Problem 1: Oscar # 1

1.1 Definitions and Parameters

Events:

- A, B: The dog is in forest A or forest B.
- D_n: The dog dies on the n^{th} evening.
- SA_n, SB_n: Oscar searches in forest A or B on day n.
- F_n: Oscar finds the dog on day n.
- $\neg F_n$: Oscar does not find the dog on day n.

Probabilities:

$$P(A) = 0.4 \qquad P(B) = 0.6$$
$$P(D_n \mid \neg F_1, \ldots, \neg F_n) = \frac{n}{n+2}$$

Search Success Rates:

$$P(F_n \mid A \wedge SA_n) = 0.25 \qquad P(\neg F_n \mid A \wedge SA_n) = 0.75$$
$$P(F_n \mid B \wedge SB_n) = 0.15 \qquad P(\neg F_n \mid B \wedge SB_n) = 0.85$$

Initial Constraints ($n = 1$):

$$P(F_1 \mid SA_1 \wedge B) = 0 \qquad P(F_1 \mid SB_1 \wedge A) = 0$$
$$P(\neg F_1 \mid SA_1 \wedge B) = 1 \qquad P(\neg F_1 \mid SB_1 \wedge A) = 1$$

Physical Constraints:

- Dog cannot move between forests.
- Oscar searches only during the daytime of day n.
- Oscar searches exactly one forest per day.
- Oscar can only switch forests at night.

1.2 Optimal Search for Day 1

We calculate the marginal probability of finding the dog on Day 1 for each location choice.

Option 1: Search Forest A

$$P(F_1 \mid SA_1) = P(A)P(F_1 \mid SA_1, A) + P(B)P(F_1 \mid SA_1, B)$$

$$\begin{aligned}P(F_1 \mid SA_1) &= 0.4(0.25) + 0.6(0) \\ &= \mathbf{0.10}\end{aligned}$$

Option 2: Search Forest B

$$P(F_1 \mid SB_1) = P(B)P(F_1 \mid SB_1, B) + P(A)P(F_1 \mid SB_1, A)$$

$$\begin{aligned}P(F_1 \mid SB_1) &= 0.6(0.15) + 0.4(0) \\ &= \mathbf{0.09}\end{aligned}$$

Conclusion: Since $0.10 > 0.09$, Oscar should search in **Forest A** to maximize the probability of finding the dog on Day 1.

1.3 Posterior Probability (Bayes' Update)

We compute the probability the dog is in Forest A, given Oscar searched A on Day 1 and failed to find the dog ($SA_1 \wedge \neg F_1$).

$$\begin{aligned}P(A \mid SA_1 \wedge \neg F_1) &= \frac{P(SA_1 \wedge \neg F_1 \mid A)P(A)}{P(SA_1 \wedge \neg F_1)} \\ &= \frac{P(SA_1 \mid A)P(\neg F_1 \mid SA_1 \wedge A)P(A)}{P(A)P(SA_1 \wedge \neg F_1 \mid A) + P(B)P(SA_1 \wedge \neg F_1 \mid B)} \\ &= \frac{1 \cdot (0.75) \cdot (0.4)}{(0.4)(1)(0.75) + (0.6)(1)(1)} \\ &= \frac{0.3}{0.3 + 0.6} = \frac{0.3}{0.9} = \mathbf{\frac{1}{3}} \approx \mathbf{0.333}\end{aligned}$$

1.4 Randomized Policy Analysis

Consider a policy where Oscar chooses the forest by a fair coin flip: $P(SA_1) = P(SB_1) = 0.5$.

Total Probability of Finding Dog (F_1):

$$\begin{aligned}P(F_1) &= P(F_1 \mid SA_1)P(SA_1) + P(F_1 \mid SB_1)P(SB_1) \\ &= (0.10)(0.5) + (0.09)(0.5) \\ &= 0.05 + 0.045 = \mathbf{0.095}\end{aligned}$$

Posterior Probability of Searching A: Given that the dog was found on Day 1, what is the probability Oscar looked in A?

$$\begin{aligned}P(SA_1 \mid F_1) &= \frac{P(F_1 \mid SA_1)P(SA_1)}{P(F_1)} \\ &= \frac{(0.10)(0.5)}{0.095} = \frac{0.05}{0.095} \approx \mathbf{0.5263}\end{aligned}$$

Problem 2: Oscar # 2 (Extended Scenario)

2.1 Setup and Parameters

Definitions:

- $\neg D_n$: The dog survives the n^{th} evening.
- D_n: The dog dies on the n^{th} evening.
- Other definitions (A, B, SA_n, F_n) remain consistent with Problem 1.

Probabilities:

$$
\begin{aligned}
P(A) &= 1/3 & P(B) &= 2/3 \\
P(\neg D_n \mid A) &= 4/5 & P(D_n \mid A) &= 1/5 \\
P(\neg D_n \mid B) &= 3/5 & P(D_n \mid B) &= 2/5 \\
P(F_n \mid A \wedge SA_n) &= 1/2 & P(\neg F_n \mid A \wedge SA_n) &= 1/2 \\
P(F_n \mid B \wedge SB_n) &= 2/5 & P(\neg F_n \mid B \wedge SB_n) &= 3/5
\end{aligned}
$$

Rewards (R):

- Finding dog alive: $+60$
- Finding dog dead: 0
- Not finding dog: -10
- Cost per search day: -3
- Penalty (must search both forests at least once): -3

Proposed Policy:

1. Day 1: Search Forest B.
2. If not found: Switch to Forest A overnight.
3. Day 2: Search Forest A, then stop.

Key Assumption The dog is alive throughout Day 1. It is assumed the dog was lost on the morning of Day 1 and cannot die during the daytime. The earliest the dog could die is the evening of Day 1 (between Day 1 and Day 2).

2.2 Expected Value Calculation

We evaluate the policy by summing the probability-weighted values of all possible outcomes.

Case 1: Dog is in Forest B ($P(B) = 2/3$)

Oscar searches B on Day 1.

outcome 1.1: Dog Found (Alive)

- Value: $60 - 3 = 57$.
- Probability: $P(B) \times P(F_1 \mid B \wedge SB_1) = \frac{2}{3} \times \frac{2}{5} = \frac{4}{15}$.

outcome 1.2: Dog Not Found (Search Failure)

- Note: Policy implies searching A on Day 2, but dog is in B. Search fails. Dog is never found.
- Value: -10 (not found) $- 3$ (Day 1) $- 3$ (Day 2) $- 3$ (Switch) $= -19$.
- Probability: $P(B) \times P(\neg F_1 \mid B \wedge SB_1) = \frac{2}{3} \times \frac{3}{5} = \frac{2}{5} = \frac{6}{15}$.

$$\mathbb{E}[\text{Value} \mid B] = \left(\frac{4}{15} \times 57\right) + \left(\frac{6}{15} \times -19\right) = 15.2 - 7.6 = \mathbf{7.6}$$

Case 2: Dog is in Forest A ($P(A) = 1/3$)

Oscar searches B on Day 1 (Finds nothing). Switch cost applied. Search A on Day 2. Base Value Calculation: $-3(\text{Day 1}) - 3(\text{Day 2}) - 3(\text{Switch}) = -9$.

outcome 2.1: Dog Survives Night $\rightarrow$ Found Alive

- Value: $60 - 9 = 51$.
- Prob: $P(A) \times P(\neg D_1 \mid A) \times P(F_2 \mid A) = \frac{1}{3} \times \frac{4}{5} \times \frac{1}{2} = \frac{2}{15}$.

outcome 2.2: Dog Survives Night $\rightarrow$ Not Found

- Value: $-10 - 9 = -19$.
- Prob: $P(A) \times P(\neg D_1 \mid A) \times P(\neg F_2 \mid A) = \frac{1}{3} \times \frac{4}{5} \times \frac{1}{2} = \frac{2}{15}$.

outcome 2.3: Dog Dies Overnight $\rightarrow$ Found Dead

- Value: $0 - 9 = -9$.
- Prob: $P(A) \times P(D_1 \mid A) \times P(F_2 \mid A) = \frac{1}{3} \times \frac{1}{5} \times \frac{1}{2} = \frac{1}{30}$.

outcome 2.4: Dog Dies Overnight $\rightarrow$ Not Found

- Value: $-10 - 9 = -19$.
- Prob: $P(A) \times P(D_1 \mid A) \times P(\neg F_2 \mid A) = \frac{1}{3} \times \frac{1}{5} \times \frac{1}{2} = \frac{1}{30}$.

$$\mathbb{E}[\text{Value} \mid A] = \left(\frac{4}{30} \cdot 51\right) + \left(\frac{4}{30} \cdot -19\right) + \left(\frac{1}{30} \cdot -9\right) + \left(\frac{1}{30} \cdot -19\right)$$
$$\mathbb{E}[\text{Value} \mid A] = \frac{204 - 76 - 9 - 19}{30} = \frac{100}{30} = \mathbf{3.33}$$

2.3 Total Expected Value

$$\mathbb{E}[\text{Total}] = \mathbb{E}[\text{Value} \mid B] + \mathbb{E}[\text{Value} \mid A] = 7.6 + 3.33 = \mathbf{10.93}$$