Problem 1: Queueing Analysis (Service Variability)

a. System Diagram

```
for 1000000 jobs
   average interarrival time =   2.00
   average wait ............ =   5.91
   average delay ........... =   4.41
   average service time .... =   1.50
   average # in the node ... =   2.95
   average # in the queue .. =   2.20
   utilization ............. =   0.75
```

Figure 1: System snapshot or simulation output.

b. Statistical Comparison

Example 3.1.3 (Original) Results:

$\bar{r}$	$\bar{w}$	$\bar{d}$	$\bar{s}$	$\bar{l}$	$\bar{q}$	$\bar{x}$
2.00	3.83	2.33	1.50	1.92	1.17	0.75

Modified (Exponential Service) Results:

$\bar{r}$	$\bar{w}$	$\bar{d}$	$\bar{s}$	$\bar{l}$	$\bar{q}$	$\bar{x}$
2.00	5.91	4.41	1.50	2.95	2.20	0.75

Observation: The average service time ($\bar{s} = 1.50$), average interarrival time ($\bar{r} = 2.00$), and utilization ($\bar{x} = 0.75$) remain identical between the two cases.

c. Impact of Variance

We compare the statistical properties of the two service distributions:

- **Uniform(1.0, 2.0):**
$$\text{Mean} = 1.5, \quad \text{Variance} = \frac{(2-1)^2}{12} \approx 0.083$$
- **Exponential(1.5):**
$$\text{Mean} = 1.5, \quad \text{Variance} = (1.5)^2 = 2.25$$

Analysis: Although the first-order statistics (means) are identical, the second-order statistics (variances) differ drastically. The Exponential distribution has a significantly higher variance (2.25 vs 0.083). Queueing performance metrics such as wait time ($\bar{w}$), delay ($\bar{d}$), and queue length ($\bar{q}$) are highly sensitive to variability (as seen in the Pollaczek-Khinchine formula for M/G/1 queues). The high variance of the Exponential distribution leads to occasional very long service times, which cause backlog accumulation and increase the average wait times significantly.

Problem 2: Service Time Distributions

a. Simulation Output

```
for 1000000 jobs
   average interarrival time =   2.00
   average wait ........... =   5.77
   average delay .......... =   4.28
   average service time .... =   1.50
   average # in the node ... =   2.89
   average # in the queue .. =   2.14
   utilization ............ =   0.75
```

Figure 2: Simulation trace or histogram.

b. Verification of Steady State

Example 3.1.4 (Theoretical/Reference) Results:

$\bar{r}$	$\bar{w}$	$\bar{d}$	$\bar{s}$	$\bar{l}$	$\bar{q}$	$\bar{x}$
2.00	5.77	4.27	1.50	2.89	2.14	0.75

Modified Simulation Results:

$\bar{r}$	$\bar{w}$	$\bar{d}$	$\bar{s}$	$\bar{l}$	$\bar{q}$	$\bar{x}$
2.00	5.77	4.28	1.50	2.89	2.14	0.75

The simulation results match the steady state statistics of Example 3.1.4 almost exactly, verifying the correctness of the implementation.

c. Comparison with Example 3.1.3

Example 3.1.3 (Uniform Service) Results:

$\bar{r}$	$\bar{w}$	$\bar{d}$	$\bar{s}$	$\bar{l}$	$\bar{q}$	$\bar{x}$
2.00	3.83	2.33	1.50	1.92	1.17	0.75

Current (Sum of Subtasks) Results:

$\bar{r}$	$\bar{w}$	$\bar{d}$	$\bar{s}$	$\bar{l}$	$\bar{q}$	$\bar{x}$
2.00	5.77	4.28	1.50	2.89	2.14	0.75

Analysis: The arrival rate, service rate, and utilization are identical. However, the wait times and queue lengths are significantly higher in the current example. In Example 3.1.3, the service time was Uniform with low variance. Here, the service time is formed by summing multiple random subtasks (or a different high-variance distribution), which increases the coefficient of variation. As established in Problem 1, increased variance in service times leads to congestion, thereby increasing $\bar{w}$, $\bar{d}$, $\bar{l}$, and $\bar{q}$.

Problem 3: Feedback and Geometric Distributions

a. Probability Mass Function

Let β be the probability a job feeds back (returns for service), and $1-\beta$ be the probability it departs. Let X be the number of times a job feeds back.

- $X = 0$: Job departs immediately. Prob = $(1-\beta)$.
- $X = 1$: Job feeds back once, then departs. Prob = $\beta(1-\beta)$.
- $X = x$: Job feeds back x times, then departs. Prob = $\beta^x(1-\beta)$.

The probability mass function is:

$$\mathbb{P}(X = x) = (1-\beta)\beta^x \quad \text{for } x = 0, 1, 2, \ldots$$

This is the definition of a **Geometric distribution** (counting the number of failures before the first success).

b. Analogy to Acceptance/Rejection

The feedback mechanism is mathematically identical to the Acceptance/Rejection method for generating random variates:

- **Acceptance/Rejection:** We perform repeated Bernoulli trials with success probability p (acceptance) and failure probability $1-p$ (rejection). The number of rejections before the first acceptance is Geometric(p).
- **Job Feedback:** We perform repeated Bernoulli trials with success probability $1-\beta$ (departure) and failure probability β (feedback). The number of feedbacks before the first departure is Geometric($1-\beta$).

Both processes rely on independent Bernoulli trials until a specific "success" condition is met, naturally producing a Geometric distribution for the count of "failures".

Problem 4: Finite Capacity Queues (Rejection Probabilities)

a. Case 1: Uniform(1.0, 2.0)

```
For capacity 1 and 1000000 jobs:
   rejection probability ........ =   0.43
   average interarrival time ...... =   2.00
   average wait .................. =   0.86
   average delay ................. =   0.00
   average service time ........... =   0.86
   average # in the node .......... =   0.43
   average # in the queue ......... =   0.00
   utilization ................... =   0.43

For capacity 2 and 1000000 jobs:
   rejection probability ........ =   0.19
   average interarrival time ...... =   2.00
   average wait .................. =   1.59
   average delay ................. =   0.37
   average service time ........... =   1.22
   average # in the node .......... =   0.80
   average # in the queue ......... =   0.19
   utilization ................... =   0.61

For capacity 3 and 1000000 jobs:
   rejection probability ........ =   0.09
   average interarrival time ...... =   2.00
   average wait .................. =   2.21
   average delay ................. =   0.85
   average service time ........... =   1.36
   average # in the node .......... =   1.11
   average # in the queue ......... =   0.43
   utilization ................... =   0.68

For capacity 4 and 1000000 jobs:
   rejection probability ........ =   0.05
   average interarrival time ...... =   2.00
   average wait .................. =   2.69
   average delay ................. =   1.27
   average service time ........... =   1.43
   average # in the node .......... =   1.35
   average # in the queue ......... =   0.63
   utilization ................... =   0.71

For capacity 5 and 1000000 jobs:
   rejection probability ........ =   0.03
   average interarrival time ...... =   2.00
   average wait .................. =   3.05
   average delay ................. =   1.59
   average service time ........... =   1.46
   average # in the node .......... =   1.53
   average # in the queue ......... =   0.80
   utilization ................... =   0.73

For capacity 6 and 1000000 jobs:
   rejection probability ........ =   0.02
   average interarrival time ...... =   2.00
   average wait .................. =   3.30
   average delay ................. =   1.82
   average service time ........... =   1.48
   average # in the node .......... =   1.65
   average # in the queue ......... =   0.91
   utilization ................... =   0.74
```


Figure 3: Rejection probability vs Queue Capacity.

Table 1: Rejection Probabilities for Uniform(1.0, 2.0)

Queue Capacity (K)	**Probability of Rejection (P_{rej})**
1	0.43
2	0.19
3	0.09
4	0.05
5	0.03
6	0.02

b. Case 2: Uniform(1.0, 3.0)

```
For capacity 1 and 1000000 jobs:
   rejection probability ........ =   0.50
   average interarrival time ...... =   2.00
   average wait .................. =   1.00
   average delay ................. =   0.00
   average service time .......... =   1.00
   average # in the node ......... =   0.50
   average # in the queue ........ =   0.00
   utilization ................... =   0.50

For capacity 2 and 1000000 jobs:
   rejection probability ........ =   0.28
   average interarrival time ...... =   2.00
   average wait .................. =   2.00
   average delay ................. =   0.55
   average service time .......... =   1.44
   average # in the node ......... =   1.00
   average # in the queue ........ =   0.28
   utilization ................... =   0.72

For capacity 3 and 1000000 jobs:
   rejection probability ........ =   0.18
   average interarrival time ...... =   2.00
   average wait .................. =   3.04
   average delay ................. =   1.40
   average service time .......... =   1.63
   average # in the node ......... =   1.52
   average # in the queue ........ =   0.70
   utilization ................... =   0.81

For capacity 4 and 1000000 jobs:
   rejection probability ........ =   0.14
   average interarrival time ...... =   2.00
   average wait .................. =   4.06
   average delay ................. =   2.34
   average service time .......... =   1.73
   average # in the node ......... =   2.03
   average # in the queue ........ =   1.17
   utilization ................... =   0.86

For capacity 5 and 1000000 jobs:
   rejection probability ........ =   0.11
   average interarrival time ...... =   2.00
   average wait .................. =   5.09
   average delay ................. =   3.30
   average service time .......... =   1.78
   average # in the node ......... =   2.54
   average # in the queue ........ =   1.65
   utilization ................... =   0.89

For capacity 6 and 1000000 jobs:
   rejection probability ........ =   0.09
   average interarrival time ...... =   2.00
   average wait .................. =   6.10
   average delay ................. =   4.28
   average service time .......... =   1.82
   average # in the node ......... =   3.05
   average # in the queue ........ =   2.14
   utilization ................... =   0.91
```

Figure 4: Rejection probability vs Queue Capacity.

Table 2: Rejection Probabilities for Uniform(1.0, 3.0)

Queue Capacity (K)	Probability of Rejection ($P_{\textbf{rej}}$)
1	0.50
2	0.28
3	0.18
4	0.14
5	0.11
6	0.09

c. Mechanism of Rejection

The probability of rejection in a finite-capacity queue ($M/G/1/K$ or similar) depends on the likelihood that the system is full ($N = K$) when a new job arrives.

- **Mean Effect:** An increase in the mean service time increases the traffic intensity $\rho = \lambda \bar{s}$. As ρ increases, the steady-state probability of having n jobs in the system increases for all n, including $n = K$.
- **Variance Effect:** Even with fixed means, higher variability leads to longer queues (as discussed in Problem 1), pushing the system state closer to capacity K more frequently.

Changing the service distribution from $U(1,2)$ to $U(1,3)$ increases both the mean (1.5 to 2.0) and the variance, drastically increasing system occupancy and thus the rejection probability.

d. Comparison of Results

The results align with theoretical expectations:

1. **Decreasing Trend with K:** As capacity K increases, there is more buffer space, so fewer jobs are rejected.
2. **Comparison of Distributions:**
   - $U(1.0, 2.0)$: Mean = 1.5, Variance ≈ 0.083.
   - $U(1.0, 3.0)$: Mean = 2.0, Variance ≈ 0.333.

The shift to $U(1.0, 3.0)$ represents a higher load on the server. For $K = 1$, rejection jumps from 43% to 50%. For larger K, the relative impact of the heavier load is even more pronounced (e.g., at $K = 6$, rejection is negligible for the first case but still nearly 10% for the second). This confirms that both higher mean service time and higher variance contribute to increased blocking probabilities.