HARMONIC OSCILLATOR ON A LATTICE

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The continuum limit of the ground state energy for the harmonic oscillator with discrete time is derived for all possible choices of the lattice derivative. The occurrence of unphysical values is shown to arise whenever the lattice laplacian is not strictly positive on its Brillouin zone. These undesirable limits can either be finite and arbitrary (multiple spectrum) or infinite (overlapping sublattices with multiple spectrum).

Corresponding to a given field theory there are many lattice models which reduce classically to the same continuum theory in the zero lattice spacing limit ($a \to 0$). This equivalence, which is a consequence of the ambiguity in the definition of the lattice derivative, is only formal and not necessarily true at the quantum level, as is well known in the case of the fermionic lattice action which induces spurious states even as $a \to 0$ (spectrum doubling). It is a merit of recent studies [1,2] to have stressed that this kind of difficulty can arise also in the bosonic case, for certain lattice actions which involve more sites than two nearest neighbours.

On the other hand, these generalised actions, which introduce additional free parameters, have been proposed [3] as to possibly improve the results obtained from the canonical one in actual computations (perturbative or Monte Carlo). It is then useful to characterise the constraints one must put on the lattice derivative, in addition to the classical ones, so as to maintain the wanted equivalence between the continuum and the lattice theory.

The purpose of this work is to study such constraints through the specific example of the harmonic oscillator with discrete time, as an application of the strong coupling expansion (SCE) method, which allows one to express the lattice ground state energy as a simple functional of the lattice action. In fact, this kind of method has already been used [4,5] to discuss the physical relevance of the continuous SCE regulations when the cut-off is removed. In the present example, it allows one to discuss an arbitrary lattice regulation and thus to confirm – and generalize – the results of ref. [1].

Our aim is to obtain E_a, the (dimensionless) ground state energy of an harmonic oscillator with discrete euclidean action $S_a\{x\}$, corresponding to the continuous action

$$S\{x\} = -\frac{1}{2}\int \mathrm{d}t_1 \mathrm{d}t_2 x(t_1)\, \partial^2\delta(t_2 - t_1)\, x(t_2) + \frac{m^2}{2}\int \mathrm{d}t\, x^2(t) . \tag{1}$$

In order to discretise $S\{x\}$, we first define the lattice derivative in a general way through:

$$-\partial_t \delta(t) \to a^{-2}\nabla = a^{-2}\Big(\beta_0 P_0 + \sum_{N\geqslant 1} (\beta_N^+ P_N^+ - \beta_{\overline{N}} P_{\overline{N}})\Big) , \tag{2}$$

where $\{P_0, P_N^\pm\}$ are the projectors on the (infinite) time lattice of sites j, spacing a:

$$P_N^\pm(t = ja) \equiv P_N^\pm(j) = \delta_{j,\pm N} , \quad P_0(j) = \delta_{0j} , \tag{3}$$

and the real numbers $\{\beta_0, \beta_N^\pm\}$ must satisfy the constraints

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$$\beta_0 + \sum_{N\geqslant 1} (\beta_N^+ - \beta_{\overline{N}}) = 0 \,, \quad \sum_{N\geqslant 1} N(\beta_N^+ + \beta_{\overline{N}}) = 1 \,, \quad \text{(4a,b)}$$

in such a way that

$$\lim_{a\to 0} \{a^{-2}\nabla * x\} = \dot{x} \quad (5)$$

which ensures the classical condition on the derivative. In expression (5), we use the lattice convolution rule

$$\nabla * x(i) \equiv a \sum_j \nabla(j-i)\, x(j) \,, \quad (6)$$

which by repeated application defines the lattice laplacian

$$-\partial^2\delta(t) \to a^{-3}\Delta = -a^{-4}\nabla * \nabla \,. \quad (7)$$

The discrete action $S_a\{x\}$ is then defined by insertion of the representation (7) in eq. (1).

It is now convenient for our purpose to express E_a as [5]

$$E_a = \left(-2m \Big/ \int \mathrm{d}t\right) \frac{\mathrm{d}}{\mathrm{d}m^2} \ln Z_a \,, \quad (8)$$

where the partition function Z_a associated to $S_a\{x\}$ is evaluated by the SCE resummation [5] to give

$$E_a = \sqrt{Z} \int_{-\pi/a}^{\pi/a} \frac{a\,\mathrm{d}q}{2\pi} \frac{1}{1 + Za^2\widetilde{\Delta}(q)} \,,$$

$$Z \equiv 1/a^2m^2 \,. \quad (9)$$

In order to go further we must express the (lattice) Fourier transform $a^2\widetilde{\Delta}(q)$ of $a^{-1}\Delta(i)$. From its definition (7), Δ can be written as

$$\Delta = \sum_{N\geqslant 0} \alpha_N P_N \,, \quad P_N \equiv \tfrac{1}{2}(P_N^+ + P_{\overline{N}}) \,, \quad N \geqslant 1 \,, \quad (10)$$

where the numbers α_N are obtained from $\{\beta_0, \beta_N^\pm\}$ if one for example uses in definition (7) the convolution rules:

$$P_0 * P_N^\pm = aP_N^\pm \,, \qquad P_N^\pm * P_0 = aP_N^\mp \,,$$

$$P_N^\pm * P_M^\pm = aP_{|M-N|}^{\pm\epsilon} \,, \qquad \epsilon \equiv \mathrm{Sign}(M-N) \,,$$

$$P_N^\pm * P_M^\mp = aP_{M+N}^\mp \,. \quad (11)$$

Then

$$a^2\widetilde{\Delta}(q) = \sum_j \exp(\mathrm{i}qja)\,\Delta(j) = \sum_{N\geqslant 0} \alpha_N \cos Nqa \,, \quad (12)$$

i.e.,

$$a^2\widetilde{\Delta}(q) = \Delta(\theta) \,, \quad \theta \equiv aq \,, \quad -\pi \leqslant \theta \leqslant \pi \,. \quad (13)$$

The continuum limit of E_a is then the Z infinite limit of

$$E(Z) = \sqrt{Z} \int_0^\pi \frac{\mathrm{d}\theta}{\pi} \frac{1}{1 + Z\Delta(\theta)} \,, \quad (14)$$

which obviously depends upon the distribution of zeroes of $\Delta(\theta)$ in the range of integration. From the classical conditions – eqs. (4a), (4b) – alone we know that $\Delta(\theta) \sim \theta^2$ as $\theta \to 0$. Thus, for some value θ_0, $\theta_0 \leqslant \pi$, it can be written

$$\Delta(\theta) = \theta^2 g(\theta) \,, \quad g(0) = 1 \,,$$

$$g(\theta) > 0 \quad \text{if} \quad 0 \leqslant \theta \leqslant \theta_0 \,, \quad (15)$$

in such a way that

$$E(Z) = \frac{\sqrt{Z}}{\pi} \int_0^{\theta_0} \frac{\mathrm{d}\theta}{1 + Z\theta^2 g(\theta)} + \frac{\sqrt{Z}}{\pi} \int_{\theta_0}^{\pi} \frac{\mathrm{d}\theta}{1 + Z\Delta(\theta)} \,. \quad (16)$$

In eq. (16), the first term can be easily evaluated (putting $u = Z\theta^2$) as

$$\frac{1}{2\pi} \int_0^{Z\theta_0^2} \frac{\mathrm{d}u}{\sqrt{u}} \frac{1}{1 + ug(u/Z^{1/2})}$$

$$\xrightarrow[Z\to+\infty]{} \frac{1}{2\pi} \int_0^\infty \frac{\mathrm{d}u}{\sqrt{u}[1 + ug(0)]} = \frac{1}{2} \,, \quad (17)$$

i.e. the expected physical value.

Now if $\Delta(\theta)$ is strictly positive in the remaining integration range $\theta_0 \leqslant \theta \leqslant \pi$, the second term in eq. (16) vanishes as $1/\sqrt{Z}$. This will no longer be the case if there is, for example, one zero θ_1 of multiplicity $2n$, (so as to preserve positivity). In such a case one has the representation:

$$\Delta(\theta) = (\theta - \theta_1)^{2n} g_1(\theta) \,,$$

$$g_1(\theta) > 0 \quad \text{if} \quad \theta_0 \leqslant \theta \leqslant \pi \,, \quad (18)$$

and the second integral in eq. (16) can be evaluated as before (i.e. putting $u = Z(\theta - \theta_1)^{2n}$, $\int_{\theta_0}^{\pi} = \int_{\theta_0}^{\theta_1} + \int_{\theta_1}^{\pi}$) to give a contribution

$$E_1(Z) = \frac{Z^{1/2-1/2n}}{n \sin(\pi/2n)} \left(\frac{1}{(2n)!} \frac{\mathrm{d}^{2n}\Delta(\theta)}{\mathrm{d}\theta^{2n}} \bigg|_{\theta=\theta_1} \right)^{-1/2n} . \quad (19)$$

More generally, each zero of $\Delta(\theta)$ contributes additively in an analogous way to the asymptotic behaviour of $E(Z)$. (An end point zero gives one half of such a contribution.) There are thus two kinds of unphysical circumstances: a finite ground state energy [but arbitrary, due to the freedom left in the derivative appearing in eq. (19); usual doubling of the spectrum imposes $n = 1$, $\frac{1}{2}\mathrm{d}^2\Delta(\theta_i)/\mathrm{d}\theta^2 = 1$], and an unbounded energy if at least one zero has a higher multiplicity.

In order to illustrate these features, one may construct a very simple example. One observes that if ∇ is an admissible lattice gradient, the whole convolution family ∇_s defined by

$$\nabla_s = -a^{-s}(1+\lambda)^{-s}\nabla[*(P_0 + \lambda P_1^{\dagger})]^s , \quad 0 \leqslant \lambda \leqslant 1 , \quad (20)$$

where s is a positive integer also fulfills the classical constraints – eqs. (4a), (4b). The associated laplacians are

$$\Delta_s = a^{-s}\Delta\{*[(1+\lambda^2)P_0 + 2\lambda P_1]\}^s(1+\lambda)^{-2s} . \quad (21)$$

and their Fourier transform is

$$\Delta_s(\theta) = \Delta(\theta)\{[(1-\lambda)^2 + 4\lambda\cos^2\tfrac{1}{2}\theta]/(1+\lambda)^2\}^s , \quad (22)$$

which shows that ∇_s is admissible (in the sense of positivity) for all s with the exception of the value $\lambda = 1$, where these operators are more and more singular as s increases, since then $\Delta_s(\theta)$ has a zero of order $2s$ at $\theta = \pi$. It is instructive to write these derivatives explicitly for the case where ∇ corresponds to the canonical choice of the nearest neighbours:

$$\nabla = P_1^{\dagger} - P_0$$

i.e.

$$\dot{x}(n) = a^{-1}[x(n+1) - x(n)] , \quad (23)$$

$$\Delta(\theta) = 4\sin^2\tfrac{1}{2}\theta .$$

Using the convolution rules [eqs. (11)], one finds (for $\lambda = 1$):

$$\nabla_1 = \tfrac{1}{2}(P_1^{\dagger} - P_{\bar{1}})$$

i.e.

$$\dot{x}(n) = (1/2a)[x(n+1) - x(n-1)] , \quad (24)$$

$$\nabla_2 = \tfrac{1}{4}(P_2^{\dagger} - P_0 + P_1^{\dagger} - P_{\bar{1}})$$

i.e.

$$\dot{x}(n) = \tfrac{1}{2}\{[x(n+2) - x(n)]/2a + [x(n+1) - x(n-1)]/2a\} . \quad (25)$$

Thus the next-to-nearest neighbour interaction [eq. (24)] has a finite ground state, but which has twice the physical value and this has been shown [1,2] to correspond to a doubling spectrum phenomenon. As for the interaction (25), which has an unbounded ground state [it diverges like $a^{-1/2}$, according to eq. (19)], it can be interpreted as a partition of the original lattice in two overlapping sublattices of the previous kind.

More generally, a derivative fulfilling $\beta_0 = 0$, $\beta_N^{\dagger} = \beta_{\bar{N}}$ with a *finite* number of terms must vanish at $\theta = \pi$ and has to be rejected. This is not the case of the SLAC [6] derivative

$$\nabla_{\mathrm{SL}} \equiv \sum_{N \geqslant 1} \frac{(-1)^{N+1}}{N} (P_N^{\dagger} - P_{\bar{N}}) , \quad (26)$$

since then $\Delta(\theta) = \theta^2$ for $0 \leqslant \theta \leqslant \pi$.

To conclude, we want to point out how apparently reasonable derivatives such as eq. (25) can produce extremely unphysical results. The strict positivity of the lattice laplacian on its first Brillouin zone is necessary in order for the lattice regulation to yield sensible results. This conjecture, made in ref. [1], on the basis of examples, has been proven here. In the framework of the strong coupling expansion method, this analysis also shows the reason for the failure of the soft continuous regulations of the inverse free propagator [4], since they must vanish at $q^2 = +\infty$, which plays in this case the role played here by the end point $\theta = \pi$ of a lattice regulation.

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