* Problem 7 :— What is the mass equivalent of energy of an antenna radiating 1000 watts for a period of 24 hrs

$$\begin{array}{r} 24 \\ 36 \\ \hline 144 \\ 72\times \\ \hline 86400 \end{array}$$

We know, $1 \text{ Watt} = 10^7 \frac{\text{erg}}{\text{sec}}$

$100 \text{ watts} = 10^{10} \text{ erg/sec}$

$1 \text{ day} = 24 \times 60 \times 60 \text{ secs} = 8.64 \times 10^4 \text{ secs}$

$E = Mc^2$

$E_{Total} = 10^{\cancel{7}\,10} \text{ erg/sec} \times 8.64 \times 10^4 \text{ sec}$

$= 8.64 \times 10^{11} \text{ erg}$

$M = \frac{E_{Total}}{c^2} = \frac{8.64 \times 10^{10} \text{ erg}}{9 \times 10^{20} (\text{cm/sec})^2} \sim 10^{-3} \text{ g}$

$\mu = 0, 1, 2, 3$ } Remember — 4 vectors

$dx^\mu = (c\,dt, -dx^i)$ where $i = 1, 2, 3$

$\Leftrightarrow (dx^i, -c\,dt)$ [sometimes, the inverse notation may be used]

4-velocity

$U^\mu = \frac{dx^\mu}{ds}$, s : proper time

where $ds^2 = \underbrace{-c^2dt^2 + dx^2 + dy^2 + dz^2}_{\text{A Lorentz-invariant}} = -c^2 \overbrace{dt'^2}^{\text{proper time}} = -c^2 d\tau^2$ (matter of notation)

dt': In the frame moving with the particle $(dx = 0 = dy = dz)$

$dt' = \frac{ds}{c} = i\frac{\sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}}{c}$

$\cancel{= c\,dt \sqrt{}}$

$= i\sqrt{\frac{c^2dt^2 - dx^2 - dy^2 - dz^2}{c^2}}$

$= i\sqrt{dt^2 - \frac{(dx^2 + dy^2 + dz^2)}{c^2}}$

$= i\,dt\sqrt{1 - \frac{(dx^2 + dy^2 + dz^2)}{dt^2\,c^2}}$

$$\boxed{dt' = i\,dt\sqrt{1 - \frac{v^2}{c^2}}}$$

$ds = i c\,dt\sqrt{1 - v^2/c^2}$ $\Rightarrow \frac{dt}{ds} = \frac{1}{i c\sqrt{1 - v^2/c^2}}$

$= \frac{-i}{c\sqrt{1 - v^2/c^2}}$

$[i^2 = -1]$

i = imaginary no.

$$u^i = \frac{dx^i}{ds} = \frac{dx^i}{c\,dt\sqrt{1-v^2/c^2}}$$

Landau's volume 3

$$= \frac{v^i}{c\sqrt{1-v^2/c^2}} \qquad [i = 1, 2, 3]$$

$$U^0 = c\frac{dt}{ds} = \frac{1}{c\sqrt{1-v^2/c^2}} \qquad \frac{1}{c\,c\sqrt{1-v^2/c^2}}$$

$$\therefore \sum_{\mu=0}^{3} (U^\mu)^2 = (U^1)^2 + (U^2)^2 + (U^3)^2 + (U^0)^2$$

$$= \frac{v_x^2 + v_y^2 + v_z^2 + 1}{c^2(1-v^2/c^2)}$$

$$c^2 - (v_x^2 + v_y^2 + v_z^2) \qquad c^2 - v_x^2 - v_y^2 - v_z^2$$

$$= \frac{v^2+1}{c^2-v^2} \qquad \frac{v^2}{c^2(1-v^2/c^2)} + \frac{1}{1-v^2/c^2}$$

$$= \frac{v_x^2+v_y^2+v_z^2}{c^2(1-v^2/c^2)} + \frac{1}{c^2} \qquad \frac{v^2/c^2 + 1}{1-v^2/c^2}$$

$$= \frac{v^2}{c^2(1-v^2/c^2)} + \frac{1}{c^2}$$

$$= \frac{v^2 + (1-v^2/c^2)}{c^2(1-v^2/c^2)}$$

$$= \frac{v^2 + c^2 - v^2}{c^2 - v^2}$$

$$\sum_{\mu=0}^{3} (U^\mu)^2 = -1$$

$$U^0 = c\frac{dt}{ds} = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\therefore \sum_{\mu=0}^{3} (U^\mu)^2 = \frac{v_x^2 + v_y^2 + v_z^2}{c^2(1-v^2/c^2)} + \frac{1}{1-v^2/c^2}$$

$$= \frac{v^2 + c^2}{c^2(1-v^2/c^2)}$$

$$= \frac{v^2/c^2 + 1}{1-v^2/c^2}$$

$\cancel{U^0 = \frac{dt}{ds}}$

$$U^i = \frac{dx^i}{ds} = \frac{dx^i}{c\,dt\sqrt{1-v^2/c^2}} , \quad U^0 = \frac{dt}{ds} = \frac{-\cancel{1}\,ic}{c\sqrt{1-v^2/c^2}} , \; i^2 = -1$$

$$= \frac{v^i}{c\sqrt{1-v^2/c^2}}$$

$$\sum_{\mu=0}^{3} (U^\mu)^2 = -1 \quad -(1)$$

Four-Accelerations:

$$a^\mu = \frac{dU^\mu}{ds} \quad -(11)$$

differentiating (1)

$$\Rightarrow \frac{d}{ds}\sum_{\mu=0}^{3}(U^\mu)^2 = 0$$

$$\Rightarrow \sum_{\mu=0}^{3} 2\,U^\mu a^\mu = 0$$

$$\Rightarrow \boxed{\sum_{\mu=0}^{3} U^\mu a^\mu = 0}$$

: The four-velocity is $\perp$ to the four-acceleration.

* I) Center of Mass - system

II) Threshold energy

suppose we have a photon,

$$\gamma \longrightarrow e^+ + e^-$$

to have energy conservation, $\boxed{E_\gamma \geq 2m_e c^2}$

now for conservation of momenta,

The centre of mass is defined as the frame where,

$\vec{P}_{e^+} + \vec{P}_{e^-} = 0$ [the sum of the momenta = 0]

but Momentum of γ must be equal to $\vec{P}_{e^+} + \vec{P}_{e}$, but that is 0.

But $\vec{P}_\gamma$ cannot be 0. For a photon, we know, $P_\gamma = \frac{E_\gamma}{c}$ [for mass = 0] $\neq 0$.

Here is a contradiction [$E^2 - p^2c^2 = m^2c^4$, if $m=0$, $E = pc$, $p = \frac{E}{c}$]

We need to have an extra particle for this reaction to take place. let there be a nucleus, with momentum P_N, then,

$$\boxed{\vec{P}_\gamma + \vec{P}_N = \vec{P}_N' + \vec{P}_{e^+} + \vec{P}_{e^-}}$$

new momentum of the nucles

[this relation holds]

then we need to consider the 1st equation two, presence of a Catalyst (Nucleus)

$$\gamma + N \longrightarrow N' + e^{+} + e^{-}$$

* let's take a proton at rest, and another proton moves towards it,

$p \longrightarrow v$ $\qquad$ p^{*} (rest)

$K_E < M_p c^2$
(non-relativistic velocity, let's suppose)

$$K_E|_{Lab} = \frac{1}{2} M_p v^2$$

If we sit in the centre of mass,

$p \longrightarrow \frac{1}{2}v$ $\qquad$ $\longleftarrow p^{*}$ $-\frac{1}{2}v$

$$K_E|_{CM} = \frac{1}{2} M_p \left(\frac{1}{2}v\right)^2 + \frac{1}{2} M_p \left(\frac{-1}{2}v\right)^2$$

$$= \frac{1}{8} M_p v^2 + \frac{1}{8} M_p v^2$$

$$= \frac{1}{4} M_p v^2$$

$$\frac{K_E|_{CM}}{K_E|_{lab}} = \frac{1}{2}$$

$\frac{1}{2} K_E|_{lab}$ of energy is available to produce new particle

* let's consider a proton accelerated to 200 MeV. Only 100 MeV is then available to make new particle

$$E^2 - p^2c^2 = E'^2 - p'^2c^2 = M^2c^4$$ [Invariant Mass, an invariant in Lorentz frame]

Lab frame

$p \longrightarrow v$ $\qquad$ p^{*} $\qquad$ $\cancel{(E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 c^2}$

centre of mass frame

$p \longrightarrow \frac{1}{2}v$ $\qquad$ $\longleftarrow p^{*}$ $\frac{1}{2}v$

as $E^2 - p^2c^2$ is a Lorentz invariant, it does not change, moving from one frame to another,

$$\underbrace{(E_1 + E_2)^2 - (\overset{\text{momenta}}{\vec{P_1} + \vec{P_2}})^2 C^2}_{\text{lab frame}} = \underbrace{(E_1 + E_2)^2 - (\cancelto{0}{\vec{P_1} + \vec{P_2}})^2 c^2}_{\text{center of mass frame}}$$

In the centre of mass frame, $\vec{P_1} + \vec{P_2} = 0$

In the Lab frame,

Proton 2 is at rest, $\therefore\ E_2 = M_p C^2$, $\vec{P_2} = 0$

$$\cancel{E^2_{(lab)} - \vec{P}^2_{1,2\,(lab)} C^2 = M_p^2 c^4}$$

$$E_{1\,(lab)}^2 - \vec{P}_{1\,(lab)}^2 c^2 = M_p^2 c^4$$

$$\therefore\ (E_1 + E_2)^2 - \vec{P}_1^2 c^2 = (E_1 + E_2)^2$$

$$\Rightarrow E_{1\,(lab)}^2 + 2E_1E_2_{\,(lab)} + E_{2\,(lab)}^2 - \vec{P}_{1\,(lab)}^2 c^2 = E_{1\,CM}^2 + 2E_1E_2_{\,CM} + E_{2\,CM}^2 = (E_{CM}^{tot})^2$$

$$\Rightarrow 2E_{1\,(lab)} M_p C^2 + M_p^2 c^4 + (M_p C^2)^2 = (E_{CM}^{tot})^2$$

$$\Rightarrow 2E_{1\,(lab)} M_p C^2 + 2M_p^2 C^4 = (E_{CM}^{tot})^2$$

and $E_{lab}^{tot} = E_{1\,(lab)} + E_{2\,(lab)} = E_{1\,(lab)} + M_p C^2$

$$\Rightarrow E_{1\,lab} = E_{lab}^{tot} - M_p C^2$$

$$\therefore\ 2(E_{lab}^{tot} - M_p C^2) M_p C^2 + 2M_p^2 C^4 = (E_{CM}^{tot})^2$$

$$\Rightarrow 2E_{lab}^{tot} M_p C^2 - 2M_p^2 C^4 + 2M_p^2 C^4 = (E_{CM}^{tot})^2$$

$$\Rightarrow \cancel{2E_{tot}^{tot} = E_{CM}} \quad \boxed{2E_{lab}^{tot} M_p C^2 = (E_{CM}^{tot})^2}$$

$$\Rightarrow \frac{E_{CM}^{tot}}{E_{lab}^{tot}} = \frac{2M_pC^2}{E_{CM}^{tot}} = \text{efficiency in producing new particle}$$

* We want to get a total energy of 20 GeV in the ~~at~~ centre of mass. For proton $m_p c^2 = 1$ GeV. ~~we need~~ to

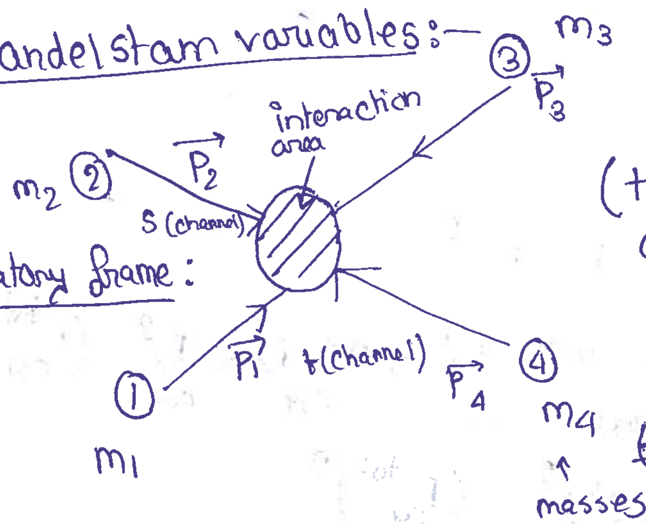
$$\therefore \frac{20}{E^{tot}_{lab}} = \frac{2}{20}$$

$$\Rightarrow E_{lab}^{tot} = \frac{400}{2} = 200 \text{ GeV}$$

~~In the laboratory frame~~ We see the efficiency is very low. This is the reason, in accelerators people use centre of mass frame in proton collisions instead of laboratory frame otherwise a great deal of energy is lost.

* **Mandelstam variables:—**

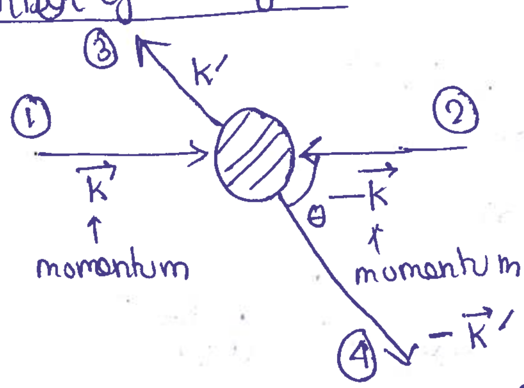
Laboratory frame:



(two new particles ③ and ④ come out, but due to convention we point them ~~toward~~ all towards the ~~see~~ interaction area) ~~(The Phy~~

A very simple model of 2 particles coming in and 2 particles coming out (we could have 30 particles come out, as the no. of particles ~~is~~ may ~~co~~ not conserved but the total momentum should be conserved)

In the center of mass frame :



a sort of rotation by angle θ

Einstein convention :

1) $x^\mu x_\mu = \sum x^\mu x^\mu$

to lower an ~~ø~~ index

~~ii)~~ $x_\mu = g_{\mu\nu} x^\nu$ where
$$g_{\mu\nu} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

$\therefore$ ii) $x^\mu x_\mu = x^\mu g_{\mu\nu} x^\nu$

$$= (x^0)^2 - (x^1)^2 - (x^2)^2 + (x^3)^2$$

$$= \sum_{\mu=0}^{3} x^\mu x^\mu$$

i) First Mandelstam variable :-

$$S = (P_1 + P_2)^{\mu} (P_1 + P_2)_{\mu}$$ [Four Momentums]

$$= (P_1^0 + P_2^0)^2 - (P_1^1 + P_2^1)^2 - (P_1^2 + P_2^2)^2 - (P_1^3 + P_2^3)^2$$

$$\boxed{S = (P_1^0 + P_2^0)^2 - (\vec{P_1} + \vec{P_2})^2}$$ [This is a Lorentz invariant quantity]

so, $(P_1' + P_2')^{\mu} (P_1' + P_2')_{\mu} = (P_1 + P_2)^{\mu} (P_1 + P_2)_{\mu}$

[3-lengths are invariant under rotation and 4-lengths are invariant under Lorentz transformation]

$\therefore$ ~~S = S cm~~ In the center of mass frame,

$\vec{P_1} + \vec{P_2} = 0$

$\therefore S_{cm} = (P_1^0 + P_2^0)^2$

$= E^2$ [let's suppose $c = 1$]

~~Scm is also~~

s is also called ~~energy~~ invariant energy

ii) Second Mandelstam variable :-

$$t = (P_1 + P_4)^{\mu} (P_1 + P_4)_{\mu}$$ [also a Lorentz invariant]

$$= \cancel{2}(P_2 + P_3)^{\mu} (P_2 + P_3)_{\mu}$$

: called invariant momentum transfer (like in the 1st figure ①'s momentum gets transferred to ④'s momentum)

~~iii) third Mandelstam variable :~~

in the center of mass frame,

$$t_{cm} = (K - K')^{\mu} (K - K')_{\mu} = t$$

iii) Third Mandelstam variable :-

$$U = (P_1 + P_3)^{\mu} (P_1 + P_3)_{\mu}$$ [Lorentz invariant]

also called exchange momentum ~~variable~~ transfer

Now s, t and u are not independent,

$(P_1^{\mu} + P_2^{\mu} + P_3^{\mu} + P_4^{\mu}) = 0$

$$s + t + u = \sum_{i=1}^{4} m_i^2$$

we know, $\boxed{P_{(i)}^{\mu} P_{i\,\mu}} = m_i^2$, $i = 1, \ldots, 4$ [mass shell relation]

equivalent to $E^2 - P^2 = m^2$ [$c = 1$]

also, $P_{(1)}^{\mu} + P_{(2)}^{\mu} + P_{(3)}^{\mu} + P_{(4)}^{\mu} = 0$

Proof: Given, $P_{(i)}^{\mu} P_{(i)\mu} = m_i^2$

and $\sum_{i=1}^{4} P_{(i)}^{\mu} = 0$

$$\therefore\ s = (P_{(1)}^{\overset{\swarrow \text{index}}{0}} + P_{(2)}^{0})^2$$

$$t = (P_{(1)}^{0} + P_{(3)}^{0})^2$$

$$u = (P_{(1)}^{0} + P_{(4)}^{0})^2$$

$$\therefore\ s+t+u = (P_{(1)}^{0} + P_{(2)}^{0})^2 + (P_{(1)}^{0} + P_{(3)}^{0})^2 + (P_{(1)}^{0} + P_{(4)}^{0})^2$$

$$= (P_{(1)}^{0})^2 + 2P_{(1)}^{0}P_{(2)}^{0} + (P_{(2)}^{0})^2 + (P_{(1)}^{0})^2 + 2P_{(1)}^{0}P_{(3)}^{0} + (P_{(3)}^{0})^2$$

$$+ (P_{(1)}^{0})^2 + 2P_{(1)}^{0}P_{(4)}^{0} + (P_{(4)}^{0})^2$$

$$= 3(P_{(1)}^{0})^2 + (P_{(2)}^{0})^2 + (P_{(3)}^{0})^2 + (P_{(4)}^{0})^2 + \cancel{2P_{(1)}^{0}(P_2^{(0)} + P_3^{(0)} + P_4^{(0)})}$$

$$+ 2P_{(1)}^{0}(P_{(2)}^{0} + P_{(3)}^{0} + P_4^{(0)})$$

$$= 3(P_{(1)}^{0})^2 + (P_{(2)}^{0})^2 + (P_{(3)}^{0})^2 + (P_{(4)}^{0})^2 + 2P_{(1)}^{0}(-P_{(1)}^{0})$$

$$[\text{as, } P_{(1)}^{\mu} + P_{(2)}^{\mu} + P_{(3)}^{\mu} + P_{(4)}^{\mu} = 0]$$

$$= (P_{(1)}^{0})^2 + (P_{(2)}^{0})^2 + (P_{(3)}^{0})^2 + (P_{(4)}^{0})^2$$

$$= m_1^2 + m_2^2 + m_3^2 + m_4^2$$

$$= \sum_{i=1}^{4} m_i^2 \quad [\text{Q.E.D}]$$

* Group - structures

a set, G with properties -

with an operation $\otimes$, such that

1) $a \in G$, $b \in G$

$a \otimes b = c \in G$

11) $\exists\ e \in G$ such $e \otimes a = a \otimes e = a$, where e is the identity or the neutral element

iii) $\exists\, a^{-1}$ such that $a \otimes a^{-1} = \cancel{e}\, a^{-1} \otimes a = e$. a^{-1} is the inverse of a

iv) In general $a \otimes b \neq b \otimes a$: a non-commutative groups
But there are groups where $a \otimes b = b \otimes a$: commutative groups

v) If $a \otimes (b \otimes c) \neq (a \otimes b) \otimes c$, with $a, b, c \in G$, G is a non-associative group.

examples :-

1. let $G = N$ (set of integer nos, +ve and -ve)
and $\otimes = +$ (addition)
then G is a group

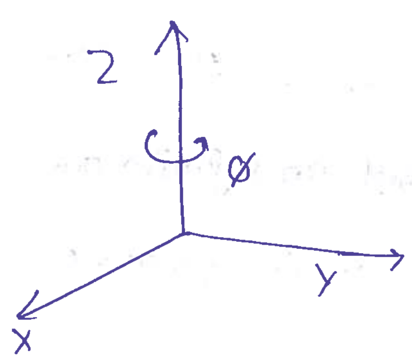
i) $3, 2 \in N$, $3 + 2 = 5 \in N$

ii) $0 + 3 = 3 + 0$, $0 \in N$

iii) $-2 \in N$, then $-2 + 2 = 2 - 2 = 0$, -2 is the inverse of $N \therefore \cancel{2}$

This proves N is a group

2. Lets take $\otimes$ a rotation along z by an angle $\varnothing$



any element $\in G$: rotation along z ~~on~~ by an arbitrary angle

i) $\cancel{a \otimes}$ $\cancel{a, b}$ $a, b \in G$ and $a \otimes b \in G$
$\cancel{R(P_1)}$ let the operation be the rotation along z by an arbitrary angle, : $R(\ )$, $R(P_1)$, $R(P_2) \in G$
~~let P_1, P_2~~ we know $R(P_1) + R(P_2) = R(P_1 + P_2) \in G$

ii) $\exists\, e$, such that $e \otimes a = a \otimes e$
here, $e = R(0) = R(2\pi)$

iii) $\exists\, a^{-1}$ such that $\cancel{e \otimes e^{-1} = e^{-1} \otimes e = e}$ $a \otimes a^{-1} = a^{-1} \otimes a = e$
here, $R(P) + R(-P) = R(e)$ [This has a group-structure]

Group- Representations : Most used in physics

1. let $G = (1, -1)$,

a) $1.1 = 1 \in G$, $1.(-1) = -1 \in G$, $(-1).(-1) = 1 \in G$, operation → $\cdot$

b) $1.1 = 1 \in G$ $(e = 1)$, $-1.1 = -1$

c) $1.(1^{-1}) = 1 \in G$ (1^{-1} is the inverse $\in G$) $[1^{-1} = 1]$, also $(-1)\cdot(-1) = 1$, $(-1)^{-1} = (-1)$

∴ G has a group structure under $\cdot$

But G does not have a group structure under +

as $1 + (-1) = 0 \notin G$

G is a discrete group, containing a finite no. of elements.

2. let $G = \mathbb{R}_{\pm}$ (all the real nos.)

It has a group structure under the property of $\cdot$

a) ~~3.2~~ $3 \cdot 2 = 6 \in \mathbb{R}_{\pm}$

$-(0.4) \cdot 3 = -1.2 \in \mathbb{R}_{\pm}$

b) $3 . 1 = 3 \in \mathbb{R}_{\pm}$, $(-0.8) \cdot 1 = -0.8 \in \mathbb{R}_{\pm}$

∴ $e = 1$

c) ~~$3.(3^{-1})$~~ $3.(1/3) = 1$ ~~$\in \mathbb{R}_+$~~ and $\frac{1}{3}, \frac{1}{0.6} \in \mathbb{R}_{\pm}$

$(0.6) \cdot (1/0.6) = 1$

∴ ~~for each~~ $\mathbb{R}_{\pm}$ has a group structure

$\mathbb{R}$ is also a discrete set, but an infinite one

3. let $\mathbb{N}_+ = \{1, 2, 3, 4, \ldots\}$, let the operation be "$-$"

if $1 - 2 = -1 \notin G$

$\mathbb{N}_+$ does not have a group structure under $-$, but

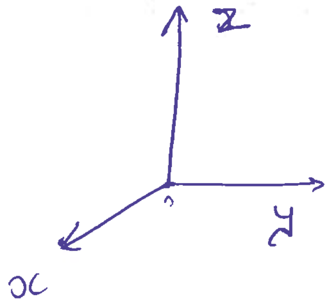
$\mathbb{N}_{\pm}$ has a group structure under operations $\pm$

* Lie Groups (by Sophus Lie) : ~~Grou~~ Continuous groups.

Elements in this group like $a, a', a'', \ldots$ can be parameterised by some parameter like the real no., a_ε, so that all of those ~~nos. a~~ nos. $a, a', a'', \ldots \in a_\varepsilon$, ~~&~~ $f : \varepsilon \to \mathbb{R}^{\pm}$

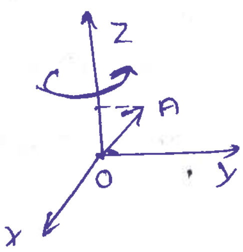
we can define the concept of continuity

In 3D - Eucledean space



, group elements $= R \in G$. Those elements belonging to the group are rotations along any axis that preserve the distance, $l^2 = x^2 + y^2 + z^2$ ~~*~~ [This is invariant for any $R \in G$]

Let's take rotation along z, and check if $R \in G$



1) $R^{z}_{\rho_1} \cdot R^{z}_{\rho_2} = R^{z}_{(\rho_1 + \rho_2)}$: ~~It has a gro~~

rotations by angles ρ_2 and then ρ_1 along z-axis

II) $R_{\rho_1} \cdot R_0 = R_{\rho_1}$ [Rotation by 0 angle, R_0 = identity, e]

~~also, $R_0 = R_{2\pi}$ [there isn't an one-to-one correspondence~~

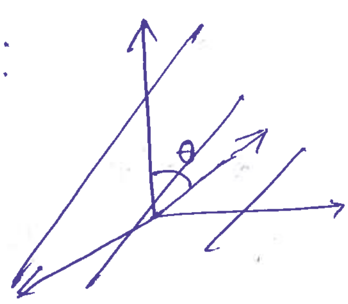
III) $R_{\rho_1} \cdot R_{-\rho_1} = R_0$

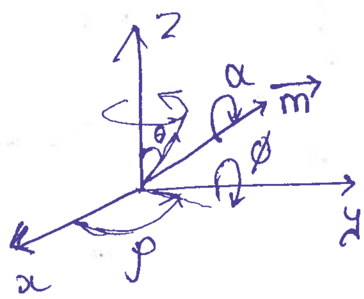
∴ G has a group structure for rotation along z. Keep in mind $R_0 = R_{2\pi}$ [!], there isn't a one-to-one correspondence b/w real nos. and Rotations. ~~does not give $0 = 2\pi$~~ but $0 \neq 2\pi$.

as $R_0 = R_{2\pi}$,

It is one-to-many.

* A generic rotation:





at an instant, the rotation is always along an axis, which can change by an angle α

$\vec{m}(\theta, \rho)$, angle: α

$\vec{m}$ is charecterised by two angles θ and ρ and is a unit vector at any direction, of an angle α

This is important in rigid dynamics. $\vec{m}$ ~~is a fun~~ and α are ~~tot~~ both functions of time. It has a group structure, and is a Lie group.

* Rotation R is a 3×3 ~~vec~~ matrix

$$\begin{pmatrix} & & \\ & R & \\ & & \end{pmatrix}_{3\times3}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

but $x^2+y^2+z^2 = x'^2+y'^2+z'^2$. ~~R is~~ This is an orthogonal group in 3D, called $O(3)$ (orthogonal) because the above relation is preserved.

Also, ~~if we inc~~ This group allows the discrete transformation,

$x \to -x$

$y \to -y$

$z \to -z$.

If we don't want to include these discrete transformation, ~~we put~~ we write $SO(3)$ [Spatial orthogonal Group].

$SO(3) \equiv$ Rotation in 3 dimension

* what is a rotation in 11-dimension?

It is a transformation that keeps invariant the length described by $x_1^2+x_2^2+x_3^2+\cdots+x_{11}^2 = l^2$

* ~~$R_z^{(\phi)}$~~ R^z_{ϕ} : Rotation of an angle ϕ along z-direction,

$$R^z_{\phi} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times3}$$

~~Ro~~ Rotation along x-direction,

$$R^x_{\rho} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\rho & -\sin\rho \\ 0 & \sin\rho & \cos\rho \end{pmatrix}_{3\times3}$$

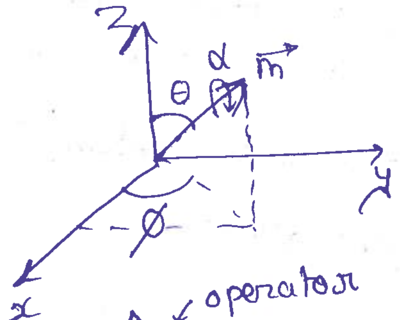
Rotation along y - direction:

$$R_\theta^z = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & \cancel{0}\,1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}_{3\times 3}$$

Rotation along a generic rotation (Rotation of an unit vector by an angle α)

these two angles find a direction
angle of rotation along the direction

$\underbrace{\theta, \phi}_{r}$, α



$$R_{\vec{m},\alpha} = e^{-i\alpha\, \vec{m}\cdot\hat{\vec{J}}}$$ operator, where. $\vec{J} = (\vec{J_x}, \vec{J_y}, \vec{J_z})$

~~where~~ $\hat{J_x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$, $\hat{J_y} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$

$\hat{J_z} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. These matrices are hermitian

if α is very smal

$R_{\vec{m},\alpha} \triangleq 1 - i\alpha\, \vec{m}\cdot\hat{\vec{J}}$ [$\hat{\vec{J}}$ is called generator of the transform -ation]

* If we have two consecutive rotations,

$R_{\vec{n},\alpha} \cdot R_{\vec{m},\beta} = R_{\vec{p},\gamma}$, we need to find $\vec{p}$ and γ

we just need to know how $\hat{J_x}, \hat{J_y}, \hat{J_z}$ commute among each others.

$R_{\vec{p},\gamma} = e^{-i(\ldots)[\hat{J_i}, \hat{J_j}]}$ (a generic structure)

$[\hat{J_x}, \hat{J_y}] = i\hat{J_z}$: This is algebra of the generator, it is not a group. From this we build a group we need to know these commutation algebra to understand which group they make up.

~~B~~/7 There is a one-to-one relation b/w the algebra of ~~get~~ a generator and a group. Lie groups $\Leftrightarrow$ algebra of the generators

$[\hat{J}_i, \hat{J}_j] = f(J_k)$

There are as many generators as parameter (θ, ϕ, α) of the group

* Algebra of the rotation group; angular momentum quantization :—

$\longleftrightarrow$ Representation of the rotation group

$[\hat{J}_x, \hat{J}_y] = i\hat{J}_z$ — (I) ; We can build an abstract group composition law (we don't represent by 3x3 matrices we can use other representations)

$J^2 = J_x^2 + J_y^2 + J_z^2 \equiv$ (total angular momentum)^2 $\equiv$ casimir operator

J^2 commutes with all the generators,

$[J^2, J_i] = 0$ — (II)

$J^2|J, m\rangle = \cancel{J(J+1)}\ \cancel{j(j+1)}\ \overset{J(J+1)}{}|J, m\rangle$ (diagonalising) (eigenvalue relation)

J is an integer or half integer

and $-J \leq m \leq J$ (It does not say J is 3)

I) if $J = 1$, $m = -1, 0, -1$ $|1, 1\rangle$; $|1, 0\rangle$ and $|1, -1\rangle$. The we have 3-D state. The matrix that represents the rotation will be a 3x3 matrix. We recover the 3D - representation. This is the vector representation of the group.

II) if $J = \frac{1}{2}$, $m = -\frac{1}{2}$ and $\frac{1}{2}$

The we have a 2D space : $|\frac{1}{2}, -\frac{1}{2}\rangle$ and $|\frac{1}{2}, \frac{1}{2}\rangle$

and J is a 2x2 matrix. This is also a rotation in 3D, [O(3)], but represented in 2 dimensions (in smaller space, an abstract space, not space - time).

III) if $J = \frac{3}{2}$, $m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

states are $|\frac{3}{2}, \frac{3}{2}\rangle$, $|\frac{3}{2}, \frac{1}{2}\rangle$, $|\frac{3}{2}, -\frac{1}{2}\rangle$, $|\frac{3}{2}, -\frac{1}{2}\rangle$. A 4-d space. This is still a rotation in 3D, represented in an abstract manner in an Hilbert space that has 4 dimensions.

Group — representations (many different Hilbert spaces)

SO(3) →
- 2d : $J = 1/2$
- 3d : $J = 1$
- 4d : $J = 3/2$
- 5d : $J = 2$

iv) if $J = 2$, $m = -2, -1, 0, 1, 2$

states are $|2,-2\rangle$, $|2,-1\rangle$, $|2,0\rangle$, $|2,1\rangle$, $|2,2\rangle$: 5d space

projecting on 3d space from Hilbert space,

$\langle x,y,z | 1,1\rangle$
$\langle x,y,z | 1,0\rangle$: spherical harmonics
$\langle x,y,z | 1,-1\rangle$

* <u>The Lorentz's Group</u>

(i) <u>Lie Group</u> : an element $R \in G$ [a generic Lie group]

$\Downarrow$

$$[J_i, J_j] = f(J_K) \qquad i,j,K = 1,\dots,n$$

(ii) algebra of generator

group element, $R = e^{i\sum_i \alpha_i J_i}$ satisfies the algebra

$\Downarrow$

(iii) find the casimir operator

$$\hat{C}(J_i, J_j, J_K) \text{ such that } [\hat{C}, J_i] = 0$$

(iv) diagonalise the casimir

$$\hat{C}|c_s \dots\rangle = C_k |c_k,\dots\rangle$$

$|c_s\dots\rangle \equiv$ the hilbert space which caries the representation of the group

· The rotation group : $\overset{SO(3)}{\wedge}$ $x^2+y^2+z^2 = l^2$ [invariant]

$$[\hat{J}_x, \hat{J}_y] = i\hat{J}_z$$

in vector representation, $J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$, $J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$

$$J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now considering the Lorentz's Group : SO(3, 1) ← ~~means~~

It means : → $\boxed{ct^2 - x^2 - y^2 - z^2 = \Delta s}$: quantity that ~~rem~~ remains invariant. Lorentz's Group is that group of transformation that leaves this invariant. The "(3, 1)" means, there are 3 "−1"s (x^2, y^2 and z^2) and one "+1" in the invariant length, also called the pseudolength

$$ct^2 - x^2 - y^2 - z^2 = ct'^2 - x'^2 - y'^2 - z'^2$$ [like the rotational invariant]

: a kind of pseudorotation in 4D

lets have a transformation : we are in 4 dimension, so let's consider

leaves the t invariant

$$\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array}\right) = L_{(R)}$$

a rotation (leaves the x, y and z invariant)

associated with the rotation

∴ It is a Lorentz transformation

$$\Rightarrow \quad t \to t' \qquad \left.\begin{array}{l} x \to x' \\ y \to y' \\ z \to z' \end{array}\right\} R$$

∴ we have 3 parameters of rotation

*Rotation for sure is a subgroup of the Lorentz Group

let's call a Lorentz transformation that changes the velocity along x

$$L_{\xi x} = \begin{array}{c} t \\ x \\ y \\ z \end{array}\begin{pmatrix} \cosh \xi_x & \sinh \xi_x & 0 & 0 \\ \sinh \xi_x & \cosh \xi_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$: changes t and x

This is a boost (change in velocity) in the x- direction, not a rotation.

now, a boost along y,

$$L_{\xi y} = \begin{array}{c} t \\ x \\ y \\ z \end{array}\begin{pmatrix} \cosh \xi_y & 0 & \sinh \xi_y & 0 \\ 0 & 1 & 0 & 0 \\ \sinh \xi_y & 0 & \cosh \xi_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$: changes t and y

$$L_{\xi_z} = \begin{array}{c} t \\ x \\ y \\ z \end{array} \begin{pmatrix} \cosh \xi_z & 0 & 0 & \sinh \xi_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \xi_z & 0 & 0 & \cosh \xi_z \end{pmatrix}$$

~~thee~~ there are 3 boosts : 3 parameters of boosts

∴ There are total of 6 parameters : 3 for rotation, 3 for boosts

→ As we have a group, we have a Lie algebra, with 6 generators $\hat{J}_x, \hat{J}_y$ and $\hat{J}_z$ from rotations. $(\hat{J}_x, \hat{J}_y, \hat{J}_z)$. Other 3 related to the boosts: $(\hat{K}_x, \hat{K}_y, \hat{K}_z)$

$(\hat{J}_x, \hat{J}_y, \hat{J}_z) \sim R = e^{i\alpha \vec{n} \cdot \vec{J_i}}$ [an element]

$(\hat{K}_x, \hat{K}_y, K_z) \sim L = e^{-\beta \vec{m} \cdot \vec{K_i}}$

$$K_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

: Representation of the six generators in 4-dimensions

1) $[J_i, J_j] = \sum_{k} i \varepsilon_{ijk} J_k$ [algebra between the rotation generators]

ii) $[K_i, K_j] = \sum_k i \varepsilon_{ijk} J_k$ [algebra b/w the Lorentz generators]
→ two boosts give back a rotation

iii) $[J_i, K_j] = i \varepsilon_{ijk} K_k$ [Rotation → Boost → Boost in another direction]

(i) (ii) and (iii) indicates the closeness of the algebra of the six generators of the Lorentz Group

*Now, we need to find the casimir :-

$$J^2 = J_x^2 + J_y^2 + J_z^2$$

$$\hat{J}^2 |J, (\ )\rangle = J(J+1)|J, (\ )\rangle$$

$$[\vec{J}^2, J_i] = 0$$

the algebra of the 6 generators

$$[J_i, J_j] = i\sum_k \varepsilon_{ijk} J_k$$

$$[K_i, K_j] = i\sum_k \varepsilon_{ijk} J_k$$

$$[J_i, K_j] = i\sum_k \varepsilon_{ijk} K_k$$

$R = e^{i\alpha \vec{n}\cdot\vec{J}}$ (Rotation)

$L = e^{-\beta \vec{m}\cdot\vec{K}}$ (Lorentz Boost)

2 Casimirs are,

$$\left.\begin{aligned}\hat{F} &= \tfrac{1}{2}(J^2 + K^2) \\ \hat{G} &= (\vec{J}\cdot\vec{K})\end{aligned}\right\}$$ they commute with all the 6 generators

diagonalizing the casimirs

$$\hat{F}|\quad\rangle = (\quad)|\quad\rangle$$

$$\hat{G}|\quad\rangle = (\quad)|\quad\rangle$$

eigenvalues of the operators [these numbers characterize the ~~operators~~ dimension of the representations]

we build the following operators to handle the strange algebra of, mixed with Rotations and boosts

$$\vec{\hat{A}} = \frac{1}{2}(\vec{J} + i\vec{K})$$

$$\vec{\hat{B}} = \frac{1}{2}(\vec{J} - i\vec{K})$$

$$\left.\begin{aligned}[\hat{A}_i, \hat{A}_j] &= i\sum_k \varepsilon_{ijk} \hat{A}_k \\ [\hat{B}_i, \hat{B}_j] &= i\sum_k \varepsilon_{ijk} \hat{B}_k \\ [\hat{A}_i, \hat{B}_j] &= 0\end{aligned}\right\}$$ We have two rotation groups each SO(3)

$\hat{J}$ and $\hat{K}$ were hermitian but $\hat{A}$ is not.

$$\hat{A}^{+} = \hat{B}$$

$\therefore$ ~~SO(3) ~ SO(3) ×~~ ~~(isomorphic)~~

$SO(3,1) \sim SO(3) \times SO(3)$

$\uparrow$ isomorphic

algebra of the Lorentz Group is isomorphic to two rotation groups, but $\hat{A}$ and $\hat{B}$ are not hermitian.

Algebra of ~~Generator of Rotation~~

$$[\hat{J}_i, \hat{J}_j] = i\sum_K \varepsilon_{ijK} \hat{J}_K \qquad i = ~~1,2,3~~\ 1,2,3$$

$$[\hat{K}_i, \hat{K}_j] = i\sum_K \varepsilon_{ijK} \hat{J}_K$$

$$[\hat{J}_i, \hat{K}_j] = i\sum_K \varepsilon_{ijK} \hat{K}_K$$

two operators that commute with all the 6 generators

$$\hat{F} = \frac{1}{2}(\vec{J}^2 + \vec{K}^2)^2$$

$$\hat{G} = \vec{J}\cdot\vec{K}$$

decoupling the algebra, new operators are introduced,

$$\hat{A} = \frac{1}{2}(\vec{J} + i\vec{K})$$

$$\hat{B} = \frac{1}{2}(\vec{J} - i\vec{K})$$

$$\left.\begin{aligned}[\hat{A}_i, \hat{A}_j] &= i\sum_K \varepsilon_{ijK} \hat{A}_K \\ [\hat{B}_i, \hat{B}_j] &= i\sum_K \varepsilon_{ijK} \hat{B}_K\end{aligned}\right\} \text{two algebra like the angular Momentum.}$$

$[A_i, B_j] = 0 \leftarrow$ decoupled

But we cannot proceed in the same procedure as that of angular Momentum as, $\hat{A}_i^\dagger = \hat{B}_i$ [They are not hermitian]

- The unitary representation of the Lorentz group are not finite dimension, due to the lack of hermiticity in the operators

$$J_x = \begin{pmatrix} 1 & \\ & l_x \end{pmatrix}_{4\times4} \leftarrow \text{Generator of rotation.} \qquad K_x = \begin{pmatrix} \ \end{pmatrix}_{4\times4}$$

$$J_y = \begin{pmatrix} 1 & \\ & l_y \end{pmatrix}_{4\times4} \qquad K_y = \begin{pmatrix} \ \end{pmatrix}_{4\times4}$$

$$J_z = \begin{pmatrix} 1 & \\ & l_z \end{pmatrix}_{4\times4} \qquad K_z = \begin{pmatrix} \ \end{pmatrix}_{4\times4}$$

: but they seem finite dimensional.

$R = \cancel{e^{i\vec{\theta}\cdot\vec{n}}}\ e^{i\alpha\vec{n}\cdot\vec{J}}$

$L = e^{|\vec{v}|\vec{m}\cdot\vec{K}}$ even if K is hermitian L is not unitary.

$\therefore$ It is a non-unitary representation of the Lorentz Group. The unitary representation is infinite.

* We know the invariant quantity under Lorentz transformation is a sort of 4-length

$$c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = c^2(\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2$$

$\Downarrow$

Relativistic analog of the invariant under rotation:

$$(\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

We can have an object with index $\mu = 0, 1, 2, 3, \cancel{4}$.

$$a^\mu = (\cancel{a^4}\ a^0, a^1, a^2, a^3) \leftarrow \text{4-vector}$$

transformation:

$$\boxed{a'^\mu = \underbrace{\Lambda^\mu{}_\gamma}_{\text{Lorentz transform}}\, a^\gamma}$$

: contravariant 4-vector [index: superscript]

$$(a^0)^2 - (a^1)^2 - (a^2)^2 - (a^3)^2 = (a'^0)^2 - (a'^1)^2 - (a'^2)^2 - (a'^3)^2$$

We introduce another notation of covariant 4-vector: index subscript

$$\boxed{a_\mu = \underbrace{g_{\mu\nu}}_{\text{Metric}}\, a^\nu}$$

$$g_{\mu\nu} = \begin{array}{c|cccc} \mu\backslash\nu & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 \\ 3 & 0 & 0 & 0 & -1 \end{array}$$

: 4-metric

4-length of the 4-vector a^μ:

is $(a^0)^2 - (a^1)^2 - (a^2)^2 - (a^3)^2 = (a'^0)^2 - (a'^1)^2 - (a'^2)^2 - (a'^3)^2$

$= g_{\mu\nu}\, \cancel{a'^\mu}\, a'^\nu$

$= \cancel{a'_\nu\, a'^\nu}$ $\quad$ [$\cancel{a_\mu = g_{\mu\nu} a^\nu}$]

$\cancel{(a^0)^2 - (a^1)^2 - (a^2)^2 - (a^3)^2 = a^\mu a_\mu a^\nu}$

$\cancel{= a'_\mu a'^\nu}$

$= g_{\mu\nu} a^\nu.$

$(a^0)^2 - (a^1)^2 - (a^2)^2 - (a^3)^2 = (a'^0)^2 - (a'^1)^2 - (a'^2)^2 - (a'^3)^2$

$= \cancel{g_{\mu\nu} a'^\mu a'^\mu} \quad g_{\mu\nu} a'^\mu a'^\nu$

$= \cancel{a'_\mu a'^\mu} \quad a'_\nu a'^\nu$

$= \cancel{a_\mu a^\mu} \quad a_\nu a^\nu$

$= \cancel{g_{\mu\nu} a^\nu a^\mu} \quad g_{\mu\nu} a^\mu a^\nu$

* $(a^0)^2 - (a^1)^2 - (a^2)^2 - (a^3)^2$

$= a^\mu a_\mu$

$= a'^\mu a'_\mu$ — Lorentz Transformation of covariant component of vectors.

$= g_{\mu\nu} a^\mu a^\nu$

now if $a'^\mu = \underbrace{\Lambda_\nu^{\ \mu}}_{\text{Lorentz transformation}} a^\nu$

then $a'_\mu = \cancel{a_\nu (\Lambda^{-1})^\nu_\mu} \quad (\Lambda^{-1})^\nu{}_\mu\, a_\nu$

proof :- $\cancel{a^\mu a_\mu = \Lambda^\mu_\nu}$

$a^\mu a_\mu$

$= a'^\mu a'_\mu$ [invariant]

$= \Lambda_\nu^{\ \mu} a^\nu a'_\mu$ [Lorentz transformation]

$\Rightarrow \therefore\ a'^\mu = \Lambda_\nu^{\ \mu} a^\nu$

$\cancel{\Rightarrow (\Lambda_\nu^{\ \mu})^{-1} (\Lambda^{-1})^\mu_\nu}$

$\cancel{\Rightarrow (\Lambda^{-1})^\nu{}_\mu a'^\mu = I a^\nu}$

$\cancel{\Rightarrow a^\nu}$

$$a^{\mu}a_{\mu} = a'^{\mu}a'_{\mu}$$

$$= \Lambda_{\nu}{}^{\mu} a^{\nu} a'_{\mu} \qquad \text{[The indices are saturated]}$$

$$\Rightarrow (\Lambda^{-1})^{\nu}{}_{\mu} a^{\mu} a_{\mu} = (\Lambda^{-1})^{\nu}{}_{\mu} (\Lambda)_{\nu}{}^{\mu} a^{\nu} a'_{\mu}$$

$$\Rightarrow a'_{\mu} a^{\nu} = \cancel{(\Lambda^{-1})^{\nu}{}_{\mu} a^{\mu} a_{\mu}} \; (\Lambda^{-1})^{\nu}{}_{\mu} a^{\mu} a_{\mu}$$

$$= \cancel{(\Lambda^{-1})^{\nu}{}_{\mu} a^{\mu} g_{\mu\nu} a^{\mu}} \; (\Lambda^{-1})^{\nu}{}_{\mu} a^{\mu} g_{\mu\nu} a^{\nu}$$

$$\Rightarrow a'_{\mu} = \cancel{(\Lambda^{-1})^{\nu}{}_{\mu} a^{\mu} g_{\mu\nu}} \; a^{\mu} g_{\mu\nu} (\Lambda^{-1})^{\nu}{}_{\mu}$$

$$\boxed{\cancel{a'_{\mu} = (\Lambda^{-1})^{\nu}{}_{\mu} a_{\nu}}} \; \cancel{(Q.E.D)} \; \cancel{a'_{\mu} = (\Lambda^{-1})^{\nu}{}_{\mu} a_{\nu}}$$

$$\boxed{a'_{\mu} = a_{\nu} (\Lambda^{-1})^{\nu}{}_{\mu}} \quad (Q.E.D)$$

$$* \quad \cancel{a_{\mu} = (\Lambda^{-1})^{\nu}{}_{\mu} a_{\nu}} \; \text{— (I)}$$

$$\cancel{\Rightarrow g_{\mu\nu} a^{\nu} = (\Lambda^{-1})^{\nu}{}_{\mu}} \quad a'_{\mu} = a_{\nu} (\Lambda^{-1})^{\nu}{}_{\mu}$$

and $a'_{\mu} = g_{\mu\rho} a'^{\rho} = g_{\mu\rho} \Lambda_{\sigma}{}^{\rho} a^{\sigma}$

$$= \cancel{g_{\mu\rho} \Lambda_{\sigma}{}^{\rho} a^{\sigma}}$$

$$= g_{\mu\rho} \Lambda_{\sigma}{}^{\rho} \underline{\underline{g^{\sigma\nu}}} a_{\nu} \quad \text{— (II)}$$

a new metric

now, $g_{\mu\nu} g^{\nu\sigma} = \underbrace{\delta_{\mu}{}^{\sigma}}$ → Identity matrix

~~comparing (I) and (II),~~

$$\cancel{(\Lambda^{-1})^{\nu}{}_{\mu} = g_{\mu\rho} \Lambda_{\sigma}{}^{\rho} g^{\sigma\nu}}$$

$$\cancel{= \Lambda_{\mu\sigma} g^{\sigma\nu}}$$

$$\cancel{= \Lambda_{\mu}{}^{\nu}} \quad \text{: \cancel{Analog to unitary property}}$$

comparing (I) and (II),

$$(\Lambda^{-1})^{\nu}{}_{\mu} = g_{\mu\rho} \Lambda_{\sigma}{}^{\rho} g^{\sigma\nu}$$

$$= \Lambda_{\mu\sigma} g^{\sigma\nu}$$

$$= \Lambda_{\mu}{}^{\nu} \quad \text{: Analog to unitary property}$$

* $\cancel{x}^{\mu} = (ct, x, y, z)$ —— (i)

$\cancel{x}_{\mu} = g_{\mu\nu} x^{\nu} = (ct, -x, -y, -z)$ —— (ii)

$\cancel{x}^{\mu}\cancel{x}_{\mu} = \sum x^{\mu} x^{\mu} = x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3$

$= (ct)^2 - x^2 - y^2 - z^2$

$= \cancel{g_{\mu\nu}}\, g^{\mu\nu} \cancel{x}_{\mu}\cancel{x}_{\mu}$ —— (iii)

the analog of momentum

$p^{\mu} = (E/c, P_x, P_y, P_z)$: contravariant

$p_{\mu} = (E/c, -P_x, -P_y, -P_z)$; Covariant

$p_{(1)}^{\mu} p_{(2)\mu} = \frac{E_{(1)}}{c} \frac{E_{(2)}}{c} - \vec{P}_{(1)} \cdot \vec{P}_{(2)}$

and $\boxed{\cancel{x} \cdot p = x^{\mu} p_{\mu} = Et - \vec{x} \cdot \vec{p}}$

* $\boxed{\hat{p}^{\mu} \equiv i\hbar \frac{\partial}{\partial x_{\mu}}}$ [defination of differential operator]

$= \Big(i\hbar \frac{\partial}{\partial (ct)}, \underbrace{-i\hbar \frac{\partial}{\partial x}, -i\hbar \frac{\partial}{\partial y}, -i\hbar \frac{\partial}{\partial z}}_{\text{space part}}\Big)$

$= i\hbar \left(\frac{\partial}{\partial (ct)}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right)$

$= i\hbar \left(\frac{\partial}{\partial (ct)}, -\vec{\nabla} \right)$

and $\hat{p}^{\mu} \hat{p}_{\mu} = -\hbar^2 \frac{\partial}{\partial x_{\mu}} \cdot \frac{\partial}{\partial x^{\mu}}$

$= -\hbar^2 \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2}, - \frac{\partial^2}{\partial x^2}, -\frac{\partial^2}{\partial y^2}, -\frac{\partial^2}{\partial z^2} \right)$

$= -\hbar^2 \underbrace{\left(\frac{1}{c^2} \frac{\partial}{\partial t^2}, -\Delta \right)}$ $\quad \Delta = \nabla^2$

$\Box \equiv$ ~~Four La~~ 4-Laplacian

* Relativistic Quantum Mechanics :-

Quantum Mechanics + Special Relativity

1) Time and space are combined : space-time, we don't talk about space and time separately

2) $\Delta x \sim \frac{\hbar}{\Delta p}$: uncertainty principle, Δx: smallest distance where a particle can be localised if the uncertainty in p is Δp.

we consider single particles here :

$$\Delta p = m_0 c \quad [\text{maximum uncertainty in } \Delta p]$$

as ~~$p^2 = m$~~ $E^2 - p^2c^2 = m_0^2c^4$

$$\Rightarrow E^2 - m_0^2c^4 = m_0^2c^4$$

$$\Rightarrow E^2 = 2m_0^2c^4$$ [This describe two particles at rest] → not any more a single particle

$\therefore$ We have to consider $\Delta p \leq m_0 c$ must be (a bound)

then, $\Delta x \sim \frac{\hbar}{m_0 c}$ = Compton wave length

$\therefore$ a single free particle cannot be confined to a distance that is less than the Compton ~~free length~~. wave-length

3) $c\Delta t \sim \Delta x \sim \frac{\hbar}{m_0 c}$

there is an uncertainty in time too (because space and time are equivalent).

$$\therefore \boxed{\Delta t \sim \frac{\hbar}{m_0 c^2}}$$

~~$\Psi(x, y, z, ct)$~~

4) $\hat{p}^\mu = i\hbar \frac{\partial}{\partial \hat{x}_\mu}$

and
$$[\hat{p}^\mu, \hat{x}^\nu] = i\hbar\left[\frac{\partial}{\partial \hat{x}_\mu}, \hat{x}^\nu\right]$$

$$= i\hbar\left[\frac{\partial}{\partial \hat{x}_\mu}, \cancel{g^{\mu\nu} x}\; g^{\nu\sigma}\hat{x}_\sigma\right]$$

$$= i\hbar\, g^{\nu\sigma}\left[\frac{\partial}{\partial \hat{x}_\mu}, \hat{x}_\sigma\right]$$

now, $\left[\frac{\partial}{\partial \hat{x}_\mu}, \hat{x}_\sigma\right] f$

$$= \frac{\partial}{\partial \hat{x}_\mu}(\hat{x}_\sigma f) - \hat{x}_\sigma \frac{\partial f}{\partial \hat{x}_\mu}$$

$$= \frac{\partial \hat{x}_\sigma}{\partial \hat{x}_\mu} f + \hat{x}_\sigma \frac{\partial f}{\partial \hat{x}_\mu} - \hat{x}_\sigma \frac{\partial f}{\partial \hat{x}_\mu}$$

$$= \frac{\partial \hat{x}_\sigma}{\partial \hat{x}_\mu} f$$

$$\therefore \boxed{\left[\frac{\partial}{\partial \hat{x}_\mu}, \hat{x}_\sigma\right] = \frac{\partial \hat{x}_\sigma}{\partial \hat{x}_\mu}}$$

$\therefore \cancel{[\hat{p}^\mu, \hat{x}_\sigma] = \frac{\partial \hat{x}_\sigma}{}}$

$$\therefore [\hat{p}^\mu, \hat{x}^\nu] = i\hbar g^{\nu\sigma}\left[\frac{\partial}{\partial \hat{x}_\mu}, \hat{x}_\sigma\right]$$

$$= i\hbar g^{\nu\sigma} \frac{\partial \hat{x}_\sigma}{\partial \hat{x}_\mu} = i\hbar g^{\nu\sigma} \delta_{\sigma\mu}$$

$$\boxed{[\hat{p}^\mu, \hat{x}^\nu] = i\hbar g^{\nu\mu}}$$

* In the non-relativistic case, the schrodinger's equation

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\hat{x})\right]\psi(x,t)$$

and $\hat{E} = \frac{\hat{p}^2}{2m} + V(\hat{x}) = i\hbar \frac{\partial}{\partial t}$

in non-interaction case, $V(\hat{x}) = 0$

$$\therefore i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi(x,t)$$

and $\hat{E} = \frac{\hat{p}^2}{2m} = i\hbar \frac{\partial}{\partial t}$

~~For Relativistic case~~ $\frac{E^2}{c^2} - \vec{p}^2 = m_0^2 c^2$

For ~~classical~~ relativistic case,

let's introduce an energy and momentum operator, $\hat{E} = \frac{\hat{p}^2}{2m}$

we have, $\hat{p}^\mu \hat{p}_\mu = m_0^2 c^2 \mathbb{I}$ $\quad \hat{p}^\mu = i\hbar \frac{\partial}{\partial x_\mu} = i\hbar\left(\frac{\partial}{\partial (ct)}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)$

$$\therefore \hat{p}^\mu \hat{p}_\mu \psi = m_0^2 c^2 \psi$$

$$\Rightarrow -\hbar^2 \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)\psi = m_0^2 c^2 \psi$$

$$\Rightarrow \boxed{\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m_0^2 c^2}{\hbar^2}\right]\psi = 0}$$

$\Rightarrow$ Klein-Gordon equation, for the free particle this is a wave equation.

there is a soln, $\psi = \exp\left(-\frac{i}{\hbar} p_\mu x^\mu\right)$

$$= \exp\left[-\frac{i}{\hbar}\left(p_0 x^0 - \vec{p}.\vec{x}\right)\right]$$

$$= \exp\left[\frac{i}{\hbar}\left(\vec{p}.\vec{x} - Et\right)\right]$$

to prove that ψ is a soln of this free equation

$$\therefore \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m_0^2 c^2}{\hbar^2}\right)\psi = 0$$

now, $$\frac{\partial^2}{\partial t^2}\psi = -\frac{iE}{\hbar}\frac{\partial}{\partial t}\exp\left(\frac{i}{\hbar}(\vec{p}.\vec{x} - Et)\right)$$

$$= -\frac{E^2}{\hbar^2}\exp\left(\frac{i}{\hbar}(\vec{p}.\vec{x} - Et)\right)$$

$$\frac{\partial^2}{\partial x^2}\exp\left[\frac{i}{\hbar}\left(\vec{p}.\vec{x} - Et\right)\right]$$

$$= -\frac{p_x^2}{\hbar^2}\exp\left[\frac{i}{\hbar}\left(\vec{p}.\vec{x} - Et\right)\right]$$

$$\therefore \left[-\frac{1}{c^2}\frac{E^2}{\hbar^2} + \frac{\vec{p}^2}{\hbar^2} + \frac{m_0^2 c^2}{\hbar^2}\right] e^{\frac{i}{\hbar}(\vec{p}.\vec{x} - Et)} = 0$$

$$\psi \neq 0 \therefore -\frac{1}{c^2}\frac{E^2}{\hbar^2} + \frac{\vec{p}^2}{\hbar^2} + \frac{m_0^2 c^2}{\hbar^2} = 0$$

$$\Rightarrow -\frac{E^2}{c^2} + \vec{p}^2 = -m_0^2 c^2$$

$$\Rightarrow \boxed{\frac{E^2}{c^2} - \vec{p}^2 = m_0^2 c^2} \leftarrow \text{relativistic constraint.}$$

ψ is a soln if $\frac{E^2}{c^2} - \vec{p}^2 = m_0^2 c^2$

* For schrodinger's equation ψ is a soln if $E = \frac{p^2}{2m}$ [non-relativistic constraint]

* Relation b/w momentum and energy : $p^\mu p_\mu = m_0^2c^2$

Promoting those to operators : $\hat{p}^\mu \hat{p}_\mu = m_0^2 c^2$

$$\therefore \hat{p}^\mu = i\hbar \frac{\partial}{\partial x_\mu}$$

$$\therefore \hat{p}^\mu \hat{p}_\mu \psi(x,t) = m_0^2 c^2 \psi(x,t)$$

$$\Rightarrow \left[(i\hbar)^2 \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}\right]\psi(x,t) = m_0^2c^2\psi(x,t)$$

$$\Rightarrow \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m_0^2c^2}{\hbar^2}\right]\psi = 0 \quad \text{—(1)}$$

We need to see that the solution describes a physical system

$$\psi = \exp\left(-\frac{i}{\hbar} p_\mu x^\mu\right) = \exp\left[-\frac{i}{\hbar}\left(p_0 x^0 - \vec{p}\cdot\vec{x}\right)\right]$$

$$= \exp\left[\frac{i}{\hbar}\left(\vec{p}\cdot\vec{x} - Et\right)\right]$$

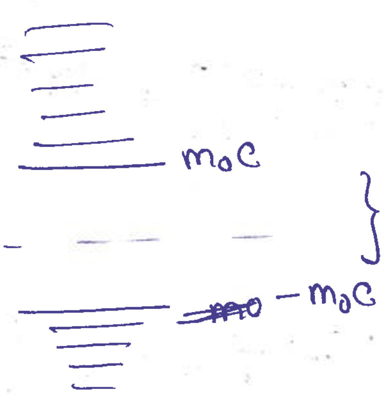
ψ ~~is~~ has usual form of non-relativistic plane wave.

plugging ψ in (1), we get $\boxed{\frac{\vec{E}^2}{c^2} - \vec{p}^2 = m_0^2c^2}$

~~A~~ A plane wave is a soln// of the free Klein Gordon equation only if it satisfies : $\frac{\vec{E}^2}{c^2} - p^2 = m_0^2c^2$ (mass - shell relation)

$\therefore$ ~~$E = \pm c\sqrt{\ }$~~ $E_\pm = \cancel{E_0} \pm c\sqrt{m_0^2c^2 + p^2}$

In the non-relativistic case there is a single solution only.



} energy is above and below this threshold → could describe ~~energy~~ system with negative energy

Klein-Gordon immediately noticed that there is ~~no~~ a negative energy, (mainly associated with potential) and tried to give a different interpretat-ion

* <u>Trouble with the probability interpretation</u> :-

$$\left(\hat{p}_\mu \hat{p}^\mu - m_0^2c^2\right)\psi = 0 \quad \text{—(i)}$$

and $(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2)\psi^* = 0$ —(ii)

$\psi^* \times (i) - \psi \times (ii)$

$\therefore \Rightarrow \psi^*(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2)\psi - \psi(\hat{p}_\mu \hat{p}^\mu - m_0^2 c^2)\psi^* = 0$

now, $\hat{p}_\mu = i\hbar \frac{\partial}{\partial x^\mu}$ and $\hat{p}^\mu = i\hbar \frac{\partial}{\partial x_\mu}$

$\Rightarrow \therefore -\psi^*(\hbar^2 \nabla_\mu \nabla^\mu + m_0^2 c^2)\psi + \psi(\hbar^2 \nabla_\mu \nabla^\mu + m_0^2 c^2)\psi^* = 0$

$\Rightarrow -\hbar^2 [-\psi \nabla_\mu \nabla^\mu \psi^* + \psi^* \nabla_\mu \nabla^\mu \psi] = 0$

$\Rightarrow \nabla_\mu [\psi^* \nabla^\mu \psi - \psi \nabla^\mu \psi^*] = 0$

$\Rightarrow \boxed{\nabla_\mu J^\mu = 0}$: conservation law : 4-components. [4-divergence]

where, $J^\mu = \frac{i\hbar}{2m_0}(\psi^* \nabla^\mu \psi - \psi \nabla^\mu \psi^*)$

this factor gives ~~a meaning~~ to the zero component the meaning of probability density

$J^0 = \frac{i\hbar}{2m_0}(\psi^* \nabla^0 \psi - \psi \nabla^0 \psi^*)$

$[J^0] = \frac{1}{L^3}$

$x_0 = ct$

$\therefore \mu = 0,$

we get, $\underbrace{\frac{\partial}{\partial t}\left[\frac{i\hbar}{2m_0 c^2}\left(\psi^* \frac{\partial}{\partial t}\psi - \psi \frac{\partial}{\partial t}\psi^*\right)\right]}_{\mu=0} + \vec{\nabla}\cdot\left(\frac{-i\hbar}{2m_0}\right)\underbrace{\left[\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right]}_{\mu=1,2,3} = 0$

$\Rightarrow \frac{\partial}{\partial t}\left[\frac{i\hbar}{2m_0 c^2}\left(\psi^* \frac{\partial}{\partial t}\psi - \psi \frac{\partial}{\partial t}\psi^*\right)\right] + \vec{\nabla}\cdot\left(\frac{-i\hbar}{2m_0}\right)\left[\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right] = 0$

$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0} \longrightarrow$ continuity equation

where $\rho = \frac{i\hbar}{2m_0 c^2}\left(\psi^* \frac{\partial}{\partial t}\psi - \psi \frac{\partial}{\partial t}\psi^*\right)$

$\vec{J} = -\frac{i\hbar}{2m_0}(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*)$

integrating over d^3x (over all volume)

$$\int_{V_\infty} d^3x \frac{\partial \rho}{\partial t} = -\int_{V_\infty} d^3x \, \vec{\nabla}.\vec{J}$$

$$= -\oiint_S \vec{J}.d\vec{\sigma} \quad \text{(using Green's function)}$$

$$= 0 \quad \text{(at infinity the wave function} \to 0)$$

$$\Rightarrow \int_{V_\infty} \cancel{d} \frac{\partial \rho}{\partial t} d^3x = 0$$

$$\Rightarrow \frac{\partial}{\partial t} \underbrace{\int_{V_\infty} d^3x \, \rho}_{\text{total probability}} = 0$$

the total probability is conserved

In Non-relativistic case, prob density $P_{NR} = \psi^* \psi = |\psi|^2 > 0$,
total probability is +ve

But here, ρ can be negative if we specify such values of ψ and $\frac{\partial \psi}{\partial t}$ which gives ρ negative. The interpretation of probability, then is in trouble.

Lorentz D'Alembertian : $(\Box^2 + m_0^2 c^2)$

* Non-Relativistic limit of the Klien Gordon-Equation :-

The KE < Mass energy. let's parameterize a general soln/

$$\psi(x, t) = \rho(r, t) \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right)$$

$$\text{where, } \rho(r,t) = \psi(x,t) \exp\left(\frac{i}{\hbar} m_0 c^2 t\right)$$

total energy of the system,

$$E' = E - m_0 c^2$$

$$\boxed{E' \ll m_0 c^2} \leftarrow \text{non-relativistic limit}$$

$$i\hbar \frac{\partial \varphi}{\partial t} \sim E' \varphi \ll m_0 c^2 \rho$$

$\rho(r,t)$ goes with the KE

$$\therefore \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) \psi = m_0^2 c^2 \psi$$

now, $\dfrac{\partial \psi}{\partial t} = \dfrac{\partial \rho}{\partial t} \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right) - \frac{i}{\hbar} m_0 c^2 \rho(r,t) \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right)$

$$\frac{\partial \psi}{\partial t} = \left(\frac{\partial \rho}{\partial t} - \frac{i m_0 c^2}{\hbar}\rho\right) \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right)$$

now, $\cancel{\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \rho}{\partial t^2} \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right) - \frac{i}{\hbar} m_0 c^2 \frac{\partial \rho}{\partial t} \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right)}$

as we know, $\frac{\partial}{\partial t} \sim E'$ (total energy),

in the non-relativistic limit, $E \ll m_0 c^2$

$$\therefore \frac{\partial \rho}{\partial t} \ll m_0 c^2$$

$\cancel{\therefore \frac{\partial \psi}{\partial t} \approx -\frac{i m_0 c^2}{\hbar} \rho \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right)}$

$\cancel{\therefore \frac{\partial^2 \psi}{\partial t^2} = -\frac{m_0^2 c^4}{\hbar^2} \rho \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right) - \frac{i m_0 c^2}{\hbar} \frac{\partial \rho}{\partial t} \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right)}$

$\cancel{= -\left[\frac{i \partial \rho}{\partial t} + \rho\right] \frac{m_0 c^2}{\hbar}}$

$\cancel{\approx \left[-\frac{i m_0 c^2}{\hbar} \frac{\partial \rho}{\partial t} - \frac{m_0^2 c^4}{\hbar^2} \rho\right] \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right)}$

$$\frac{\partial^2 \psi}{\partial t^2} = \left(\frac{\partial^2 \rho}{\partial t^2} - \frac{i m_0 c^2}{\hbar} \frac{\partial \rho}{\partial t}\right) \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right) + -\frac{i}{\hbar} m_0 c^2 \left(\frac{\partial \rho}{\partial t} - \frac{i m_0 c^2}{\hbar}\rho\right) \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right)$$

$$= \left[\frac{\partial^2 \rho}{\partial t^2} - \frac{2i}{\hbar} m_0 c^2 \frac{\partial \rho}{\partial t} - \frac{m_0^2 c^4}{\hbar^2} \rho\right] \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right)$$

neglecting $\frac{\partial^2 \rho}{\partial t^2}$,

$$\frac{\partial^2 \psi}{\partial t^2} \approx -\left[\frac{2i}{\hbar} m_0 c^2 \frac{\partial \rho}{\partial t} + \frac{m_0^2 c^4}{\hbar^2} \rho\right] \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right)$$

$\cancel{\Rightarrow \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = \frac{1}{c^2} \left[\frac{2i}{\hbar} m_0 c^2 \frac{\partial \rho}{\partial t}\right]}$

now, the original eqn was $(\hat{p}^\mu \hat{p}_\mu - m_0^2 c^2)\psi = 0$

$$\psi(x, t) = \rho(x, t) \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right)$$

$$\therefore -\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = \left(-\nabla^2 + \frac{m_0^2 c^2}{\hbar}\right)\psi \quad \text{[from KG equation]}$$

$\cancel{\Rightarrow \frac{1}{c^2}\left[\frac{2i}{\hbar} m_0 c^2 \frac{\partial \rho}{\partial t} + \frac{m_0^2 c^4}{\hbar^2} \rho\right] = -\nabla^2 \psi + \frac{m_0^2 c^2}{\hbar} \rho}$

$$\Rightarrow \frac{1}{c^2}\left[\frac{2i}{\hbar} m_0 c^2 \frac{\partial \rho}{\partial t} + \frac{m_0^2 c^4}{\hbar^2}\rho\right] \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right) = -\nabla^2 \psi + \frac{m_0^2 c^2}{\hbar}\rho \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right)$$

$$\Rightarrow \cancel{\frac{1}{c^2}\, \frac{2i}{\hbar} m_0 \frac{\partial \rho}{\partial t} + \frac{m_0^2 c^2}{\hbar}\rho}$$

$$\Rightarrow \frac{2i}{\hbar} m_0 \frac{\partial \rho}{\partial t} \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right) = -\nabla^2 \psi$$

$$\Rightarrow \nabla^2 \psi = -\frac{2i}{\hbar} m_0 \frac{\partial \rho}{\partial t} \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right)$$

~~and as~~ $\cancel{\frac{\partial \rho}{\partial t} \ll m_0 c^2 \rho}$

$\cancel{\frac{1}{\hbar}\partial \rho}$ now, $\nabla^2 \psi = \left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2}\right) \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right)$

$$\therefore \left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2}\right) \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right) = -\frac{2i}{\hbar} m_0 \frac{\partial \rho}{\partial t} \exp\left(-\frac{i}{\hbar} m_0 c^2 t\right)$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = \frac{i\hbar}{2m_0}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\rho$$

$$\Rightarrow \boxed{i\hbar \frac{\partial \rho}{\partial t} = -\frac{\hbar^2}{4m_0}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\rho}$$

: usual schrodinger's equation

This is the Non-relativistic limit.

This describes a spin 0 particle.

KG is also for a spin 0 particle.

~~KG~~

* The free-wave equation was: $(\hat{p}^\mu \hat{p}_\mu - m_0^2 c^2)\psi = 0$

where, $\hat{p}_\mu = i\hbar \frac{\partial}{\partial x^\mu}$

$x^0 = ct$

$p^0 = E/c$

and $\psi = A \exp\left(-\frac{i}{\hbar} \hat{p}_\mu x^\mu\right) = A \exp\left[\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - E_p t)\right]$

and $E^2 = c^2(\vec{p}^2 + m_0^2 c^2)$

Given, $\vec{p}$, $E_p^{\pm} = \pm c\sqrt{\vec{p}^2 + m_0^2 c^2}$

$\therefore \psi_{\pm} = A \exp\left[\frac{i}{\hbar}(\vec{p}\cdot\vec{x} - E_p^{\pm} t)\right]$ [two solutions]

the probability density, was,

$$\rho = \frac{i\hbar}{2m_0 c^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)$$

$$\vec{J} = -\frac{i\hbar}{2m_0}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right)$$

we introduce a charge e,

$$\rho' = \frac{i\hbar e}{2m_0c^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) \leftarrow \text{charge density}$$

$$\vec{J}' = \frac{ie\hbar}{2m_0}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \leftarrow \text{charge-current density}$$

things might get fixed, as ρ' can now be negative or positive. KG ~~soln~~ soln// can describe a positive or negative charged particle.

we can interpret ψ not only as a wave, but a field too.

field = $f(x, t)$ [a function]

~~and charge:~~ $(\hat{p}^\mu \hat{p}_\mu - m_0^2c^2)\psi(x,t) = 0$ ← can be interpreted as a classical equation of a field, that is not yet quantized

now, $\boxed{\rho'_\pm = \pm\frac{e|E_p|}{m_0c^2}\psi^*_\pm\psi_\pm}$