Zero-point length: introduction of the Quantum Metric and its implications

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1 Introduction

The construction of a quantum theory of gravity is one of the most important issue of today's physics. As far as we know the most accurate and simplest theory of gravity is Einstein's general relativity (GR) which is formulated in a geometrical language: gravitational field is nothing but the spacetime metric itself, regarded as a 2-rank tensor defined on every points on a pseudo-riemann manifold. GR is a classical theory valid up to Planck scales: we need an UV completion which could take in account the quantum nature of gravity. Several attempts has been made in order to unify GR and quantum mechanics: string theory, loop quantum gravity, canonical quantization etc..., but the results are yet unsatisfactory. Despite the differences among them, many of the quantum gravity formulations are agree on the fact that the quantum structure of spacetime is characterized by the presence of a zero-point length [7] due to intrinsic quantum uncertainty: this minimal length, which is of the order of the Planck scales, would acts as a universal regulator in quantum field theories and could avoid the formation of spacetime singularities [4]. We can ask what happen at the semiclassical level if we start directly with the assumption of the existence of this minimal length trying to define an effective metric which incorporates it. This line of research lead to the elaboration of the so called quantum metric or simply qmetric[5]. Technically the qmetric is a bitensor: it depends on two different spacetime points (it is a non-local object), and it is defined in a way such that spacetime intervals are bounded from below. The introduction of this qmetric has many interesting results: it naturally leads to the emergent gravity variational principle seen as a relic of the quantum spacetime [6] and can avoid spacetime singularities [2].

2 Bitensors

The quantum metric is technically a second-rank bitensor. A bitensor is a tensorial function of two points of spacetime[10]: a *base point* x' and a *field point* x to which we assign respectively primed indeces $\alpha', \beta', etc.$, and unprimed indeces $\alpha, \beta, etc.$ In this section we briefly review the formalism of such mathematical object and we introduce two biscalars which plays a fundamental role in the construction of the qmetric (they are also interesting in their own): the *Synge's world function* and the *Van Vleck's determinant.*

2.1 Synge's world function

Let be x' a point (base point) of the spacetime. We consider the point x (field point) in the *normal convex neighbourhood* of x' which is the set of points linked to x' by a unique geodesic. The geodesics segment β linking x and x' is described by $z^\mu(\lambda)$ where λ is the affine parameter of the geodesic that ranges from λ_0 and λ_1 such that $z(\lambda_0) = x'$ and $z(\lambda_1) = x$. Given an arbitrary point $z \in \beta$ we assign to it unprimed indeces $\mu, \nu, etc.$ We define the tangent vector to β as

$$t^\mu = \frac{dz^\mu}{d\lambda} \tag{1}$$

which satisfies the geodesic equation

$$t^\mu t^\nu{}_{;\mu} = t^\mu \nabla_\mu t^\nu = 0. \tag{2}$$

The situation is illustrated in Figure 1.

We define the Synge's world function as a scalar function both of x and x' [10] in the following

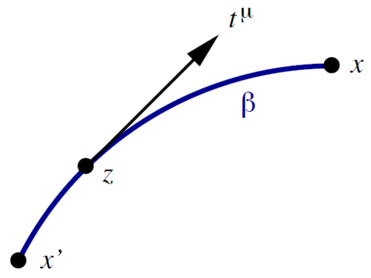

Figure 1: The base point x', the field point x, and the geodesic segment β that links them. The geodesic is described by parametric relations $z^\mu(\lambda)$ and $= t^\mu = dz^\mu/d\lambda$ is its tangent vector.

way

$$\Omega(x,x') = \frac{1}{2}(\lambda_1 - \lambda_0)\int_{\lambda_0}^{\lambda_1} g_{\mu\nu}(z)t^\mu t^\nu \, d\lambda \tag{3}$$

where the integral is performed along the geodesic segment β. By virtue of (2) the quantity $\epsilon \equiv g_{\mu\nu}(z)t^\mu t^\nu$ is constant along β. Thus numerically $\Omega(x,x') = 1/2\epsilon(\Delta\lambda)^2 \equiv 1/2\sigma(x,x')^2$ which is nothing but the half squared geodesic distances between x and x'. In particular $\epsilon = -1, +1, 0$ respectively for timelike, spacelike, null geodesic segment.

2.2 Differentiation

Given $\Omega(x,x')$ defined as in (3) (but the following is true for generic biscalar functions) we can differentiate it with respect both to x and to x'. We indicate with primed indeces derivation w.r.t. x' and with unprimed indeces derivation w.r.t. x as well. For example we can have $\Omega_\alpha = \partial\Omega/\partial x^\alpha = \Omega_{;\alpha}$ or $\Omega_{\alpha'} = \partial\Omega/\partial x'^\alpha = \Omega_{;\alpha'}$ We have that Ω_α is a 1-rank tensor w.r.t. tensorial operations carried out in x and a scalar w.r.t. to the ones in x'. In analogy $\Omega_{\alpha'}$ is a 1-rank tensor w.r.t. tensorial operations carried out in x' and a scalar w.r.t. to the ones in x. Iterating this procedure we can have $\Omega_{\alpha\beta} = \Omega_{\alpha;\beta}$ which is a 2-rank tensor in x and a scalar in x', $\Omega_{\alpha\beta'} = \Omega_{\alpha;\beta'}$ which is a 1-rank tensor both in x and in x' and so on. From the properties of partial and covariant derivatives we have some symmetries such as $\Omega_{\alpha\beta} = \Omega_{\beta\alpha}$ or $\Omega_{\alpha\beta'} = \Omega_{\beta'\alpha}$.
We can now explicitly compute Ω_α. In order to do so we need to evaluate $\delta_x\Omega = \Omega(x+\delta x, x') - \Omega(x,x')$ which tell us how the Synge's world function varies when the field point x moves. Under the shift δx the segment β is shifted to the unique geodesic segment $\beta + \delta\beta$ linking $x + \delta x$ to x', parametrized by $z^\mu(\lambda) + \delta z^\mu(\lambda)$ with the affine parameter λ rescaled such that it still runs from λ_0 to λ_1. Notice that $\delta z(\lambda_0) = 0$ and $\delta z(\lambda_1) = \delta x$. Expanding $\Omega(x+\delta x, x')$ at the first order in the variations we get

$$\delta_x\Omega = \Delta\lambda \int_{\lambda_0}^{\lambda_1} d\lambda \left(g_{\mu\nu}\dot{z}^\mu \delta\dot{z}^\nu + \frac{1}{2} g_{\mu\nu,\lambda}\dot{z}^\mu\dot{z}^\nu\delta z^\lambda \right) \tag{4}$$

where $\Delta\lambda = \lambda_1 - \lambda_0$, the overdot indicates differentiation w.r.t. λ and the metric and its derivatives are evaluated on β. We can integrate by parts the first term:

$$g_{\mu\nu}\dot{z}\delta\dot{z}^\nu = \frac{d}{d\lambda}\left(g_{\mu\nu}\dot{z}^\mu\delta z^\nu\right) - g_{\mu\nu}\ddot{z}^\mu\delta z^\nu - g_{\mu\nu,\lambda}\dot{z}^\mu\dot{z}^\lambda\delta z^\nu \tag{5}$$

and 4 becomes after a suitable relabelling of indeces

$$\delta_x\Omega = \Delta\lambda \left[g_{\mu\nu}\dot{z}^\mu\delta z^\nu\right]_{\lambda_0}^{\lambda_1} - \Delta\lambda\int_{\lambda_0}^{\lambda_1} d\lambda \left[g_{\mu\nu}\ddot{z}^\nu + \left(g_{\mu\nu,\lambda} - \frac{1}{2} g_{\lambda\nu,\mu} \right) \dot{z}^\nu \dot{z}^\lambda \right] \delta z^\mu \tag{6}$$

With some algebraic passages it can be shown the integrand is equivalent to $g_{\mu\nu}(\dot{z}^\lambda \dot{z}^\nu_{;\lambda})$ which is vanishing because of (2). Thus we have :

$$\delta_x\Omega = \Delta\lambda g_{\mu\nu}\dot{z}^\mu\delta x^\nu = \Delta\lambda g_{\mu\nu}t^\mu\delta x^\nu \tag{7}$$

hence

$$\Omega_\mu(x,x') = \frac{\delta_x\Omega}{\delta x^\mu} = \Delta\lambda g_{\mu\nu}(x)t^\nu(x) \tag{8}$$

We see that for any points $z \in \beta$ the vector $\Omega^\mu(z,x') = (\lambda - \lambda_0)t^\mu$ can be thought as a rescaled tangent vector. A similar computation leads to:

$$\Omega_{\mu'}(x,x') = -\Delta\lambda g_{\mu\nu}(x')t^\nu(x') \tag{9}$$

It is interesting to compute the norm $\Omega_\mu\Omega^\mu = (\Delta\lambda^2)t^\mu t^\nu = (\Delta\lambda)^2\epsilon$ which we can rewrite as:

$$g^{\mu\nu}(x)\partial_\mu\Omega\partial_\nu\Omega = 2\Omega \tag{10}$$

and similarly

$$g^{\mu\nu}(x')\partial_{\mu'}\Omega\partial_{\nu'}\Omega = 2\Omega \tag{11}$$

2.3 Geodesics congruence

If the base point x' is kept fixed [10] the world function is an ordinary scalar function of x: in this view (10) becomes a differential equation which defines the world function itslef. A second differentiation of that equation brings to

$$\Omega^\mu{}_\nu\Omega^\nu \equiv \Omega^\mu{}_{;\nu}\Omega^\nu = \Omega^\mu \tag{12}$$

which is the geodesic equation in a non-affine parametrization with λ as parameter. We restrict our attention to the case of timelike geodesics with $\Omega < 0$. If we define:

$$u^\alpha = \frac{\Omega^\alpha}{\sqrt{|2\Omega|}} \tag{13}$$

we have a unit norm vector satisfying affine geodesic equation $u^\alpha_{;\beta}u^\beta = 0$ parametrized by parameter τ. We then have the relation $d\lambda/d\tau = |2\Omega|^{-1/2}$ showing that λ parametrization is singular in $\Omega = 0$. We now define equi-geodesic surfaces $\Sigma_{x'} = \{x \text{ s.t. } \Omega(x,x') = \mathit{const.}\}$ and the transverse metric $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. The intrinsic and extrinsic geometry of these hypersurfaces are described respectively by the projection of Ricci scalar R_Σ and by the extrinsic curvature $K_{\mu\nu}$. The extrinsic curvature is given by:

$$K_{\mu\nu} = \nabla_\mu u_\nu = \frac{\nabla_\mu\nabla_\nu\Omega - \epsilon u_\mu u_\nu}{\sqrt{|2\Omega|}} \tag{14}$$

We can compute the geodesics expansion in the affine parametrization as the trace of extrinsic curvature:

$$\theta = K^\mu{}_\mu = \frac{\Omega^\mu{}_\mu - 1}{\sqrt{|2\Omega|}} = \frac{\theta^*}{\sqrt{|2\Omega|}} \tag{15}$$

with $\theta^* = h_{\mu\nu}\Omega^{\mu\nu} = \Omega^\mu{}_\mu - 1$ is the expansion in the original non affine parameter. We see that as $\Omega \to 0$ the expansion θ becomes singular while we will see in the next section $\theta^* \to 3$.

2.4 Coincidence limit

We now introduce the coincidence limit procedure which allow us to investigate the bitensors behaviour in the limit x tends to x'. This will be a non trivial procedure once we introduce the zero-point length. We use the following notation for the limit of a generic bitensor $[T_{\dots}(x,x')$ [10]:

$$[T_{\dots}(x,x')] \equiv \lim_{x\to x'} T_{\dots}(x,x') = T_{\dots}(x') \tag{16}$$

where $T_{\dots}(x')$ is now a tensor in x'. We assume that the coincidence limit is a unique tensorial function of the base point x', independent of the direction in which the limit is taken. Strictly speaking if the limit is computed by letting $\lambda_1 \to \lambda_0$ on a precise geodesic segment β the result is independet of the geodesic choice. Considering the world function, from (3),(8) and (9) we have:

$$[\Omega] = [\Omega_\mu] = [\Omega_{\mu'}] = 0 \tag{17}$$

We can rewrite (12) once we use (8) as

$$(g_{\mu\nu} - \Omega_{\mu\nu})t^\nu = 0 \tag{18}$$

From the assumption that the coincidence limit must be independent from the direction in which is computed, namely t^μ, we get:

$$[\Omega_{\mu\nu}] = g_{\mu'\nu'} \equiv g_{\mu\nu}(x') \tag{19}$$

and we also have $g_{\mu'\nu'} = [\Omega_{\mu'\nu'}] = -[\Omega_{\mu'\nu}] = -[\Omega_{\mu\nu'}]$. Thus we obtain that $\left[\Omega^{\mu}_{\mu}\right] = 4$ and we see that $\lim_{\Omega\to 0}\theta^* = \left[\Omega^{\mu}_{\mu} - 1\right] = 3$ as stated before. We can continue to differentiate the world function and compute all the coincidence limits using Synge's rule (see [10]). We have the following interesting result:

$$[\Omega_{\alpha\beta\gamma\delta}] = -\frac{1}{3}\left(R_{\alpha'\gamma'\beta'\delta'} + R_{\alpha'\delta'\beta'\gamma'}\right) \tag{20}$$

where it appears the Riemann tensor.

2.5 Tetrads and parallel propagator

We introduce an orthonormal basis e^{μ}_{a}, with index a run from 0 to 3, on the geodesic segment linking x to x' and we assume this basis to be parallel transported along the geodesic:

$$e^{\mu}_{a}e^{\nu}_{b}g_{\mu\nu} = \eta_{ab} \qquad\qquad e^{\mu}_{a;\nu}t^{\nu} = 0 \tag{21}$$

where $\eta_{ab} = diag(-1,1,1,1)$ is the Minkowski metric which we use to raise and lower latin indeces. We have the completeness relation:

$$g^{\mu\nu} = \eta_{ab}e^{\mu}_{a}e^{\nu}_{b} \tag{22}$$

We define the dual tetrad as:

$$e^{a}_{\mu} = \eta^{ab}g_{\mu\nu}e^{\nu}_{b} \tag{23}$$

and the completenss relation takes the form:

$$g_{\mu\nu} = \eta_{ab}e^{a}_{\mu}e^{b}_{\nu} \tag{24}$$

It's easy to see that $e^{a}_{\mu}e^{\mu}_{b} = \delta^{a}_{b}$ and $e^{a}_{\mu}e^{\nu}_{a} = \delta^{\mu}_{\nu}$.
Now consider a vector A^{μ} which is parallel transported along the geodesic segment β. On β the vector can be decomposed in the tetrad frame as $A^{\mu} = A^{a}e^{\mu}_{a}$ with $A^{a} = A^{\mu}e^{a}_{\mu}$. The fact that the vector is parallel transported implies that coefficients A^{a} are constants along β. Then we can express $A^{\alpha} = (A^{alpha'}e^{a}_{\alpha'})e^{\alpha}_{a}$ or

$$A^{\alpha}(x) = \Pi^{\alpha}_{\alpha'}(x,x')A^{\alpha}(x') \qquad\qquad \Pi^{\alpha}_{\alpha'}(x,x') \equiv e^{a}_{\alpha'}(x')e^{\alpha}_{a}(x) \tag{25}$$

where $\Pi^{\alpha}_{\alpha'}$ is called the *parallel propagator*: it takes a vector in x' and parallel transport it to x along the unique geodesic segment linking the two points. We also have the inverse parallel propagator such that $\Pi^{\alpha}_{\alpha'}\Pi^{\alpha'}_{\beta} = \delta^{\alpha}_{\beta}$ and $\Pi^{\alpha'}_{\alpha}\Pi^{\alpha}_{\beta'} = \delta^{\alpha'}_{\beta'}$. The action of the parallel propagator can be extendend on tensor of arbitrary rank: we have the occurance of one parallel propagator for each tensorial index. We can compute the coincidence limit of the parallel propagator leading to $\left[\Pi^{\alpha}_{\beta'}\right] = \delta^{\alpha'}_{\beta'}$.

2.6 Van Vleck determinant

We're now ready to define the *Van Vleck determinant* (VVD) $\Delta(x,x')$. It is defined as:

$$\Delta(x,x') \equiv det\left|\Delta^{\alpha'}{}_{\beta'}(x,x')\right| \qquad\qquad \Delta^{\alpha'}{}_{\beta'}(x,x') \equiv -\Pi^{\alpha'}_{\alpha}(x,x')\Omega^{\alpha}_{\beta'} \tag{26}$$

It can be shown [10] that this definition is equivalent to:

$$\Delta(x,x') = \frac{det|\Omega_{\alpha\beta'}(x,x')|}{\sqrt{g(x)g(x')}} \tag{27}$$

where $g(x)$ and $g(x')$ are the metric determinant evaluated respectively in x and x'. At coincidence we have $\left[\Delta^{\alpha'}{}_{\beta'}\right] = \delta^{\alpha'}_{\beta'}$ and so $[\Delta] = 1$. Moreover we can expand $\Delta^{\alpha'}{}_{\beta'}$ near coincidence (see appendix A) obtaining:

$$\Delta^{\alpha'}{}_{\beta'} = \delta^{\alpha'}{}_{\beta'} + \frac{1}{6}R^{\alpha'}{}_{\gamma'\beta'\delta'}\Omega^{\gamma'}\Omega^{\delta'} + o(\epsilon^3) \tag{28}$$

We can use the approximation [10] $det|1+\mathbf{a}| = 1 + Tr[\mathbf{a}] + o(\mathbf{a}^2)$ to get the expansion of the Van Vleck determinant in x'

$$\Delta = 1 + \frac{1}{6}R_{\alpha'\beta'}\Omega^{\alpha'}\Omega^{\beta'} + o(\epsilon^3) \tag{29}$$

with $R_{\alpha\beta}$ the Ricci tensor. We can relate the behaviour of Van Vleck determinant to the behaviour of the geodesics congruences. In fact it can be shown[10] that VVD satisfies the following differential equation:

$$\frac{1}{\Delta}(\Delta\Omega^{\alpha})_{;\alpha} = 4 \tag{30}$$

which can be rewritten as

$$(\ln\Delta)_{;\alpha}\Omega^{\alpha} = 4 - \Omega^{\alpha}{}_{\alpha} \tag{31}$$

$$\frac{d}{d\lambda}(\ln\Delta) = 3 - \theta^{*} \tag{32}$$

The expansion θ^* of geodesics emanating from x' governs the VVD behaviour: if $\theta* < 3$ the congruence expands less rapidly than it would in flat spacetime and Δ increases along the geodesics; if $\theta* > 3$ the congruence expands more rapidly than it would in Minkowski and Δ decreases along the geodesic. In other words $\Delta > 1$ indicates a *focusing* of geodesics while $\Delta < 1$ indicates a *defocusing*. Moreover (29) illustrates the relation between VVD and the strong energy condition: the sign of $\Delta - 1$ near x' is determined by $R_{\alpha'\beta'}\Omega^{\alpha'}\Omega^{\beta'}$. The VVD is important also in quantum field theory in curved spacetime: in fact in any arbitrary spacetime of d dimensions the leading singular structure of the two points function associated to the d'Alambertian operator$\Box_{xx'}$ is given by the Hadamard form [11]

$$G(x,x') = \frac{\sqrt{\Delta}}{(\sigma^2)^{\frac{d-2}{2}}}(1 + \text{ smooth therms}) \tag{33}$$

thus Δ must carry information about curvature. We can also write the following identities which are usefull in the computations:

$$(\ln\Delta)_{;\alpha}u^{\alpha} = \frac{d-1}{\sqrt{\epsilon\sigma^2}} - K \tag{34}$$

$$(\ln\Delta)_{;\alpha;\beta}u^{\alpha}u^{\beta} = -\frac{d-1}{\epsilon\sigma^2} + K_{\mu\nu}K^{\mu\nu} + R_{\mu\nu}u^{\mu}u^{\nu} \tag{35}$$

where $K = \theta$ is the trace of extrinsic curvature $K_{\mu\nu}$.

3 The Quantum Metric

Physics must deal with measurements: our conception of spacetime itself is constructed from measurements and observations of physical phenomena by means of "rods and clocks" which we use as probes [5]. The quantum behaviour of our probes should characterize the quantum structure of spacetime. In General Relativity we use local tensors as the metric, the Riemann tensor etc.., that describe very well motion and measurements. However at quantum level local tensorial functions might not make much sense. Moreover if the quantum effects are non-analytic and non-local then the limit in which we recover locality might not commute to the $\hbar \to 0$ limit and hence the classical limit might carry a vestige of an inherently quantum spacetime [5].

3.1 Zero point length

Most of the attempts to combine quantum physics and general relativity are agree on at least two results:

1. Thermal nature of Rindler Horizons which is manifest in Hawking and Unruh radiation;

2. Existence of a minimal length scale, i.e. a lower bound on spacetime intervals, arising when we implement Heisenberg uncertainty principle in gravitational physics and when we take in account the limitness of our probes.

The first result is the starting point of the so called emergent gravity paradigm [8] which we will see has an intriguing connection with the existence of a zero-point length. The simplest way in which we can implement a lower bound on spacetime intervals in our description of spacetime is to work directly with the distance function $d(x,x')$ between spacetime events instead of the metric tensor. In Lorentzian geometries is better to use the lorentz invariant squared geodesic distance $\sigma^2(x,x')$, which is two times the Synge's world function $\Omega(x,x')$ defined in (3), related to

$d(x,x')$ via $d(x,x') = \sqrt{|\sigma^2(x,x')|}$. A minimal length scale l_0 can be implemented directly with a modification of the geodesic distance (here and in the following we consider only the case in which x, x' are timelike/spacelike separated):

$$\sigma^2(x,x') \longrightarrow S(\sigma^2) \text{ such that } S(0) = \pm l_0^2 \neq 0 \tag{36}$$

thus in the coincidence limit $x \to x'$ the modified geodesic distance is different from zero. This means that we can't localize spacetime events with an accuracy better than l_0: clearly we are breaking the first axiom of metric spaces, i.e. $d(x,x) = 0$. Thus we expect that at semiclassical level spacetime is no more described by a metric tensor. We saw in (19) and (20) that we can reconstruct spacetime geometry by means of the coincidence limits of differentiation of the Synge's world function. With the introduction of a minimal length clearly the coincidence limit will be affected and we can use this fact to define the effective quantum metric at mesoscopic scales.

3.2 Construction of the qmetric

In order to formally define the quantum metric $q_{\mu\nu}$ we need two mathematical inputs[11]:

1. Geodesic distances are modified in order to have a lorentz invariant lower bound. As already stated this is summarized in the replacement $\sigma^2(x,x') \to S(\sigma^2)$ such that $S(0) = \pm l_0^2 \neq 0$. The precise structure of $S(\sigma^2)$ must be determined by a complete theory of quantum gravity and so we need to be the most general as we can.

2. The modified d'Alambertian operator $\tilde{\Box}$ obtained from the qmetric must yields to the modification of two-point functions (green functions) in flat spacetime $G(x,x') = G(\sigma^2) \to \tilde{G}(x,x') = \tilde{G}(\sigma^2) = G(S(\sigma^2)$.

The point (2) is required by the fact that a minimal length in spacetime distances will act as a universal regulator in UV divergences which affect quantum field theory. To implement the first point we must start from a modification of (10):

$$g^{\mu\nu}(x)\partial_\mu\Omega\partial_\nu\Omega = 2\Omega \longrightarrow q^{\mu\nu}\partial_\mu S\partial_\nu S = 4S \tag{37}$$

In this sense the qmetric $q_{\mu\nu}$ is that "metric" which gives S as geodesic distance. The general form of the qmetric can be written as [4]:

$$q^{\mu\nu} = A^{-1}g^{\mu\nu} + \epsilon Q u^\mu u^\nu = A^{-1}h^{\mu\nu} + \epsilon(Q + A^{-1})u^\mu u^\nu \tag{38}$$

with the corresponding covariant components:

$$q_{\mu\nu} = A g_{\mu\nu} - \epsilon B u_\mu u_\nu \tag{39}$$

where $\epsilon = u^\alpha u_\alpha$, $B \equiv QA/(A^{-1} + Q)$, $h^{\mu\nu} = g^{\mu\nu} - \epsilon u^\mu u^\nu$ is the induced metric on $\Sigma_{x'}$ and A, Q are functions of x, x' to be determined. This ansatz is motivated by the fact that the qmetric must be s symmetric 2-rank bitensor depending on the metric tensor $g_{\mu\nu}$ and on the vector tangent to the geodesic segment u^μ.
Substituting the ansatz in the equation (37) we can partially fix[11] (for more details refer to appendix C)

$$\alpha \equiv A^{-1} + Q = \frac{1}{\sigma^2}\frac{S_{l_0}(\sigma^2)}{S'^2_{l_0}(\sigma^2)} \tag{40}$$

where a prime index means differentiation w.r.t. σ^2. As we said before we don't want to fix exactly the function S_{l_0}, we only require that:

(i) Minimal length condition : $S_{l_0} = \pm l_0^2$

(ii) $S_0(\sigma^2) = \sigma^2$

(iii) $\frac{S_{l_0}}{S'^2_{l_0}}(\sigma^2 = 0) < \infty$

At this point we have:

$$q^{\mu\nu} = A^{-1}h^{\mu\nu} + \epsilon\left(\frac{1}{\sigma^2}\frac{S_{l_0}(\sigma^2)}{S'^2_{l_0}(\sigma^2)}\right)u^\mu u^\nu \tag{41}$$

The input (2) will fix completely the qmetric[11] determining the value of A. We need to use the following relation:

$$\sqrt{|\det\, q|} = \frac{A^{\frac{d-1}{2}}}{\sqrt{A^{-1}+Q}}\sqrt{|g|} \tag{42}$$

following from the application of the *matrix determinant lemma*[6] $\det(\mathbf{M}+\mathbf{u}\mathbf{v}^{\mathbf{t}}) = \det(M)\times(1+\mathbf{v}^{\mathbf{t}}\mathbf{M}^{-1}\mathbf{u})$ with $\mathbf{M}$ being an invertible matrix and $\mathbf{u},\mathbf{v}$ column vectors (of the same dimension of $\mathbf{M}$) with the qmetric in the form given by (39) (more details in appendix C). At this point one can compute $\tilde{\Box}G(S(\sigma^2)$ and imposing this quantity equal to zero when $x \neq x'$. This would end up with the following differential equation[11]:

$$\frac{d}{d\sigma^2}\ln\left(\frac{A}{S_{l_0}/\sigma^2}\left(\frac{\Delta_s}{\Delta}\right)^{\frac{2}{d-1}}\right) = 0 \tag{43}$$

whose solution is given by

$$A = \frac{S_{l_0}}{\sigma^2}\left(\frac{\Delta}{\Delta_S}\right)^{\frac{2}{d-1}} \tag{44}$$

where the constant of integration is fixed by the condition $A = 1$ when $S_{l_0}(\sigma^2) = \sigma^2$. Thus we can write the complete form of the qmetric form timelike/spacelike separated events as:

$$q_{\mu\nu} = \frac{S_{l_0}}{\sigma^2}\left(\frac{\Delta}{\Delta_S}\right)^{\frac{2}{d-1}} g_{\mu\nu} + \epsilon\left[\frac{\sigma^2 S_{l_0}'^2}{S_{l_0}} - \frac{S_{l_0}}{\sigma^2}\left(\frac{\Delta}{\Delta_S}\right)^{\frac{2}{d-1}}\right]u_\mu u_\nu \tag{45}$$

$$q^{\mu\nu} = \frac{\sigma^2}{S_{l_0}}\left(\frac{\Delta}{\Delta_S}\right)^{-\frac{2}{d-1}} g^{\mu\nu} + \epsilon\left[\frac{S_{l_0}}{\sigma^2 S_{l_0}'^2} - \frac{\sigma^2}{S_{l_0}}\left(\frac{\Delta}{\Delta_S}\right)^{-\frac{2}{d-1}}\right]u^\mu u^\nu \tag{46}$$

where $u_\mu = g_{\mu\nu}u^\nu$ and Δ_S defined as Δ with the replacement $\sigma^2 \to S(\sigma^2)$. We notice that the qmetric has a singular behaviour in the limit $\sigma^2 \to 0$ while in the limit $l_0 \to 0$ it reduces to $q_{\mu\nu}(x,x') \to g_{\mu\nu}(x')$.

3.3 The Ricci biscalar

Once we have introduced the qmetric we can build biscalar quantities such as the Ricci biscalar $\tilde{R}(x,x')$ which bears the same algebraic relation to the qmetric like the usual Ricci scalar $R(x)$ has with respect to the standard metric. This is the simplest curvature invariant associated to any spacetime [2]. First of all we notice that the qmetric (45) is *disformally* coupled to the metric. In [3] relations between geometrical quantities belonging to disformally coupled metrics are explorated. We use the same relations between Ricci scalars belonging to disformally coupled metrics (see appendix D):

$$\tilde{R}(x,x') = A^{-1}R(x') + \epsilon(\alpha - A^{-1})\xi_d - \epsilon\alpha\xi_c \tag{47}$$

with

$$\xi_d = 2R_{\mu\nu}u^\mu u^\nu + K_{\mu\nu}K^{\mu\nu} - K^2 = \epsilon(R - R_\Sigma) \tag{48}$$

$$\xi_c = \epsilon\left[2(d-1)A^{-\frac{1}{2}}\Box A^{\frac{1}{2}} + (d-1)(d-4)A^{-1}(\nabla\sqrt{A})^2\right] + (K + (d-1)u^\alpha\nabla_\alpha \ln\sqrt{A})u^\beta\nabla_\beta \ln\alpha A \tag{49}$$

Now the exact form of A and α can be substituted and using the identities (34) we get the final form:

$$\begin{aligned}\tilde{R}(x,x') = &\underbrace{\left[\frac{\sigma^2}{S_{l_0}}\left(\frac{\Delta}{\Delta_S}\right)^{-2/(d-1)} R_\Sigma - \frac{(d-1)(d-2)}{S_{l_0}} + 4(d+1)\frac{d\ln\Delta_S}{dS_{l_0}}\right]}_{Q_0} \\ &\underbrace{-\frac{S_{l_0}}{\lambda^2 S_{l_0}'^2}\left[K_{\mu\nu}K^{\mu\nu} - \frac{1}{d-1}K^2\right]}_{Q_K} + \underbrace{4S_{l_0}\left[-\frac{d}{d-1}\left(\frac{d\ln\Delta_S}{dS_{l_0}}\right)^2 + 2\left(\frac{d^2\ln\Delta_S}{dS_{l_0}^2}\right)\right]}_{Q_\Delta}\end{aligned} \tag{50}$$

This is an exact result, in the sense that no Taylor expansions have been used: this expression holds the key to understand non-perturbative effects of a zero point length[2]. We notice that the Ricci biscalar is completely determined by the geodesic structure of the spacetime, namely by the intrinsic curvature R_Σ, the extrinsic curvature $K_{\mu\nu}$ and the Van Vleck determinant Δ. We also have that the Ricci biscalar reduces to the usual ricci scalar in the $l_0 \to 0$ limit.

3.4 The $l_0 \to 0$ limit

The qmetric is singular in $\sigma^2 \to 0$ limit: it is unclear if local scalars constructed from $q_{\mu\nu}$ reduces in the limit $l_0 \to 0$ to the scalars constructed from $g_{\mu\nu}$. From the Ricci biscalars we can construct an "effective" ricci scalar by taking the coincidence limit $\tilde{R}(x') = \left[\tilde{R}(x,x')\right] = lim_{x\to x'}\tilde{R}(x,x')$. At this point we can ask if in the $l_0 \to 0$ limit $\tilde{R}(x')$ reduces to the usual ricci scalar $R(x')$ computed classicaly from $g_{\mu\nu}(x')$. This will be true if:

$$\tilde{R}(x') = R(x') + o(l_0) \tag{51}$$

This would be the case if the two limits $x \to x'$ and $l_0 \to 0$ commute. However this turn out to not be the case. In order to compute the coincidence limit we need to consider smooth regions of spacetime since covariant taylor expansions are needed (see appendix (B)). We notice that in (50) there are three distinct terms Q_0, Q_K and Q_Δ We see separately how to evaluate their coincidence limits. For the Q_0 limit we use the taylor expansion in appendix B. We have that:

$$\lim_{l_0\to 0}\lim_{\sigma^2\to 0}\frac{d}{dS_{l_0}}\ln\Delta_S = \frac{1}{6}\epsilon\left[R_{\mu\nu}u^\mu u^\nu\right](x') \tag{52}$$

while using the fact $\Delta(0) = 1$

$$\lim_{\sigma^2\to 0}\left[\frac{\sigma^2}{S_{l_0}}\left(\frac{\Delta}{\Delta_S}\right)^{-2/(d-1)}R_\Sigma - \frac{(d-1)(d-2)}{S_{l_0}}\right] = \frac{(d-1)(d-2)}{S_{l_0}(0)}\left(\Delta_{l_0}^{\frac{2}{d-1}} - 1\right) \tag{53}$$

where $\Delta_{l_0}^{1/2} = 1 + \frac{1}{12}\epsilon l_0^2\left[R_{\mu\nu}u^\mu u^\nu\right](x') + \ldots$. To evaluate $l_0 \to 0$ limit we use *de l'hopital rule*:

$$\lim_{l_0\to 0}\frac{(d-1)(d-2)}{S_{l_0}(0)}\left(\Delta_{l_0}^{\frac{2}{d-1}} - 1\right) = \lim_{l_0\to 0}\frac{(d-1)(d-2)}{\partial_{l_0^2}S_{l_0}(0)}\partial_{l_0^2}\Delta_{l_0}^{\frac{2}{d-1}} = \frac{1}{3}(d-2)\epsilon\left[R_{\mu\nu}u^\mu u^\nu\right](x') \tag{54}$$

Thus we have that:

$$\lim_{l_0\to 0}Q_0 = \epsilon\left[\frac{4(d-1)}{6} + \frac{d-2}{3}\right]\left[R_{\mu\nu}u^\mu u^\nu\right](x') = \epsilon d\left[R_{\mu\nu}u^\mu u^\nu\right](x') \tag{55}$$

It can be shown Q_K is zero in coincidence limit as we can see in appendix B. The final term Q_Δ provides only terms at order $o(l_0^2)$ thus it does not contribute in the $l_0 \to 0$ limit. We can conclude that the effective Ricci scalar $\tilde{R}(x')$ does not reduces to $R(x')$ when $l_0 \to 0$ limit is taken:

$$\lim_{l_0\to 0}\tilde{R}(x') = \epsilon d R_{\mu\nu}(x')u^\mu u^\nu \tag{56}$$

4 Discussion and implications

We saw how to incorporate the existence of a minimal length, say l_0, in a semiclassical description of the spacetime. In order to do so we introduced the notion of bitensors which are able to naturally incorporate a lower bound on spacetime intervals with a particular attention to the Synge's world function and the Van Vleck determinant(VVD) which turned out to play an important role in this attempt: in fact the spacetime geometry can be reconstructed by its geodesics behaviour and bitensorial quantities associated to it. In absence of a minimal length this description turns out to reduce to the standard one of tensorial quantities defined on a manifold. However with the introduction of l_0 it seems that at mesoscopic scales the structure of the spacetime is better described by a 2-rank bitensor $q_{\mu\nu}$ called quantum metric, or simply qmetric, which depends on the world function and on the VVD. In such a description, trying to keep things as general as possible, the minimal length is implemented as a modification of the distance function between spacetime points, without resorting to the details of a more fundamental and complete theory of quantum gravity. Having introduced such a minimal length means that we can't localize spacetime events with an accuracy better than l_0 which is what we expect from quantum uncertainty: it seems that this non-local deformation of spacetime at small scales is due to the quantum fluctuations of yet unknown degrees of freedom characterizing quantum gravity. In fact because of quantum fluctuations at small scales we can't expect that any local tensorial object can be able to describe the quantum structure of spacetime [5].

4.1 Implication for the emergent gravity paradigm

Once we have introduced the qmetric we saw the expression for the Ricci biscalar $\tilde{R}(x, x')$ (which bears the same algebraic relation to qmetric as the usual Ricci scalar does to the usual metric [6]) and for its coincidence limit (a kind of effective Ricci scalar) $\tilde{R}(x')$ which in the "classical" limit $l_0 \to 0$ does not coincide to the classical Ricci scalar but it is proportional to:

$$\lim_{l_0 \to 0} \tilde{R}(x') \propto R_{\mu\nu}(x')u^{\mu}(x')u^{\nu}(x') \tag{57}$$

with $u^{\mu}$is an arbitrary vector field with constant norm. The quantity on the right handside is nothing but the *entropy functional* introduced in the emergent gravity paradigm. The emergent paradigm provides an alternative view on gravity's nature: basically it describes gravity as an emergent phenomenum like fluid dynamics or elasticity. The standard description of GR regards the metric as the dynamical variable of the theory: gravitational fields equation can be determined from a variational principle involving the Einstein-Hilbert action built with the gravitational lagrangian $\mathcal{L}_{EH} = \sqrt{-g}R$, with R the Ricci scalar (in presence of matter we need to add matter lagrangian). In the emergent paradigm gravitational field equations are derived from a variational principle based on the entropy functional $R_{\mu\nu}n^{\mu}n^{\nu}$: extremizing this quantity with respect to all vector fields n^{α} leads to a constraint on the background metric which are the Einstein equations themselves (if there's matter we need to add matterentropy functional) [4]. In this view the standard gravitational dynamics arises as the thermodynamic limit of some underlying statistical physics which deals with the yet unknown spacetime micro degrees of freedom. Adopting this view, in which the metric does not play the role of a dynamical variable even at the classical level, any attempt to construct a quantum gravity theory based on E.H. lagrangian is not different from trying to directly quantize fluidodynamics or elasticity: it's reasonable that metric has no role in a quantum theory of gravity [4]. The problem with emergent gravity is that it is a top-down approach. However the result (57) seems to promote this paradigm. In fact, once we implement l_0 as a lower bound on spacetime intervals, we can think to derive a dynamical variational principle starting from the Ricci biscalar $\tilde{R}(x, x')$ associated to the qmetric as in the standard view we use the Ricci scalar associated to the metric. In order to get a local variational principle we compute the coincidence limit and we get the effective Ricci scalar $\tilde{R} = kR_{\alpha\beta}n^{\alpha}n^{\beta} + o(l_0^2)$ with k proportionality constant. Thus in the $l_0 \to 0$ limit it is reasonable to use the entropy functional as an effective lagrangian at classical level instead of the Einstein Hilbert one

$$\mathcal{L}_{EH} = R \longrightarrow \mathcal{L}_{eff} = kR_{\alpha\beta}n^{\alpha}n^{\beta} \tag{58}$$

where the vector field n^{α} is the quantity to be varied. Thus with the introduction of quantum metric the variational principle of emergent gravity seems to naturally arises higlighting the robustness of the model: emergent nature of gravitational dynamics, gravitational action, and space and time itself, might be an unmistakable relic of a quantum spacetime endowed with a zero point length[5]. Moreover the emergent approach was motivated in the first place by the thermal behaviour of Rindler horizons: this paradigm nicely incorporate the two main results of semiclassical gravity. It's also worth saying that emergent gravity paradigm has a direct application on the cosmological constant problem as described in [9].

4.2 Spacetime singularities: modification of Raychaudhuri equation

Raychaudhury equation governs the flow of the geodesics in a given spacetime manifold: it is purely a geometric equation which determines the rate of change of area/volume along a geodesics congruence connecting it to the shear and twist tensors and to the projection of the Ricci tensor along the geodesics flow. Only when we try to relate the Ricci tensor to the stress-energy tensor gravitational equations come into play[2]. In particular under the strong energy condition(SEC) it can be shown that geodesics converge (in the past or in the future) to a singular focal point (caustic) due to the attractiveness of gravity. Under certain conditions (which are reassumed in the general Hawking-Penrose singularity theorem[1]) this caustics lead to the formation of spacetime singularities. It is reasonable to ask if a minimal length in spacetime could avoid caustics formation with possible avoidance of spacetime singularities formation. This issue is addressed in [2] where it is presented a modified Raychaudhuri equation built from the qmetric both for timelike/spacelike and null geodesics (here we brielfy resume the results for timelike case).

The Raychaudhuri equation tells us the evolution of the expansion of a geodesics congruence θ along the geodesic congruence itself. The expansion is defined as the trace of the extrinsic curvature

$\theta = K^{\mu}{}_{\mu} = u^{\mu}_{;\mu}$ where u^{μ} is the vector tangent to the congruence. Classically the Raychaudhury equation in a d-dimensional spacetime, in absence of twist, reads as:

$$\frac{d\theta}{d\sigma} = -\frac{1}{d-1}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - R_{\mu\nu}u^{\mu}u^{\nu} \tag{59}$$

where $\sigma_{\mu\nu} = (K_{\mu\nu} + K_{\nu\mu})/2 - \theta h_{\mu\nu}/(d-1)$ is the shear thensor with $h_{\mu\nu} = g_{\mu\nu} + u_{\mu}u_{\nu}$ the transverse metric. In the coincidence limit both θ and its derivative diverges. We can see what happens if we use quantity built from the qmetric. For the expansion we have(see appendix D):

$$\theta_q = \sqrt{\alpha}\left[\theta + (d-1)\frac{d}{d\sigma}\ln\sqrt{A}\right] \tag{60}$$

with α defined in (40), A defined in (44) and σ the geodesic distance. Now $\theta_q = \nabla^{(q)}_{\alpha} n^{\alpha}_{(q)}$ with $\nabla^{(q)}_{\alpha}$ the covariant derivative associated to the qmetric leading to its own affine connection $\Gamma^{(q)\mu}_{\alpha\beta} = \Gamma^{\mu}_{\alpha\beta} + \frac{1}{2}q^{\mu\gamma}\left[q_{\gamma\beta,\alpha} + q_{\alpha\gamma,\beta} - q_{\alpha\beta,\gamma}\right]$ where $\Gamma^{\mu}_{\alpha\beta}$ is the affine connection compatible with $g_{\mu\nu}$ and $n^{\alpha}_{(q)} = (1/2\sqrt{\epsilon S}\partial S)$ is the tangent to the geodesic according to the qmetric affine parametrization (S is the modified geodesic distance function). Expressing $\alpha = (d\sqrt{\epsilon S}/d\sigma)^{-2}$ we can write

$$\left(\frac{d\theta}{d\sigma}\right)_q = \frac{d\theta_q}{d\sqrt{\epsilon S}} = \alpha\frac{d\theta}{d\sigma} + (d-1)\alpha\frac{d^2\ln\sqrt{A}}{d\sigma^2} + \frac{1}{2}\frac{d\alpha}{d\sigma}\left[\theta + (d-1)\frac{d}{d\sigma}\ln\sqrt{A}\right] \tag{61}$$

Using the expression for A and the relation between VVD and the expansion θ in 34 we get:

$$\left(\frac{d\theta}{d\sigma}\right)_q = -\frac{d-1}{(\sqrt{\epsilon S})^2} - \frac{d^2\ln\Delta_S}{d\sqrt{\epsilon S}^2} \tag{62}$$

Thus we can relate the rate of expansion of space-like/time-like geodesics in the presence of zero point length with the modified geodesic distance and modified Van Vleck determinant associated with the qmetric[2]. Now we can write the generalization of the Raychaudhuri equation:

$$-\frac{d-1}{(\sqrt{\epsilon S})^2} - \frac{d^2\ln\Delta_S}{d\sqrt{\epsilon S}^2} = \left(\frac{d\theta}{d\sigma}\right)_q = -\frac{1}{d-1}\theta_q^2 - \sigma^{(q)}_{\mu\nu}\sigma^{\mu\nu}_{(q)} - R^{(q)}_{\mu\nu}n^{\mu}_{(q)}n^{\nu}_{(q)} \tag{63}$$

It can be proven [2]that both sides of modified Raychaudhuri equation remain finite in the coincidence limit: it seems in the context of qmetric geodesic convergence can be avoided. However this result relies on taylor expansion, thus in spacetime regions which are enough smooth. Near an already formed singularity one would need the exact form of Synge's world function and Van vleck determinant which is not an easy task. In [2] it is suggested to find a non-covariant expansion in a suitable set of coordinates near the singularity.

A Expansion of bitensors near coincidence

We want to express a bitensor $B_{\alpha'\beta'}(x,x')$ near coincidence as an expansion in terms of $\Omega^{\alpha'}$:

$$B_{\alpha'\beta'}(x,x') = A_{\alpha'\beta'} + A_{\alpha'\beta'\gamma'}\Omega^{\gamma'} + \frac{1}{2}A_{\alpha'\beta'\gamma'\delta'}\Omega^{\gamma'}\Omega^{\delta'} + \dots \tag{64}$$

where all $A_{\dots}$ are tensorial functions in x'. They can be determined via coincidence limits[10]:

$$\begin{aligned} A_{\alpha'\beta'} &= [B_{\alpha'\beta'}] \\ A_{\alpha'\beta'\gamma'} &= [B_{\alpha'\beta';\gamma'}] - A_{\alpha'\beta';\gamma'} \\ A_{\alpha'\beta'\gamma'\delta'} &= [B_{\alpha'\beta';\gamma';\delta'}] - A_{\alpha'\beta';\gamma';\delta'} - A_{\alpha'\beta'\gamma';\delta'} - A_{\alpha'\beta'\delta';\gamma'} \end{aligned}$$

For example we have:

$$\begin{aligned} \Omega_{\alpha'\beta'} &= g_{\alpha'\beta'} - \frac{1}{3}R_{\alpha'\gamma'\beta'\delta'}\Omega^{\gamma'}\Omega^{\delta'} + \dots \\ \Omega_{\alpha\beta'} &= \Pi^{\alpha'}_{\alpha}\left(g_{\alpha'\beta'} - \frac{1}{6}R_{\alpha'\gamma'\beta'\delta'}\Omega^{\gamma'}\Omega^{\delta'}\right) + \dots \\ \Omega_{\alpha\beta} &= \Pi^{\alpha'}_{\alpha}\Pi^{\beta'}_{\beta}\left(g_{\alpha'\beta'} - \frac{1}{3}R_{\alpha'\gamma'\beta'\delta'}\Omega^{\gamma'}\Omega^{\delta'}\right) + \dots \end{aligned} \tag{65}$$

Moreover taking the trace of the last one we get $\Omega^{\mu}{}_{\mu} = 4 - \frac{1}{3}R_{\alpha'\beta'}\Omega^{\alpha'}\Omega^{\beta'} + \dots$ which implies $\theta^* = 3 - \frac{1}{3}R_{\alpha'\beta'}\Omega^{\alpha'}\Omega^{\beta'} + \dots$. From 65 we can infer VVD expansion 29.

B Usefull taylor expansions

Here are reported taylor expansions of geometrical quantity needed in the computation of the coincidence limit of the Ricci biscalar. Consider two points x, x' linked by a geodesic parametrized by λ whose tangent vector is:

$$u_\alpha = \frac{\nabla_\alpha \sigma^2}{2\sqrt{\epsilon\sigma^2}} \tag{66}$$

ehere σ^2 is the squared geodesic distance. The extrinsic curvature of the surface normal to u_α is given by:

$$K_{\mu\nu} = \nabla_\mu u_\nu = \frac{\nabla_\mu \nabla_\nu (\sigma^2/2) - \epsilon u_\mu u_\nu}{\sqrt{\epsilon\sigma^2}} \tag{67}$$

The key expansion is the following [11]:

$$\nabla_\mu \nabla_\nu (\sigma^2/2) = g_{\mu\nu} - \frac{\lambda^2}{3}\xi_{\mu\nu} + \frac{\lambda^3}{12} u^\alpha \nabla_\alpha \xi_{\mu\nu} - \frac{\lambda^4}{60}\left(u^\alpha u^\beta \nabla_\alpha \nabla_\beta \xi_{\mu\nu} + \frac{4}{3}\xi_{\gamma\mu}\xi^\gamma{}_\nu\right) + o(\lambda^5) \tag{68}$$

where $\xi_{\mu\nu} = R_{\mu\alpha\nu\beta}u^\alpha u^\beta$. From this expansion it follows that in d dimensions[11]:

$$\Delta^{\frac{1}{2}} = 1 + \frac{1}{12}\lambda^2 R_{\mu\nu}u^\mu u^\nu + o(\lambda^3) \tag{69}$$

$$K_{\mu\nu} = \frac{1}{\lambda}h_{\mu\nu} - \frac{1}{3}\lambda\xi_{\mu\nu} + \frac{1}{12}\lambda^2 u^\alpha \nabla_\alpha \xi_{\mu\nu} - \frac{1}{60}\lambda^3 F_{\mu\nu} + o(\lambda^4) \tag{70}$$

$$K = \frac{d-1}{\lambda} - \frac{1}{3}\lambda\xi + \frac{1}{12}\lambda^2 u^\alpha \nabla_\alpha \xi - \frac{1}{60}\lambda^3 F + o(\lambda^4) \tag{71}$$

$$R_\Sigma = \frac{\epsilon(d-1)(d-2)}{\lambda^2} + R - \frac{2\epsilon(d+1)}{3}\xi + o(\lambda) \tag{72}$$

where $F_{\mu\nu} = u^\alpha u^\beta \nabla_\alpha \nabla_\beta \xi_{\mu\nu} + \frac{4}{3}\xi_{\mu\tau}\xi^\tau{}_\nu$. From the expansions above we can construct the following expression:

$$\begin{aligned} K^2_{\mu\nu} - \eta K^2 = (1-\eta(d-1))\left[\frac{d-1}{\lambda^2} - \frac{2}{3}\xi + \frac{1}{6}\lambda u^\alpha \nabla_\alpha \xi - \frac{1}{30}\lambda^2\left(u^\alpha u^\beta \nabla_\alpha \nabla_\beta \xi - \frac{4}{3}\xi^2_{\mu\nu}\right)\right] \\ + \frac{1}{9}\lambda^2\left(\xi^2_{\mu\nu} - \eta\xi^2\right) + o(\lambda^3) \end{aligned}$$

We notice that in the limit $\lambda \to 0$ the above quantity is zero only if $\eta = \frac{1}{d-1}$. This is indeed the case when the coincidence limit of the ricci biscalar is taken providing a non divergent result.

C Details of qmetric derivation

Here there are more details of the computations of the bifunctions α and A appearing in the final form of the qmetric. To determine α we substitute in 37 the ansatz for the qmetric 38 getting

$$\begin{aligned}
&\left(A^{-1}g^{\mu\nu} + \epsilon Q u^\mu u^\nu\right)\partial_\mu S \partial_\nu S = 4S \\
&A^{-1}g^{\mu\nu}\partial_\mu S\partial_\nu S + \epsilon Q u^\mu u^\nu \partial_\mu S \partial_\nu S = 4S \\
&A^{-1}g^{\mu\nu}\partial_\mu\sigma^2\partial_\nu\sigma^2\left(\frac{\partial S}{\partial(\sigma^2)}\right)^2 + \epsilon Q u^\mu u^\nu \partial_\mu \sigma^2 \partial_\nu \sigma^2 \left(\frac{\partial S}{\partial(\sigma^2)}\right)^2 = 4S \\
&4A^{-1}g^{\mu\nu}\Omega_\mu\Omega_\nu\left(\frac{\partial S}{\partial(\sigma^2)}\right)^2 + 4\epsilon Q\frac{\Omega^\mu}{\sqrt{2\epsilon\Omega}}\frac{\Omega^\nu}{\sqrt{2\epsilon\Omega}}\Omega_\mu\Omega_\nu\left(\frac{\partial S}{\partial(\sigma^2)}\right)^2 = 4S \\
&2A^{-1}\Omega\left(\frac{\partial S}{\partial(\sigma^2)}\right)^2 + 2Q\Omega\left(\frac{\partial S}{\partial(\sigma^2)}\right)^2 = S \\
&(A^{-1}+Q)\sigma^2\left(\frac{\partial S}{\partial(\sigma^2)}\right)^2 = S \\
&\alpha \equiv (A^{-1}+Q) = \frac{S}{\sigma^2\left(\frac{\partial S}{\partial(\sigma^2)}\right)^2}
\end{aligned}$$

A direct appication of the *matrix determinant lemma* to the q metric in the form 39 gives:

$$det(\mathbf{q}) = det|A\mathbf{g}|(1 - \frac{B}{A}\epsilon g^{\mu\nu}u_\mu u_\nu) \tag{73}$$

Using that $g^{\mu\nu}u_\mu u_\nu = \epsilon$ and $B = \frac{QA^2}{1+QA}$ we get

$$det(\mathbf{q}) = A^d det|\mathbf{g}| \left(1 - \frac{Q}{Q + A^{-1}}\right) =$$
$$= \frac{A^{d-1}}{A^{-1} + Q} det|\mathbf{g}|$$

Using this one could compute $\tilde{\Box} = \frac{1}{\sqrt{|q|}}\partial_\mu(\sqrt{|q|}q^{\mu\nu}\partial_\nu)$. It is simpler to consider maximally symmetric spaces in which we have a simplified form [11]:

$$\tilde{\Box} = \alpha\Box_g + 2\alpha\sigma^2 \left[\ln \alpha A^{d-1}\right]' \frac{\partial}{\partial\sigma^2} \tag{74}$$

with in maximally symmetric spacetimes:

$$\Box_g = \frac{\partial^2}{\partial\sigma^2} + \left(\frac{\partial}{\partial\sigma}\ln\Delta^{-1} + \frac{d-1}{\sigma}\right)\frac{\partial}{\partial\sigma} \tag{75}$$

Then it could be imposed the condition $\tilde{\Box}G(S(\sigma^2)) = 0$ when $\Box_g G(\sigma^2) = 0$ (i.e. when $x \neq x'$) founding the differential equation 43.

D Relations between geometrical quantities belonging to disformally coupled metrics

Consider two metrics $g_{\mu\nu}, \tilde{g}_{\mu\nu}$ on a given manifold. We say that they are dispformally coupled if they are related in the following way[3]:

$$\tilde{g}_{\mu\nu} = Ag_{\mu\nu} - \epsilon Bu_\mu u_\nu \tag{76}$$

where u_μ is of the form 13. In [3] are derived relations between geometrical quantities belonging to disformally coupled metrics which are important in the derivation of quantity associated to the qmetric. In particular:

$$\tilde{h}_{\mu\nu} = Ah_{\mu\nu} \text{ induced metrics on orthogonal surfaces to } u^\alpha \tag{77}$$
$$\tilde{R}_\Sigma = A^{-1}R_\Sigma \text{ Induced Ricci scalar} \tag{78}$$
$$\tilde{K}_{\mu\nu} = A\sqrt{\alpha}\left[K_{\mu\nu} + (u^\alpha\nabla_\alpha \ln\sqrt{A})h_{\mu\nu}\right] \tag{79}$$
$$\tilde{K} = \sqrt{\alpha}\left[K + (d-1)u^\alpha\nabla_\alpha \ln\sqrt{A}\right] \tag{80}$$

Using Gauss-Codazzi equations we can reconstruct $\tilde{R}$ [3]:

$$\tilde{R} = \tilde{R}_\Sigma - \epsilon\left(\tilde{K}^2 + \tilde{K}^2_{\mu\nu}\right) - 2\epsilon U^\alpha\tilde{\nabla}_\alpha\tilde{K} + 2\epsilon\tilde{\nabla}_\alpha\tilde{a}^\alpha \tag{81}$$

where $U^\alpha = \sqrt{\alpha}u^\alpha$ and $\tilde{a}^\alpha = U^\beta\tilde{\nabla}_\beta U^\alpha$. After some algebra one get the form 47.

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