1.

a.

Solve $Fc = y$ by finding $c = F_4^{-1} y$.

$$n = 2 \quad (\text{so } F = F_4), \quad y = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \omega_4 = e^{2\pi i/4} = i, \quad D_2 = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$F_4 = \begin{bmatrix} I_2 & D_2 \\ I_2 & -D_2 \end{bmatrix} \begin{bmatrix} F_2 & 0 \\ 0 & F_2 \end{bmatrix} P$$

Step 1:

$$\begin{bmatrix} I_2 & D_2 \\ I_2 & -D_2 \end{bmatrix} \begin{bmatrix} \underbrace{F_2 \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}}_{r} \\ \underbrace{F_2 \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}}_{s} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \implies \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = r + D_2 s, \quad \begin{bmatrix} y_3 \\ y_4 \end{bmatrix} = r - D_2 s$$

$$r = \tfrac{1}{2}\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} y_3 \\ y_4 \end{bmatrix} \right), \qquad D_2 s = \tfrac{1}{2}\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} y_3 \\ y_4 \end{bmatrix} \right)$$

Plug in $y = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$, $D_2 = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$:

$$r = \tfrac{1}{2}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad D_2 s = \tfrac{1}{2}\begin{bmatrix} 2 \\ 0 \end{bmatrix} - \tfrac{1}{2}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Rightarrow s = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Step 2:

$$F_2 \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = r = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 + x_3 = 0 \\ x_1 - x_3 = 0 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$F_2 \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = s = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_2 + x_4 = 2 \\ x_2 - x_4 = 0 \end{cases} \Rightarrow \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus

$$\begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Step 3:

$$P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} \implies c = P^{-1} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

b.

Forward verification with $c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Step 1:

$$P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying to c (with $x_1 = 0,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1$):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Step 2:

$$\begin{bmatrix} F_2 & 0 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} F_2 \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \\ F_2 \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$F_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + 1 \cdot 0 \\ 1 \cdot 0 + (-1) \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$F_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 \\ 1 \cdot 1 + (-1) \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} F_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ F_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

Step 3:

$$y = \begin{bmatrix} I_2 & D_2 \\ I_2 & -D_2 \end{bmatrix} \begin{bmatrix} F_2 \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \\ F_2 \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

Block form: $\quad y = \begin{bmatrix} F_2 \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + D_2 F_2 \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \\ F_2 \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} - D_2 F_2 \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \end{bmatrix}$

Compute $D_2 F_2 \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$: $\quad \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 0 \cdot 0 \\ 0 \cdot 2 + i \cdot 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Top block: $\quad F_2 \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + D_2 F_2 \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$$\text{Bottom block:} \quad F_2 \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} - D_2\, F_2 \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$y = \begin{bmatrix} I_2 & D_2 \\ I_2 & -D_2 \end{bmatrix} \begin{bmatrix} F_2 \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \\ F_2 \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$$

2.

$$2v_1 + 2v_2 + v_3 = 18$$

For the ℓ^1 norm:

$$\min \|v\|_1 = |v_1| + |v_2| + |v_3|$$

The solution occurs when the ℓ^1 ball first touches the plane on one of its coordinate faces, where one component becomes zero. Suppose $v_3 = 0$. Then

$$2v_1 + 2v_2 + v_3 = 18$$

reduces to

$$2v_1 + 2v_2 = 18 \implies v_1 + v_2 = 9$$

The ℓ^1-norm at this point is

$$\|v\|_1 = |v_1| + |v_2| + |v_3| = |v_1| + |v_2|$$

$$\|v\|_1 = v_1 + v_2 = 9$$
$$\|v^{(1)}\|_1 = 9$$

For the ℓ^2 norm:

The solution occurs when the circle touches the plane at a point perpendicular to it.

$$2v_1 + 2v_2 + v_3 = 18$$

The normal to the plane is

$$n = (2, 2, 1)$$

and its length is

$$\|n\|_2^2 = 2^2 + 2^2 + 1^2 = 9$$

The closest point to the origin lies along the normal, so let $v = \alpha n$. Substitute into the plane equation:

$$n \cdot v = 18 \implies \alpha(n \cdot n) = 18 \implies \alpha = \frac{18}{9} = 2$$

$$v^{(2)} = \alpha n = 2(2, 2, 1) = (4, 4, 2)$$
$$\|v^{(2)}\|_2 = 2\|n\|_2 = 6$$

For ℓ^∞ norm:

The solution occurs when this cube first touches the plane at one of its corners, where all coordinates are equal.

$$2v_1 + 2v_2 + v_3 = 18$$

Let

$$v_1 = v_2 = v_3 = c.$$

Substitute into the plane equation:

$$2c + 2c + c = 5c = 18 \implies c = \frac{18}{5}$$

$$v^{(\infty)} = \left(\frac{18}{5}, \frac{18}{5}, \frac{18}{5}\right)$$

$$\|v^{(\infty)}\|_\infty = \frac{18}{5}$$

3.

a.

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.5 & 0.25 & 0 \\ 0 & 0.25 & 0.5 & 0.25 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

To find the eigenvalues λ, we solve

$$\det(\lambda I - P) = 0$$

$$\lambda I - P = \begin{bmatrix} \lambda - 0.5 & -0.5 & 0 & 0 \\ -0.25 & \lambda - 0.5 & -0.25 & 0 \\ 0 & -0.25 & \lambda - 0.5 & -0.25 \\ 0 & 0 & -0.5 & \lambda - 0.5 \end{bmatrix}.$$

The first row has only two non-zero entries $(\lambda - 0.5)$ and -0.5 so expanding the determinant along the first row:

$$\det(\lambda I - P) = (\lambda - 0.5)C_{11} - (-0.5)C_{12} = (\lambda - 0.5)M_{11} + 0.5M_{12}$$

since cofactors alternate in sign and $C_{11} = M_{11},\ C_{12} = -M_{12}$.

Compute M_{11} - Delete row 1 and column 1:

$$M_{11} = \begin{vmatrix} \lambda - 0.5 & -0.25 & 0 \\ -0.25 & \lambda - 0.5 & -0.25 \\ 0 & -0.5 & \lambda - 0.5 \end{vmatrix}$$

Expand along the first row:

$$M_{11} = (\lambda - 0.5)\begin{vmatrix} \lambda - 0.5 & -0.25 \\ -0.5 & \lambda - 0.5 \end{vmatrix} - (-0.25)\begin{vmatrix} -0.25 & -0.25 \\ 0 & \lambda - 0.5 \end{vmatrix}$$

$$\begin{vmatrix} \lambda - 0.5 & -0.25 \\ -0.5 & \lambda - 0.5 \end{vmatrix} = (\lambda - 0.5)^2 - (0.125)$$

$$\begin{vmatrix} -0.25 & -0.25 \\ 0 & \lambda - 0.5 \end{vmatrix} = (-0.25)(\lambda - 0.5)$$

$$\begin{aligned} M_{11} &= (\lambda - 0.5)\left[(\lambda - 0.5)^2 - 0.125\right] + 0.25[-0.25(\lambda - 0.5)] \\ &= (\lambda - 0.5)\left[(\lambda - 0.5)^2 - 0.125 - 0.0625\right] \\ &= (\lambda - 0.5)\left[(\lambda - 0.5)^2 - 0.1875\right] \end{aligned}$$

Compute M_{12} - Delete row 1 and column 2:

$$M_{12} = \begin{vmatrix} -0.25 & -0.25 & 0 \\ 0 & \lambda - 0.5 & -0.25 \\ 0 & -0.5 & \lambda - 0.5 \end{vmatrix}$$

Expand along the first row:

$$\begin{aligned} M_{12} &= (-0.25) \begin{vmatrix} \lambda - 0.5 & -0.25 \\ -0.5 & \lambda - 0.5 \end{vmatrix} - (-0.25) \begin{vmatrix} 0 & -0.25 \\ 0 & \lambda - 0.5 \end{vmatrix} \\ &= -0.25\big[(\lambda - 0.5)^2 - 0.125\big] \end{aligned}$$

Therefore,

$$\begin{aligned} \det(\lambda I - P) &= (\lambda - 0.5)M_{11} + 0.5M_{12} \\ &= (\lambda - 0.5)\Big[(\lambda - 0.5)\big((\lambda - 0.5)^2 - 0.1875\big)\Big] + 0.5\big[-0.25\big((\lambda - 0.5)^2 - 0.125\big)\big] \\ &= (\lambda - 0.5)^2\big[(\lambda - 0.5)^2 - 0.1875\big] - 0.125\big[(\lambda - 0.5)^2 - 0.125\big] \\ &= (\lambda - 0.5)^4 - 0.1875(\lambda - 0.5)^2 - 0.125(\lambda - 0.5)^2 + 0.015625 \\ &= (\lambda - 0.5)^4 - 0.3125(\lambda - 0.5)^2 + 0.015625 \end{aligned}$$

Let $x = \lambda - 0.5$

$$x^4 - 0.3125x^2 + 0.015625 = 0$$

Let $y = x^2$:

$$y^2 - 0.3125y + 0.015625 = 0$$

Solve for y using the quadratic formula:

$$\begin{aligned} y &= \frac{0.3125 \pm \sqrt{(0.3125)^2 - 4(0.015625)}}{2} \\ y_1 &= \frac{0.3125 + 0.1875}{2} = 0.25 \\ y_2 &= \frac{0.3125 - 0.1875}{2} = 0.0625 \end{aligned}$$

Hence, as $y = x^2$:

$$x = \pm\sqrt{0.25}, \quad \pm\sqrt{0.0625} \quad \Rightarrow \quad x = \pm 0.5,\ \pm 0.25$$

Hence, as $x = \lambda - 0.5$:

$$\begin{aligned} x = 0.5 \quad &\Rightarrow \quad \lambda \ = \ x + 0.5 \ = \ 0.5 + 0.5 \ = \ 1 \\ x = -0.5 \quad &\Rightarrow \quad \lambda \ = \ x + 0.5 \ = \ 0.5 - 0.5 \ = \ 0 \\ x = 0.25 \quad &\Rightarrow \quad \lambda \ = \ x + 0.5 \ = \ 0.5 + 0.25 \ = \ 0.75 \\ x = -0.25 \quad &\Rightarrow \quad \lambda \ = \ x + 0.5 \ = \ 0.5 - 0.25 \ = \ 0.25 \end{aligned}$$

For each eigenvalue λ, we solve $(P - \lambda I)v = 0$.

For $\lambda = 1$:

$$P - I = \begin{bmatrix} -0.5 & 0.5 & 0 & 0 \\ 0.25 & -0.5 & 0.25 & 0 \\ 0 & 0.25 & -0.5 & 0.25 \\ 0 & 0 & 0.5 & -0.5 \end{bmatrix}.$$

We solve $(P - I)v = 0$

$$-0.5v_1 + 0.5v_2 = 0 \quad \Rightarrow \; v_1 = v_2$$

$$0.25v_1 - 0.5v_2 + 0.25v_3 = 0 \quad \Rightarrow \; v_1 = v_2 = v_3$$

$$0.25v_2 - 0.5v_3 + 0.25v_4 = 0 \quad \Rightarrow \; v_3 = v_4$$

Thus $v_1 = v_2 = v_3 = v_4$.

$$v^{(1)} = [1,\, 1,\, 1,\, 1]^T$$

For $\lambda = 0.75$:

$$P - 0.75I = \begin{bmatrix} -0.25 & 0.5 & 0 & 0 \\ 0.25 & -0.25 & 0.25 & 0 \\ 0 & 0.25 & -0.25 & 0.25 \\ 0 & 0 & 0.5 & -0.25 \end{bmatrix}.$$

We solve $(P - 0.75I)v = 0$:

$$-0.25v_1 + 0.5v_2 = 0 \quad \Rightarrow \; v_1 = 2v_2$$

$$0.25v_1 - 0.25v_2 + 0.25v_3 = 0 \quad \Rightarrow \; v_3 = -v_2$$

$$0.25v_2 - 0.25v_3 + 0.25v_4 = 0 \quad \Rightarrow \; v_4 = -2v_2$$

Hence $v_1 : v_2 : v_3 : v_4 = 2 : 1 : -1 : -2$

$$v^{(2)} = [2,\, 1,\, -1,\, -2]^T$$

For $\lambda = 0.25$:

$$P - 0.25I = \begin{bmatrix} 0.25 & 0.5 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0 \\ 0 & 0.25 & 0.25 & 0.25 \\ 0 & 0 & 0.5 & 0.25 \end{bmatrix}$$

We solve $(P - 0.25I)v = 0$:

$$0.25v_1 + 0.5v_2 = 0 \quad \Rightarrow \; v_1 = -2v_2$$

$$0.25v_1 + 0.25v_2 + 0.25v_3 = 0 \Rightarrow \; -0.5v_2 + v_2 + v_3 = 0 \Rightarrow v_3 = -v_2$$

$$0.25v_2 + 0.25v_3 + 0.25v_4 = 0 \Rightarrow \; v_4 = 2v_2$$

Thus $v_1 : v_2 : v_3 : v_4 = 2 : -1 : -1 : 2$

$$v^{(3)} = [2,\, -1,\, -1,\, 2]^T$$

For $\lambda = 0$:

$$P - 0I = P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.5 & 0.25 & 0 \\ 0 & 0.25 & 0.5 & 0.25 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

We solve $Pv = 0$:

$$0.5v_1 + 0.5v_2 = 0 \quad \Rightarrow \; v_1 = -v_2$$

$$0.25v_1 + 0.5v_2 + 0.25v_3 = 0 \Rightarrow \; v_3 = -v_2$$

$$0.25v_2 + 0.5v_3 + 0.25v_4 = 0 \Rightarrow \; v_4 = -v_3 = v_2$$

Thus $v_1 : v_2 : v_3 : v_4 = 1 : -1 : 1 : -1$

$$v^{(4)} = [1,\, -1,\, 1,\, -1]^T$$

Collecting Eigenvectors

$$V = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -2 & 2 & -1 \end{bmatrix}$$

b.

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.5 & 0.25 & 0 \\ 0 & 0.25 & 0.5 & 0.25 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}, \qquad x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 = 1, \qquad v^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 0.75, \quad v^{(2)} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix}$$

$$\lambda_3 = 0.25, \quad v^{(3)} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\lambda_4 = 0, \qquad v^{(4)} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

The initial state $x(0)$ can be written as a linear combination of eigenvectors:

$$x(0) = c_1 v^{(1)} + c_2 v^{(2)} + c_3 v^{(3)} + c_4 v^{(4)}.$$

Substituting and solving the linear system:

$$\begin{aligned} c_1 + 2c_2 + 2c_3 + c_4 &= 1 \\ c_1 + c_2 - c_3 - c_4 &= 0 \\ c_1 - c_2 - c_3 + c_4 &= 0 \\ c_1 - 2c_2 + 2c_3 - c_4 &= 0 \end{aligned}$$

$$\begin{aligned} c_1 + c_2 &= c_3 + c_4 \\ c_1 - c_2 &= c_3 - c_4 \end{aligned} \quad \Rightarrow \quad c_1 = c_3,\ c_2 = c_4$$

$$c_1 - 2c_2 + 2c_1 - c_2 = 0 \ \Rightarrow\ 3c_1 - 3c_2 = 0 \ \Rightarrow\ c_1 = c_2$$

$$c_1 = c_2 = c_3 = c_4 = c$$

$$c + 2c + 2c + c = 6c = 1 \ \Rightarrow\ c = \tfrac{1}{6}$$

$$c_1 = c_2 = c_3 = c_4 = \tfrac{1}{6}$$

Plugging in eigenvalues the state vector at time t is:

$$x(t) = c_1(1)^t v^{(1)} + c_2(0.75)^t v^{(2)} + c_3(0.25)^t v^{(3)} + c_4(0)^t v^{(4)}$$

$$x(t) = \tfrac{1}{6}\left(v^{(1)} + 0.75^{\,t} v^{(2)} + 0.25^{\,t} v^{(3)}\right)$$

As $t \to \infty$, the terms containing eigenvalues $|\lambda| < 1$ disappear, so the component corresponding to $\lambda_1 = 1$ dominates:

$$\lim_{t\to\infty} x(t) = \tfrac{1}{6} v^{(1)} = \tfrac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

To obtain a probability distribution for the stationary (equilibrium) distribution, we normalize so that all entries sum to 1:

$$x_{eq} = \frac{\frac{1}{6}}{4(\frac{1}{6})}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0.25\\0.25\\0.25\\0.25\end{bmatrix}$$

Therefore, proportion of time, the system is in State 1 in the long run = 0.25 = 25%.

4.

a.

$$A = \begin{bmatrix}0 & 4 & -1\\1 & -2 & -2\\2 & 1 & 0\end{bmatrix}$$

$$A^T A = \begin{bmatrix}0 & 1 & 2\\4 & -2 & 1\\-1 & -2 & 0\end{bmatrix}\begin{bmatrix}0 & 4 & -1\\1 & -2 & -2\\2 & 1 & 0\end{bmatrix} = \begin{bmatrix}5 & 0 & -2\\0 & 21 & 0\\-2 & 0 & 5\end{bmatrix}$$

By definition:

$$A^T A v_i = \lambda_i v_i \quad \sigma_i = \sqrt{\lambda_i}$$

For $v_1 = (0, 1, 0)$:

$$A^T A v_1 = \begin{bmatrix}5 & 0 & -2\\0 & 21 & 0\\-2 & 0 & 5\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\21\\0\end{bmatrix} = 21\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\Rightarrow \lambda_1 = 21 \quad , \sigma_1 = \sqrt{21}$$

For $v_2 = (-1, 0, 1)$:

$$A^T A v_2 = \begin{bmatrix}5 & 0 & -2\\0 & 21 & 0\\-2 & 0 & 5\end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}-5+(-2)\\0\\2+5\end{bmatrix} = \begin{bmatrix}-7\\0\\7\end{bmatrix} = 7\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$\Rightarrow \lambda_2 = 7 \quad , \sigma_2 = \sqrt{7}$$

For $v_3 = (1, 0, 1)$:

$$A^T A v_3 = \begin{bmatrix}5 & 0 & -2\\0 & 21 & 0\\-2 & 0 & 5\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}5+(-2)\\0\\-2+5\end{bmatrix} = \begin{bmatrix}3\\0\\3\end{bmatrix} = 3\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\Rightarrow \lambda_3 = 3 \quad , \sigma_3 = \sqrt{3}$$

b.

$$\lambda_1 = 21 \text{ for } v_1 = (0,1,0), \quad \lambda_2 = 7 \text{ for } v_2 = (-1,0,1), \quad \lambda_3 = 3 \text{ for } v_3 = (1,0,1)$$

$$\sigma_1 = \sqrt{21}, \qquad \sigma_2 = \sqrt{7}, \qquad \sigma_3 = \sqrt{3}$$

$$\hat{v}_1 = \frac{v_1}{\|v_1\|} = (0,1,0), \quad \hat{v}_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}(-1,0,1), \quad \hat{v}_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{2}}(1,0,1)$$

$$V = \begin{bmatrix}\hat{v}_1 & \hat{v}_2 & \hat{v}_3\end{bmatrix} = \begin{bmatrix}0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\1 & 0 & 0\\0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 4 & -1 \\ 1 & -2 & -2 \\ 2 & 1 & 0 \end{bmatrix}$$

For $\hat{v}_1 = (0, 1, 0)$:

$$A\hat{v}_1 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \qquad u_1 = \frac{A\hat{v}_1}{\sigma_1} = \frac{1}{\sqrt{21}} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{\sqrt{21}} \\ -\frac{2}{\sqrt{21}} \\ \frac{1}{\sqrt{21}} \end{bmatrix}$$

For $\hat{v}_2 = \frac{1}{\sqrt{2}}(-1, 0, 1)$:

$$A\hat{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}, \qquad u_2 = \frac{A\hat{v}_2}{\sigma_2} = \frac{1}{\sqrt{2}\sqrt{7}} \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{14}} \\ -\frac{3}{\sqrt{14}} \\ -\frac{2}{\sqrt{14}} \end{bmatrix}$$

For $\hat{v}_3 = \frac{1}{\sqrt{2}}(1, 0, 1)$:

$$A\hat{v}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \qquad u_3 = \frac{A\hat{v}_3}{\sigma_3} = \frac{1}{\sqrt{2}\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

Thus,

$$U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{\sqrt{21}} & -\frac{1}{\sqrt{14}} & -\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{21}} & -\frac{3}{\sqrt{14}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{21}} & -\frac{2}{\sqrt{14}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

Therefore,

$$A = U\,\Sigma\,V^T$$

with

$$U = \begin{bmatrix} \frac{4}{\sqrt{21}} & -\frac{1}{\sqrt{14}} & -\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{21}} & -\frac{3}{\sqrt{14}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{21}} & -\frac{2}{\sqrt{14}} & \frac{2}{\sqrt{6}} \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sqrt{21} & 0 & 0 \\ 0 & \sqrt{7} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \quad V = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

c.

$$A = U\Sigma V^T = \sum_{i=1}^{3} \sigma_i\, u_i v_i^T, \qquad A_k = \sum_{i=1}^{k} \sigma_i\, u_i v_i^T, \qquad A_1 = \sigma_1 u_1 v_1^T$$

$$\sigma_1 = \sqrt{21}, \qquad v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{21}} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

$$u_1 v_1^T = \frac{1}{\sqrt{21}} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{21}} \begin{bmatrix} 0 & 4 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A_1 = \sigma_1\, u_1 v_1^T = \sqrt{21} \cdot \frac{1}{\sqrt{21}} \begin{bmatrix} 0 & 4 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

d.

$$A_2 = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$$

$$\sigma_1 = \sqrt{21}, \quad v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_1 = \tfrac{1}{\sqrt{21}} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \qquad \sigma_2 = \sqrt{7}, \quad v_2 = \tfrac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad u_2 = \tfrac{1}{\sqrt{14}} \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix}.$$

$$\sigma_1 u_1 v_1^T = \sqrt{21}\,\frac{1}{\sqrt{21}} \begin{bmatrix} 0 & 4 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$u_2 v_2^T = \frac{1}{\sqrt{14}\sqrt{2}} \begin{bmatrix} -1 \\ -3 \\ -2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{28}} \begin{bmatrix} 1 & 0 & -1 \\ 3 & 0 & -3 \\ 2 & 0 & -2 \end{bmatrix}$$

$$\sigma_2 u_2 v_2^T = \frac{\sqrt{7}}{\sqrt{28}} \begin{bmatrix} 1 & 0 & -1 \\ 3 & 0 & -3 \\ 2 & 0 & -2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \\ 3 & 0 & -3 \\ 2 & 0 & -2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 4 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \\ 3 & 0 & -3 \\ 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0.7071 & 4 & -0.7071 \\ 2.1213 & -2 & -2.1213 \\ 1.4142 & 1 & -1.4142 \end{bmatrix}$$

5.

Let A be a real $m \times n$ matrix with SVD

$$A = U\,\Sigma\,V^T$$

Let $A \in \mathbb{R}^{m\times n}$ have SVD

$$A = U\,\Sigma\,V^T,$$

Taking transpose:

$$A^T = (U\,\Sigma\,V^T)^T = V\,\Sigma^T\,U^T$$

General form:

$$A^T = U'\,\Sigma'\,V'^T$$

Matching terms with the general form:

$$U' = V, \qquad \Sigma' = \Sigma^T, \qquad V' = U$$

Therefore, an SVD of A^T is $V\,\Sigma^T\,U^T$.

Verification with $A = U\,\Sigma\,V^T$:

$$A^T = (U\,\Sigma\,V^T)^T = V\,\Sigma^T\,U^T$$

$$A^T A = (V\,\Sigma^T\,U^T)(U\,\Sigma\,V^T) = V\,(\Sigma^T U^T U \Sigma)\,V^T = V\,(\Sigma^T\Sigma)\,V^T, \text{since } U^T U = I$$

The columns of V are eigenvectors of $A^T A$, and the eigenvalues are the diagonal entries of $\Sigma^T\Sigma$, i.e. σ_i^2.

$$AA^T = (U\,\Sigma\,V^T)(V\,\Sigma^T\,U^T) = U\,(\Sigma V^T V \Sigma^T)\,U^T = U\,(\Sigma\Sigma^T)\,U^T, \text{since } V^T V = I$$

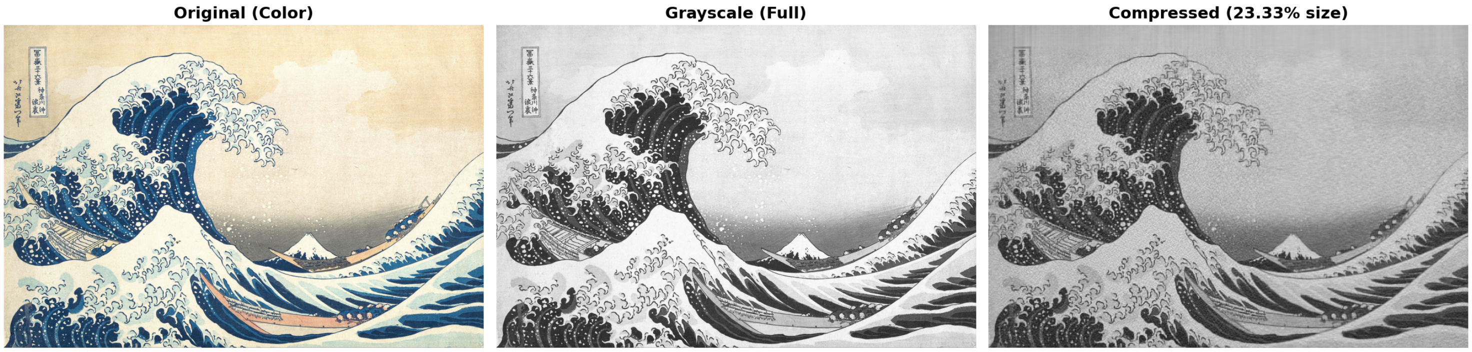
The columns of U are eigenvectors of AA^T, and the eigenvalues are the diagonal entries of $\Sigma\Sigma^T$, i.e. σ_i^2.

Since $A^T A$ and AA^T share the same nonnegative eigenvalues σ_i^2, it follows that A and A^T have the same singular values.

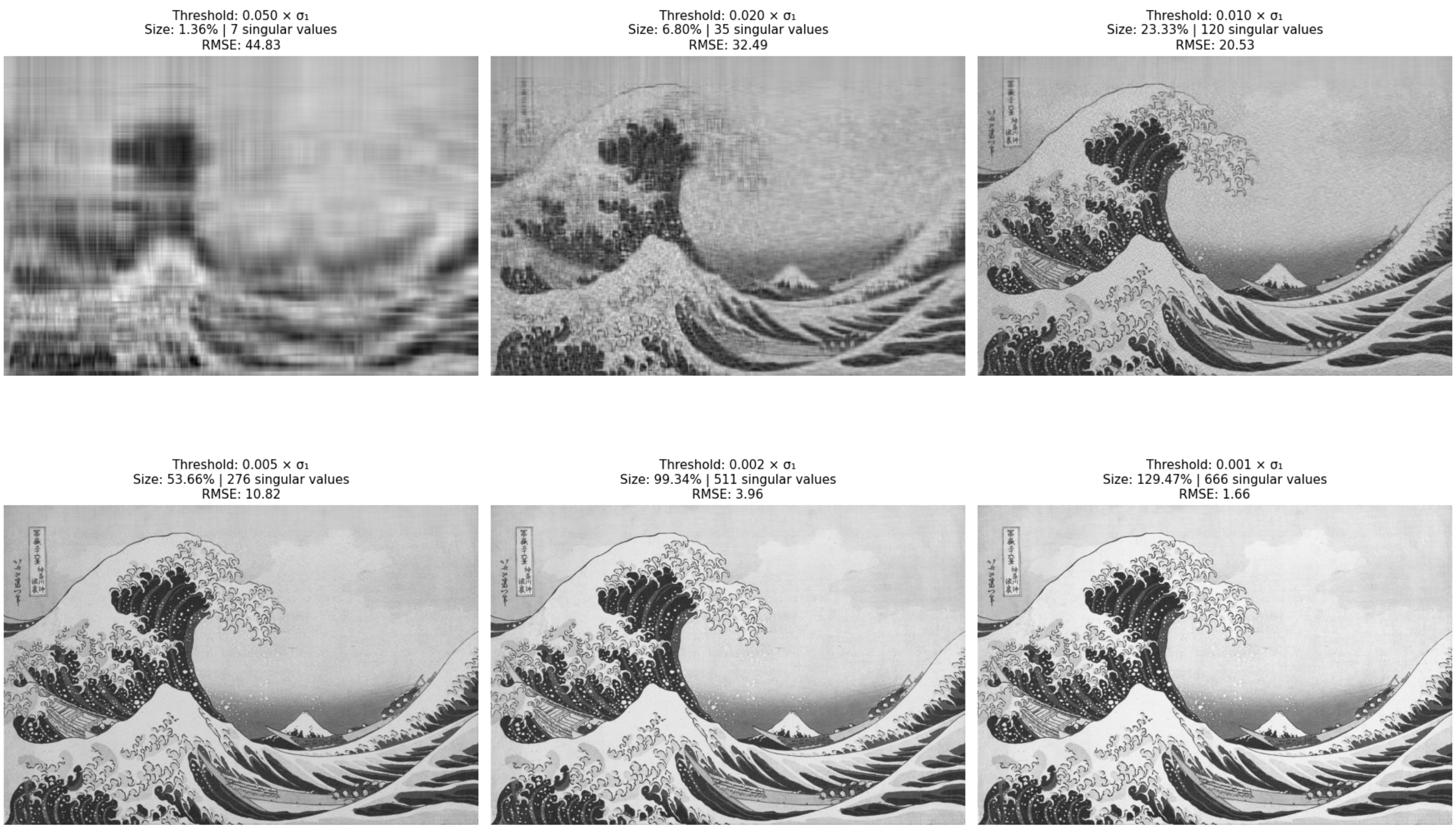
Therefore, an SVD of A^T is $A^T = V\,\Sigma^T\,U^T$ where U and V are orthogonal matrices, and Σ contains the singular values of both A and A^T.

6.

a.



b.

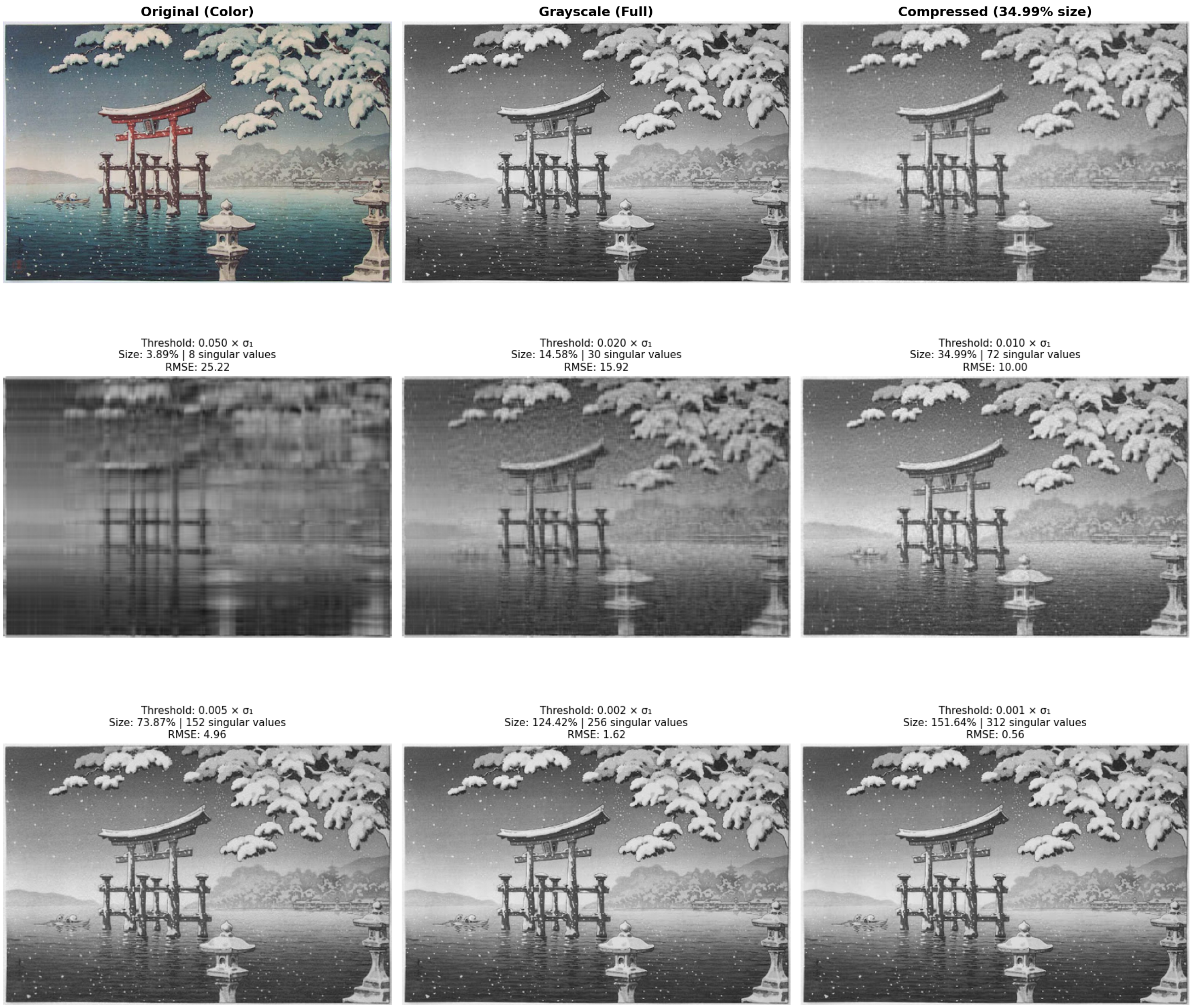


The best compromise configuration is:

Threshold:	$0.005 \times \sigma_1$
Number of singular values used:	276 out of 860
Compression percentage:	53.66%
Compression ratio:	1.86 : 1
Space saved:	46.34%

This threshold achieves nearly a $2:1$ compression while maintaining excellent visual quality, providing an optimal balance between file size and image quality.

c.



The best compromise configuration is:

Threshold:	$0.005 \times \sigma_1$
Number of singular values used:	152 out of 344
Compression percentage:	73.87%
Compression ratio:	$1.35:1$
Space saved:	26.13%

This threshold maintains excellent reconstruction quality while providing a moderate level of compression, balancing storage efficiency and image quality.