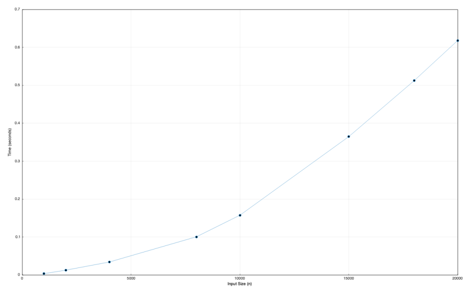
Question 1: Empirical Analysis of Sorting Algorithms

We compare the worst-case running time (reverse-sorted input) of Insertion Sort ($\mathcal{O}(n^2)$) and Merge Sort ($\mathcal{O}(n \log n)$).

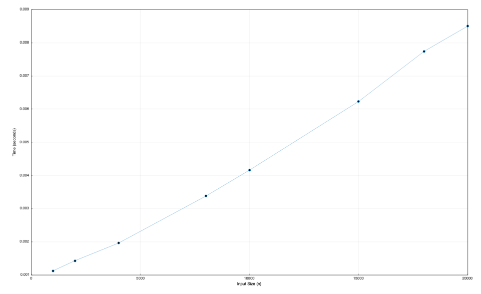
Table 1: Worst-Case Execution Times

Input Size (n)	**Insertion Sort (s)**	**Merge Sort (s)**
1000	0.0043	0.0011
2000	0.0136	0.0014
4000	0.0347	0.0020
8000	0.1020	0.0034
10000	0.1585	0.0042
15000	0.3512	0.0062
18000	0.4958	0.0077
20000	0.6255	0.0085

Analysis: Merge Sort consistently outperforms Insertion Sort for $n \geq 1000$. The quadratic growth of Insertion Sort is evident as doubling the input size (e.g., $10k \to 20k$) results in an approximate $4\times$ increase in time ($0.15s \to 0.62s$). In contrast, Merge Sort exhibits near-linear growth consistent with $n \log n$. The crossover point occurs at very small n (likely $n < 1000$ given the overhead of recursion versus simple loops), but for the tested range, Merge Sort is strictly superior.



(a) Insertion Sort Growth



(b) Merge Sort Growth

Figure 1: Empirical Runtime Analysis

Question 2: Quicksort Stability and Input Sensitivity

We analyze Quicksort on sorted inputs versus shuffled inputs ($n = 10,000$).

- **Worst-Case (Sorted Input):** 0.205513 seconds
- **Average-Case (Shuffled Input):** 0.000922 seconds

Analysis: The standard Quicksort implementation (using the last element as the pivot) degrades to $\mathcal{O}(n^2)$ when the input is already sorted, as the partition partitions the array into size $n-1$ and 0. This explains the high runtime of $\approx 0.2s$. Shuffling the array prior to sorting restores the probabilistic guarantee of balanced partitions, reducing the complexity to expected $\mathcal{O}(n \log n)$ and the runtime to $\approx 0.0009s$.

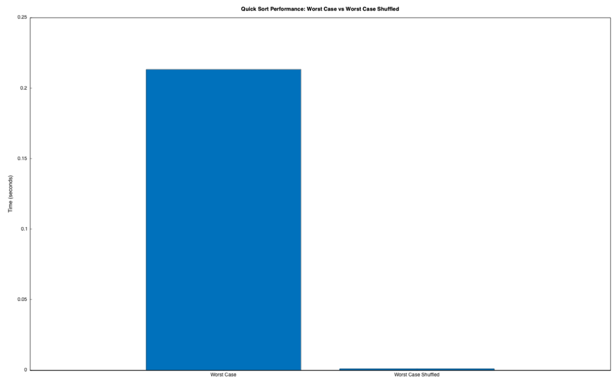


Figure 2: Impact of Input Distribution on Quicksort Performance

Question 3: Asymptotic Analysis

3.1. Prove $n + 3 \in \Omega(n)$

Definition: $f(n) \in \Omega(g(n))$ if $\exists c, n_0 > 0$ such that $0 \le c \cdot g(n) \le f(n)$ for all $n \ge n_0$.

Let $f(n) = n + 3$ and $g(n) = n$. We require $n + 3 \ge c \cdot n$. Choosing $c = 1$ and $n_0 = 1$:

$$n + 3 \ge 1 \cdot n \implies 3 \ge 0 \quad \text{(True for all } n \ge 1\text{)}.$$

Thus, constants $c = 1, n_0 = 1$ satisfy the condition. $\square$

3.2. Prove $n + 3 \in \mathcal{O}(n^2)$

Definition: $f(n) \in \mathcal{O}(g(n))$ if $\exists c, n_0 > 0$ such that $0 \le f(n) \le c \cdot g(n)$ for all $n \ge n_0$.

We require $n + 3 \le c \cdot n^2$. Dividing by n^2: $\frac{1}{n} + \frac{3}{n^2} \le c$. The LHS is a decreasing function for $n \ge 1$. At $n = 1$, LHS $= 4$. At $n = 3$, LHS $= 1/3 + 1/3 \approx 0.66 \le 1$. Choosing $c = 1$ and $n_0 = 3$:

$$n + 3 \le n \cdot n + 3 \le n^2 \quad \text{False for general } c.$$

Let's strictly evaluate $n + 3 \le 1 \cdot n^2$ for $n \ge 3$: $3^2 = 9, 3 + 3 = 6$. $6 \le 9$. Since n^2 grows quadratically and n linearly, this holds. Thus, $c = 1, n_0 = 3$ satisfy the condition. $\square$

3.3. Prove/Disprove $n + 3 \in \Theta(n^2)$

Definition: $f(n) \in \Theta(g(n))$ if $f(n) \in \mathcal{O}(g(n))$ AND $f(n) \in \Omega(g(n))$.

From 3.2, we know $n + 3 \in \mathcal{O}(n^2)$. We check if $n + 3 \in \Omega(n^2)$. This requires $n + 3 \ge c \cdot n^2$ for large n.

$$\lim_{n \to \infty} \frac{n + 3}{n^2} = 0.$$

Since the limit is 0, $n + 3$ is strictly asymptotically smaller than n^2. No constant $c > 0$ exists to satisfy the lower bound for large n. Therefore, $n + 3 \notin \Omega(n^2)$, implying $n + 3 \notin \Theta(n^2)$. $\square$

3.4. Prove/Disprove $2^{n+1} \in \mathcal{O}(n+1)$

We compare exponential growth 2^{n+1} to linear growth $n+1$.

$$\lim_{n\to\infty} \frac{2^{n+1}}{n+1} = \lim_{n\to\infty} \frac{2 \cdot 2^n}{n+1} = \infty \quad \text{(by L'Hopital's rule)}.$$

Since the ratio diverges, 2^{n+1} is not upper-bounded by $c(n+1)$. False. $\square$

3.5. Prove $2^{n+1} \in \Theta(2^n)$

We require $c_1 2^n \leq 2^{n+1} \leq c_2 2^n$. Rewrite $f(n)$: $2^{n+1} = 2 \cdot 2^n$. This is exactly $2 \cdot g(n)$. By choosing $c_1 = 2$ and $c_2 = 2$ (or $c_1 = 1, c_2 = 3$), the condition holds for all $n \geq 1$. Thus, $2^{n+1} \in \Theta(2^n)$. $\square$

Question 4: The Master Theorem

The Master Theorem applies to recurrences of the form $T(n) = aT(n/b) + f(n)$. Here, $a = 8, b = 2$. The critical exponent is $\log_b a = \log_2 8 = 3$.

4.1. $f(n) = n$

$f(n) = n^1 = \mathcal{O}(n^{\log_b a - \epsilon})$ with $\epsilon = 2$. Case 1 applies: $T(n) = \Theta(n^{\log_b a}) = \Theta(n^3)$.

4.2. $f(n) = n^2$

$f(n) = n^2 = \mathcal{O}(n^{3-\epsilon})$ with $\epsilon = 1$. Case 1 applies: $T(n) = \Theta(n^3)$.

4.3. $f(n) = n^3$

$f(n) = \Theta(n^{\log_b a})$. Case 2 applies: $T(n) = \Theta(n^{\log_b a} \log n) = \Theta(n^3 \log n)$.

4.4. $f(n) = n^4$

$f(n) = \Omega(n^{3+\epsilon})$ with $\epsilon = 1$. Regularity condition: $af(n/b) = 8(n/2)^4 = 8(n^4/16) = \frac{1}{2}n^4 \leq cn^4$ holds for $c = 1/2 < 1$. Case 3 applies: $T(n) = \Theta(f(n)) = \Theta(n^4)$.

Question 5: Recurrence Tree and Substitution Method

Recurrence: $T(n) = 8T(n/2) + n$.

5.1 Recursion Tree Analysis

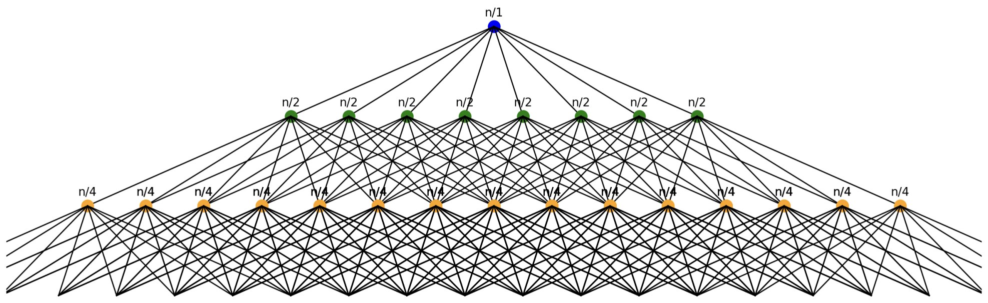


Figure 3: Recursion Tree for $T(n) = 8T(n/2) + n$

The cost at depth k is the number of nodes (8^k) times the cost per node ($\frac{n}{2^k}$).

$$\text{Cost}_k = 8^k \cdot \frac{n}{2^k} = \left(\frac{8}{2}\right)^k n = 4^k n.$$

The tree height is $L = \log_2 n$. The total cost is the sum over all levels:

$$T(n) = \sum_{k=0}^{\log_2 n} 4^k n = n \sum_{k=0}^{\log_2 n} 4^k.$$

This is a geometric series $\sum_{k=0}^{L} r^k = \frac{r^{L+1}-1}{r-1}$ with $r = 4$.

$$T(n) = n\left(\frac{4^{\log_2 n+1}-1}{4-1}\right) = \frac{n}{3}\left(4 \cdot 4^{\log_2 n} - 1\right).$$

Note that $4^{\log_2 n} = (2^2)^{\log_2 n} = (2^{\log_2 n})^2 = n^2$.

$$T(n) = \frac{n}{3}(4n^2 - 1) = \frac{4}{3}n^3 - \frac{1}{3}n.$$

Thus, the guess is $T(n) = \Theta(n^3)$.

5.2 Proof via Substitution Method

We wish to prove $T(n) \le cn^3$ for some constant c. However, direct substitution fails for the exact form cn^3 because of the linear term. Assume a stronger hypothesis to handle lower-order terms: $T(n) \le cn^3 - dn$ for constants $c, d > 0$.

Inductive Step: Assume $T(k) \le ck^3 - dk$ for $k < n$.

$$\begin{aligned} T(n) &= 8T(n/2) + n \\ &\le 8\left(c\left(\frac{n}{2}\right)^3 - d\left(\frac{n}{2}\right)\right) + n \\ &= 8\left(\frac{cn^3}{8} - \frac{dn}{2}\right) + n \\ &= cn^3 - 4dn + n \\ &= cn^3 - (4d-1)n. \end{aligned}$$

We require this to be $\leq cn^3 - dn$.

$$\begin{aligned} cn^3 - (4d-1)n &\leq cn^3 - dn \\ -(4d-1)n &\leq -dn \\ 4d - 1 &\geq d \\ 3d &\geq 1 \implies d \geq 1/3. \end{aligned}$$

This holds for any $d \geq 1/3$ and any $c > 0$ sufficient to cover base cases. Therefore, $T(n) = \mathcal{O}(n^3)$.