1.

a.

$$T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}, \qquad T(p) = \int_{-1}^{1} p(x)\,dx.$$

The domain is $\mathcal{P}_2(\mathbb{R})$ (polynomials of degree ≤ 2) with standard basis $(1, x, x^2)$:

$$\mathcal{P}_2(\mathbb{R}) = \{\, p(x) = ax^2 + bx + c : a, b, c \in \mathbb{R} \,\}.$$

i.

From the definition of a linear map, we must check:

$$T(u+v) = T(u) + T(v), \qquad T(cu) = cT(u),$$

for all $u, v \in \mathcal{P}_2(\mathbb{R})$ and scalars $c \in \mathbb{R}$.

$$T(u+v) = \int_{-1}^{1} (u+v)(x)\,dx = \int_{-1}^{1} u(x)\,dx + \int_{-1}^{1} v(x)\,dx = T(u) + T(v),$$

$$T(cu) = \int_{-1}^{1} c\,u(x)\,dx = c\int_{-1}^{1} u(x)\,dx = c\,T(u).$$

Thus T satisfies the linear conditions and is a linear transformation.

ii.

By definition, for a linear map $T : V \to W$:

$$\ker T = \{\, x \in V : T(x) = 0 \,\}.$$

By definition:

$$\ker T = \{\, p \in \mathcal{P}_2 : T(p) = 0 \,\}$$

Let $p(x) = ax^2 + bx + c$. Then

$$T(p) = a\int_{-1}^{1} x^2\,dx + b\int_{-1}^{1} x\,dx + c\int_{-1}^{1} 1\,dx = \tfrac{2}{3}a + 0 + 2c = \tfrac{2}{3}a + 2c$$

$$\begin{aligned} T(p) &= 0 \\ \Rightarrow \tfrac{2}{3}a + 2c &= 0 \\ \Rightarrow a &= -3c \end{aligned}$$

Therefore any polynomial in the kernel has the form:

$$p(x) = ax^2 + bx + c = bx + c(1 - 3x^2).$$

Thus:

$$\ker T = \operatorname{span}\{x,\ 1 - 3x^2\}, \qquad \dim(\ker T) = 2$$

By definition, for a linear map $T : V \to W$:

$$\operatorname{im} T = \{\, T(x) : x \in V \,\}$$

By definition:

$$\operatorname{im} T = \{\, T(p) : p \in \mathcal{P}_2 \,\} \subseteq \mathbb{R}$$

Evaluating on the basis elements, $T(1) = 2, T(x) = 0, T(x^2) = \frac{2}{3}$.The outputs are real numbers, and since both 2 and $\frac{2}{3}$ are nonzero scalars, they span the same one-dimensional subspace of $\mathbb{R}$ as 1. Any nonzero scalar $r \in \mathbb{R}$ spans the same one-dimensional subspace as 1 because $1 = (1/r)\,r$ and $r = r \cdot 1$. The image of T is a subspace of W in $T : V \to W$. The image of a linear map is always a subspace of the codomain, and in this case it is the whole codomain $\mathbb{R}$. Thus:

$$\operatorname{im} T = \operatorname{span}\{1\} = \mathbb{R}, \qquad \dim(\operatorname{im} T) = 1$$

iii.

Rank–Nullity Theorem. For a linear map $T : V \to W$,

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T).$$

$$\dim(\ker T) + \dim(\operatorname{im} T) = 2 + 1 = 3 = \dim \mathcal{P}_2(\mathbb{R}),$$

so the Rank-Nullity Theorem holds.

b.

$$T : \mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^2, \qquad T(p) = \big(p(0),\, -p(0)\big).$$

The domain is $\mathcal{P}_3(\mathbb{R})$ (polynomials of degree ≤ 3) with standard basis $(1, x, x^2, x^3)$:

$$\mathcal{P}_3(\mathbb{R}) = \{\, p(x) = ax^3 + bx^2 + cx + d : \ a, b, c, d \in \mathbb{R} \,\}.$$

i.

From the definition of a linear map, we must check:

$$T(u + v) = T(u) + T(v), \qquad T(cu) = cT(u),$$

for all $u, v \in \mathcal{P}_3(\mathbb{R})$ and scalars $c \in \mathbb{R}$.

$$T(u + v) = \big((u + v)(0), -(u + v)(0)\big) = \big(u(0) + v(0), -(u(0) + v(0))\big) = T(u) + T(v),$$

$$T(cu) = \big((cu)(0), -(cu)(0)\big) = \big(cu(0), -cu(0)\big) = c\,T(u).$$

Thus T satisfies the linear conditions and is a linear transformation.

ii.

By definition, for a linear map $T : V \to W$:

$$\ker T = \{\, x \in V : T(x) = 0 \,\}.$$

By definition:

$$\ker T = \{\, p \in \mathcal{P}_3 : T(p) = 0 \,\}$$

Let $p(x) = ax^3 + bx^2 + cx + d$. Then

$$\begin{aligned} T(p) &= \big(p(0),\, -p(0)\big) \\ &= \big(a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d,\, -(a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d)\big) \\ &= (d,\, -d). \end{aligned}$$

For p to be in the kernel, we require $T(p) = (0,0)$. Thus,

$$(d, -d) = (0,0) \iff d = 0.$$

Therefore the kernel condition is $d = 0$, and any polynomial in the kernel has the form

$$p(x) = ax^3 + bx^2 + cx + d = ax^3 + bx^2 + cx.$$

Thus,

$$\ker T = \operatorname{span}\{x,\, x^2,\, x^3\}, \qquad \dim(\ker T) = 3.$$

By definition, for a linear map $T : V \to W$:

$$\operatorname{im} T = \{\, T(x) : x \in V \,\}$$

By definition:

$$\operatorname{im} T = \{\, T(p) : p \in \mathcal{P}_3 \,\} \subseteq \mathbb{R}^2$$

For $T(p) = (p(0), -p(0)) = (d, -d)$ as d ranges over all real numbers, every output vector is of the form $(d, -d) = d(1, -1)$. When $d = 0$, we obtain the zero vector, which is always part of a subspace. For nonzero d, the outputs lie on the line through the origin spanned by $(1, -1)$. Thus:

$$\operatorname{im} T = \operatorname{span}\{(1, -1)\}, \qquad \dim(\operatorname{im} T) = 1$$

The image of a linear map is always a subspace of the codomain because it is closed under vector addition and scalar multiplication.

iii.

Rank–Nullity Theorem. For a linear map $T : V \to W$,

$$\dim V = \dim(\ker T) + \dim(\operatorname{im} T).$$

$$\dim(\ker T) + \dim(\operatorname{im} T) = 3 + 1 = 4 = \dim \mathcal{P}_3(\mathbb{R}),$$

so the Rank–Nullity Theorem holds.

2.

a.

i.

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \qquad T(x, y, z) = (y, z, x)$$

Let $\alpha = \beta = \{v_1, v_2, v_3\}$ be the standard ordered bases of $V = W = \mathbb{R}^3$.

$$v_1 = (1,0,0), \quad v_2 = (0,1,0), \quad v_3 = (0,0,1)$$

Applying T to each basis vector:

$$T(v_1) = T(1,0,0) = (0,0,1) = v_3$$
$$T(v_2) = T(0,1,0) = (1,0,0) = v_1$$
$$T(v_3) = T(0,0,1) = (0,1,0) = v_2$$

Theorem: Let $T : V \to W$ be a linear map, and $\alpha = \{v_1, \ldots, v_n\}$ and $\beta = \{w_1, \ldots, w_m\}$ be ordered bases for V and W. Then $[T(x)]_\beta = [T]_\alpha^\beta [x]_\alpha$, where the matrix $[T]_\alpha^\beta$ of T relative to α and β is the $m \times n$ matrix such that the jth column is $[T(v_j)]_\beta$.

From the Matrix Representation Theorem:

$$[T(x)]_\beta = [T]_\alpha^\beta\, [x]_\alpha, \quad \textit{where} \quad [T]_\alpha^\beta = \big[\, [T(v_1)]_\beta \;\; [T(v_2)]_\beta \;\; [T(v_3)]_\beta \,\big].$$

Thus,

$$[T]_\alpha^\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

ii.

Theorem: For any square matrix $A \in M_{n\times n}(F)$, the matrix A is invertible if $\det(A) \neq 0$. If $\det(A) = 0$, the matrix is singular, meaning it is not invertible. A matrix A is invertible when its determinant is nonzero.

For a 3×3 matrix $[T]_\alpha^\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$,

$$\det([T]_\alpha^\beta) = 0\,(0\cdot 0 - 1\cdot 0) - 1\,(0\cdot 0 - 1\cdot 1) + 0\,(0\cdot 0 - 0\cdot 1) = 0 - \big(0-1\big) + 0 = 1 \neq 0$$

From the Invertibility Theorem, a square matrix is invertible if its determinant is nonzero and invertibility of the matrix implies invertibility of the corresponding linear map. As $\det([T]_\alpha^\beta) \neq 0$, the matrix $[T]_\alpha^\beta$ is invertible.

Corollary: Let α and β be ordered bases for V and W, and let $T : V \to W$ be a linear map. Then T has an inverse $T^{-1} : W \to V$ if and only if $[T]_\alpha^\beta$ is invertible, in which case $[T^{-1}]_\beta^\alpha = ([T]_\alpha^\beta)^{-1}$.

From the Invertibility of Linear Maps Corollary, if $[T]_\beta^\alpha$ is invertible, then T is invertible. Its inverse has the representation $[T^{-1}]_\beta^\alpha = ([T]_\alpha^\beta)^{-1}$. Therefore, T has an inverse.

Theorem: A linear map $T : V \to W$ is an isomorphism if T is one-to-one and onto. Two vector spaces are isomorphic if there is an isomorphism between them. It follows from general function theory that a linear map $T : V \to W$ has an inverse $T^{-1} : W \to V$ such that $TT^{-1} = T^{-1}T = I$ if and only if T is one-to-one and onto, that is, an isomorphism.

From the Isomorphisms and Inverses Theorem, a linear map is an isomorphism if it is one-to-one and onto and having an inverse implies that T is both one-to-one and onto.Therefore, T is an isomorphism.

Therefore, matrix invertibility implies nonzero determinant, invertible matrix implies invertible linear map and invertible linear map implies isomorphism. Since $\det([T]_\alpha^\beta) = 1 \neq 0$, the map T is invertible, hence bijective, and therefore an isomorphism.

$$[T^{-1}]_\beta^\alpha = ([T]_\alpha^\beta)^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

b.

i.

$$T : M_{2\times 2}(\mathbb{R}) \;\to\; M_{2\times 2}(\mathbb{R}), \quad T(A) = LA, \quad \text{where } L = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Let $\alpha = \beta = \{v_{11}, v_{12}, v_{21}, v_{22}\}$ be the standard ordered bases of $V = W = M_{2\times 2}(\mathbb{R})$.

$$v_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad v_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad v_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad v_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

By applying $T(A) = LA$ to each basis vector:

$$T(v_{11}) = Lv_{11} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = v_{11}$$

$$T(v_{12}) = Lv_{12} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = v_{12}$$

$$T(v_{21}) = Lv_{21} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = 2\,v_{11}$$

$$T(v_{22}) = Lv_{22} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = 2\,v_{12}$$

Theorem: Let $T : V \to W$ be a linear map, and $\alpha = \{v_1, \ldots, v_n\}$ and $\beta = \{w_1, \ldots, w_m\}$ be ordered bases for V and W. Then $[T(x)]_\beta = [T]_\alpha^\beta [x]_\alpha$, where the matrix $[T]_\alpha^\beta$ of T relative to α and β is the $m \times n$ matrix such that the jth column is $[T(v_j)]_\beta$.

From the Matrix Representation Theorem:

$$[T(x)]_\beta = [T]_\alpha^\beta \, [x]_\alpha, \quad where \quad [T]_\alpha^\beta = \big[\, [T(v_{11})]_\beta \;\; [T(v_{12})]_\beta \;\; [T(v_{21})]_\beta \, [T(v_{22})]_\beta \,\big]$$

Each column is the coordinate vector of $T(v_{ij})$ relative to β:

$$[T(v_{11})]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [T(v_{12})]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, [T(v_{21})]_\beta = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [T(v_{22})]_\beta = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ii.

For any $n \times n$ matrix $A = (a_{ij})$, the Laplace expansion along row i is

$$\det(A) = \sum_{j=1}^{n} a_{ij} \, (-1)^{i+j} \det(A_{ij}).$$

Choosing row 3 from $[T]_\alpha^\beta$, since it has the most number of zeros to make the determinant computation simpler. In the Laplace expansion formula,

$$\det([T]_\alpha^\beta) = \sum_{j=1}^{4} m_{3j} \, (-1)^{3+j} \det\big(([T]_\alpha^\beta)_{3j}\big),$$

every term involves m_{3j}, the entries of row 3. Since each $m_{3j} = 0$, all four terms vanish immediately. So, $a_{31} = a_{32} = a_{33} = a_{34} = 0$. Therefore, each term in the Laplace expansion is 0:

$$\det([T]_\alpha^\beta) = 0 \cdot (-1)^{3+1} \det\big(([T]_\alpha^\beta)_{31}\big) + 0 \cdot (-1)^{3+2} \det\big(([T]_\alpha^\beta)_{32}\big) + 0 \cdot (-1)^{3+3} \det\big(([T]_\alpha^\beta)_{33}\big) + 0 \cdot (-1)^{3+4} \det\big(([T]_\alpha^\beta)_{34}\big) = 0.$$

Theorem: For any square matrix $A \in M_{n\times n}(F)$, the matrix A is invertible if $\det(A) \neq 0$. If $\det(A) = 0$, the matrix is singular, meaning it is not invertible. A matrix A is invertible when its determinant is nonzero.

From the Invertibility Theorem, a square matrix is invertible if its determinant is nonzero and invertibility of the matrix implies invertibility of the corresponding linear map. As $\det([T]_\alpha^\beta) = 0$, the matrix $([T]_\alpha^\beta)$ is not invertible.

From the Invertibility of Linear Maps Corollary, if $[T]_\alpha^\beta$ is invertible, then T is invertible. If not, then T is not invertible and hence not an isomorphism. Since $\det([T]_\alpha^\beta) = 0$, the map T is not invertible. Therefore:

$$T \text{ is not an isomorphism, and } [T^{-1}]_\beta^\alpha \text{ does not exist.}$$

By applying the matrix representation theorem, determinant test, and invertibility corollary, we conclude that the matrix representation of T is singular, so the linear map T is not invertible, not bijective, and not an isomorphism.

3.

$$T : \mathbb{R}^n \to \mathbb{R}^n, \qquad T(x) = x * y, \qquad \text{where } (T(x))_k = (x * y)_k = \sum_{i=1}^{n} x_i \, y_{k-i}$$

From $y_0 = y_n, y_{-1} = y_{n-1}$, it can be inferred that every term y_{k-i} is in $\{y_1, y_2, \ldots, y_n\}$.
A map $T : V \to W$ is linear if:

$$T(x+u) = T(x) + T(u), \qquad T(cx) = cT(x)$$

By definition of convolution map T, for any $x \in \mathbb{R}^n$,

$$T(x)_k = (x * y)_k = \sum_{i=1}^{n} x_i\, y_{k-i}$$

Taking two vectors $x = (x_1, \ldots, x_n),\ u = (u_1, \ldots, u_n) \in \mathbb{R}^n$. Then

$$T(x+u)_k = \big((x+u) * y\big)_k$$

$$\begin{aligned} T(x+u)_k &= \sum_{i=1}^{n} (x_i + u_i)\, y_{k-i} \\ &= \sum_{i=1}^{n} \big(x_i y_{k-i} + u_i y_{k-i}\big) \\ &= \sum_{i=1}^{n} x_i y_{k-i} \ + \ \sum_{i=1}^{n} u_i y_{k-i} \qquad \Big(\text{since } \sum_{i=1}^{n} x_i y_{k-i} = (x*y)_k = T(x)_k \text{ and } \sum_{i=1}^{n} u_i y_{k-i} = (u*y)_k = T(u)_k\Big) \\ &= T(x)_k + T(u)_k \end{aligned}$$

Since this equality holds for every component k, we conclude $T(x+u) = T(x) + T(u)$.

Taking a scalar $c \in \mathbb{R}$ and a vector $x \in \mathbb{R}^n$. By the definition of convolution map T,

$$T(cx)_k = \big((cx) * y\big)_k = \sum_{i=1}^{n} (cx_i)\, y_{k-i}$$

$$\begin{aligned} T(cx)_k &= \sum_{i=1}^{n} (cx_i)\, y_{k-i} \\ &= \sum_{i=1}^{n} c\,(x_i y_{k-i}) \\ &= c \sum_{i=1}^{n} x_i y_{k-i} \qquad \Big(\text{by the definition of convolution map, } \sum_{i=1}^{n} x_i y_{k-i} = (x*y)_k = T(x)_k\Big) \\ &= c\, T(x)_k \end{aligned}$$

Since this holds for every component k, we conclude $T(cx) = cT(x)$.

Since both conditions hold for all k, convolution map T is a linear map.

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

Let $\alpha = \beta = \{v_1, v_2, \ldots, v_n\}$ be the standard ordered bases of $V = W = \mathbb{R}^n$.
In the standard basis, each v_i has exactly one coordinate equal to 1 in the i-th position and all others equal to 0. Applying T to each basis vector:

$$T(v_i)_k = \sum_{j=1}^{n} (v_i)_j\, y_{k-j} = y_{k-i}$$

By definition, $[T(v_i)]_\beta$ is the coordinate vector of $T(v_i)$ in the basis β:

$$[T(v_i)]_\beta = \begin{bmatrix} y_{1-i} \\ y_{2-i} \\ \vdots \\ y_{n-i} \end{bmatrix}$$

Thus, the i-th column of the matrix $[T]_\alpha^\beta$ is $[T(v_i)]_\beta$. From Matrix Representation Theorem:

$$[T]_\alpha^\beta = \begin{bmatrix}[T(v_1)]_\beta & [T(v_2)]_\beta & \cdots & [T(v_n)]_\beta\end{bmatrix} = \begin{bmatrix} y_{1-1} & y_{1-2} & \cdots & y_{1-n} \\ y_{2-1} & y_{2-2} & \cdots & y_{2-n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n-1} & y_{n-2} & \cdots & y_{n-n} \end{bmatrix}$$

Entry-wise, this means $[T]_\alpha^\beta = \left(y_{k-i}\right)_{1\le k,i\le n}$ where indices are mod n (e.g. $y_0 = y_n$, $y_{-1} = y_{n-1}$, etc.).

4.

a.

$$\alpha = \{1,\ 1+x,\ (1+x)^2\}, \qquad \beta = \{1,\ x,\ x^2\}.$$

The change-of-basis matrix $[I]_\alpha^\beta$ has columns $[\alpha_j]_\beta$ and each coordinate vector is obtained by writing α_j as a linear combination of the β-basis elements.

For $\alpha_1 = 1$: $1 = c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2 \quad \Rightarrow \quad c_1 = 1,\ c_2 = 0,\ c_3 = 0$
For $\alpha_2 = 1+x$: $1 + x = c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2 \quad \Rightarrow \quad c_1 = 1,\ c_2 = 1,\ c_3 = 0$
For $\alpha_3 = (1+x)^2$: $1 + 2x + x^2 = c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2 \quad \Rightarrow \quad c_1 = 1,\ c_2 = 2,\ c_3 = 1$

$$[\alpha_1]_\beta = \begin{bmatrix}1\\0\\0\end{bmatrix}, [\alpha_2]_\beta = \begin{bmatrix}1\\1\\0\end{bmatrix}, [\alpha_3]_\beta = \begin{bmatrix}1\\2\\1\end{bmatrix}$$

From the Matrix Representation Theorem, the change-of-basis matrix is:

$$[I]_\alpha^\beta = \begin{bmatrix}[\alpha_1]_\beta & [\alpha_2]_\beta & [\alpha_3]_\beta\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}$$

The inverse change-of-basis matrix is:

$$[I]_\beta^\alpha = \left([I]_\alpha^\beta\right)^{-1} = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix}^{-1} = \begin{bmatrix}1 & -1 & 1\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}$$

b.

For $\beta_1 = 1$:

$$1 = a_1 \cdot 1 + a_2 \cdot (1+x) + a_3 \cdot (1+x)^2$$
$$= (a_1 + a_2 + a_3) + (a_2 + 2a_3)x + a_3x^2.$$

Comparing coefficients with $1 + 0x + 0x^2$, we obtain the system: $a_1 + a_2 + a_3 = 1, a_2 + 2a_3 = 0, a_3 = 0$. Solving gives $a_3 = 0, \quad a_2 = 0, \quad a_1 = 1$. Therefore,

$$[\beta_1]_\alpha = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

For $\beta_2 = x$:

$$x = a_1 \cdot 1 + a_2 \cdot (1+x) + a_3 \cdot (1+x)^2$$
$$= (a_1 + a_2 + a_3) + (a_2 + 2a_3)x + a_3x^2.$$

Comparing coefficients with $0 + 1x + 0x^2$ we obtain the system: $a_1 + a_2 + a_3 = 0, a_2 + 2a_3 = 1, a_3 = 0$. Solving gives $a_3 = 0, \quad a_2 = 1, \quad a_1 = -1$. Therefore,

$$[\beta_2]_\alpha = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

For $\beta_3 = x^2$:

$$x^2 = a_1 \cdot 1 + a_2 \cdot (1 + x) + a_3 \cdot (1 + x)^2$$

$$= (a_1 + a_2 + a_3) + (a_2 + 2a_3)x + a_3x^2.$$

Comparing coefficients with $0 + 0x + 1x^2$ we obtain the system: $a_1 + a_2 + a_3 = 0, a_2 + 2a_3 = 0, a_3 = 1$. Solving gives $a_3 = 1, \quad a_2 = -2, \quad a_1 = 1$. Therefore,

$$[\beta_3]_\alpha = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

c.

$T(p) = p + x\,p''$. Let $p, q \in \mathcal{P}_2$, $c \in \mathbb{R}$.

$$T(p + q) = (p + q) + x(p + q)'' = (p + q) + x(p'' + q'') = T(p) + T(q)$$

$$T(cp) = cp + x(cp)'' = cp + xcp'' = c(p + xp'') = cT(p)$$

Therefore $T = p + xp''$ is linear. Therefore for the ordered basis $\beta = \{1, x, x^2\}$ of $V = W = \mathcal{P}_2(\mathbb{R})$, the matrix representation theorem applies:

$$[T]_\beta^\beta = \big[\, [T(1)]_\beta \;\; [T(x)]_\beta \;\; [T(x^2)]_\beta \,\big]$$

$p(x) = 1 \Rightarrow p''(x) = 0$

$$T(1) = 1 + x \cdot 0 = 1, \qquad [\,T(1)\,]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$p(x) = x \Rightarrow p''(x) = 0$

$$T(x) = x + x \cdot 0 = x, \qquad [\,T(x)\,]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$p(x) = x^2 \Rightarrow p''(x) = 2$

$$T(x^2) = x^2 + x \cdot 2 = x^2 + 2x, \qquad [\,T(x^2)\,]_\beta = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$[T]_\beta^\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

d.

From the computation in the standard basis $\beta = \{1, x, x^2\}$,

$$[T]_\beta^\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $[T]_\beta^\beta$ is an upper triangular matrix, its determinant is the product of its diagonal entries:

$$\det([T]_\beta^\beta) = (1)(1)(1) = 1 \neq 0.$$

From the Invertibility Theorem, a square matrix is invertible if its determinant is nonzero, and invertibility of the matrix implies invertibility of the corresponding linear map. As $\det([T]_\beta^\beta) \neq 0$, the matrix $[T]_\beta^\beta$ is invertible. A linear map T has an inverse if its matrix (in any basis) is invertible; in that case T is one-to-one and onto, i.e., an isomorphism. Therefore, T is invertible and hence an isomorphism.

The inverse matrix is

$$([T]_\beta^\beta)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

This corresponds to the operator

$$S : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R}), \qquad S(p) = p - xp''.$$

Using a polynomial in the standard basis $\beta = \{1, x, x^2\}$ such as

$$p(x) = a + bx + cx^2 \iff [p]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Since $p'' = 2c$, we have

$$S(p) = p - xp'' = a + bx + cx^2 - 2cx = a + (b - 2c)x + cx^2$$

In coordinates,

$$[p]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \longmapsto [S(p)]_\beta = \begin{bmatrix} a \\ b - 2c \\ c \end{bmatrix}.$$

That is exactly the transformation effected by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b - 2c \\ c \end{bmatrix}.$$

Therefore $([T]_\beta^\beta)^{-1}$ implements the map $S(p) = p - xp''$ which is the inverse of T. Since T has an inverse linear map, T is invertible, and hence T is an isomorphism.

e.

For $p, q \in \mathcal{P}_2$, $c \in \mathbb{R}$,

$$T(p + q) = (p + q) + x(p + q)'' = (p + q) + x(p'' + q'') = T(p) + T(q),$$

$$T(cp) = cp + x(cp)'' = cp + x(cp'') = c(p + xp'') = cT(p).$$

T is linear. Therefore, the matrix representation theorem applies:

$$[T]_\alpha^\alpha = \big[\, [T(1)]_\alpha \;\; [T(1 + x)]_\alpha \;\; [T((1 + x)^2)]_\alpha \,\big].$$

For $\alpha_1 = 1$:

$$p(x) = 1 \quad \Rightarrow \quad p''(x) = 0,$$

$$T(1) = 1, \qquad [T(1)]_\alpha = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\alpha_2 = 1 + x$:

$$p(x) = 1 + x \quad \Rightarrow \quad p''(x) = 0,$$

$$T(1+x) = 1 + x, \qquad [T(1+x)]_\alpha = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\alpha_3 = (1+x)^2 = 1 + 2x + x^2$:

$$p(x) = (1+x)^2 \quad \Rightarrow \quad p''(x) = 2,$$

$$T((1+x)^2) = (1 + 2x + x^2) + 2x = 1 + 4x + x^2.$$

$$\begin{aligned} & 1 + 4x + x^2 = a \cdot 1 + b \cdot (1+x) + c \cdot (1 + 2x + x^2) \\ \Rightarrow\ & 1 + 4x + x^2 = (a + b + c) + (b + 2c)x + cx^2 \end{aligned}$$

Matching coefficients with $1 + 4x + x^2$:

$$a + b + c = 1, \qquad b + 2c = 4, \qquad c = 1.$$

So $c = 1$, $b = 2$, $a = -2$ Hence

$$[T((1+x)^2)]_\alpha = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Corollary (Section 3.5). Let $T : V \to V$ be a linear map, and let α and β be bases of V.Then:

$$[T]_\beta^\beta = [I]_\alpha^\beta \, [T]_\alpha^\alpha \left([I]_\alpha^\beta\right)^{-1} = [I]_\alpha^\beta \, [T]_\alpha^\alpha \, [I]_\beta^\alpha$$

Multiplying both sides by $[I]_\beta^\alpha$:

$$\begin{aligned} & [I]_\beta^\alpha \, [T]_\beta^\beta = \left([I]_\beta^\alpha [I]_\alpha^\beta\right) [T]_\alpha^\alpha \, [I]_\beta^\alpha \\ \Rightarrow\ & [I]_\beta^\alpha \, [T]_\beta^\beta = I \, [T]_\alpha^\alpha \, [I]_\beta^\alpha \\ \Rightarrow\ & [I]_\beta^\alpha \, [T]_\beta^\beta = [T]_\alpha^\alpha \, [I]_\beta^\alpha \end{aligned}$$

Multiplying both sides by $[I]_\alpha^\beta$

$$\begin{aligned} & [I]_\beta^\alpha \, [T]_\beta^\beta = [T]_\alpha^\alpha \, [I]_\beta^\alpha \\ \Rightarrow\ & [I]_\beta^\alpha \, [T]_\beta^\beta \, [I]_\alpha^\beta = [T]_\alpha^\alpha \left([I]_\beta^\alpha [I]_\alpha^\beta\right) \\ \Rightarrow\ & [I]_\beta^\alpha \, [T]_\beta^\beta \, [I]_\alpha^\beta = [T]_\alpha^\alpha \, I \\ \Rightarrow\ & [I]_\beta^\alpha \, [T]_\beta^\beta \, [I]_\alpha^\beta = [T]_\alpha^\alpha \end{aligned}$$

Therefore,

$$[T]_\alpha^\alpha = [I]_\beta^\alpha \, [T]_\beta^\beta \, [I]_\alpha^\beta$$

Applying the corollary to $T(p) = p + x\,p''$ on $\mathcal{P}_2(\mathbb{R})$:

$$[I]_\alpha^\beta = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [I]_\beta^\alpha = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [T]_\beta^\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T]_\alpha^\alpha = [I]_\beta^\alpha \, [T]_\beta^\beta \, [I]_\alpha^\beta = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

5.

Given V, W are vector spaces over a field F, and $T : V \to W$ is linear and invertible. Then the inverse map $T^{-1} : W \to V$ is also linear.

A linear map $T : V \to W$ has an inverse $T^{-1} : W \to V$ with $T^{-1}T = I_V$ and $TT^{-1} = I_W$ if T is one-to-one and onto; i.e., T is an isomorphism.

Linearity of T^{-1}: To prove that the map $T^{-1} : W \to V$ is linear, we must show that for all $u, v \in W$ and all $c \in F$, we have $T^{-1}(u + v) = T^{-1}(u) + T^{-1}(v)$ and $T^{-1}(cu) = c \cdot T^{-1}(u)$.

Let $u, v \in W$ and $c \in F$. Since $T : V \to W$ is invertible, the inverse map $T^{-1} : W \to V$ exists. The invertibility of T implies that T is bijective: T is injective (if $T(x_1) = T(x_2)$, then applying T^{-1} gives $x_1 = T^{-1}(T(x_1)) = T^{-1}(T(x_2)) = x_2$), and T is surjective (for any $u \in W$, we have $T(T^{-1}(u)) = u$). Therefore, there exist unique $x, y \in V$ such that $T(x) = u$ and $T(y) = v$. By definition of inverse, $T^{-1}(u) = x$ and $T^{-1}(v) = y$.

As T is linear, $T(x + y) = T(x) + T(y)$.

Substituting $T(x) = u$ and $T(y) = v$:

$$T(x + y) = u + v$$

Applying T^{-1} to both sides:

$$T^{-1}(T(x + y)) = T^{-1}(u + v)$$

By the property of inverses ($T^{-1}T = I_V$),

$$T^{-1}(T(x + y)) = (T^{-1}T)(x + y) = I_V(x + y) = x + y$$

Therefore,

$$T^{-1}(u + v) = x + y = T^{-1}(u) + T^{-1}(v)$$

As T is linear,

$$T(cx) = c \cdot T(x).$$

Substitute $T(x) = u$:

$$T(cx) = c \cdot u.$$

Applying T^{-1}:

$$T^{-1}(c \cdot u) = T^{-1}(T(cx)).$$

By the property of inverses,

$$T^{-1}(T(cx)) = (T^{-1}T)(cx) = cx.$$

Therefore,

$$T^{-1}(c \cdot u) = c \cdot x = c \cdot T^{-1}(u).$$

As T^{-1} preserves both addition and scalar multiplication:

$$T^{-1}(u + v) = T^{-1}(u) + T^{-1}(v), \qquad T^{-1}(cu) = c\,T^{-1}(u).$$

Therefore, T^{-1} is linear.