Galilean relativity principle :- The form of the laws of motions do not change in going from one frame to another moving with constant velocity

$x' = x - vt$ ← position of particle transformed

$w' = w - v$ ← velocity of the particle transformed

But,

1) The Maxwell's equations do not remain invariant under Galilean transformation and seem to violate the relativity principle

2) Expt. by Michelson - Morley,

velocity $c' = c$ (speed of light) ← do not change from frame to frame

This does not comply with $w' = w - v$

* Lorentz - Einstein Relativity principle :- (New Relativity principle)

I) The form of the equations of motion remain the same in moving from one frame to another, if the second frame is moving with constant velocity wrt the first

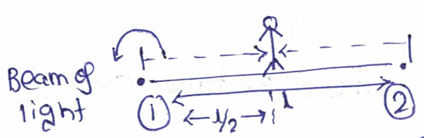
new transform: $x' = \dfrac{x - vt}{\sqrt{1 - v^2/c^2}}$

II) c does not change from one frame to another

We go from Lorentz - Einstein to Galilean Relativity when $v \ll c$

* Consequence of (II) : Events which are simultaneous in one frame are not ~~only~~ any more simultaneous in another inertial frame

event is charecterised by position and time

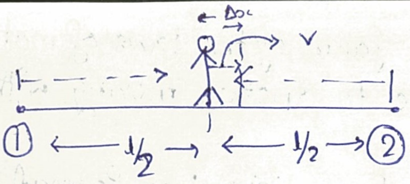


| frame 1,

time from (1) to the person, $\Delta t_1 = \dfrac{l/2}{c}$

time from (2) to the person, $\Delta t_2 = \dfrac{l/2}{c}$

now, $\Delta t_1 = \Delta t_2$: the events are simultaneous



frame 2, moves with constant velocity v wrt frame 1

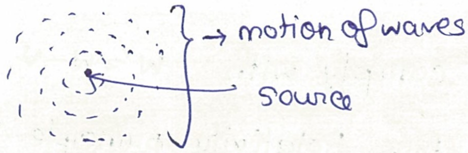
$$\therefore \Delta t_1 = \frac{l/2}{c} + \frac{\Delta x}{v}$$

$$\Delta t_2 = \frac{l/2}{c} - \frac{\Delta x}{v}$$

$$\therefore \Delta t_1 \neq \Delta t_2$$

those events are not simultaneous

* Let's consider a spherical wave front whose equations of motion are analysed



equation of the front spherical wave (light) : $x^2+y^2+z^2 = c^2t^2$

$$\left.\begin{aligned} \therefore\ x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \text{ If frame 2 moves in the } x\text{- direction}$$

We want the spherical wave in the 1st frame to go into the spherical wave $x'^2 + y'^2 + z'^2 = c^2t'^2$ in the Galilean transformation according to the 1st postulate

* $x'^2 + y'^2 + z'^2 = c^{*2}t'^2$ [c remains the same]

$\Rightarrow (x - vt)^2 + y^2 + z^2 = c^2t^2$

$\Rightarrow x^2 \underbrace{- 2xvt + v^2t^2}_{\text{these two extra pieces, contain } t} + y^2 + z^2 = c^2t^2$

we could try a new transformation, where we take care of time,

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t + fx \end{aligned}$$

, f needs to be determined

now, $x^2 - 2xvt + v^2t^2 + y^2 + z^2 = c^2t^2 + 2c^2ftx + c^2f^2x^2$

if we choose, $f = -v/c^2$, $f^2 = v^2/c^4$

$$\therefore x^2 - 2\cancel{xvt} + v^2t^2 + y^2 + z^2 = c^2t^2 - 2\cancel{vtx} + x^2v^2/c^2$$

$$\Rightarrow x^2(1 - v^2/c^2) + y^2 + z^2 = c^2t^2(1 - v^2/c^2)$$

we still have extra pieces, $-x^2v^2/c^2$ and $-v^2t^2$

We write, x' as, $x' = \dfrac{x - vt}{\sqrt{1 - v^2/c^2}}$

and $t' = \dfrac{t + fx}{\sqrt{1 - v^2/c^2}}$ } t is not an absolute variable anymore

∴ and we get rid of those extra terms

Now, the correct transformation,

$$\boxed{\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned}}$$

Lorentz - Einstein transformation

if $v \ll c$, $x' = x - vt + \theta(v^2/c^2)$

$t' = t + O(v/c^2)$

* Now, let's check simultaneity :

we take, $t' = \dfrac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$. In the frame at rest (1st frame), in Galilean transform, Δt was $0 \rightarrow$ simultaneous, and $\Delta x \neq 0$

here, $\Delta t' = \dfrac{\Delta t - \frac{v}{c^2}\Delta x}{\sqrt{1 - v^2/c^2}}$

$$= \frac{-v/c^2 \, \Delta x}{\sqrt{1 - v^2/c^2}}$$

But in the Lorentzian case, $\Delta t' \neq 0$

* If $v \ll c$, $\Delta t \rightarrow 0$: Galilean Transform

* Composition of velocities :

$\underbrace{\omega'}_{\text{velocity of particle in frame 2}} = \underbrace{\omega}_{\text{velocity of particle in frame 1}} - v$ → velocity of frame 2 wrt frame 1 : Galilean way

Now composition of velocities in the Lorentzian way, to check if ~~the~~ the velocity of a photon, c remains consistent with the ~~theorem~~ postulate

~~x'~~= now in Lorentzian ~~frame~~ transforms,

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - v^2/c^2}}$$

$$dx' = \frac{dx - v\,dt}{\sqrt{1 - v^2/c^2}} \qquad dt' = \frac{dt - \frac{v}{c^2}dx}{\sqrt{1 - v^2/c^2}}$$

$$\therefore \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx}$$

$$= \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$, now, $\omega = \frac{dx}{dt}$, velocity of particle

$$\Rightarrow \boxed{\omega' = \frac{\omega - v}{1 - \frac{v}{c^2}\omega}}$$: Composition of velocity by Lorentz-Einstein

if $v \ll c$, $\omega' = \omega - v$: we go to the Galilean way

Now lets, consider the particle to be a photon that moves with velocity c in the laboratory frame

$$\omega = c$$

$$\therefore \omega' = \frac{c - v}{1 - \frac{v}{c^2}c} = \frac{c - v}{1 - \frac{v}{c}} = \frac{c - v}{c - v}\cdot c = c$$

$\therefore$ the new velocity, $\omega' = c$

$\therefore$ the velocity does not change → consistent with the second postulate of relativity.

* Composition of transverse velocity :-

$$\omega_y = \frac{dy}{dt}$$

now, ~~t~~ $t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - v^2/c^2}} \Rightarrow t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - v^2/c^2}}$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

proof : $t = t'\sqrt{1 - v^2/c^2} + \frac{v}{c^2}x$

$$= t'\sqrt{1 - v^2/c^2} + \frac{v}{c^2}\left[x'\sqrt{1 - v^2/c^2} + vt\right]$$

$$t = \left(t' + \frac{v}{c^2}x'\right)\sqrt{1-v^2/c^2} + \frac{v^2}{c^2}t$$

$$\Rightarrow t\left(1 - v^2/c^2\right) = \left(t' + \frac{v}{c^2}x'\right)\sqrt{1-v^2/c^2}$$

$$\Rightarrow \boxed{t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1-v^2/c^2}}}$$

now, $$dt = \frac{dt' + \frac{v}{c^2}dx'}{\sqrt{1-v^2/c^2}}$$

and $y' = y$, $\Rightarrow dy' = dy$

now, $$\omega_y = \frac{dy}{dt} = \frac{dy'}{\frac{dt' + \frac{v}{c^2}dx'}{\sqrt{1-v^2/c^2}}} = \frac{dy'/dt'}{\frac{1 + \frac{v}{c^2}\frac{dx'}{dt'}}{\sqrt{1-v^2/c^2}}}$$

$$= \frac{\omega_y'}{1 + \frac{v}{c^2}\omega_x'}\sqrt{1-v^2/c^2}$$

$$\boxed{\omega_y = \frac{\omega_y'}{1 + \frac{v}{c^2}\omega_x'}\sqrt{1-v^2/c^2}}$$, transverse velocity changes too because time changes

Even if the frame moves in the x-direction, the velocity changes in the y-direction too.

* Contraction of length :– (Another consequence of the Lorentzian transform)

We consider rest frame as the laboratory frame.

$L_0 = x_2 - x_1$

In the moving frame $\xrightarrow{v}$, $L = x_2'(t') - x_1'(t')$

now, $$x' = \frac{x - vt}{\sqrt{1-v^2/c^2}} \Rightarrow x = x'\left(\sqrt{1-v^2/c^2}\right) + vt$$

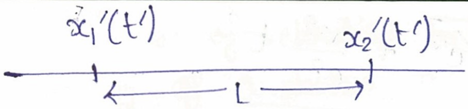
$$= x'\sqrt{1-v^2/c^2} + v\left[t'\sqrt{1-v^2/c^2} + \frac{v}{c^2}x\right]$$

$$= x'\sqrt{1-v^2/c^2} + vt'\sqrt{1-v^2/c^2} + \frac{v^2}{c^2}x$$

$$\Rightarrow x\left(1 - \frac{v^2}{c^2}\right) = (x' + vt')\sqrt{1-v^2/c^2}$$

$$\Rightarrow \boxed{x = \frac{x' + vt'}{\sqrt{1-v^2/c^2}}}$$

and $$\boxed{t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1-v^2/c^2}}}$$



, we measure the length L at time t' for the moving frame

now, $x_2 = \dfrac{x_2'(t') + vt'}{\sqrt{1-v^2/c^2}}$

and $x_1 = \dfrac{x_1'(t') + vt'}{\sqrt{1-v^2/c^2}}$

$$\therefore \quad x_2 - x_1 = \frac{x_2'(t') - x_1'(t')}{\sqrt{1-v^2/c^2}}$$

$\Rightarrow L_0 = \dfrac{L}{\sqrt{1-v^2/c^2}}$ ← length in the moving frame

↑ length in the laboratory frame

$\Rightarrow \boxed{L = L_0\sqrt{1-v^2/c^2}}$ since, $\sqrt{1-v^2/c^2} << 1$

$\boxed{L << L_0}$ ← length contraction

length was absolute in the absolute case but isn't so in the Lorentzian case.

c is the maximum ~~length~~ ~~distance~~ velocity one can reach, otherwise, $\sqrt{1-v^2/c^2}$ << 1 would be imaginary which isn't physical

* Time dilation :—

$\cancel{dt = dt' + \frac{v}{c^2}x}$ $\cancel{\Delta t = \dfrac{\Delta t' + \frac{v}{c^2}\Delta x'}{\sqrt{1-v^2/c^2}}}$

let's suppose some event takes place at a particular ~~space~~ point one after the another : one ~~in~~ at time t_1 and another at time t_2

$\therefore$ ~~$\Delta x = 0$~~ $\Delta x' = 0$ ~~xxx~~ instantly

$\therefore$ $\cancel{\boxed{\Delta t = \dfrac{\Delta t'}{\sqrt{1-v^2/c^2}}}}$

$\Delta t' = \dfrac{\Delta t - \frac{v}{c^2}\Delta x}{\sqrt{1-v^2/c^2}}$, $\Delta x = 0$

$\underbrace{\Delta t'}_{\text{moving frame}} = \dfrac{\Delta t}{\sqrt{1-v^2/c^2}}$ ← interval of time in the rest frame; $\sqrt{1-v^2/c^2}$ ← << 1

$\therefore \boxed{\Delta t' > \Delta t}$ time dilation

* <u>Invariant Quantities</u>: In the Galilean case, $t = t'$, $\Delta t = \Delta t'$ and $\Delta x = \Delta x'$ but position changes

In the Lorentzian case, [Minkowski's ~~di~~ work]

$$x^2 - c^2t^2 = x'^2 - c^2t'^2$$ [a combination of space and time that do not change under Lorentzian transformation]

now, $x'^2 - c^2t'^2$

$$= \left(\frac{x - vt}{\sqrt{1 - v^2/c^2}}\right)^2 - c^2\left(\frac{t - \frac{v}{c^2}x}{\sqrt{1 - v^2/c^2}}\right)^2$$

$$= \frac{x^2 - 2xvt + v^2t^2 - c^2\left(t^2 - \frac{2vt}{c^2}x + \frac{v^2}{c^4}x^2\right)}{1 - v^2/c^2}$$

$$= \frac{x^2 - 2xvt + v^2t^2 - c^2t^2 + 2xvt - \frac{v^2x^2}{c^2}}{1 - v^2/c^2}$$

$$= \frac{x^2 - c^2t^2 - \frac{x^2v^2}{c^2} + v^2t^2}{1 - v^2/c^2}$$

$$= \frac{(x^2 - c^2t^2) - \frac{v^2}{c^2}(x^2 - c^2t^2)}{1 - v^2/c^2}$$

$$= \frac{(x^2 - c^2t^2)(1 - v^2/c^2)}{1 - v^2/c^2}$$

$$= x^2 - c^2t^2$$

$$\therefore \boxed{x'^2 - c^2t'^2 = x^2 - c^2t^2}$$ (Proved)

This is an absolute quantity in the Lorentz - Einstein Relativity

$$\therefore (\Delta x)^2 - c^2(\Delta t)^2 = (\Delta x)'^2 - c^2(\Delta t)'^2$$

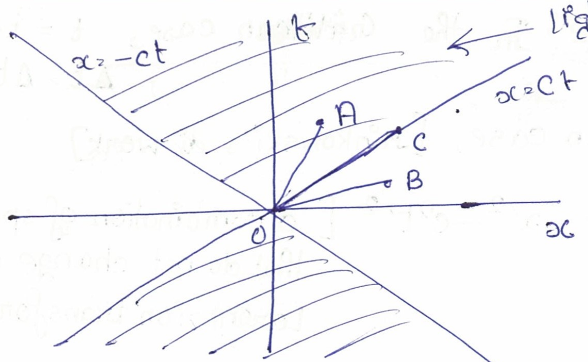
$$\Rightarrow \sqrt{(\Delta x)^2 - c^2(\Delta t)^2} = \sqrt{(\Delta x)'^2 - c^2(\Delta t)'^2}$$

This has dimension of Δx, also called "proper length",

Proper length $= \sqrt{(\Delta x)^2 - c^2(\Delta t)^2}$

and Proper time $= \sqrt{(\Delta t)^2 - \frac{(\Delta x)^2}{c^2}}$

} Lorentz invariant



~~OB = space like distance~~

OB = space like distance, because, in the proper length, in this region, $\Delta x > c\Delta t$

OC : $\Delta x = c\Delta t$: Light distance

OA : $\Delta x < c\Delta t$: time like distance

to reach A from O, one needs to move at a speed ~~>C~~ <C, to reach C from O, one needs to move at a speed = C, B can never be reached as, one needs to move at a speed ~~≤C~~ >C. OB is out of the light cone. O and B ~~OB~~ are causally disconnected.

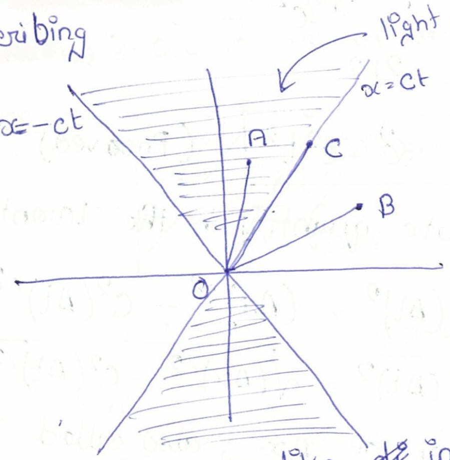
$x^2 - c^2t^2 = x'^2 - c^2t'^2 = x''^2 - c^2t''^2$: Going from one frame to another, this quantity is invariant

also, $(\Delta x)^2 - c^2(\Delta t)^2 = (\Delta x)'^2 - c^2(\Delta t)'^2$: a sort of space-time transform

~~proper distance~~

if $\Delta x = 0$, proper time $= \Delta t$

* proper time is the time we measure, once we move with the particle whose motion we are describing



OB $= (\Delta x)^2 - c^2(\Delta t)^2 > 0$: space like ~~dt~~ interval. O and B are causally disconnected. As a signal needs to travel ~~at least~~ at a ~~dr speed~~ velocity $> c$ from O to reach B.

OC $= (\Delta x)^2 - c^2(\Delta t)^2 = 0$: light like interval. signal needs to move at a velocity equal to speed of light.

$OA = (\Delta x)^2 - c^2(\Delta t)^2 = 0$: Time like distance. C can be reached from O by travelling at a velocity $< c$.

*In 3D,

$$(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2, \text{ is invariant under rotation}$$

and $(\Delta x)^2 - (c\Delta t)^2$, is invariant under Lorentz boost

Δx is a pseudo-distance in space-time.

metric concept :

$$\underbrace{\Delta L}_{\text{distance}} = \sqrt{\underbrace{g^{ij}}_{\text{a matrix}} \Delta x_i \Delta x_j}$$

$i = 1 \ldots n$ (n-dimensional space)

g^{ij} = symmetric $n \times n$ matrix called metric

now, lets, $n = 3$

$$(\Delta L)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \qquad (i,j = 1, 2, 3)$$

$$= g^{ij} \Delta L_i \Delta L_j$$

where $g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

now, the proper length,

$$\Delta s^2 = g^{\mu\nu} \Delta L_\mu \Delta L_\nu$$

$$g^{ij} = \begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

where, $\Delta L_1 = \Delta x$
$\Delta L_2 = \Delta y$
$\Delta L_3 = \Delta z$
$\Delta L_4 = c\Delta t$

considering the proper time, $(\Delta \tau)^2 = (\Delta t)^2 - \left(\frac{\Delta x}{c}\right)^2$

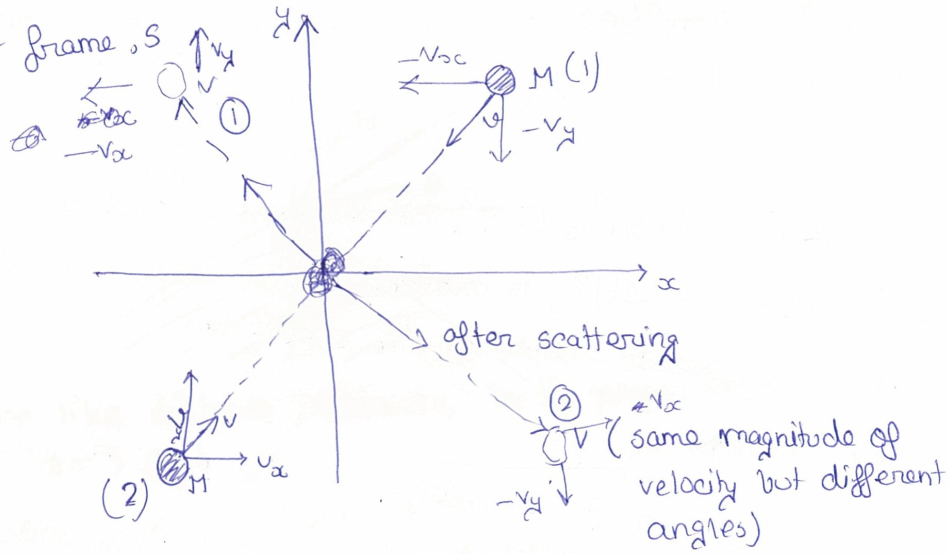
$(\Delta\tau)^2 = \tilde{g}^{\mu\nu} \Delta L_\mu \Delta L_\nu$, where, $\tilde{g}^{\mu\nu} = \begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$

* Momentum :-

i) $\vec{P}_{\text{relativistic}} \xrightarrow{v \ll c} \vec{P}_{\text{no-relativistic}}$

ii) $\vec{p}$ must be conserved (related to symmetry) ~~not~~

In the rest frame, S



$$\vec{p}^{\,Total}_{t=0} = 0$$

$$\vec{p}^{\,Total}_{t=T\,(\text{after scattering})} = 0$$

$$\Delta P_y^{(1)} = 2M v_y^{(1)}$$

$$\Delta P_y^{(2)} = -2M v_y^{(2)}$$

but $v^{(1)} = v^{(2)}$

$$\therefore \Delta P_y^{(1)} + \Delta P_y^{(2)} = 0$$

we will have this conservation of momentum, when we change frame the frame ~~is moving with velocity v~~ , S'. [S' moves with velocity v_x]

$V = v_x$

We know, $$\omega_y' = \frac{\omega_y}{1-\frac{\omega_x V}{c^2}}\sqrt{(1-V^2/c^2)}$$ $\hookrightarrow$ velocity of moving frame

In the relativistic case, even if the frame moves in x-direction, there is change in velocity of the particle in the y-direction

$$\therefore v_y' = \frac{v_y}{1-\frac{v_x V}{c^2}}\sqrt{1-V^2/c^2}$$

V: velocity of frame 2 wrt frame 1.

① $M v_y'(1)$ $\qquad$ $M v_y'(1)$

② $M v_y'(2)$ $\qquad$ $M v_y'(2)$

$t=0$ $\qquad$ $t=T$

$v_x^{(1)} = -v_x \qquad v_y^{(1)} = -v_y$ [In the old frame]

$$\therefore v_y'(1) = \frac{v_y(1)}{1-\frac{v_x(1)V}{c^2}}\sqrt{1-V^2/c^2} = \frac{-v_y}{1+\frac{v_x V}{c^2}}\sqrt{1-V^2/c^2} = \frac{-v_y}{1+\frac{v_x^2}{c^2}}\sqrt{1-v_x^2/c^2}$$

$[V = v_x]$

$v_y^{(2)} = v_y$ and $v_x^{(2)} = v_x$, in the old frame

$$\therefore v_y'(2) = \frac{v_y(2)}{1 - \frac{v_x^{(2)} v}{c^2}} \sqrt{1 - v^2/c^2}$$

$$= \frac{v_y}{1 - \frac{v_x v}{c^2}} \sqrt{1 - v^2/c^2}$$

$$= \frac{v_y}{1 - \frac{v_x^2}{c^2}} \sqrt{1 - v_x^2/c^2} \quad [v = v_x]$$

$\therefore$ ~~$v_y'(1) = -v_y$~~

$$\therefore \boxed{\begin{aligned} v_y'(1) &= \frac{-v_y}{1 + \frac{v_x^2}{c^2}} \sqrt{1 - v_x^2/c^2} \\ v_y'(2) &= \frac{v_y}{1 - \frac{v_x^2}{c^2}} \sqrt{1 - v_x^2/c^2} \end{aligned}}$$

Total change in momentum for the particle 2,

$$\Delta P_y'^{(2)} = 2 M v_y'(2)$$

$$= \frac{2 M v_y}{1 - v_x^2/c^2} \sqrt{1 - v_x^2/c^2}$$

for particle 1,

$$\Delta P_y'^{(1)} = \frac{-2 M v_y}{1 + v_x^2/c^2} \sqrt{1 - v_x^2/c^2}$$

$\therefore \Delta P_y'^{(1)} + \Delta P_y'^{(2)} \neq 0$ [No conservation in momentum]

$\therefore$ ~~The w~~ with the naive definition of momentum we have used, momentum seems ~~is~~ not to be conserved in the Lorentzian Boost. But it must be conserved, and we have to ~~bos~~ devise some modifications.

* While changing frame, we also need to change time:-

new time in the moving frame,

$$\Delta \tau = \Delta t \sqrt{1 - v^2/c^2}$$

$$\therefore v_y = \frac{\Delta y}{\Delta \tau} = \frac{\Delta y}{\Delta t \sqrt{1 - v^2/c^2}}$$

$$= \cancel{\frac{\Delta y}{\Delta t \sqrt{1 - v_x^2/c^2}}}$$

momentum $P_y = M V_y = M \frac{\Delta y}{\Delta \tau}$

$$= \cancel{\frac{M V_y}{\sqrt{1-V_y^2/c^2}}} \quad \frac{M V_y}{\sqrt{1-V^2/c^2}}$$

$$= M(V)\, V_y \quad , \text{ where, } M(V) = \frac{M}{\sqrt{1-V^2/c^2}}$$

$\Delta\tau$ is the proper time, and ~~rela~~ relativistic invariant. Using this, we can show that the momentum is conserved. The momentum is a vector in this framework

$$P = \frac{MV}{\sqrt{1-V^2/c^2}}$$

$$\Rightarrow P^2 = \frac{M^2V^2}{1-V^2/c^2} \quad \text{—(i)}$$

We can build something that is invariant

now, identity: $\frac{1}{1-V^2/c^2} - \frac{V^2/c^2}{1-V^2/c^2} = 1 \quad \text{—(ii)}$

$(ii) \times M^2c^4$

$$\Rightarrow \frac{M^2c^4}{1-V^2/c^2} - \frac{M^2V^2c^2}{1-V^2/c^2} = M^2c^4$$

$$\Rightarrow \frac{M^2c^4}{1-V^2/c^2} - P^2c^2 = M^2c^4 \quad \text{[from (i)]} \quad \text{—(iii)}$$

This has dimension of energy

$\therefore$ we define Relativistic energy, $E^2 \equiv \frac{M^2c^4}{1-V^2/c^2}$

$$\cancel{\simeq M^2c^4\left(1+\frac{V^2}{c^2}\right)^{-1}}$$

$$\cancel{\simeq M^2c^4\left(1+\frac{V^2}{c^2}\right) + O\left(\frac{V^2}{c^2}\right)}$$

$$\cancel{= M^2c^2 + \frac{1}{2}M^2}$$

$$\Rightarrow E \equiv \frac{Mc^2}{\sqrt{1-V^2/c^2}}$$

$$= Mc^2\left(1-\frac{V^2}{c^2}\right)^{-1/2}$$

$$\simeq Mc^2\left(1+\frac{V^2}{2c^2}\right) + O\left(\frac{V^2}{c^2}\right)$$

$$\Rightarrow E \simeq \underbrace{\frac{1}{2}Mv^2}_{\text{Relativistic KE}} + \underbrace{Mc^2}_{\text{constant term (Mass energy)}} + \underbrace{O\left(v^2/c^2\right)}_{\text{lower order corrections}}$$

~~(iii)~~

Now, (iii) $\Rightarrow$ $E^2 - p^2c^2 = M^2c^4$

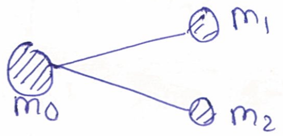
~~and E~~ and $E = \frac{1}{2}Mv^2 + Mc^2 + O(v^2/c^2)$

if particle is at rest,

$$\boxed{E = Mc^2 + O(\cancelto{0}{v^2/c^2}) + \cdots}$$

and from (iii) if $P = 0$, $\boxed{E = Mc^2}$ ← due to the mass

In phenomenon like nuclear fission, ~~to~~



$m_1 + m_2 < m_0$ [some energy is lost due to conversion to mass energy]

The equation: $E^2 - p^2c^2 = M^2c^4$ is analog to the invariant quantity ~~Δx~~ $(\Delta t)^2 - \frac{(\Delta x)^2}{c^2} = (\Delta\tau)^2$ under Lorentz transformation. We see ~~$E^2 \Delta t$~~ E is analogous to Δt, and M^2c^4 is constant.

and PC analog to $\frac{\Delta x}{c}$

* We should note that $\frac{1}{2}Mv^2$ is not the only term that contributes to the relativistic KE, but the correction terms do too

if $v \rightarrow c$, $M = \frac{M_0}{\sqrt{1 - v^2/c^2}} \simeq \infty$

* $E^2 - p^2c^2 = E'^2 - p'^2c^2 = E''^2 - p''^2c^2$ [from frame to frame]

E and P should change in a proper manner so that $E'^2 - p'^2c^2$ remains invariant in the moving frame.

$P_x = M\frac{dx}{d\tau}$ ← invariant, $P_y = M\frac{dy}{d\tau}$, $P_z = M\frac{dz}{d\tau}$

~~$E = Mc^2$~~ we can write, $E = Mc^2\frac{dt}{d\tau}$

as $\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}}$ ~~as $\left[(dt)^2 - \frac{(dx)^2}{c^2} = (d\tau)^2 \Rightarrow \left(\frac{dt}{d\tau}\right)^2 = 1 - \frac{dx}{d\tau}\right]$~~

as, $(dt)^2 - \frac{(dx)^2}{c^2} = (d\tau)^2$

$$\Rightarrow 1 - \left(\frac{dx}{dt}\right)^2 \cdot \frac{1}{c^2} = \left(\frac{d\tau}{dt}\right)^2$$

$$\Rightarrow 1 - \frac{v^2}{c^2} = \left(\frac{d\tau}{dt}\right)^2 \Rightarrow \frac{d\tau}{dt} = \sqrt{1 - v^2/c^2}$$

$$\Rightarrow \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$\therefore E = Mc^2 \frac{dt}{d\tau}$ (This energy is conserved in a scattering experiment)

~~$P_x' =$ new momenta,~~ now, $P_x = \cancel{M \frac{dx}{d\tau}}$

$\cancel{P_x' = M \frac{dx'}{d\tau}}$ $= M$

new momenta, and E,

$$\boxed{\begin{aligned} P_x' &= \frac{P_x - \frac{EV}{c^2}}{\sqrt{1 - V^2/c^2}} \\ P_y' &= P_y \\ P_z' &= P_z \\ E' &= \frac{E - P_x V}{\sqrt{1 - V^2/c^2}} \end{aligned}}$$

Derivations:—

* ~~x, y, z and ct form the 4 vector~~

1) as $x' = \frac{x - Vt}{\sqrt{1 - V^2/c^2}}$

$$P_x' = M \frac{dx'}{d\tau} = \frac{M\frac{dx}{d\tau} - MV\frac{dt}{d\tau}}{\sqrt{1 - V^2/c^2}}$$

$$= \frac{P_x - \left(M\frac{dt}{d\tau}\right)V}{\sqrt{1 - V^2/c^2}}$$, now, $E = Mc^2 \frac{dt}{d\tau} \Rightarrow M\frac{dt}{d\tau} = \frac{E}{c^2}$

$$= \frac{P_x - \frac{EV}{c^2}}{\sqrt{1 - V^2/c^2}}$$

ii) $y' = y$

$P_y' = M\frac{dy'}{d\tau} = M\frac{dy}{d\tau} = P_y$, similar for P_z

iii) $t' = \dfrac{t - \frac{v}{c^2}x}{\sqrt{1-v^2/c^2}}$

$\Rightarrow E' = Mc^2\frac{dt'}{d\tau} = \cancel{Mc^2\left[\frac{dt}{d\tau}\right.}$

$= \dfrac{Mc^2\frac{dt}{d\tau} - Mv\frac{dx}{d\tau}}{\sqrt{1-v^2/c^2}}$

$= \dfrac{E - \left(M\frac{dx}{d\tau}\right)v}{\sqrt{1-v^2/c^2}} = \dfrac{E - P_x v}{\sqrt{1-v^2/c^2}}$

* x, y, z and ct form 4-vector. And P_x, P_y, P_z and EC forms another 4-vector

$\Delta s = \sqrt{g^{\mu\nu}\Delta x_\mu \Delta x_\nu}$ ← proper time ~~distance distance~~

$\boxed{\mu = \begin{cases} i = 1,2,3 \\ 0 = 4 \end{cases}}$

: In the Relativistic case

multiplied by this vector

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{4\times4}, \begin{pmatrix} \Delta x_0 \\ \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{pmatrix}, \boxed{\begin{matrix} \Delta x_1 = \Delta x \\ \Delta x_2 = \Delta y \\ \Delta x_3 = \Delta z \\ \Delta x_0 = c\Delta t \end{matrix}}$$

In the non-relativistic case

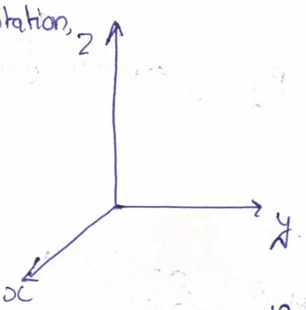
$\Delta L = \cancel{\sqrt{g^{ij}\Delta L_i \Delta L_j}}\ \sqrt{g^{ij}\Delta L_i \Delta L_j}$

$= \frac{E}{c^2}$

Where, $g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times3}$

$\Delta L = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$

* A non-relativistic rotation,



if we wanted a transformation that preserves the length,

eg. rotation along z :

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

$$z' = z$$

but ~~it preserve~~ here, $\cancel{x^2+y^2+z^2}$ ~~is~~ $x^2+y^2+z^2$ remains invariant, i.e, $x'^2+y'^2+z'^2 = x^2+y^2+z^2$

* Δs is invariant under Lorentz transformation in the relativistic case,

$$x' = \frac{x - vt}{\sqrt{1-v^2/c^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-v^2/c^2}}$$

$\Rightarrow \sqrt{t^2 - \frac{x^2}{c^2}}$ is invariant, has dimension of length.

It has a '$-$' sign which can be characterised by the negative signs in the 4-vector

The forms :

$$\boxed{\begin{aligned} x' &= x\cosh\xi + t\sinh\xi \\ y' &= y \\ z' &= z \\ t' &= -x\sinh\xi + t\cosh\xi \end{aligned}}$$

where, $\cosh\xi = \frac{1}{\sqrt{1-v^2/c^2}}$

*, $$\left.\begin{aligned} \Delta s &= \sqrt{g^{\mu\nu}\Delta x_\mu \Delta x_\nu} \\ &= \sqrt{(\Delta x_1)^2 - (\Delta x_2)^2 - (\Delta x_3)^2 - (\Delta x_4)^2} \\ &= \sqrt{(c\Delta t)^2 - \{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\}} \end{aligned}\right\}$$ Keep this in mind

* proper time,

$$\Delta\tau\, c = \sqrt{c^2(\Delta t)^2 - (\Delta x)^2}$$

now, $t' = \dfrac{t - \frac{v}{c^2}x}{\sqrt{1 - v^2/c^2}}$

$$\Delta t' = \frac{\Delta t - \frac{v}{c^2}\Delta x}{\sqrt{1 - v^2/c^2}}$$

now, if we are running with the particle moving, $\Delta x = 0$

$$\therefore \Delta t' = \frac{\Delta t}{\sqrt{1 - v^2/c^2}}$$

* Rotation (along z) :

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

to prove that in the Lorentz transformation,

$$\boxed{\begin{aligned} d\tilde{\tau} &= d\tau \cosh\xi - dx \sinh\xi \\ d\tilde{x} &= dx \cosh\xi + d\tau \sinh\xi \end{aligned}}$$

, this τ has nothing to do with proper time, it is just a definition [proper time do not transform ☺]

and $\dfrac{d\tilde{x}}{d\tilde{\tau}} = \dfrac{v}{c}$, $\cosh\xi = \dfrac{1}{\sqrt{1 - v^2/c^2}}$

we know, ~~$d\tilde{x}$~~ $\Delta t' = \dfrac{\Delta t - \frac{v}{c^2}\Delta x}{\sqrt{1 - v^2/c^2}}$, $\Delta x' = \dfrac{\Delta x - v\Delta t}{\sqrt{1 - v^2/c^2}}$

Let's define, $\tilde{\tau} = t'c$, $\tau = tc$

now, $\Delta t' c = \dfrac{\Delta t c - \frac{v}{c}\Delta x}{\sqrt{1 - v^2/c^2}}$ — (I)

and $\Delta x' = \dfrac{\Delta x - \frac{v}{c}\Delta\tau}{\sqrt{1 - v^2/c^2}}$ — (II) $[\Delta\tau = \Delta t c]$

$\cancel{\frac{(I)}{(II)} \Rightarrow \frac{\Delta\tilde{\tau}}{\Delta\tilde{x}} = \frac{c}{v}}$

$$(I) \Rightarrow \Delta\tilde{\tau} = \frac{\Delta\tau - \frac{v}{c}\Delta x}{\sqrt{1 - v^2/c^2}}$$

$$\Delta\tilde{\tau} = \frac{\Delta\tau - \frac{v}{c}\Delta x}{\sqrt{1-v^2/c^2}} \quad \text{and} \quad \Delta x' = \frac{\Delta x - \frac{v}{c}\Delta\tau}{\sqrt{1-v^2/c^2}}$$

$$\cosh\frac{\varrho}{a} = \frac{1}{\sqrt{1-v^2/c^2}} \quad \text{and} \quad \frac{d\tilde{x}}{d\tilde{\tau}} = \frac{v}{c}$$

$$\therefore \Delta\tilde{\tau} = \frac{\Delta\tau}{\sqrt{1-v^2/c^2}} - \frac{\frac{v}{c}\Delta x}{\sqrt{1-v^2/c^2}}$$

$$= \Delta\tau\cosh\frac{\varrho}{a} - \frac{\frac{v}{c}\Delta x}{\sqrt{1-v^2/c^2}}$$

also, $\cancel{\cosh^2(\frac{\varrho}{a}) - \sinh^2(x) = 1}$

$$\cosh^2\left(\frac{\varrho}{a}\right) - \sinh^2\left(\frac{\varrho}{a}\right) = 1$$

$$\Rightarrow \frac{1}{1-v^2/c^2} - \sinh^2\left(\frac{\varrho}{a}\right) = 1$$

$$\Rightarrow \sinh^2\left(\frac{\varrho}{a}\right) = \frac{1}{1-v^2/c^2} - 1 = \frac{1-1+v^2/c^2}{1-v^2/c^2}$$

$$= \frac{v^2/c^2}{1-v^2/c^2}$$

$$\Rightarrow \sinh\frac{\varrho}{a} = \frac{v/c}{\sqrt{1-v^2/c^2}}, \quad \text{(neglecting the −ve term)}$$

$$\therefore \boxed{\Delta\tilde{\tau} = \Delta\tau\cosh\frac{\varrho}{a} - \Delta x\sinh\frac{\varrho}{a}}$$

also,
$$\Delta x' = \frac{\Delta x}{\sqrt{1-v^2/c^2}} - \frac{v/c}{\sqrt{1-v^2/c^2}}\Delta\tau$$

$$\boxed{\Delta x' = \Delta x\cosh\frac{\varrho}{a} - \Delta\tau\sinh\frac{\varrho}{a}} \quad \text{(proved)}$$

(Note the matter of conventions)

* Also, note, $\cosh(ix) = \cos x$
$\sinh(ix) = i\sin x$

$$\sinh(ix) = \frac{e^{ix} - e^{-ix}}{2} = i \cdot \frac{e^{ix} - e^{-ix}}{2i}$$

$$= i\sin(x) \quad \text{~~(proved)~~}$$

and
$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2} = \cos(x) \quad \text{(proved)}$$

also, $\cosh(x) = \cos(-ix) = \cos(ix)$

as, $\cos(-ix) = \cancel{\frac{e^{-ix}+e^{ix}}{2}} = \cancel{\cos}$

$$= \frac{e^{i(-ix)} + e^{-i(-ix)}}{2}$$

$$= \frac{e^{x} + e^{-x}}{2} = \cosh(x) \text{ (proved)}$$

and $\sinh(x) = i\sin(-ix) = -i\sin(ix)$

as, $i\sin(-ix) = i\,\frac{e^{i(-ix)} - e^{-i(-ix)}}{2i}$

$$= \frac{e^{x} - e^{-x}}{2}$$

$$= \sinh(x)$$

and $-i\sin(ix) = -i\,\frac{e^{i(ix)} - e^{-i(ix)}}{2i}$

$$= -\frac{e^{-x} - e^{x}}{2}$$

$$= \frac{e^{x} - e^{-x}}{2}$$

$$= \sinh(x)$$

$\therefore$

$$\boxed{\begin{aligned} \Delta\tilde{\tau} &= \Delta\tau\cosh\xi - \Delta x \sinh\xi \\ &= \Delta\tau\cos(i\xi) + i\,\Delta x\sin(i\xi) \\ \Delta\tilde{x} &= \Delta x\cos(i\xi) + i\,\Delta\tau\sin(i\xi) \end{aligned}}$$

* problems.

The proper mean life time of π^{+}-mesons is 2.5×10^{-8} secs. Given, $\frac{v}{c} = \beta = 0.73$. i) What is the mean life in this frame.

Time dilation, $\Delta t' = \frac{\Delta t}{\sqrt{1-\beta^2}} = \frac{2.5\times10^{-8}}{\sqrt{1-(0.73)^2}}$

$$= 3.66\times10^{-8} \text{ sec}$$

ii) what is the distance travelled by the π^{+} mesons?

distance $= \beta c\,\Delta t'$

$$= 0.73\times 3\times10^{10}\,\text{cm/sec} \times 3.66\times10^{-8}\,\text{sec}$$

$$\simeq \cancel{8.02\text{ cm}}\ 802\text{cm}$$

iii) what is the distance travelled without relativistic effect? [No time dilation]

$D = \beta c\,\Delta t = 0.73\times3\times10^{10}\,\text{cm/sec}\times 2.5\times10^{-8}\,\text{sec}$

$$\cancel{= 5.48\text{ cm}}\ \simeq 548\text{ cm}$$

$73 \times 3 = 219$; $219 \times 366 = 80154$

Problem 2

* ~~Anot~~ ~~A~~ μ - meson decays. Proper mean life time = ~~2×16~~ 2×10^{-6} sec. velocity = 0.99c. collision in atmosphere is negligible. 1% reach the earth. 1) What is the height?

$$N(t) = N(0)\, e^{-t/\tau}$$, where τ : proper mean life time

$N(t)$ = no. of particles at time t.

As 1% survives, $N(t) = 0.01\, N(0)$

$$\therefore 0.01 = e^{-t/\tau} = e^{-t/(2\times10^{-6})}$$

$$\Rightarrow t = 9.22\times10^{-6}\text{ sec}$$ (time in the proper frame

We have to transform to t_{earth}.

$\therefore$ ~~to now, time dilation~~,

$$\cancel{t = t_{earth}\sqrt{1-v^2/c^2}}$$

$$\cancel{\sqrt{1-v^2/c^2}}$$

Now, due to time dilation,

$$t_{earth} = \frac{t}{\sqrt{1-v^2/c^2}} = \frac{9.22\times10^{-6}}{\sqrt{1-(0.99)^2}}$$

$$\simeq 6.54\times10^{-5}\text{ sec}$$

$$D = 0.99\times3\times10^{10}\times6.54\times10^{-5}$$

$$= 19.5\times10^{5}\text{ cm}$$ [In the frame

II) distance seen from the muon,

$$d = D\sqrt{1-v^2/c^2}$$

* Problem 3:

Two Galaxies A, B ~~rece~~ recedes in two opposite directions

If we see at ~~fr~~ frame B, what's the speed we will observe, with which frame A recedes?

Composition formula:

$$V_x' = \frac{V_x - V}{1 - \frac{V_x}{c^2}V}$$

V_x : velocity of particle in old frame

V_x' = velocity of particle in new frame

V = velocity with which new frame moves

$$V_x^A = -V_x = -0.3c$$

$$V = V_x^B = 0.3c$$

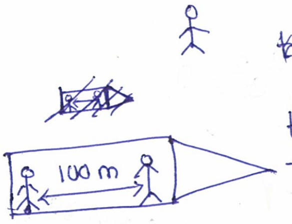
$$\therefore \quad V_x'^A = \frac{V_x^A - V}{1 - \frac{V_x^A}{c^2}V}$$

$$= \frac{-0.3c - 0.3c}{1 - \frac{(-0.3c)(0.3c)}{c^2}}$$

$$= \frac{-0.6c}{1 + 0.09} = \frac{-0.6c}{1.09}$$

$$= \cancel{0.55c}$$

* Problem 4 : The proper length of a spaceship is 100 m. An ~~Astraunut~~ astronaut is outside the space ship and realises that the spaceship takes a time interval of $\Delta\tau = 2\times10^{-6}$ to pass.



$\xrightarrow[v]{t = 2\times10^{-6}}$ (to pass the outside ~~man~~ astronaut)

* ~~the length the outside~~ The outside ~~austor~~ astronaut is in a moving frame wrt the astronauts inside the spaceship. So he observes a contracted length,

$$l = l_0\sqrt{1 - v^2/c^2}$$

$$= 100\times10^2\sqrt{1 - \frac{v^2}{(3\times10^{10})^2}} \quad \text{cm}$$

$$\Delta t = \frac{l}{v} = \frac{10^4\sqrt{1 - \frac{v^2}{9\times10^{20}}}}{v} = 2\times10^{-6}$$

$$\Rightarrow 1 - \frac{v^2}{9\times10^{20}} = v^2(4\times10^{-20}) \quad 1 - \frac{v^2}{9\times10^{20}} = 4v^2\times10^{-20}$$

$$\Rightarrow \frac{v^2}{10^{20}}\left(4 + \frac{1}{9}\right) = 1 \Rightarrow \boxed{v = \frac{c}{2}}$$

* Problem 5 : an e^- has $\beta = \frac{v}{c} = 0.99$. what's the KE ?

we know, $E_{total} = \frac{mc^2}{\sqrt{1-\beta^2}}$

$E_{Kin} = E_{Total} - mc^2$ (just subtract the mass energy from ~~that's~~ the total energy)

$= mc^2 \left[\frac{1}{\sqrt{1-(0.99)^2}} - 1 \right]$

$= mc^2 (6.07)$

$= \cancel{0.511 \frac{MeV}{c^2} \times (3 \times 10^{10})^2 cm^2 \times 6.07}$

$= \cancel{0.55}\ 0.511 \frac{MeV}{c^2} \times \cancel{3\times} c^2 \times 6.07$

$= 3.10$ MeV

* Problem 6 : We have nucleus of iron,

$Fe^{*57} \xrightarrow{emits} \leftarrow \gamma$ (photon) $+ Fe^{57} \rightarrow$
(moves in opposite direction)

what is the recoil momentum of Fe in the laboratory due to the emission of 14 KeV γ ray

* The mass - energy relation for photon,

$E^2 - p^2c^2 = m^2c^4$

but here, $m_\gamma = 0$ $\therefore E_\gamma^2 - P_\gamma^2 c^2 = 0$

$\Rightarrow P_\gamma = \frac{E}{c}$ [1 KeV $= 1.6 \times 10^{-9}$ erg]

$= \frac{14 \times 1.6 \times 10^{-9} \text{ erg}}{3 \times 10^{10} \text{ cm/sec}}$

$= 7.5 \times 10^{-19}$ g/(cm/sec)

$M_{Fe^{57}} = \cancel{57}\ 9.7 \times 10^{-24}$ g

$\left(\frac{dx}{d\tau}\right) M = 7.5 \times 10^{-19}$ g $= P_\gamma$

$\Rightarrow \frac{dx}{d\tau} = v = \frac{P_\gamma}{M} = \frac{7.5 \times 10^{-19}}{9.7 \times 10^{-24}}$ cm/s

$= 7.7 \times 10^{3}$ cm/s [Not relativistic order of velocity]