Progress of Theoretical Physics, Vol. V, No. 2, March~April, 1950.

The Statistics of Honeycomb and Triangular Lattice. I.

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(Received December 1, 1949)

Summary

After the matrix method in crystal statistics was introduced by R. Kubo[1] Kramers and Wannier[2], and others, Onsager[3], using the abstract algebraic method, got the exact solution in the case of plane square net. His rather complicated method was simplified by Nambu[6] considerably. We, independenty, attained the simplification of Onsager's method and have applied to the honeycomb lattice. And the exact solution has been obtained. From this, by the so called dual transformation we can get the partition function for triangular net.

The Curie point occurs at ch $2H=2$ in the honeycomb lattice and at $\exp(4H)=3$ in the triangular net. These coincide with the results obtained by the dual and star-triangle transformations[4]. The specific heat becomes logarithmically infinite but the energy itself remains continuous at this temperature. In the antiferromagnetic case, the honeycomb lattice behaves similar to the former case but the triangular net exhibits no phase change.

1. Eigenvalue Operator

In applying the matrix method, we take as one tier the vertical set of points $(1, 2, 3 \ldots n)$ and the next tier $(1', 2', \ldots n')$. (See fig. 1) To avoid the edge effect, the n-th atoms are considered to be connected to the first atoms. Then in this case n must be even. As the most simple case, we use the Ising model. Namely each lattice point is occupied by the atoms having $(+)$ or $(-)$ spin and the interaction energy of the nearest neighbor is $J/2$ or $-J/2$ according as the neighboring spins are antiparallel or parallel. The interactions except the nearest are neglected. By μ_i, which takes the value 1 or -1, we designate the spin on the i-th atom in the tier. Then putting matrix V as follows

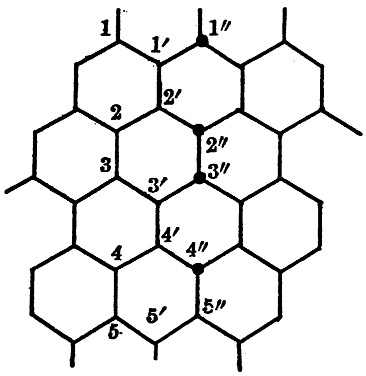

Fig. 1.

$$V(\mu_1, \mu_2, \cdots \mu_n;\ \mu_1'', \mu_2'' \cdots \mu_n'') = \sum_{<\mu'>} \exp H(\mu_2\mu_3 + \mu_4\mu_5 + \cdots) \exp H(\mu_1\mu_1' + \mu_2\mu_2' + \cdots)$$
$$\times \exp H(\mu_1'\mu_2' + \mu_3'\mu_4' + \cdots) \exp H(\mu_1'\mu_1'' + \mu_2'\mu_2'' + \cdots)$$

where $H=J/2kT$. The eigenvalue problem becomes

$$\sum_{<\mu''>} V(\mu_1, \mu_2, \cdots \mu_n;\ \mu_1'', \mu_2'' \cdots \mu_n'')\ \varphi(\mu_1'', \cdots \mu_n'') = \lambda \varphi(\mu_1, \cdots \mu_n).$$

The partition function becomes by their eigenvalues $\lambda_1, \lambda_2, \cdots$

$$f = \lambda_1^N + \lambda_2^N + \cdots$$

where N is the number of tiers divided by 2. By the theorem of Frobenius the maximum eigenvalue of a matrix, whose elements are all positive, is positive, simple and the eigenfunction is even. If λ_1 is assumed to be maximum, then as $N \to \infty$, $f = \lambda_1^N$. So that to get the partition function, we must solve the eigenvalue problem of V. As in Onsager's case, we introduce the operators c_i and s_i as follows

$$c_i f(\mu_1, \cdots, \mu_i, \cdots \mu_n) = f(\mu_i, \cdots, (-)\mu_i, \cdots \mu_n),$$
$$s_i f(\mu_1, \cdots, \mu_i, \cdots \mu_n) = \mu_i f(\mu, \cdots, \mu_i, \cdots \mu_n).$$

Then it is evident that

$$c_i^2 = s_i^2 = 1,\ c_i s_i = -s_i c_i,\ c_i c_k = c_k c_i,\ s_i s_k = s_k s_i,\ c_i s_k = s_k c_i \quad (i \neq k)$$

By using these, V can be written as

$$V = (2 \operatorname{sh} 2H)^n \exp H(s_2 s_3 + s_4 s_5 + \cdots) \exp H^*(c_1 + c_2 + \cdots + c_n)$$
$$\times \exp H(s_1 s_2 + s_3 s_4 + \cdots) \exp H^*(c_1 + c_2 + \cdots + c_n).$$

Here H^* is connected with H as follows

$$e^{2H^*} = \tanh \cdot H.$$

The relation between H^* and H is symmetrical.

Thus the problem is to diagonalise the operator V in the space of the direct products of n quaternion algebras $(1, c_i, s_i, c_i s_i)\ (i=1, 2, 3 \cdots n)$. This algebra Π_{2n} is also generated by $2n$ p's and q's defined as follows which appear in the theory of the second quantization or in the spinor theory(5).

$$\begin{aligned} p_k &= s_k c_{k-1} c_{k-2} \cdots c_2 c_1 \\ q_k &= c_1 c_2 \cdots c_{k-1} (i c_k s_k). \end{aligned} \qquad (k=1, 2, \cdots n)$$

These operators satisfy the simple commutation relations.

$$[p_i, q_k]_+ = 0, \quad [q_i, q_k]_+ = 2\delta_{ik}, \quad [p_i, p_k]_+ = 2\delta_{ik},$$

where $[A, B]_+$ denotes $AB + BA$.
Or in a single relation

$$\Big[\sum_{k=1}^{n} (x_k p_k + y_k q_k)\Big]^2 = \sum_{k=1}^{n} (x_k^2 + y_k^2)$$

where x_k and y_k are any complex numbers. This relation suggests that if we carry the linear orthogonal transformation in (p, q) system, the above relation remains unchanged. So that this transformation gives an automorphism of the algebra Π_{2n}. If we define $C=C_1 C_2 C_3 \cdots C_n$ which reverses all spins at once, then C anticommutes with all p's and q's and commutes with the products of even number of these. And as

$$c_k = i p_k q_k, \quad s_k s_{k+1} = -i p_{k+1} q_k$$

C also commutes with V. If we extend the suffices of p and q over n

$$p_{i+n} = -C p_i, \quad q_{i+n} = C \cdot q_i, \quad p_{i+2n} = p_i, \quad q_{i+2n} = q_i.$$

Hence p_i and q_i have period $2n$ in their suffices. Making use of these, we define the following operators which are invariant or alternating under the transformation $p_k \to p_{k+1}$, $q_k \to q_{k+1}$;

$$A_m = \frac{1}{2} \sum_{k=1}^{2n} p_{k+m} q_k = \sum_{k=1}^{n} p_{k+m} q_k,$$

$$A_m' = \frac{1}{2} \sum_{k=1}^{2n} p_{k+m} q_k (-)^k = \sum_{k=1}^{n} p_{k+m} q_k (-)^k.$$

Then V is written as

$$V = (2\,\mathrm{sh}\,2H)^n \exp\{-iH(A_1 + A_1')/2\} \exp\{iH^* A_0\}$$
$$\times \exp\{-iH(A_1 - A_1')/2\} \exp\{iH^* A_0\}.$$

As A_m nnd A_m' commute with C, V also commute with C. Then V lies in the subalgebra G of Π_{2n} which commute with C which accordingly divided into two ideals $G(1+C)/2$ and $G(1-C)/2$. Here we shall define $2n$ (ξ, η) as the linear orthogonal transformation of (p, q)

$$\xi_0 = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} p_k, \qquad \xi_r = \sqrt{\frac{2}{n}} \sum_{k=1}^{n} \cos \frac{r\pi}{n} k \cdot p_k,$$

$$\eta_0 = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} q_k, \qquad \xi_r^+ = \sqrt{\frac{2}{n}} \sum_{k=1}^{n} \sin \frac{r\pi}{n} k \cdot p_k,$$

$$\xi_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (-)^k p_k, \qquad \eta_r = \sqrt{\frac{2}{n}} \sum_{k=1}^{n} \cos \frac{r\pi}{n} k \cdot q_k,$$

$$\eta_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (-)^k q_k, \qquad \eta_r^+ = \sqrt{\frac{2}{n}} \sum_{k=1}^{n} \sin \frac{r\pi}{n} k \cdot q_k.$$

Here $\quad r = 0, 2, 4 \ldots\ldots\ldots 2n-2 \quad$ or

$\quad r = 1, 3, 5 \ldots\ldots\ldots 2n-1,$

We take exclusively even or exclusinely odd series from 0 to $2n-1$. Then as $\xi_{-r} = \xi_{2n-r} = \xi_r$, $\xi_{n+r} = \xi_{n-r}$, $\xi_{-r}^+ = -\xi_r^+$, $\xi_{n+r}^+ = -\xi_{n-r}^+$ and similarly for η and η^+, there are $2n$ linearly independent (ξ, η) in both cases. The commutation relations are summarized to

$$[\sum(x_r\xi_r+x_r^+\xi_r^++y_r\eta_r+y_r^+\eta_r^+)]^2=\sum(x_r^2+x_r^{+2}+y_r^2+y_r^{+2}).$$

where summation extend from 0 to n (even series) or from 1 to $n-1$ (odd series), and x_r, x_r^+, y_r, y_r^+ are any complex numbers. Namely, these $2n$ (ξ, η) are anticommuting set isomorphic to (p, q) system. So that the algebra generated by these (ξ, η) is isomorphic to Π_{2n} which is generated by (p, q). Or in other word, this transformation gives the automorphism of the algebra Π_{2n}. In these (ξ, η) system too, C has the invariant character.

Since $\sum p_k q_k=\sum(\xi_r\eta_r+\xi_r^+\eta_r^+)$, it follows

$$\Pi(p_kq_k)=\exp\frac{\pi}{2}\sum p_kq_k=\exp\frac{\pi}{2}(\sum\xi_r\eta_r+\xi_r^+\eta_r^+)=\Pi(\xi_r\eta_r)(\xi_r^+\eta_r^+).$$

So $$C=i^n\prod_{k=1}^{n}(p_kq_k)=i^n\prod_{r:\,\mathrm{odd}}(\xi_r\eta_r)(\xi_r^+\eta_r^+)=i^n\prod_{r:\,\mathrm{even}}(\xi_r\eta_r)(\xi_r^+\eta_r^+)_{\text{factors}}.$$

C also commute with the products of even number of (ξ, η). Here let us introduce the following operators having the propeities that they are invariant or alternant under the transformation

$$p_k\to p_{k+1},\ q_k\to q_{k+1}.\quad (k=1, 2, 3\ldots\ldots\ldots 2n)$$

$$X_r=\frac{1}{2n}\sum_{k,l=1}^{2n}\cos\frac{r\pi}{n}(k-l)\,p_kq_e=\frac{1-(-)^rC}{n}\sum_{k,l-1}^{n}\cos\frac{r\pi}{n}(k-l)p_kq_e$$

$$=\frac{I_r}{2}(\xi_r\eta_r+\xi_r^+\eta_r^+),\quad X_0=I_0\xi_0\eta_0,\quad X_n=I_n\xi_n\eta_n.$$

$$Y_r=\frac{I_r}{2}(\xi_r^+\eta_r-\xi_r\eta_r^+),$$

$$U_r=\frac{I_r}{2}(\xi_r\eta_{n+r}+\xi_r^+\eta_{n+r}^+)\quad U_0=I_0\cdot\xi_0\eta_n,\quad U_n=I_0\cdot\xi_n\eta_0,$$

$$V_r=\frac{I_r}{2}(\xi_r^+\eta_{n+r}-\xi_r\eta_{n+r}^+),$$

$$M_r=\frac{-1}{2n}\sum_{k,l=1}^{2n}\cos\frac{r\pi}{n}(k-l)\,p_xq_kp_xq_l(-)^l=\frac{2I_r}{n}\sum_{k,l=1}^{n}\cos\frac{r\pi}{n}(k-l)\,q_kq_l(-)^l$$

$$=\frac{I_r}{2}(\eta_r\eta_{n+r}+\eta_r^+\eta_{n+r}^+),\quad M_0=I_0\cdot\eta_0\eta_n=-M_n.$$

$$N_r=\frac{\bar{I}_r}{2}(\eta_r^+\eta_{n+r}-\eta_r\eta_{n+r}^+).$$

$$M_r'=\frac{I_r}{2}(\xi_r\xi_{n+r}+\xi_r^+\xi_{n+r}^+),\quad M_0'=I_0\xi_0\xi_n=-M_n',$$

$$N_r' = \frac{I_r}{2}(\xi_r^+\xi_{n+r} - \xi_r\xi_{n+r}^+),$$

$$j_r = I_r\xi_r^+\xi_r, \qquad \left[I_r = \frac{1}{2}(1-(-)^r C)\right].$$

$$k_r = I_r\eta_r^+\eta_r.$$

As seen by inspection, these operators having the suffices $(r, -r, n+r, n-r)$ contain only $(\xi_r, \xi_r^+, \eta_r, \eta_r^+, \xi_{n+r}, \xi_{n+r}^+, \eta_{n+r}, \eta_{n+r}^+)$ beside C.

These eight (ξ, η) generate the simple algebra Π_8 of order 2^8.

When $r=0$, $r+n=n$, they contain $(\xi_0, \eta_0, \xi_n, \eta_n)$ and when $r=n/2$, $r+n=3n/2$, they contain $(\xi_{n/2}, \eta_{n/2}, \xi_{n/2}^+, \eta_{n/2}^+)$, each set generating a simple algebra Π_4 of order 16.

In our algebra, (ξ, η) always appear in even products, so that we confine ourselves to the subalgebra $\boldsymbol{F}_r$ of Π_8 which commutes with C. This semi-simple algebra is of order 2^7 with the exceptional cases of $r=0$ and $r=n/2$.

Since $\boldsymbol{F}_1, \boldsymbol{F}_3, \boldsymbol{F}_5 \cdots\cdots \boldsymbol{F}_{n/2-1}$ have no common (ξ, η) and elements of one F are all commutative with o her F. So that we can consider the algebra of direct products of these. (Hereafter we assume $n/2$ is even to avoid unnecessary complexity)

$$\boldsymbol{D} = \boldsymbol{F}_1 \times \boldsymbol{F}_3 \times \cdots \times \boldsymbol{F}_{n/2-1}.$$

Similarly we consider

$$D' = \boldsymbol{F}_0 \times \boldsymbol{F}_2 \times \cdots \times \boldsymbol{F}_{n/2}.$$

$\boldsymbol{D}$ also commute with C, so that $\boldsymbol{D}$ can be divided into two ideals

$$\boldsymbol{D}(1+C)/2, \quad \boldsymbol{D}(1-C)/2.$$

The algebra $\boldsymbol{H}_r$ generated by $(X_r, U_{n+r}, M_r \cdots\cdots)$ are subalgebra of $\boldsymbol{D}(1-(-)^r C)/2$ and isomorphic to a subalgebra $\overline{\boldsymbol{H}}_r$ of $\boldsymbol{F}_r$. We shall investigate the properties of $\boldsymbol{H}_r$ in next chapter. In chapter 3, we shall see, how the eigenvalue operator V will be factorised into the operators each belonging to $\boldsymbol{H}_r$.

2. The Abstract Algebra $\boldsymbol{H}_r$

In this chapter, we shall examine the properties of the algebra $\boldsymbol{H}_r$. Omitting suffix r, and putting $r+n=s$, $n-r=-s$, for brevity, the generating elements run as follows

$$X = \frac{I}{2}(\xi\eta + \xi^+\eta^+), \qquad Y = \frac{I}{2}(\xi^+\eta - \xi\eta^+),$$

$$U = \frac{I}{2}(\xi\eta_s + \xi^+\eta_s^+), \qquad V = \frac{I}{2}(\xi^+\eta_s - \xi\eta_s^+),$$

$$M=\frac{I}{2}(\eta\eta_s+\eta^+\eta_s^+), \qquad N=\frac{I}{2}(\eta^+\eta_s-\eta\eta_s^+),$$

$$M'=\frac{I}{2}(\xi\xi_s+\xi^+\xi_s^+), \qquad N'=\frac{I}{2}(\xi^+\xi_s-\xi\xi_s^+),$$

$$X_s=\frac{I}{2}(\xi_s\eta_s+\xi_s^+\eta_s^+), \qquad Y_s=\frac{I}{2}(\xi_s^+\eta_s-\xi_s\eta_s^+),$$

$$U_s=\frac{I}{2}(\xi_s\eta+\xi_s^+\eta^+), \qquad V_s=\frac{I}{2}(\xi_s^+\eta-\xi_s\eta^+),$$

$$j=I\xi^+\xi, \qquad k=I\cdot\eta^+\eta,$$

$$j_s=I\xi_s^+\xi_s, \qquad k_s=I\cdot\eta_s^+\eta.$$

From these, we can construct the following 70 linearly independent elements

$$X,\ Y,\ k_sX,\ k_sY,\ j_sX,\ j_sY,\ j_sk_sX,\ j_sk_sY,$$
$$X_s,\ Y_s,\ kX_s,\ kY_s,\ jX_s,\ jY_s,\ jkX_s,\ jkY_s,$$
$$U,\ V,\ kU,\ kV,\ j_sU,\ j_sV,\ j_skU,\ j_skV,$$
$$U_s,\ V_s,\ k_sU_s,\ k_sV_s,\ jU_s,\ jV_s,\ jk_sU,\ jk_sV_s,$$
$$M,\ N,\ jM,\ jN,\ j_sM,\ j_sN,\ jj_sM,\ jj_sN,$$
$$M',\ N',\ k_sM',\ k_sN',\ kM',\ kN',\ K_skM',\ k_skN',$$
$$i,\ k,\ j_s,\ k_s,\ jk,\ j_sk_s,\ j_sk,\ j_sj,\ k_sk,\ jk_s,$$
$$jk_sk,\ j_sk_sk,\ j_sjk,\ j_sjk_s,\ jj_skk_s,\ I,$$
$$XX_s,\ YY_s,\ XY_s,\ X_sY,\ U_sU+V_sV,\ U_sV-UV_s.$$

1) The unity of $\boldsymbol{H}_r$ is $I_r=(1-(-)^rC)/2$.

2) Commutable subalgebra

$j,\ k,\ j_s,\ k_s$ are commutable each other and

$$j^2=k^2=j_s^2=k_s^2=-I_r.$$

3) Quaternion subalgebra

a)	$E=(I+jk)/2,$	$X,$	$Y,$	$XY=(j-k)/2.$
b)	$E'=(I+jk_s)/2,$	$U,$	$V,$	$UV=(j-k_s)/2.$
c)	$E''=(I+kk_s)/2,$	$M,$	$N,$	$MN=(k-k_s)/2.$
d)	$E'''=(I+jj_s)/2,$	$M',$	$N',$	$M'N'=(j-j_s)/2.$
e)	$E_s=(I+j_sk_s)/2,$	$X_s,$	$Y_s,$	$X_sY_s=(j_s-{}_sk)/2.$
f)	$E_s'=(I+j_sk)/2,$	$U_s,$	$V_s,$	$U_sV_s=(j_s-k)/2.$

These are six quaternions, namely they satisfy the following relations

a) $X^2=Y^2=(XY)^2=-E,\ XY=-YX,\ XE=EX=X,\ YE=EY=Y,\ E^2=E.$

And similar relations for others.

And there holds following table of multiplication.

fore \ rear	X	Y	U	V	M'	N'	M	N	X_s	Y_s	U_s	V_s
j	Y	$-X$	V	$-U$	N'	$-M'$						
k	$-Y$	X					N	$-M$			$-V_s$	U_s
j_s					$-N'$	M'			$+Y_s$	$-X_s$	V_s	$-U_s$
k_s			$-V$	U			$-N$	M	$-Y_s$	X_s		

The blanc spaces give the commutable products such as kV, jX_s etc. which are linearly independent bases. Other than these are anticommutative, e.g. $jX=-Xj=Y_s$, $k_sU=-Uk_s=-V$ etc.

4) i) $XM'=YN'=(U_s-jV_s)/2,\qquad M'X=N'Y=-(U_s+jV_s)/2,$

$YM'=-XN'=(V_s+jU_s)/2,\qquad M'Y=-N'X=-(V_s-jU_s)/2.$

ii) $M'U=N'V=-(X_s+jY_s)/2,\qquad UM'=VN'=(X_s-jY_s)/2,$

$N'U=-M'V=(Y_s-jX_s)/2,\qquad UN'=-(Y_s+jX_s)/2=-VM'.$

iii) $XU_s=YV_s=-(M'+kN')/2,\qquad UX_s=VY_s=(M'-kN')/2,$

$YU_s=-XV_s=-(N'-kM')/2,\qquad UY_s=-VX_s=(N'+kM')/2.$

iv) $X_sU=Y_sV=(M'-k_sN')/2,\qquad UX_s=VY_s=(-)(M'+k_sN')/2,$

$Y_sU=-X_sV=-(N'+k_sM')/2,\qquad UY_s=-VX_s=(N'-k_sM')/2.$

v) $X_sM'=-Y_sN'=-(U-j_sV)/2,\qquad M'X_s=-N'Y_s=(U+j_sV)/2,$

$Y_sM'=X_sN'=-(V+j_sV)/2,\qquad M'Y_s=N'X_s=(V-j_sU)/2.$

vi) $M'U=-N'V=(X+jY)/2,\qquad UM'=-VN'=(X-jY)/2,$

$N'U=M'V=(Y-jX)/2,\qquad UN'=VM'=-(Y+jX)/2.$

vii) $X_sU_s=Y_sV_s=(M-j_sN)/2,\qquad U_sX_s=V_sY_s=(-)(M+j_sN)/2,$

$Y_sU_s=-X_sV_s=(N+j_sM)/2,\qquad U_sY_s=-V_sX_s=-(N-j_sM)/2.$

viii) $XU=YV=(-)(M+jN)/2,\qquad UX=VY=+(M-jN)/2,$

$YU=-XV=(N-jM)/2,\qquad UY=-VX=-(N+jM)/2.$

ix) $XM=-YN=(U+kV)/2,\qquad MX=-NY=-(U-kV)/2,$

$YM=XN=(V-kU)/2,\qquad MY=NX=-(V+kU)/2.$

x) $X_sM=Y_sN=(-)(U_s+k_sV)/2,\qquad MX_s=NY_s=(U_s-k_sV_s)/2,$

$$Y_s M = -X_s N = -(V_s - k_s U_s)/2, \quad MY_s = (-)NX = +(V_s + k_s U_s)/2.$$

xi) $MU = NV = (X - k_s Y)/2, \quad UM = VN = -(X + k_s Y)/2,$

$UN = -VM = (Y - k_s X)/2, \quad NU = -MV = -(Y + k_s X)/2.$

xii) $U_s M = -V_s N = (X_s + kY_s)/2, \quad MU_s = -NV_s = -(X_s - kY_s)/2,$

$U_s N = V_s M = (Y_s - kX_s)/2, \quad NU_s = MV_s = -(Y_s + kX_s)/2.$

5) (X, Y) commute with (X_s, Y_s) and (U, V) commute with (U_s, V_s) and (M, N) commute with (M', N').

6) $UU_s - VV_s = -(XX_s - YY_s), \quad UV_s + VU_s = -(XY_s + X_s Y),$

$XX_s + YY_s = -(MM' + N'N), \quad XX_s - X_s Y = -(MN' - M'N),$

$MM' - NN' = U_s U + V_s V, \quad MN' + NM' = U_s V - UV_s.$

These complete the multiplication table of algebra $\boldsymbol{H}_r$.

In the special case of $r=0$, the algebra $\boldsymbol{H}_0$ made up of X, X_n, U, U_n, M', M, I, and $XX_n = X_n X = -UU_n = -U_n U = -MM' = -M'M$.

The multiplication table becomes

fore \ rear	X	X_n	U	U_n	M	M'
X	$-I$	XX_n	$-M$	$-M'$	U	U_n
X_n	XX_n	$-I$	M'	M	$-U_n$	$-U$
U	M	$-M'$	$-I$	$-X_n X$	$-X$	X_n
U_n	M'	$-M$	$-XX_n$	$-I$	X_n	$-X$
M	$-U$	U_n	X	$-X_n$	$-I$	$-XX_n$
M'	$-U_n$	U	$-X_n$	X	$-XX_n$	$-I$

This algebra is of order 8, and the center is composed of (I, XX_n). Then the algebra is divided into the direct sum of two simple ideals of order 4, i.e. two quaternions

$$(I - XX_n)/2, \quad (X + X_n)/2, \quad (U - U_n)/2, \quad (M - M')/2 \quad \text{and}$$

$$(I + XX_n)/2, \quad (X - X_n)/2, \quad (U + U_n)/2, \quad (M + M')/2.$$

Nextly the algebra $\boldsymbol{H}_{n/2}$ is composed of 8 elements, as the direct sum of two quaternions (X, Y, XY, E) and (U, V, UV, E').

3. Eigenvalue Problem

In the preceding chapter, we have completed the construction of Algebra $\boldsymbol{H}_r$. Here we shall resolve the eigenoperator V into the product of $\bar{V}_r$ each belonging to algebra $\boldsymbol{H}_r$. Since

$$X_r=\frac{1}{2n}\sum_{k,l=1}^{2n}\cos\frac{r\pi}{n}(k-l)\,p_k q_l=\frac{1}{2n}\sum_{m=1}^{2n}\cos\frac{r\pi m}{n}\sum_{k=1}^{2n}p_{k+m}q_k$$

$$=\frac{1}{n}\sum_{m=1}^{2n}A_m\cos\frac{r\pi}{n}m,$$

$$Y_r=\frac{1}{n}\sum_{m=1}^{2n}A_m\sin\frac{r\pi}{n}m,$$

we have

$$A_m=\sum_{r=0}^{2n-1}\left[X_r\cos\frac{r\pi}{n}m+Y_r\sin\frac{r\pi}{n}m\right].$$

Similarly

$$A_m'=\sum_{r=0}^{2n-1}\left[U_r\cos\frac{r\pi}{n}m+V_r\sin\frac{r\pi}{n}m\right].$$

Especially

$$A_0=X_0+X_1+X_2+\cdots\cdots+X_{2n-1}=2(X_1+X_2+\cdots\cdots+X_{n-1})+X_0+X_n,$$

$$A_1=2\,(X_1^*+X_2^*+X_3^*+\cdots\cdots+X_{n-1}^*)+X_0-X_n.$$

Similarly

$$A_0'=2\,(U_1+U_2+U_3+\cdots\cdots+U_{n-1})+U_0+U_n,$$

$$A_1'=2\,(U_1^*+U_2^*+U_3^*+\cdots\cdots+U_{n-1}^*)+U_0-U_n,$$

where

$$X_r^*=X_r\cos\frac{r\pi}{u}+Y_r\sin\frac{r\pi}{n},$$

$$Y_r^*=X_r\sin\frac{r\pi}{n}-Y_r\cos\frac{r\pi}{n}.$$

Similar relation holds for U_r^*, V_r^*.
Then V becomes

$$V=(2\,\mathrm{sh}\,2H)^n\exp\{-iH/2\sum_{r=0}^{2n-1}(X_r^*+U_r^*)\}\exp(iH^*\sum_{r=0}^{2n-1}X_r)$$

$$X\exp\{-iH/2\sum_{r=0}^{2n-1}(X_r^*-U_r^*)\}\exp(iH^*\sum_{r=0}^{2n-1}X_r)$$

which is factorised into the product of $\bar{V}_r$:

$$V=(2\,\mathrm{sh}\,2H)^n\,\bar{V}_0\bar{V}_1\bar{V}_2\cdots\cdots\bar{V}_{\frac{n}{2}-1}\bar{V}_{n/2}$$

where

$$\bar{V}_r=\exp\{-iH(X_r^*+X_s^*)-iH\cdot(U_r^*+U_s^*)\}\exp\{2H^*i(X_r+X_s)\}$$

$$\times \exp\{-iH(X_r^*+X_s^*)+iH\cdot(U_r^*+U_s^*)\}\exp\{2H^*i(X_r+X_s)\},$$
$$(r=1,2,3\ldots\ldots n/2-1)$$

$$\bar{V}_0=\exp\{-iH/2(X_0-X_n)-iH/2(U_0-U_n)\}\exp\{iH^*(X_0+X_n)\}$$
$$\times\exp\{-iH/2(X_0-X_n)+iH/2(U_0-U_n)\}\exp\{iH^*(X_0+X_n)\},$$

$$\bar{V}_{n/2}=\exp\{-iH(Y_{n/2}+V_{n/2})\}\exp\{2H^*iX_{n/2}\}\exp\{-iH(Y_{n/2}-V_{n/2})\}$$
$$\times\exp((2H^*iX_{n/2}).$$

And

$$\frac{1+C}{2}V=(2\,\mathrm{sh}\,2H)^n\frac{1+C}{2}\cdot\bar{V}_1\bar{V}_3\bar{V}_5\cdots\cdots\bar{V}_{\frac{n}{2}-1},$$

$$\frac{1-C}{2}V=(2\,\mathrm{sh}\,2H)^n\frac{1-C}{2}\cdot\bar{V}_0\bar{V}_2\bar{V}_4\cdots\cdots\bar{V}_{n/2}.$$

Hence $(1+C)/2\cdot V$ lies in the algebra $\boldsymbol{D}(1+C)/2$ where

$$\boldsymbol{D}=\boldsymbol{F}_1\times\boldsymbol{F}_3\times\boldsymbol{F}_5\times\cdots\cdots\times\boldsymbol{F}_{\frac{n}{2}-1}.$$

And $(1-C)/2\cdot V$ lies in the algebra $\boldsymbol{D}'(1-C)/2$ where

$$\boldsymbol{D}'=\boldsymbol{F}_0\times\boldsymbol{F}_2\times\boldsymbol{F}_4\times\cdots\cdots\times\boldsymbol{F}_{n/2}.$$

And $\bar{V}_r$ lies in $\boldsymbol{H}_r$ which is isomorphic to the subalgebra $\bar{\boldsymbol{H}}_r$ of $\boldsymbol{F}_r$.

If we can find the eigenvalues of $\bar{V}_r$ in $\boldsymbol{F}_r$, by making product of these in suitable selection, we can get the eigenvalues of V in the subspace $C=1$ or $C=-1$, according as the odd series or even series are adopted.

In the next section we shall find the eigenvalues of $\bar{V}_r$. (*to be continued*)

References

1) R. Kubo; Busseiron Kenkyu 1, 1943.
2) H. A. Kramers & G. H. Wannier; Phys. Rev. 60 (1941), 252.
N. W. Montroll: J. Chem. Phy 9 (1941), 706, 10 (1942), 61.
E. N. Lassettre & J. P. Howe; J. Chem. Phy. 9 (1941), 747. 801.
3) L. Onsager, Phys. Rev. 65 (1944), 117.
4) G. H. Wannier; Rev. Mod. Phys. 17 (1945), 50.
5) K. Husimi, *The Quantum Statistical Mechanics*, (1948), 336.
6) Y. Nambu; Prog. Theor. Phys. 5 (1950), 1.