

Problem 1: Queueing Analysis

a. System Diagram

```

for 1000 jobs
  average interarrival time = 9.87
  average service time .... = 7.12
  average delay ..... = 18.59
  average wait ..... = 25.72
  maximum delay ..... = 118.76
  number of jobs in the service node at t=400 = 7
  proportion of jobs delayed = 0.72
  server utilization = 0.72

```

Figure 1: System snapshot or timeline.

b. Maximum Delay

The maximum delay experienced is **118.76**.

c. Number of Jobs in System

The number of jobs in the service node at $t = 400$ is 7.

Little's Law Relation: Each job i spends exactly $(c_i - a_i)$ units of time in the system from arrival a_i to departure c_i . The number of jobs in the service node at time t is denoted by the indicator function:

$$l(t) = \sum_{i=1}^n \mathbb{I}[a_i \leq t < c_i].$$

Theorem 1.2.1 states:

$$\int_0^{c_n} l(t) dt = \sum_{i=1}^n (c_i - a_i).$$

Proof by summing across all jobs:

$$\int_0^{c_n} l(t) dt = \int_0^{c_n} \sum_{i=1}^n \mathbb{I}[a_i \leq t < c_i] dt = \sum_{i=1}^n \int_0^{c_n} \mathbb{I}[a_i \leq t < c_i] dt = \sum_{i=1}^n (c_i - a_i).$$

d. Utilization and Delays

- **Proportion of delayed jobs:** 0.72
- **Server Utilization:** 0.72

Observation: The server is busy 72% of the time. Consequently, 72% of jobs arrive to find the server busy and must wait in the queue. The fraction of delayed jobs matches the utilization because utilization measures the long-term fraction of time the server is non-idle; any arrival during a non-idle period results in a delay.

Problem 2: Service Time Calculations

```

for 500 jobs
  average interarrival time = 4.08
  average service time .... = 3.03
  average delay ..... = 4.54
  average wait ..... = 7.57
  server utilization      = 0.74
  traffic intensity      = 0.74

```

a. Statistics

- **Average service time:** 3.03
- **Server utilization:** 0.74
- **Traffic intensity:** 0.74

b. Derivation of Service Time s_i

The service time s_i can be derived from arrival times a_i and completion times c_i by considering two cases based on the state of the server upon arrival.

Case 1: Job arrives while server is busy ($a_i < c_{i-1}$) The job must wait for the previous job to finish. Execution starts at c_{i-1} and ends at c_i .

$$s_i = c_i - c_{i-1}$$

Case 2: Job arrives while server is idle ($a_i \geq c_{i-1}$) The job starts executing immediately upon arrival. Execution starts at a_i and ends at c_i .

$$s_i = c_i - a_i$$

Combined Expression:

$$s_i = c_i - \max\{a_i, c_{i-1}\}$$

Problem 3: Simulation of Unfair Dice

a. Event Definition

The event of interest is that the sum of the two up faces equals 7: $\{X + Y = 7\}$, where X and Y are the random variables representing the outcomes of the first and second die, respectively. The random components are the individual rolls following the specified non-uniform distribution.

Table 1: Simulation Results for $P(X + Y = 7)$

Seed	$P(X + Y = 7)$
1	0.14224
42	0.14168
2023	0.14209
40599	0.14195
99999	0.14261

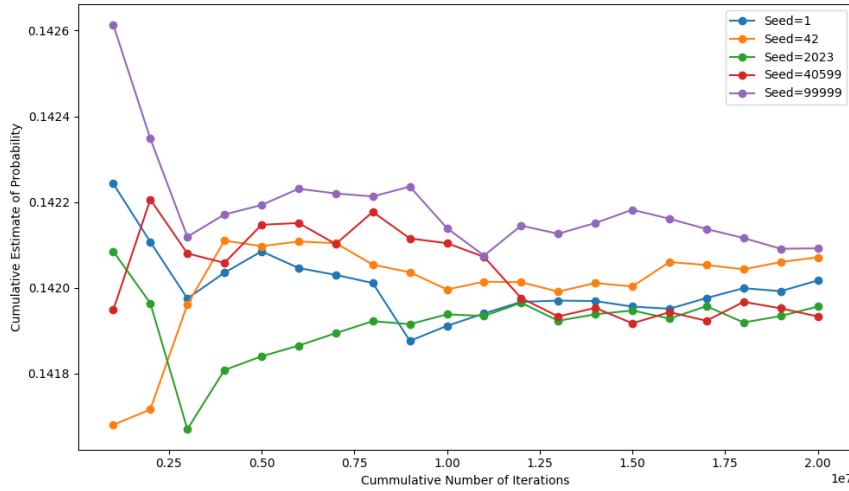


Figure 2: Histogram of simulated sums.

b. Analytical Verification

The possible pairs (X, Y) summing to 7 are:

$$(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)$$

Given probabilities:

$$P(1) = \frac{1}{13}, \quad P(6) = \frac{4}{13}, \quad P(2) = P(3) = P(4) = P(5) = \frac{2}{13}$$

The total probability is:

$$\begin{aligned}
 P(X + Y = 7) &= 2[P(1)P(6)] + 2[P(2)P(5)] + 2[P(3)P(4)] \\
 &= 2 \left[\left(\frac{1}{13} \right) \left(\frac{4}{13} \right) \right] + 2 \left[\left(\frac{2}{13} \right) \left(\frac{2}{13} \right) \right] + 2 \left[\left(\frac{2}{13} \right) \left(\frac{2}{13} \right) \right] \\
 &= 2 \left[\frac{4}{169} \right] + 2 \left[\frac{4}{169} \right] + 2 \left[\frac{4}{169} \right] \\
 &= \frac{8}{169} + \frac{8}{169} + \frac{8}{169} = \frac{24}{169} \approx \mathbf{0.1420}
 \end{aligned}$$

The simulation results closely approximate this analytical value.

Problem 4: Geometric Probability (Bertrand's Paradox Variant)

a. Event Definition

The event is that the chord length between two randomly chosen points on the circumference is greater than the circle's radius ρ .

$$\text{Event : distance}(\theta_1, \theta_2) > \rho$$

The chord length is given by $2\rho \sin\left(\frac{|\theta_1 - \theta_2|}{2}\right)$. The random components are the angles $\theta_1, \theta_2 \sim U[0, 2\pi)$.

Table 2: Simulation Results for Chord Length > Radius

Seed	Probability
1	0.66671
42	0.66659
2023	0.66686
40599	0.66770
99999	0.66653

b. Radius Independence

We show that the probability is independent of ρ . The condition is:

$$2\rho \sin\left(\frac{|\theta_1 - \theta_2|}{2}\right) > \rho$$

Dividing both sides by ρ (assuming $\rho > 0$):

$$2 \sin\left(\frac{|\theta_1 - \theta_2|}{2}\right) > 1 \implies \sin\left(\frac{|\theta_1 - \theta_2|}{2}\right) > \frac{1}{2}$$

Since ρ cancels out, the probability depends solely on the distribution of angles θ_1 and θ_2 .