

1. Algorithm Design: Max-Min Grouping

We define the recurrence for the Dynamic Programming approach. Let $C[i][j]$ be the maximum possible value of the minimal group sum when partitioning the first i elements into j groups.

$$C[i][j] = \max_{j-1 \leq k < i} \left\{ \min \left(C[k][j-1], \sum_{m=k+1}^i A[m] \right) \right\}$$

Algorithm 1 Max-Min Grouping DP

```

1: function MAXMINGROUPING( $A, N, M$ )
2:    $P[0] \leftarrow 0$ 
3:   for  $i \leftarrow 1$  to  $N$  do                                 $\triangleright$  Prefix Sums
4:      $P[i] \leftarrow P[i-1] + A[i]$ 
5:   end for
6:   Initialize  $C[0\dots N][0\dots M] \leftarrow -\infty$ 
7:    $C[0][0] \leftarrow \infty$ 
8:   for  $i \leftarrow 1$  to  $N$  do                                 $\triangleright$  Base Case: 1 Group
9:      $C[i][1] \leftarrow P[i]$ 
10:     $K[i][1] \leftarrow 0$ 
11:   end for
12:   for  $j \leftarrow 2$  to  $M$  do                                 $\triangleright$  Iterate over group counts
13:     for  $i \leftarrow j$  to  $N$  do                                 $\triangleright$  Iterate over array length
14:       for  $k \leftarrow j-1$  to  $i-1$  do                                 $\triangleright$  Iterate split points
15:          $current\_sum \leftarrow P[i] - P[k]$ 
16:          $min\_val \leftarrow \min(C[k][j-1], current\_sum)$ 
17:         if  $min\_val > C[i][j]$  then
18:            $C[i][j] \leftarrow min\_val$ 
19:            $K[i][j] \leftarrow k$                                  $\triangleright$  Store split point for backtracking
20:         end if
21:       end for
22:     end for
23:   end for
24:   Backtrack to find group sizes  $G$  using  $K$  table
25:   return  $G$ 
26: end function

```

2. Running Time Analysis

Time Complexity

1. **Prefix Sum Calculation:** Iterates 1 to N . Complexity: $\Theta(N)$.
2. **DP Initialization ($j = 1$):** Iterates 1 to N . Complexity: $\Theta(N)$.
3. **DP Table Population:** The nesting structure is:
 - Outer loop (Groups j): $2\dots M$ ($\approx M$ iterations)

- Middle loop (Elements i): $j \dots N$ ($\approx N$ iterations)
- Inner loop (Split points k): $j - 1 \dots i - 1$ ($\approx i$ iterations, worst case $O(N)$)

Summation: $\sum_{j=2}^M \sum_{i=j}^N (i - j + 1) \approx \sum_{j=2}^M \frac{N^2}{2} = \Theta(M \cdot N^2)$.

4. **Backtracking:** Iterates M times. Complexity: $\Theta(M)$.

Total Time Complexity: $\Theta(M \cdot N^2)$.

Space Complexity

We store two tables C and K of size $(N + 1) \times (M + 1)$. **Total Space Complexity:** $\Theta(N \cdot M)$.

3. Execution Traces

Example 1

Input: $A = \{3, 9, 7, 8, 2, 6, 5, 10, 1, 7, 6, 4\}$, $N = 12$, $M = 3$.

DP Table $C[i][j]$ (Selected Values):

$j \setminus i$	1	2	3	4	5	6	7	8	9	10	11	12
1	3	12	19	27	29	35	40	50	51	58	64	68
2	-	3	7	12	12	16	19	23	24	29	29	33
3	-	-	3	7	7	8	12	15	16	18	19	19

Optimal Value: $C[12][3] = 19$. **Backtracking via K Table:**

- $j = 3, i = 12 \rightarrow K[12][3] = 7$. Group 3 is $A[8 \dots 12]$ (Sum 28).
- $j = 2, i = 7 \rightarrow K[7][2] = 3$. Group 2 is $A[4 \dots 7]$ (Sum 21).
- $j = 1, i = 3 \rightarrow K[3][1] = 0$. Group 1 is $A[1 \dots 3]$ (Sum 19).

Result: Group Sizes [3, 4, 5], Sums [19, 21, 28], Min Sum = 19.

```

Optimal Grouping Sizes (G[1..M]):
Group 1: 3 elements
Group 2: 4 elements
Group 3: 5 elements

Group Details:
Group 1: A[1...3] = [3, 9, 7], Sum = 19
Group 2: A[4...7] = [8, 2, 6, 5], Sum = 21
Group 3: A[8...12] = [10, 1, 7, 6, 4], Sum = 28
Optimal Grouping G: 3 4 5

```

Figure 1: Trace visualization for Example 1

Example 2**Input:** $A = \{4, 2, 5, 1, 6, 7, 3, 8, 2, 4\}$, $N = 10$, $M = 3$.**DP Table $C[i][j]$:**

$j \setminus i$	1	2	3	4	5	6	7	8	9	10
1	4	6	11	12	18	25	28	36	38	42
2	-	2	5	6	7	12	12	18	18	18
3	-	-	2	2	6	7	7	11	12	12

Optimal Value: $C[10][3] = 12$. **Groups:** $A[1\dots 4]$ (Sum 12), $A[5\dots 6]$ (Sum 13), $A[7\dots 10]$ (Sum 17). **Result:** Group Sizes [4,2,4], Min Sum = 12.

Optimal Grouping Sizes (G[1..M]):

Group 1: 4 elements

Group 2: 2 elements

Group 3: 4 elements

Group Details:

Group 1: $A[1\dots 4] = [4, 2, 5, 1]$, Sum = 12Group 2: $A[5\dots 6] = [6, 7]$, Sum = 13Group 3: $A[7\dots 10] = [3, 8, 2, 4]$, Sum = 17

Optimal Grouping G: 4 2 4

Figure 2: Trace visualization for Example 2

Example 3**Input:** $A = \{5, 2, 4, 7, 1, 3, 6, 8, 2, 4, 9\}$, $N = 11$, $M = 4$. **Optimal Value:** $C[11][4] = 11$. **Groups:** Sums [11,11,14,15]. **Result:** Group Sizes [3,3,2,3].

Optimal Grouping Sizes (G[1..M]):

Group 1: 3 elements

Group 2: 3 elements

Group 3: 2 elements

Group 4: 3 elements

Group Details:

Group 1: $A[1\dots 3] = [5, 2, 4]$, Sum = 11Group 2: $A[4\dots 6] = [7, 1, 3]$, Sum = 11Group 3: $A[7\dots 8] = [6, 8]$, Sum = 14Group 4: $A[9\dots 11] = [2, 4, 9]$, Sum = 15

Optimal Grouping G: 3 3 2 3

Figure 3: Trace visualization for Example 3

Example 4

Input: $A = \{3, 1, 4, 1, 5, 9, 2, 6, 5, 3\}$, $N = 10$, $M = 3$. **Optimal Value:** $C[10][3] = 11$. **Groups:** Sums [14, 11, 14]. **Result:** Group Sizes [5, 2, 3].

```

Optimal Grouping Sizes (G[1..M]):
Group 1: 5 elements
Group 2: 2 elements
Group 3: 3 elements

Group Details:
Group 1: A[1...5] = [3, 1, 4, 1, 5], Sum = 14
Group 2: A[6...7] = [9, 2], Sum = 11
Group 3: A[8...10] = [6, 5, 3], Sum = 14
Optimal Grouping G: 5 2 3
-
```

Figure 4: Trace visualization for Example 4

4. C++ Implementation

```

1 #include <iostream>
2 #include <vector>
3 #include <algorithm>
4 #include <climits>
5 #include <numeric>
6 using namespace std;
7
8 vector<int> max_min_grouping(const vector<int>& A, int N, int M) {
9     // DP table: C[i][j] = max min sum for partitioning first i elements into j
10    groups
11    vector<vector<int>> C(N + 1, vector<int>(M + 1, INT_MIN));
12    // Path table for reconstruction
13    vector<vector<int>> K(N + 1, vector<int>(M + 1, -1));
14
15    // Prefix sums for O(1) range sum queries
16    vector<int> P(N + 1, 0);
17    for (int i = 1; i <= N; ++i) P[i] = P[i - 1] + A[i - 1];
18
19    // Base case: j=1 (1 group)
20    for (int i = 1; i <= N; ++i) {
21        C[i][1] = P[i];
22        K[i][1] = 0;
23    }
24
25    // Fill DP Table
26    for (int j = 2; j <= M; ++j) {
27        for (int i = j; i <= N; ++i) {
28            // Try all split points k
29            for (int k = j - 1; k < i; ++k) {
30                int current_group_sum = P[i] - P[k];
31                int min_of_split = min(C[k][j - 1], current_group_sum);
32
33                if (min_of_split > C[i][j]) {
34                    C[i][j] = min_of_split;
35                    K[i][j] = k;
36                }
37            }
38        }
39    }
40
41    return C[N][M];
42}

```

```
35         }
36     }
37 }
38
39 // Backtrack to find group sizes
40 vector<int> G(M, 0);
41 int idx = N;
42 for (int j = M; j >= 1; --j) {
43     int k = K[idx][j];
44     G[j - 1] = idx - k;
45     idx = k;
46 }
47 return G;
48 }
49
50 int main() {
51     vector<int> A = {3,1,4,1,5,9,2,6,5,3};
52     vector<int> result = max_min_grouping(A, 10, 3);
53     cout << "Optimal Group Sizes: ";
54     for(int s : result) cout << s << " ";
55     cout << endl;
56     return 0;
57 }
```

Listing 1: Max-Min Grouping Implementation