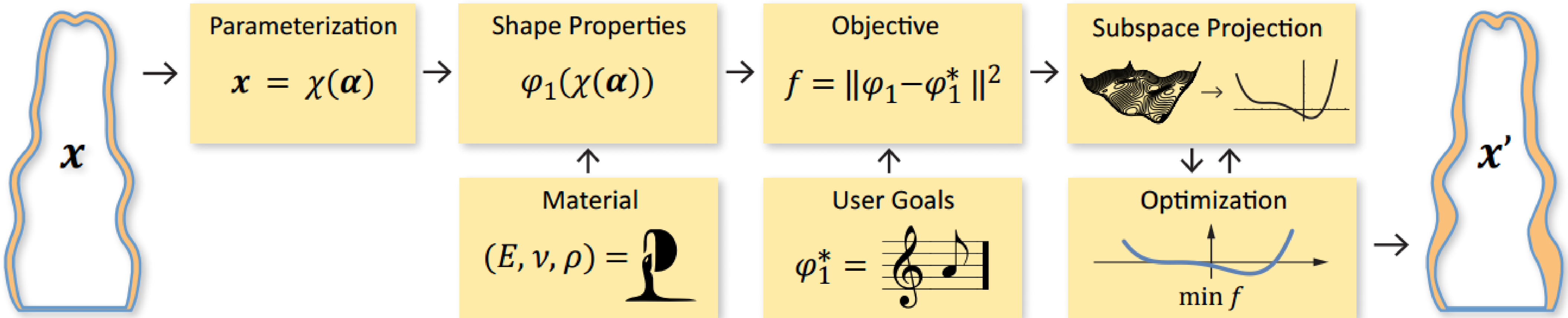
- For a particular δ = (k,k,….k), we can visualize the offset surface obtained by projecting onto just one of the basis =>

$$\boldsymbol{\delta^j = \langle b_j, \delta \rangle b_j}$$

- We have a frequency domain representation of any scalar function defined on a graph.



Where we stand



Optimization using local subspace projection

- Take m manifold harmonic bases. The parameter space is now m dimensional. Typically, m is around 64. Still too many parameters.
- The shape properties space is low dimensional however.
- Surmise that locally, at parameter value $\boldsymbol{\alpha}$**,** there exists a lower dimensional subspace of dimension d (typically just 3-4) and a parameter vector $\boldsymbol{\beta}$ where each dimension β_k dominantly controls one of the shape properties and does not influence other shape properties much (want to find a local parameter space where the local variation of shape properties is decorrelated). We will find the best linear transform **B** (around $\boldsymbol{\alpha}$) such that:
  - $\mathbf{B}^{\mathbf{mXd}}$:$R^d$->$R^m$ such that $\boldsymbol{\alpha} = \mathbf{B}\boldsymbol{\beta}$

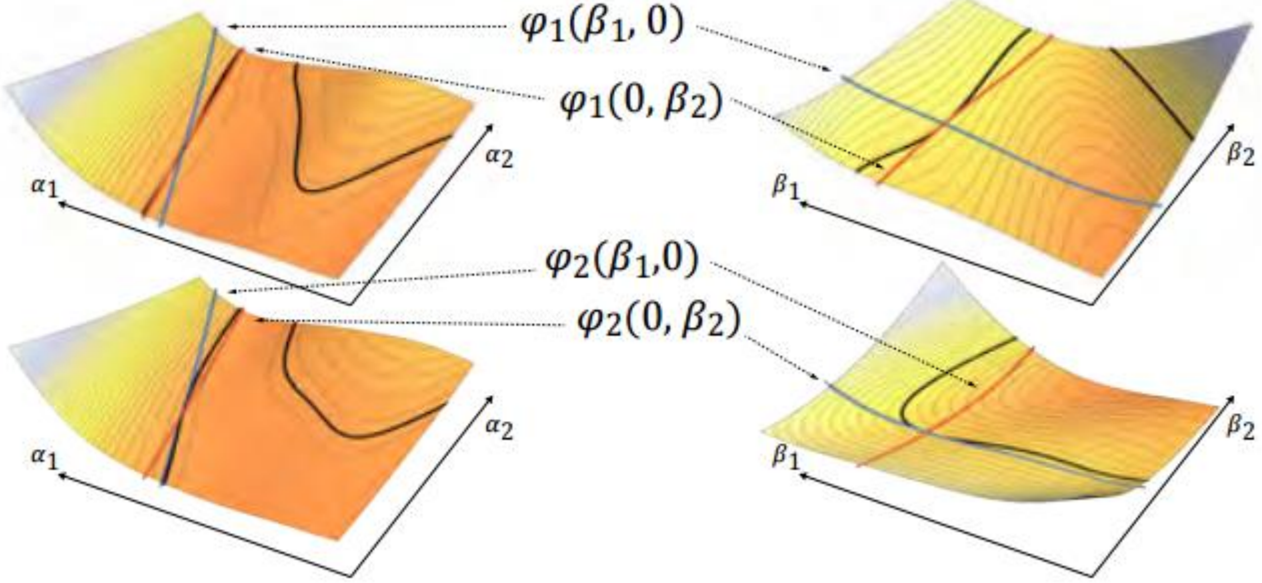


Figure 4: *Illustration of the shape property decorrelation on an $\mathbb{R}^2 \mapsto \mathbb{R}^2$ example. Left: two property functions $\varphi_1(\alpha_1, \alpha_2)$ and $\varphi_2(\alpha_1, \alpha_2)$ are mapped to $\varphi_1(\beta_1, \beta_2)$ and $\varphi_2(\beta_1, \beta_2)$.*

Optimization using local subspace projection

- **φ = x(α(β)); α = Bβ**

$$\frac{\partial\varphi}{\partial\beta} = \frac{\partial\varphi}{\partial\chi}\frac{\partial\chi}{\partial\alpha}\frac{\partial\alpha}{\partial\beta} = \frac{\partial\varphi}{\partial\chi}\frac{\partial\chi}{\partial\alpha}\mathbf{B}$$

- The differential should be a diagonal matrix for maximum de-correlation of new shape parameters. Regularized least norm solution for B.

$$\frac{\partial\varphi}{\partial\chi}\mathbf{X}\,\mathbf{B} = \mathrm{diag}(\gamma_1,\ldots,\gamma_k)$$

$$\min \| \tilde{\mathbf{B}} \|_2^2 \quad \text{s.t.} \quad \left[\frac{\partial\varphi}{\partial\chi}\frac{\partial\chi}{\partial\alpha}\right]\tilde{\mathbf{B}} = \mathbf{I}_k$$

Optimization using local subspace projection

- The iterative numerical optimization scheme runs two nested loops. In the outer loop, the shape parameter transform matrix B is computed for the current design space location α (or, equivalently, for the current intermediate shape $\chi(\alpha)$). In the inner loop, gradient-based iterations are applied to optimize the new (local) shape parameters β. The inner loop stops if either a local minimum is found or if the diagonal dominance of the gradient matrix falls below a certain threshold τ.

Optimization using local subspace projection

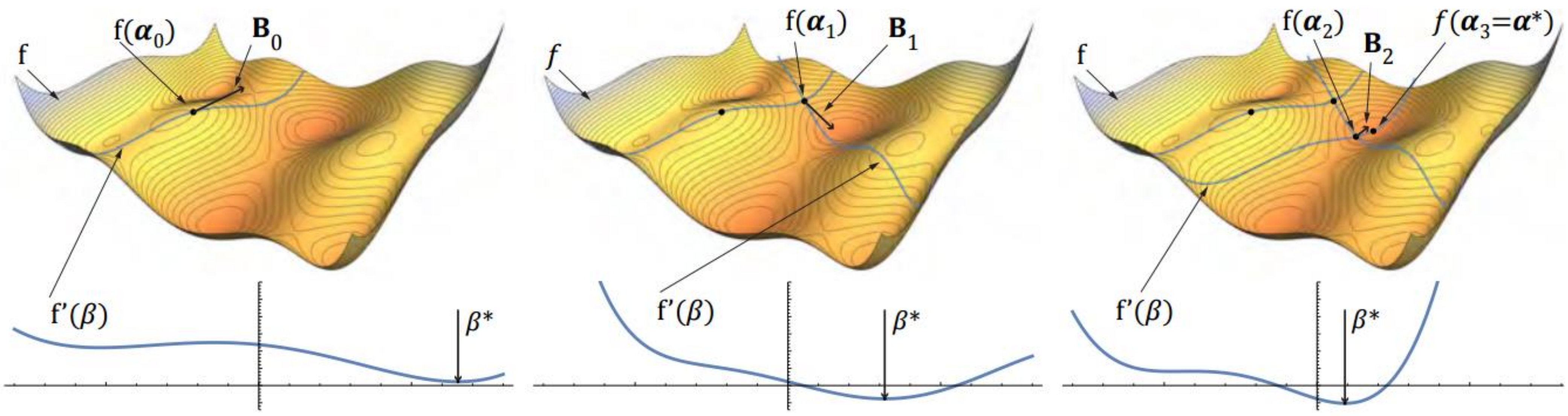


Figure 3: *Iterative subspace projection depicted illustratively on a map* $\mathbb{R}^2 \mapsto \mathbb{R}$. *Starting with an initial value* $\boldsymbol{\alpha}_0$, *our method computes an optimal subspace* $\boldsymbol{\alpha}_0 = \mathbf{B}_0\boldsymbol{\beta}$, *depicted as the vector* $\mathbf{B}_0$, *and the optimization is performed using a line-search method in the subspace until a minimum* β^* *it found, or the subspace does not decorrelate the variables anymore (left). Then, a new subspace is computed and the procedure is repeated (middle), until a local minimum* $\boldsymbol{\alpha}^*$ *is found (right). Please note that for the purpose of visualization we assume that the particular property function* $\boldsymbol{\varphi}$ *is at the same time the actual objective* f, *which is not the case in general. Moreover, for illustration purposes, we run the inner local optimization until it finds a local minimum* $\boldsymbol{\beta}^*$ *in* $\mathcal{R}$.

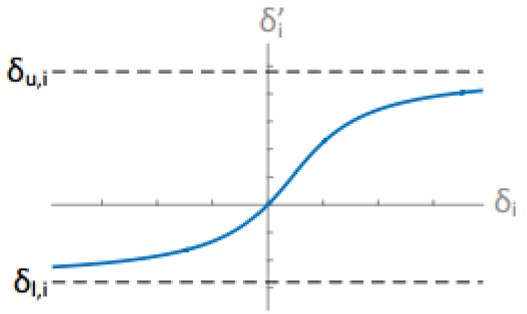
Constraint removal using constraint mapping

- A number of geometric constraints are box constraints to prevent self-intersection of shape:

$$\delta_{l,i} < \delta_i < \delta_{u,l}$$

- Clamp the current value within bound using a $\tan^{-1}$ function.

$$\delta_i' = \frac{\delta_{u,i} - \delta_{l,i}}{\pi} \tan^{-1}(\delta_i - o_i) + o_i, \text{ with}$$
$$o_i = \frac{\delta_{u,i} + \delta_{l,i}}{2}.$$



- Now, the internal surface: $\underline{\mathbf{x}}_i = \overline{\mathbf{x}}_i + \delta_i' \mathbf{v}_i$

Thanks!