Matrices

- **Matrix**
  - Array of numerical values, e.g.:

$$\mathbf{A} = \begin{bmatrix} -7 & 0 & 1 & 4 \\ 4 & -2 & 9 & 5 \\ 8 & 3 & 4 & 0 \end{bmatrix}$$

  - The variable, $\mathbf{A}$, is a ***matrix***
- An $m \times n$ matrix has m ***rows*** and n ***columns***
- These are the ***dimensions*** of the matrix
  - $\mathbf{A}$ is a 3×4 matrix

Matrix Dimensions and Indexing

- An $m \times n$ matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- Use indices to refer to individual elements of a matrix
  - a_{ij}: the element of $\mathbf{A}$ in the i^{th} row and the j^{th}column

Vectors

- ***Vectors***
  - A matrix with one dimension equal to one
  - A matrix with ***one row*** or ***one column***
- ***Row vector***
  - One row – a $1 \times n$ matrix, e.g.:

$$x = \begin{bmatrix} -9 & 1 & -4 \end{bmatrix}$$

  - A 1×3 row vector
- ***Column vector***
  - One column – an $m \times 1$ matrix, e.g.:

$$x = \begin{bmatrix} 5 \\ 1 \\ 8 \end{bmatrix}$$

  - A 3×1 column vector

Scalars

- **Scalar**
  - A 1×1 matrix
  - The numbers we are we are familiar with, e.g.:

$$b = 4, \qquad x = -3 + j5.8, \qquad y = -1 \times 10^{-9}$$

- We understand simple mathematical operations involving scalars
  - Can add, subtract, multiply, or divide any pair of scalars
  - Not true for matrices
    - Depends on the matrix dimensions

Mathematical Matrix Operations

Matrix Addition and Subtraction

- As long as matrices have the ***same dimensions***, we can add or subtract them
  - ***Addition*** and ***subtraction*** are done ***element-by-element***, and the ***resulting matrix is the same size***

$$\begin{bmatrix} 4 & 8 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 6 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 8 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -4 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ -6 & 4 \end{bmatrix}$$

- We can also add ***scalars*** to (or subtract from) matrices

$$\begin{bmatrix} 1 & -4 \\ 6 & -1 \end{bmatrix} + 5 = \begin{bmatrix} 6 & 1 \\ 11 & 4 \end{bmatrix}$$

Matrix Addition and Subtraction

- If matrices are not the same size, and neither is a scalar, addition/subtraction are not defined
  - The following operations cannot be done

$$\begin{bmatrix} 4 & 8 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -4 & 6 \\ 6 & -1 & 9 \end{bmatrix} = ?$$

$$\begin{bmatrix} 8 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & -4 \\ 6 & -1 \end{bmatrix} = ?$$

- Addition is commutative (order does not matter):

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} = \mathbf{C}$$

$$\begin{bmatrix} 4 & 8 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 6 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 8 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 6 & 2 \end{bmatrix}$$

Matrix Multiplication

- In order to multiply matrices, their ***inner dimensions*** must agree
- We can multiply $\mathbf{A} \cdot \mathbf{B}$ only if the ***number of columns*** of $\mathbf{A}$ is equal to the ***number of rows*** of $\mathbf{B}$
- Resulting Matrix has same number of rows as $\mathbf{A}$ and same number of columns as $\mathbf{B}$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$$

$$(\mathrm{m} \times \mathrm{n}) \cdot (\mathrm{n} \times \mathrm{p}) = (\mathrm{m} \times \mathrm{p})$$

Matrix Multiplication – $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{bmatrix}$$

- The (i, j^{th}) entry of $\mathbf{C}$ is the ***dot product*** of the i^{th} row of $\mathbf{A}$ with the j^{th} column of $\mathbf{B}$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

- Consider the multiplication of two 2×2 matrices:

$$\begin{bmatrix} \boxed{a_{11} \quad a_{12}} \\ a_{21} \quad a_{22} \end{bmatrix} \cdot \begin{bmatrix} \boxed{\begin{matrix} b_{11} \\ b_{21} \end{matrix}} & \begin{matrix} b_{12} \\ b_{22} \end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{a_{11}b_{11} + a_{12}b_{21}} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Matrix Multiplication – Examples

- A 2×2 and a 2×3 yield a 2×3

$$\begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -1 & 5 \\ 6 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 27 & 7 & 5 \\ 12 & 0 & 10 \end{bmatrix}$$

- A 3×3 and a 3×1 result in a 3×1

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 4 & 8 \\ 2 & 7 & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 20 \\ 25 \end{bmatrix}$$

Matrix Multiplication – Properties

- ***Matrix multiplication is not commutative***
  - Order matters
  - Unlike scalars
- In general,

$$\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$$

- If A and/or B is not square then one of the above operations may not be possible anyway
  - Inner dimensions may not agree for both product orders

Matrix Multiplication – Properties

- ***Matrix multiplication is associative***
  - Insertion of parentheses anywhere within a product of multiple terms does not affect the result:

$$(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{D}$$

$$\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) = \mathbf{D}$$

- ***Matrix multiplication is distributive***
  - Multiplication distributes over addition
  - Must maintain correct order, i.e. left- or right-multiplication

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$$

$$(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$$

Identity Matrix

- Multiplication of a scalar by 1 results in that scalar

$$a \cdot 1 = 1 \cdot a = a$$

- The matrix version of 1 is the ***identity matrix***
  - Ones along the diagonal, zeros everywhere else
  - Square $(n \times n)$ matrix
  - Denoted as $\mathbf{I}$ or $\mathbf{I_n}$, where $\mathbf{n}$ is the matrix dimension, e.g.

$$\mathbf{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Left- or right-multiplication by an identity matrix results in that matrix, unchanged

$$\mathbf{A} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{A} = \mathbf{A}$$

Identity Matrix

- Right-multiplication of an $n \times n$ matrix by an $n \times n$ identity matrix, $\mathbf{I_n}$

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 4 & 8 \\ 2 & 7 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 4 & 8 \\ 2 & 7 & 3 \end{bmatrix}$$

- Same result if we left-multiply by $\mathbf{I_n}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 & 0 \\ 0 & 4 & 8 \\ 2 & 7 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 4 & 8 \\ 2 & 7 & 3 \end{bmatrix}$$

Identity Matrix

- Right-multiplication of an $m \times n$ matrix by an $n \times n$ identity matrix

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 4 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 4 & 8 \end{bmatrix}$$

- Same result if we left-multiply the $m \times n$ matrix by an $m \times m$ identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 & 0 \\ 0 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 4 & 8 \end{bmatrix}$$

Vector Multiplication

- Vectors *are* matrices, so inner dimensions must agree
- Two types of vector multiplication:
- ***Inner product*** (***dot product***)
  - Result is a scalar

$$\begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \cdot \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{21}$$

- ***Outer product***
  - Result for n-vectors is an n x n matrix

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{21}b_{11} & a_{21}b_{12} \end{bmatrix}$$

Exponentiation

- As with scalars, raising a matrix to the power, n, is the multiplication of that matrix by itself n times

$$\mathbf{A}^3 = \mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A}$$

- What must be true of a matrix for exponentiation to be allowable?
  - Inner matrix dimensions must agree
  - Rows of $\mathbf{A}$ must equal columns of $\mathbf{A}$ – n x n
  - ***Matrix must be square***

Matrix Transpose

- The ***transpose*** of a matrix is that matrix with ***rows and columns swapped***
  - First row becomes the first column, second row becomes the second column, and so on
- For example:

$$\mathbf{A} = \begin{bmatrix} 0 & 9 \\ 2 & 7 \\ 6 & 3 \end{bmatrix} \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 0 & 2 & 6 \\ 9 & 7 & 3 \end{bmatrix}$$

- Row vectors become column vectors and vice versa

$$\mathbf{x} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix} \qquad \mathbf{x}^{\mathrm{T}} = \begin{bmatrix} 7 & -1 & -4 \end{bmatrix}$$

Why Do We Use Matrices?

- Vectors and matrices are used extensively in many engineering fields, for example:
  - Modeling, analysis, and design of dynamic systems
  - Controls engineering
  - Image processing
  - Etc. …

- Very common usage of vectors and matrices is to represent ***systems of equations***
  - These regularly occur in *all* fields of engineering

Systems of Equations

- Consider a system of three equations with three unknowns:

$$3x_1 + 5x_2 - 9x_3 = 6$$
$$-3x_1 + 7x_3 = -2$$
$$-x_2 + 4x_3 = 8$$

- Can represent this in ***matrix form***:

$$\begin{bmatrix} 3 & 5 & -9 \\ -3 & 0 & 7 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 8 \end{bmatrix}$$

- Or, more compactly as:

$$\mathbf{Ax} = \mathbf{b}$$

- Perform algebra operations as we would if $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{b}$ were scalars
  - Observing matrix-specific rules, e.g. multiplication order, etc.