EECE7205 – Project 1

1. Pseudocode

Max-min-grouping(A, N, M)

$C[0 \dots N][0 \dots M] \leftarrow -\infty$ $\quad // \ DP\ table\ to\ track\ max - min\ sum\ for\ each\ partitioning\ of elements$

$K[0 \dots N][0 \dots M] \leftarrow -1$ $\quad // \ Partition\ points\ table\ for\ backtracking\ the\ optimal\ grouping$

$C[0][0] \leftarrow \infty$

$P[0 \dots N] \leftarrow 0$

for $i \leftarrow 1$ **to** $N:$

  $P[i] \leftarrow P[i-1] + A[i]$

for $i \leftarrow 1$ **to** $N:$

  $C[i][1] \leftarrow P[i]$ $\quad // \ DP\ base\ case:\ if\ only\ 1\ group,\ use\ total\ sum\ up\ to\ i$

  $K[i][1] \leftarrow 0$ $\quad // \ Group\ starts\ from\ index\ 0\ when\ there\ is\ only\ one\ group$

for $j \leftarrow 2$ **to** $M:$ $\quad // \ Loop\ over\ number\ of\ groups,\ starting\ from\ 2\ up\ to\ M$

  for $i \leftarrow j$ **to** $N:$ $\quad // \ Loop\ over\ elements\ to\ find\ maximum\ min - sum\ for\ each\ partition$

    for $k \leftarrow j-1$ **to** $i-1:$ $\quad // \ Loop\ to\ test\ different\ partition\ points\ for\ the\ (j-1) - th\ group$

      $\text{sum} \leftarrow P[i] - P[k]$

      $\text{min\_sum} \leftarrow \min(C[k][j-1], \text{sum})$ $\quad //Calculate\ min\ between\ previous\ max - min\ sum\ and\ current\ sum$

      if $\text{min\_sum} > C[i][j]$ **then** $\quad //If\ min - sum\ is\ greater\ update\ DP\ table$

        $K[i][j] \leftarrow k$ $\quad Update\ partition\ point\ in\ parition\ table\ k\ for\ backtracking$

$G[1 \dots M] \leftarrow 0$

$\text{idx} \leftarrow N$

for $j \leftarrow M$ **down to** $1:$ $\quad //Backtrack\ to\ find\ group\ sizes\ for\ each\ partition$

  $k \leftarrow K[\text{idx}][j]$ $\quad //Retrieve\ parition\ point\ k\ for\ the\ j - th\ group$

  $G[j] \leftarrow \text{idx} - k$

  $\text{idx} \leftarrow k$

return $G[1 \dots M]$

2. Running time analysis

$$P[i] = \sum_{m=1}^{i} A[m].$$

As each iteration requires a single addition, this step takes N operations, as the loop runs from $i = 1$ to N. The $\Theta(N)$ notation represents both the upper and lower bounds, since the loop must always execute N times.

Time Complexity: $\Theta(N)$

DP Table Base Case:

$$C[i][1] = P[i] \quad \text{for } i = 1 \text{ to } N.$$

As each assignment is $O(1)$, this step takes N operations, as the loop runs from $i = 1$ to N. The $\Theta(N)$ notation again represents both the upper and lower bounds, as the loop must always execute N times.

Time Complexity: $\Theta(N)$

DP Table Population:
Loop 1 (over groups j) runs from $j = 2$ to M, resulting in M iterations.
Loop 2 (over elements i) runs from $i = j$ to N resulting in N iterations.
For each combination of i and j, loop 3 (over partition points k) runs from $k = j - 1$ to $k = i - 1$, resulting in $O(N)$ iterations.

The three nested loops result in combined time complexity:

$$\sum_{j=2}^{M} \sum_{i=j}^{N} \sum_{k=j-1}^{i-1} O(1)$$

Loop 3 Sum (over k): For each fixed i and j, the number of iterations of k ranges from $j - 1$ to $i - 1$, which is $i - j + 1$:

$$\sum_{k=j-1}^{i-1} O(1) = O(i - j + 1)$$

Loop 2 Sum (over i): Summing $O(i - j + 1)$ over i from j to N:

$$\sum_{i=j}^{N} O(i - j + 1) = O\left(\sum_{i=j}^{N} (i - j + 1)\right) = O(N^2)$$

Loop 1 Sum (over j): Summing over j from 2 to M:

$$\sum_{j=2}^{M} O(N^2) = O(M \cdot N^2)$$

Time Complexity: $\Theta(M \cdot N^2)$

Backtracking to Retrieve the Optimal Partition:
The optimal grouping is constructed using the partitions table $K[i][j]$. Since each iteration requires only constant-time operations, this step has a time complexity of $O(M)$. The loop runs from $j = M$ down to 1, resulting in M iterations.

Time Complexity: $\Theta(M)$

Total running time complexity of algorithm:

$$\Theta(M \cdot N^2)$$

3. Input example results

1.

```
A = {3, 9, 7, 8, 2, 6, 5, 10, 1, 7, 6, 4}
N = 12
M = 3
```

- **Initialize DP Table for Base case $j = 1$ (Single group):**

$$C[i, 1] = \sum_{m=1}^{i} A[m]$$

$$C[1,1] = \sum_{m=1}^{1} A[m] = 3, \quad C[2,1] = \sum_{m=1}^{2} A[m] = 12, \quad C[3,1] = \sum_{m=1}^{3} A[m] = 19$$

$$C[4,1] = \sum_{m=1}^{4} A[m] = 27, \quad C[5,1] = \sum_{m=1}^{5} A[m] = 29, \quad C[6,1] = \sum_{m=1}^{6} A[m] = 35$$

$$C[7,1] = \sum_{m=1}^{7} A[m] = 40, \quad C[8,1] = \sum_{m=1}^{8} A[m] = 50, \quad C[9,1] = \sum_{m=1}^{9} A[m] = 51$$

$$C[10,1] = \sum_{m=1}^{10} A[m] = 58, \quad C[11,1] = \sum_{m=1}^{11} A[m] = 64, \quad C[12,1] = \sum_{m=1}^{12} A[m] = 68$$

- **Compute DP Table for $j = 2$:**

$$C[i,2] = \max_{1 \le k \le i-1} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{i} A[m] \right) \right\} \quad \text{for } i = 2 \text{ to } 12$$

$$C[1,2] = \times \quad \text{(For } i = 1 \text{ and } j = 2, \text{ cannot partition one element into two groups. So } C[1,2] \text{ is not applicable.)}$$

$$C[2,2] = \max_{1 \le k \le 1} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{2} A[m] \right) \right\} = \min(3,9) = 3 \quad \text{(Selected } k = 1\text{)}$$

$$C[3,2] = \max_{1 \le k \le 2} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{3} A[m] \right) \right\} = \max(3,7) = 7 \quad \text{(Selected } k = 2\text{)}$$

$$C[4,2] = \max_{1 \le k \le 3} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{4} A[m] \right) \right\} = \max(3,12,8) = 12 \quad \text{(Selected } k = 2\text{)}$$

$$C[5,2] = \max_{1 \le k \le 4} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{5} A[m] \right) \right\} = \max(3,12,10,2) = 12 \quad \text{(Selected } k = 2\text{)}$$

$$C[6,2] = \max_{1 \le k \le 5} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{6} A[m] \right) \right\} = \max(3,12,16,8,6) = 16 \quad \text{(Selected } k = 3\text{)}$$

$$C[7,2] = \max_{1 \le k \le 6} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{7} A[m] \right) \right\} = \max(3,12,19,13,11,5) = 19 \quad \text{(Selected } k = 3\text{)}$$

$$C[8,2] = \max_{1 \le k \le 7} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{8} A[m] \right) \right\} = \max(3,12,19,23,21,15,10) = 23 \quad \text{(Selected } k = 4\text{)}$$

$$C[9,2] = \max_{1 \le k \le 8} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{9} A[m] \right) \right\} = \max(3,12,19,24,22,16,11,1) = 24 \quad \text{(Selected } k = 4\text{)}$$

$$C[10,2] = \max_{1 \le k \le 9} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{10} A[m] \right) \right\} = \max(3,12,19,27,29,23,18,8,7) = 29 \quad \text{(Selected } k = 5\text{)}$$

$$C[11,2] = \max_{1 \le k \le 10} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{11} A[m] \right) \right\} = \max(3,12,19,27,29,29,24,14,13,6) = 29 \quad \text{(Selected } k = 5\text{)}$$

$$C[12,2] = \max_{1 \le k \le 11} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{12} A[m] \right) \right\} = \max(3,12,19,27,29,33,28,18,17,10,4) = 33 \quad \text{(Selected } k = 6\text{)}$$

- Compute DP Table for $j = 3$:

$$C[i,3] = \max_{2\le k\le i-1}\left\{\min\left(C[k,2], \sum_{m=k+1}^{i} A[m]\right)\right\} \quad \text{for } i = 3 \text{ to } 12$$

$C[1,3] = \times$ (For $i = 1$ and $j = 3$, cannot partition one element into three groups. So $C[1,3]$ is not applicable.)
$C[2,3] = \times$ (For $i = 2$ and $j = 3$, cannot partition two elements into three groups. So $C[2,3]$ is not applicable.)

$$C[3,3] = \max_{2\le k\le 2}\left\{\min\left(C[k,2], \sum_{m=k+1}^{3} A[m]\right)\right\} = \min(3,7) = 3 \quad \text{(Selected } k = 2)$$

$$C[4,3] = \max_{2\le k\le 3}\left\{\min\left(C[k,2], \sum_{m=k+1}^{4} A[m]\right)\right\} = \max(3,7) = 7 \quad \text{(Selected } k = 3)$$

$$C[5,3] = \max_{2\le k\le 4}\left\{\min\left(C[k,2], \sum_{m=k+1}^{5} A[m]\right)\right\} = \max(3,7,2) = 7 \quad \text{(Selected } k = 3)$$

$$C[6,3] = \max_{2\le k\le 5}\left\{\min\left(C[k,2], \sum_{m=k+1}^{6} A[m]\right)\right\} = \max(3,7,8,6) = 8 \quad \text{(Selected } k = 4)$$

$$C[7,3] = \max_{2\le k\le 6}\left\{\min\left(C[k,2], \sum_{m=k+1}^{7} A[m]\right)\right\} = \max(3,7,12,11,5) = 12 \quad \text{(Selected } k = 4)$$

$$C[8,3] = \max_{2\le k\le 7}\left\{\min\left(C[k,2], \sum_{m=k+1}^{8} A[m]\right)\right\} = \max(3,7,12,12,15,10) = 15 \quad \text{(Selected } k = 6)$$

$$C[9,3] = \max_{2\le k\le 8}\left\{\min\left(C[k,2], \sum_{m=k+1}^{9} A[m]\right)\right\} = \max(3,7,12,12,16,11,1) = 16 \quad \text{(Selected } k = 6)$$

$$C[10,3] = \max_{2\le k\le 9}\left\{\min\left(C[k,2], \sum_{m=k+1}^{10} A[m]\right)\right\} = \max(3,7,12,12,16,18,8,7) = 18 \quad \text{(Selected } k = 7)$$

$$C[11,3] = \max_{2\le k\le 10}\left\{\min\left(C[k,2], \sum_{m=k+1}^{11} A[m]\right)\right\} = \max(3,7,12,12,16,19,14,13,6) = 19 \quad \text{(Selected } k = 7)$$

$$C[12,3] = \max_{2\le k\le 11}\left\{\min\left(C[k,2], \sum_{m=k+1}^{12} A[m]\right)\right\} = \max(3,7,12,12,16,19,18,17,10,4) = 19 \quad \text{(Selected } k = 7)$$

Table 1: DP Table $C[i][j]$

$j\backslash i$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	3	12	19	27	29	35	40	50	51	58	64	68
2	–	3	7	12	12	16	19	23	24	29	29	33
3	–	–	3	7	7	8	12	15	16	18	19	19

Table 2: Partition Points $K[i][j]$

$j\backslash i$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	0	0	0	0	0	0	0	0	0	0	0	0
2	–	1	2	2	2	3	3	4	4	5	5	6
3	–	–	2	3	3	4	4	6	6	7	7	7

- Maximum minimal sum achievable by partitioning A[1...12] into $M = 3$ groups is $C[12,3] = 19$.

- Backtracking through the DP table to determine the optimal grouping $G[1 \dots M]$ by finding partition points k that result in $C[12, 3] = 19$.

  Partitioning A[1...12] into optimal groups based on k=7 and k=3:

  - Group 1: $A[1..3] = [3, 9, 7]$

  - Group 2: $A[4..7] = [8, 2, 6, 5]$

  - Group 3: $A[8..12] = [10, 1, 7, 6, 4]$

  Grouping array $G[1 \dots 3]$: $G = [3, 4, 5]$

- Determine summation of the elements in the j-th group of array A.

  - Group 1 ($A[1 \dots 3]$): $B[1] = 3 + 9 + 7 = 19$
  - Group 2 ($A[4 \dots 7]$): $B[2] = 8 + 2 + 6 + 5 = 21$
  - Group 3 ($A[8 \dots 12]$): $B[3] = 10 + 1 + 7 + 6 + 4 = 28$

  Sum of elements in each group $B[1 \dots 3]$: $B = [19, 21, 28]$

- The minimal group sum:

$$\min_{1 \le j \le 3} B[j] = \min(19, 21, 28) = 19$$

  The minimal group sum matches $C[12, 3] = 19$, confirming that the minimal group sum is maximized.

```
Optimal Grouping Sizes (G[1..M]):
Group 1: 3 elements
Group 2: 4 elements
Group 3: 5 elements

Group Details:
Group 1: A[1...3] = [3, 9, 7], Sum = 19
Group 2: A[4...7] = [8, 2, 6, 5], Sum = 21
Group 3: A[8...12] = [10, 1, 7, 6, 4], Sum = 28
Optimal Grouping G: 3 4 5
```

2.

```
A = {4,2,5,1,6,7,3,8,2,4}
N = 10
M = 3
```

- **Initialize DP Table for Base case** $j = 1$ **(Single group):**

$$C[i, 1] = \sum_{m=1}^{i} A[m]$$

$$C[1, 1] = \sum_{m=1}^{1} A[m] = 4, \quad C[2, 1] = \sum_{m=1}^{2} A[m] = 6, \quad C[3, 1] = \sum_{m=1}^{3} A[m] = 11,$$
$$C[4, 1] = \sum_{m=1}^{4} A[m] = 12, \quad C[5, 1] = \sum_{m=1}^{5} A[m] = 18, \quad C[6, 1] = \sum_{m=1}^{6} A[m] = 25,$$
$$C[7, 1] = \sum_{m=1}^{7} A[m] = 28, \quad C[8, 1] = \sum_{m=1}^{8} A[m] = 36, \quad C[9, 1] = \sum_{m=1}^{9} A[m] = 38,$$
$$C[10, 1] = \sum_{m=1}^{10} A[m] = 42.$$

- **Compute DP Table for $j = 2$:**

$$C[i,2] = \max_{1 \le k \le i-1} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{i} A[m] \right) \right\}, \quad \text{for } i = 2 \text{ to } 10$$

$$C[1,2] = \times \quad \text{(For } i = 1 \text{ and } j = 2, \text{ cannot partition one element into two groups. So } C[1,2] \text{ is not applicable.)}$$

$$C[2,2] = \max_{1 \le k \le 1} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{2} A[m] \right) \right\} = \min(2,5) = 2 \quad \text{(Selected } k = 1\text{)}$$

$$C[3,2] = \max_{1 \le k \le 2} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{3} A[m] \right) \right\} = \max(4,5) = 5 \quad \text{(Selected } k = 2\text{)}$$

$$C[4,2] = \max_{1 \le k \le 3} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{4} A[m] \right) \right\} = \max(4,6,1) = 6 \quad \text{(Selected } k = 2\text{)}$$

$$C[5,2] = \max_{1 \le k \le 4} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{5} A[m] \right) \right\} = \max(4,6,7,6) = 7 \quad \text{(Selected } k = 3\text{)}$$

$$C[6,2] = \max_{1 \le k \le 5} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{6} A[m] \right) \right\} = \max(4,6,11,12,7) = 12 \quad \text{(Selected } k = 4\text{)}$$

$$C[7,2] = \max_{1 \le k \le 6} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{7} A[m] \right) \right\} = \max(4,6,11,12,10,3) = 12 \quad \text{(Selected } k = 4\text{)}$$

$$C[8,2] = \max_{1 \le k \le 7} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{8} A[m] \right) \right\} = \max(4,6,11,12,18,11,8) = 18 \quad \text{(Selected } k = 5\text{)}$$

$$C[9,2] = \max_{1 \le k \le 8} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{9} A[m] \right) \right\} = \max(4,6,11,12,18,13,10,2) = 18 \quad \text{(Selected } k = 5\text{)}$$

$$C[10,2] = \max_{1 \le k \le 9} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{10} A[m] \right) \right\} = \max(4,6,11,12,18,17,14,6,4) = 18 \quad \text{(Selected } k = 5\text{)}$$

- **Compute DP Table for $j = 3$:**

$$C[i,3] = \max_{k=2}^{i-1} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{i} A[m] \right) \right\}, \quad \text{for } i = 3 \text{ to } 10$$

$C[1,3] = \times \quad$ (For $i = 1$ and $j = 3$, cannot partition one element into three groups. So $C[1,3]$ is not applicable.)

$C[2,3] = \times \quad$ (For $i = 2$ and $j = 3$, cannot partition two elements into three groups. So $C[2,3]$ is not applicable.)

$$C[3,3] = \max_{2 \le k \le 2} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{3} A[m] \right) \right\} = \min(2,5) = 2 \quad \text{(Selected } k = 2\text{)}$$

$$C[4,3] = \max_{2 \le k \le 3} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{4} A[m] \right) \right\} = \max\{2,1\} = 2 \quad \text{(Selected } k = 2\text{)}$$

$$C[5,3] = \max_{2 \le k \le 4} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{5} A[m] \right) \right\} = \max\{2,5,6\} = 6 \quad \text{(Selected } k = 4\text{)}$$

$$C[6,3] = \max_{2 \le k \le 5} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{6} A[m] \right) \right\} = \max\{2,5,6,7\} = 7 \quad \text{(Selected } k = 5\text{)}$$

$$C[7,3] = \max_{2 \le k \le 6} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{7} A[m] \right) \right\} = \max\{2,5,6,7,3\} = 7 \quad \text{(Selected } k = 5\text{)}$$

$$C[8,3] = \max_{2 \le k \le 7} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{8} A[m] \right) \right\} = \max\{2,5,6,7,11,8\} = 11 \quad \text{(Selected } k = 6\text{)}$$

$$C[9,3] = \max_{2 \le k \le 8} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{9} A[m] \right) \right\} = \max\{2,5,6,7,12,10,2\} = 12 \quad \text{(Selected } k = 6\text{)}$$

$$C[10,3] = \max_{2 \le k \le 9} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{10} A[m] \right) \right\} = \max\{2,5,6,7,12,12,6,4\} = 12 \quad \text{(Selected } k = 6\text{)}$$

Table 3: DP Table $C[i][j]$

$j \backslash i$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	4	6	11	12	18	25	28	36	38	42
2	–	2	5	6	7	12	12	18	18	18
3	–	–	2	2	6	7	7	11	12	12

Table 4: Partition Points $K[i][j]$

$j \backslash i$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0	0	0	0	0	0	0	0	0	0
2	–	1	2	2	3	4	4	5	5	5
3	–	–	2	2	4	5	5	6	6	6

- Maximum minimal sum achievable by partitioning A[1...10] (i=10) into $M = 3$ (j=3) groups is $C[10,3] = 12$.
- Backtracking through the DP table to determine optimal grouping $G[1 \dots M]$ by finding partition points k that leads to $C[10,3] = 12$.
  Partitioning A[1...10] into optimal groups based on k=4 and k=6:
  - Group 1: $A[1 \dots 4] = [4,2,5,1]$
  - Group 2: $A[5 \dots 6] = [6,7]$
  - Group 3: $A[7 \dots 10] = [3,8,2,4]$

  Grouping array $G[1 \dots 3]$**:** $G = [4,2,4]$

- Determine summation of the elements in the j-th group of array A.
  - Group 1 ($A[1 \ldots 4]$): $B[1] = 4 + 2 + 5 + 1 = 12$
  - Group 2 ($A[4 \ldots 7]$): $B[2] = 6 + 7 = 13$
  - Group 3 ($A[8 \ldots 12]$): $B[3] = 3 + 8 + 2 + 4 = 17$

  Sum of elements in each group $B[1 \ldots 3]$: $B = [12, 13, 17]$

- The minimal group sum:

$$\min_{1 \le j \le 3} B[j] = \min(12, 13, 17) = 12$$

  The minimal group sum matches $C[10, 3] = 12$, confirming that the minimal group sum is maximized.

```
Optimal Grouping Sizes (G[1..M]):
Group 1: 4 elements
Group 2: 2 elements
Group 3: 4 elements

Group Details:
Group 1: A[1...4] = [4, 2, 5, 1], Sum = 12
Group 2: A[5...6] = [6, 7], Sum = 13
Group 3: A[7...10] = [3, 8, 2, 4], Sum = 17
Optimal Grouping G: 4 2 4
```

3.

```
A = {5,2,4,7,1,3,6,8,2,4,9}
N = 11
M = 4
```

- **Initialize DP Table for Base case $j = 1$ (Single group):**

$$C[i, 1] = \sum_{m=1}^{i} A[m]$$

$$C[1,1] = \sum_{m=1}^{1} A[m] = 5, \quad C[2,1] = \sum_{m=1}^{2} A[m] = 7, \quad C[3,1] = \sum_{m=1}^{3} A[m] = 11,$$
$$C[4,1] = \sum_{m=1}^{4} A[m] = 18, \quad C[5,1] = \sum_{m=1}^{5} A[m] = 19, \quad C[6,1] = \sum_{m=1}^{6} A[m] = 22,$$
$$C[7,1] = \sum_{m=1}^{7} A[m] = 28, \quad C[8,1] = \sum_{m=1}^{8} A[m] = 36, \quad C[9,1] = \sum_{m=1}^{9} A[m] = 38,$$
$$C[10,1] = \sum_{m=1}^{10} A[m] = 42 \quad C[11,1] = \sum_{m=1}^{11} A[m] = 51$$

- **Compute DP Table for $j = 2$:**

$$C[i, 2] = \max_{1 \le k \le i-1} \left\{ \min \left(C[k, 1], \sum_{m=k+1}^{i} A[m] \right) \right\}, \quad \text{for } i = 2 \text{ to } 11$$

$$C[1,2] = \times \quad \text{(For } i=1 \text{ and } j=2, \text{ cannot partition 1 element into 2 groups. So } C[1,2] \text{ is not applicable.)}$$

$$C[2,2] = \max_{1 \le k \le 1} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{2} A[m] \right) \right\} = \min(5,2) = 2 \quad \text{(Selected } k=1\text{)}$$

$$C[3,2] = \max_{1 \le k \le 2} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{3} A[m] \right) \right\} = \max(5,4) = 5 \quad \text{(Selected } k=1\text{)}$$

$$C[4,2] = \max_{1 \le k \le 3} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{4} A[m] \right) \right\} = \max(5,7,7) = 7 \quad \text{(Selected } k=2\text{)}$$

$$C[5,2] = \max_{1 \le k \le 4} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{5} A[m] \right) \right\} = \max(5,7,8,1) = 8 \quad \text{(Selected } k=3\text{)}$$

$$C[6,2] = \max_{1 \le k \le 5} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{6} A[m] \right) \right\} = \max(5,7,11,4,3) = 11 \quad \text{(Selected } k=3\text{)}$$

$$C[7,2] = \max_{1 \le k \le 6} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{7} A[m] \right) \right\} = \max(5,7,11,10,9,6) = 11 \quad \text{(Selected } k=3\text{)}$$

$$C[8,2] = \max_{1 \le k \le 7} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{8} A[m] \right) \right\} = \max(5,7,11,18,17,14,8) = 18 \quad \text{(Selected } k=4\text{)}$$

$$C[9,2] = \max_{1 \le k \le 8} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{9} A[m] \right) \right\} = \max(5,7,11,18,19,16,10,2) = 19 \quad \text{(Selected } k=5\text{)}$$

$$C[10,2] = \max_{1 \le k \le 9} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{10} A[m] \right) \right\} = \max(5,7,11,18,19,20,14,6,4) = 20 \quad \text{(Selected } k=6\text{)}$$

$$C[11,2] = \max_{1 \le k \le 10} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{11} A[m] \right) \right\} = \max(5,7,11,18,19,22,23,15,13,9) = 23 \quad \text{(Selected } k=7\text{)}$$

- **Compute DP Table for $j=3$:**

$$C[i,3] = \max_{2 \le k \le i-1} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{i} A[m] \right) \right\}, \quad \text{for } i=3 \text{ to } 11$$

$$C[1,3] = \times \quad \text{(For } i = 1 \text{ and } j = 3, \text{ cannot partition one element into three groups. So } C[1,3] \text{ is not applicable.)}$$

$$C[2,3] = \times \quad \text{(For } i = 2 \text{ and } j = 3, \text{ cannot partition two elements into three groups. So } C[2,3] \text{ is not applicable.)}$$

$$C[3,3] = \max_{2 \le k \le 2} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{3} A[m] \right) \right\} = \min(2,4) = 2 \quad \text{(Selected } k = 2\text{)}$$

$$C[4,3] = \max_{2 \le k \le 3} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{4} A[m] \right) \right\} = \max(2,5) = 5 \quad \text{(Selected } k = 3\text{)}$$

$$C[5,3] = \max_{2 \le k \le 4} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{5} A[m] \right) \right\} = \max(2,5,1) = 5 \quad \text{(Selected } k = 3\text{)}$$

$$C[6,3] = \max_{2 \le k \le 5} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{6} A[m] \right) \right\} = \max(2,5,4,3) = 5 \quad \text{(Selected } k = 3\text{)}$$

$$C[7,3] = \max_{2 \le k \le 6} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{7} A[m] \right) \right\} = \max(2,5,7,8,6) = 8 \quad \text{(Selected } k = 5\text{)}$$

$$C[8,3] = \max_{2 \le k \le 7} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{8} A[m] \right) \right\} = \max(2,5,7,8,11,8) = 11 \quad \text{(Selected } k = 6\text{)}$$

$$C[9,3] = \max_{2 \le k \le 8} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{9} A[m] \right) \right\} = \max(2,5,7,8,11,10,2) = 11 \quad \text{(Selected } k = 6\text{)}$$

$$C[10,3] = \max_{2 \le k \le 9} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{10} A[m] \right) \right\} = \max(2,5,7,8,11,11,6,4) = 11 \quad \text{(Selected } k = 6\text{)}$$

$$C[11,3] = \max_{2 \le k \le 10} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{11} A[m] \right) \right\} = \max(2,5,7,8,11,11,15,13,9) = 15 \quad \text{(Selected } k = 8\text{)}$$

- **Compute DP Table for $j = 4$:**

$$C[i,4] = \max_{3 \le k \le i-1} \left\{ \min \left(C[k,3], \sum_{m=k+1}^{i} A[m] \right) \right\}, \quad \text{for } i = 4 \text{ to } 11$$

$C[1,4] = \times$ (For $i = 1$ and $j = 4$, cannot partition 1 element into 2 groups. So $C[1,3]$ is not applicable.)
$C[2,4] = \times$ (For $i = 2$ and $j = 4$, cannot partition 2 elements into 3 groups. So $C[2,3]$ is not applicable.)
$C[3,4] = \times$ (For $i = 3$ and $j = 4$, cannot partition 3 elements into 4 groups. So $C[3,3]$ is not applicable.)

$$C[4,4] = \max_{3\leq k\leq 3}\left\{\min\left(C[k,3], \sum_{m=k+1}^{4} A[m]\right)\right\} = 2 \quad \text{(Selected } k = 3)$$

$$C[5,4] = \max_{3\leq k\leq 4}\left\{\min\left(C[k,3], \sum_{m=k+1}^{5} A[m]\right)\right\} = \max(2,1) = 2 \quad \text{(Selected } k = 3)$$

$$C[6,4] = \max_{3\leq k\leq 5}\left\{\min\left(C[k,3], \sum_{m=k+1}^{6} A[m]\right)\right\} = \max(2,4,3) = 4 \quad \text{(Selected } k = 4)$$

$$C[7,4] = \max_{3\leq k\leq 6}\left\{\min\left(C[k,3], \sum_{m=k+1}^{7} A[m]\right)\right\} = \max(2,5,5,5) = 5 \quad \text{(Selected } k = 4)$$

$$C[8,4] = \max_{3\leq k\leq 7}\left\{\min\left(C[k,3], \sum_{m=k+1}^{8} A[m]\right)\right\} = \max(2,5,5,5,8) = 8 \quad \text{(Selected } k = 7)$$

$$C[9,4] = \max_{3\leq k\leq 8}\left\{\min\left(C[k,3], \sum_{m=k+1}^{9} A[m]\right)\right\} = \max(2,5,5,5,8,2) = 8 \quad \text{(Selected } k = 7)$$

$$C[10,4] = \max_{3\leq k\leq 9}\left\{\min\left(C[k,3], \sum_{m=k+1}^{10} A[m]\right)\right\} = \max(2,5,5,5,8,6,4) = 8 \quad \text{(Selected } k = 7)$$

$$C[11,4] = \max_{3\leq k\leq 10}\left\{\min\left(C[k,3], \sum_{m=k+1}^{11} A[m]\right)\right\} = \max(2,5,5,5,8,11,11,9) = 11 \quad \text{(Selected } k = 8)$$

Table 5: DP Table $C[i][j]$

$j\backslash i$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	5	7	11	18	19	22	28	36	38	42	51
2	–	2	5	7	8	11	11	18	19	20	23
3	–	–	2	5	5	5	8	11	11	11	15
4	–	–	–	2	2	4	5	8	8	8	11

Table 6: Partition Points $K[i][j]$

$j\backslash i$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	0	0	0	0	0	0	0	0	0	0	0
2	–	1	1	2	3	3	3	4	5	6	7
3	–	–	2	3	3	3	5	6	6	6	8
4	–	–	–	3	3	4	4	7	7	7	8

- Maximum minimal sum achievable by partitioning A[1...11] into $M = 4$ groups is $C[11,4] = 11$.
- Backtracking through the DP table to determine optimal grouping $G[1 \dots M]$ by finding partition points k that leads to $C[11,4] = 11$.
  Partitioning A[1...11] into optimal groups based on k=3, k=6 and k=8:
  - Group 1: $A[1 \dots 3] = [5,2,4]$
  - Group 2: $A[4 \dots 6] = [7,1,3]$

– Group 3: $A[7 \ldots 8] = [6, 8]$

– Group 4: $A[9 \ldots 11] = [2, 4, 9]$

Grouping array $G[1 \ldots 3]$**:** $G = [3, 3, 2, 3]$

- Determine summation of the elements in the j-th group of array A.
  - Group 1 ($A[1 \ldots 3]$): $B[1] = 5 + 2 + 4 = 11$
  - Group 2 ($A[4 \ldots 6]$): $B[2] = 7 + 1 + 3 = 11$
  - Group 3 ($A[7 \ldots 8]$): $B[3] = 6 + 8 = 14$
  - Group 3 ($A[9 \ldots 11]$): $B[4] = 2 + 4 + 9 = 15$

  Sum of elements in each group $B[1 \ldots 4]$: $B = [11, 11, 14, 15]$

- The minimal group sum:

$$\min_{1 \leq j \leq 4} B[j] = \min(11, 11, 14, 15) = 11$$

The minimal group sum matches $C[11, 4] = 11$, confirming that the minimal group sum is maximized.

```
Optimal Grouping Sizes (G[1..M]):
Group 1: 3 elements
Group 2: 3 elements
Group 3: 2 elements
Group 4: 3 elements

Group Details:
Group 1: A[1...3] = [5, 2, 4], Sum = 11
Group 2: A[4...6] = [7, 1, 3], Sum = 11
Group 3: A[7...8] = [6, 8], Sum = 14
Group 4: A[9...11] = [2, 4, 9], Sum = 15
Optimal Grouping G: 3 3 2 3
```

4.

```
A = {3,1,4,1,5,9,2,6,5,3}
N = 10
M = 3
```

- **Initialize DP Table for Base case** $j = 1$ **(Single group):**

$$C[i, 1] = \sum_{m=1}^{i} A[m]$$

$$C[1,1] = \sum_{m=1}^{1} A[m] = 3, \quad C[2,1] = \sum_{m=1}^{2} A[m] = 4, \quad C[3,1] = \sum_{m=1}^{3} A[m] = 8,$$
$$C[4,1] = \sum_{m=1}^{4} A[m] = 9, \quad C[5,1] = \sum_{m=1}^{5} A[m] = 14, \quad C[6,1] = \sum_{m=1}^{6} A[m] = 23,$$
$$C[7,1] = \sum_{m=1}^{7} A[m] = 25, \quad C[8,1] = \sum_{m=1}^{8} A[m] = 31, \quad C[9,1] = \sum_{m=1}^{9} A[m] = 36,$$
$$C[10,1] = \sum_{m=1}^{10} A[m] = 39$$

- **Compute DP Table for $j = 2$:**

$$C[i,2] = \max_{1 \le k \le i-1} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{i} A[m] \right) \right\}, \quad \text{for } i = 2 \text{ to } 10$$

$$C[1,2] = \times \quad \text{(For } i = 1 \text{ and } j = 2, \text{ cannot partition 1 element into 2 groups. So } C[1,2] \text{ is not applicable.)}$$

$$C[2,2] = \max_{1 \le k \le 1} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{2} A[m] \right) \right\} = \min(3,1) = 1 \quad \text{(Selected } k = 1\text{)}$$

$$C[3,2] = \max_{1 \le k \le 2} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{3} A[m] \right) \right\} = \max(3,4) = 4 \quad \text{(Selected } k = 2\text{)}$$

$$C[4,2] = \max_{1 \le k \le 3} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{4} A[m] \right) \right\} = \max(3,4,1) = 4 \quad \text{(Selected } k = 2\text{)}$$

$$C[5,2] = \max_{1 \le k \le 4} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{5} A[m] \right) \right\} = \max(3,4,6,5) = 6 \quad \text{(Selected } k = 3\text{)}$$

$$C[6,2] = \max_{1 \le k \le 5} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{6} A[m] \right) \right\} = \max(3,4,8,9,9) = 9 \quad \text{(Selected } k = 4\text{)}$$

$$C[7,2] = \max_{1 \le k \le 6} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{7} A[m] \right) \right\} = \max(3,4,8,9,11,2) = 11 \quad \text{(Selected } k = 5\text{)}$$

$$C[8,2] = \max_{1 \le k \le 7} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{8} A[m] \right) \right\} = \max(3,4,8,9,14,8,6) = 14 \quad \text{(Selected } k = 5\text{)}$$

$$C[9,2] = \max_{1 \le k \le 8} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{9} A[m] \right) \right\} = \max(3,4,8,9,14,13,11,5) = 14 \quad \text{(Selected } k = 5\text{)}$$

$$C[10,2] = \max_{1 \le k \le 9} \left\{ \min \left(C[k,1], \sum_{m=k+1}^{10} A[m] \right) \right\} = \max(3,4,8,9,14,16,14,8,3) = 16 \quad \text{(Selected } k = 6\text{)}$$

- **Compute DP Table for $j = 3$:**

$$C[i,3] = \max_{2 \le k \le i-1} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{i} A[m] \right) \right\}, \quad \text{for } i = 3 \text{ to } 10$$

$C[1,3] = \times \quad$ (For $i = 1$ and $j = 3$, cannot partition one element into three groups. So $C[1,3]$ is not applicable.)
$C[2,3] = \times \quad$ (For $i = 2$ and $j = 3$, cannot partition two elements into three groups. So $C[2,3]$ is not applicable.)

$$C[3,3] = \max_{2 \le k \le 2} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{3} A[m] \right) \right\} = \min(1,4) = 1 \quad \text{(Selected } k = 2\text{)}$$

$$C[4,3] = \max_{2 \le k \le 3} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{4} A[m] \right) \right\} = \max(1,1) = 1 \quad \text{(Selected } k = 2\text{)}$$

$$C[5,3] = \max_{2 \le k \le 4} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{5} A[m] \right) \right\} = \max(1,4,4) = 4 \quad \text{(Selected } k = 3\text{)}$$

$$C[6,3] = \max_{2 \le k \le 5} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{6} A[m] \right) \right\} = \max(1,4,4,6) = 6 \quad \text{(Selected } k = 5\text{)}$$

$$C[7,3] = \max_{2 \le k \le 6} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{7} A[m] \right) \right\} = \max(1,4,4,6,2) = 6 \quad \text{(Selected } k = 5\text{)}$$

$$C[8,3] = \max_{2 \le k \le 7} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{8} A[m] \right) \right\} = \max(1,4,4,6,8,6) = 8 \quad \text{(Selected } k = 6\text{)}$$

$$C[9,3] = \max_{2 \le k \le 8} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{9} A[m] \right) \right\} = \max(1,4,4,6,9,11,5) = 11 \quad \text{(Selected } k = 7\text{)}$$

$$C[10,3] = \max_{2 \le k \le 9} \left\{ \min \left(C[k,2], \sum_{m=k+1}^{10} A[m] \right) \right\} = \max(1,4,4,6,9,11,8,3) = 11 \quad \text{(Selected } k = 7\text{)}$$

Table 7: DP Table $C[i][j]$ (Maximum Minimal Sums)

$j\backslash i$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	3	4	8	9	14	23	25	31	36	39
2	–	1	4	4	6	9	11	14	14	16
3	–	–	1	1	4	6	6	8	11	11

Table 8: Partition Points $K[i][j]$

$j\backslash i$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0	0	0	0	0	0	0	0	0	0
2	–	1	2	2	3	4	5	5	5	6
3	–	–	2	2	3	5	5	6	7	7

- Maximum minimal sum achievable by partitioning A[1...10] into $M = 3$ groups is $C[10,3] = 11$.
- Backtracking through the DP table to determine optimal grouping $G[1 \dots M]$ by finding partition points k that leads to $C[10,3] = 11$.
  Partitioning A[1...10] into optimal groups based on k=5, and k=7:
  - Group 1: $A[1 \dots 5] = [3,1,4,1,5]$
  - Group 2: $A[6 \dots 7] = [9,2]$
  - Group 3: $A[8 \dots 10] = [6,5,3]$

  Grouping array $G[1 \dots 3]$**:** $G = [5,2,3]$

- Determine summation of the elements in the j-th group of array A.
  - Group 1 ($A[1 \ldots 5]$): $B[1] = 3 + 1 + 4 + 1 + 5 = 14$
  - Group 2 ($A[6 \ldots 7]$): $B[2] = 9 + 2 = 11$
  - Group 3 ($A[8 \ldots 10]$): $B[3] = 6 + 5 + 3 = 14$

  Sum of elements in each group $B[1 \ldots 3]$: $B = [14, 11, 14]$

- The minimal group sum:

$$\min_{1 \leq j \leq 3} B[j] = \min(14, 11, 14) = 11$$

The minimal group sum matches $C[10, 3] = 11$, confirming that the minimal group sum is maximized.

```
Optimal Grouping Sizes (G[1..M]):
Group 1: 5 elements
Group 2: 2 elements
Group 3: 3 elements

Group Details:
Group 1: A[1...5] = [3, 1, 4, 1, 5], Sum = 14
Group 2: A[6...7] = [9, 2], Sum = 11
Group 3: A[8...10] = [6, 5, 3], Sum = 14
Optimal Grouping G: 5 2 3
```

4. Implementation

```cpp
#include <iostream>
#include <vector>
#include <algorithm>
#include <climits>
#include <numeric>
using namespace std;

vector<int> max_min_grouping(const vector<int>& A, int N, int M) {
    // Initialize DP table C[i][j] which stores the maximum
    // minimal sum that can be achieved by partitioning the i elements into j groups
    vector<vector<int>> C(N + 1, vector<int>(M + 1, INT_MIN));
    // K table to store the partition points for reconstructing the optimal groups
    vector<vector<int>> K(N + 1, vector<int>(M + 1, -1));
    // Base case: For 0 elements and 0 groups, the value max infinity as maximizing the minimum values
    C[0][0] = INT_MAX;
    // Sum array P to quickly calculate sums over any subarray of A
    vector<int> P(N + 1, 0);
    for (int i = 1; i <= N; ++i) {
        P[i] = P[i - 1] + A[i - 1];
    }
    // DP base case: Only 1 group (j=1)
    for (int i = 1; i <= N; ++i) {
        C[i][1] = P[i];
        K[i][1] = 0;
    }

    // DP table populate using the recurrence relation for each possible number of groups j (from 2 to M) and elements i (from j to N)
    // Try multiple partition points from k until (j-1) to partition the first i elements into j groups by considering different
    for (int j = 2; j <= M; ++j) {
        for (int i = j; i <= N; ++i) {
            for (int k = j - 1; k < i; ++k) {
                // Sum of elements from A[k+1] to A[i]
                int sum = P[i] - P[k];
```

```
                // Minimum value from partitioning the first k elements into (j-1) groups and including the new sum
                int min_sum = min(C[k][j - 1], sum);
                // Update the DP table if minimum value is larger minimum sum
                if (min_sum > C[i][j]) {
                    C[i][j] = min_sum;
                    K[i][j] = k; // Store the partition point for backtracking
                }
            }
        }
    }

    // Backtrack to reconstruct the optimal grouping G[1..M] by using the partition points stored in K
    vector<int> G(M, 0);
    int idx = N;
    for (int j = M; j >= 1; --j) {
        int k = K[idx][j]; // Get the partition point for group
        G[j - 1] = idx - k; // Store the size of the j-th group
        idx = k; // Move to the previous partition point
    }

    // Return the grouping sizes G[1..M], where each G[j] represents the number of elements
    return G;
}

int main() {
    vector<int> A = {3,1,4,1,5,9,2,6,5,3};
    int N = A.size();
    int M = 3;

    vector<int> G = max_min_grouping(A, N, M);

    cout << "Optimal Grouping G: ";
    for (int g : G) {
        cout << g << " ";
    }
    cout << endl;

    return 0;
}
```