1.

a.

Given

$$V = \text{span}\{1, x, x^2\}$$

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

$$T(f) = f(0)$$

We have to find $g(x) = a + bx + cx^2 \in V$ such that $T(f) = \langle f, g \rangle$ for all $f \in V$

We need to find $g \in V$ such that:

$$T(f) = \langle f, g \rangle$$
$$\Rightarrow f(0) = \int_{-1}^{1} f(x)g(x)\,dx \quad \text{for all } f \in V$$

The orthogonal basis for V is:

$$w_1 = 1, \quad w_2 = x, \quad w_3 = x^2 - \frac{1}{3}$$

with norms:

$$\|w_1\|^2 = 2, \quad \|w_2\|^2 = \frac{2}{3}, \quad \|w_3\|^2 = \frac{8}{45}$$

The corresponding orthonormal basis is:

$$e_1 = \frac{1}{\sqrt{2}}, \quad e_2 = \sqrt{\frac{3}{2}}\,x, \quad e_3 = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

If $\{e_1, e_2, e_3\}$ is an orthonormal basis of V, then the unique element $g \in V$ satisfying $T(f) = \langle f, g \rangle$ for all $f \in V$ is given by:

$$g = T(e_1)e_1 + T(e_2)e_2 + T(e_3)e_3$$

Since $T(f) = f(0)$ so $T(e_k) = e_k(0)$

$$T(e_1) = e_1(0) = \frac{1}{\sqrt{2}}$$

$$T(e_2) = e_2(0) = \sqrt{\frac{3}{2}} \cdot 0 = 0$$

$$T(e_3) = e_3(0) = \sqrt{\frac{45}{8}}\left(0^2 - \frac{1}{3}\right) = -\frac{1}{3}\sqrt{\frac{45}{8}}$$

$$\begin{aligned}
g &= T(e_1)e_1 + T(e_2)e_2 + T(e_3)e_3 \\
&= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 0 - \frac{1}{3}\sqrt{\frac{45}{8}} \cdot \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \\
&= \frac{1}{2} - \frac{45}{24}\left(x^2 - \frac{1}{3}\right) \\
&= \frac{1}{2} - \frac{15}{8}x^2 + \frac{5}{8} \\
&= \frac{9}{8} - \frac{15}{8}x^2
\end{aligned}$$

$$\therefore g(x) = \frac{9}{8} - \frac{15}{8}x^2$$

We verify that $T(f) = \langle f, g\rangle$ for $f \in \{1, x, x^2\}$.

For $f = 1$:

$$\langle 1, g\rangle = \int_{-1}^{1} 1 \cdot \left(\frac{9}{8} - \frac{15}{8}x^2\right) dx = \frac{9}{8} \cdot 2 - \frac{15}{8} \cdot \frac{2}{3} = \frac{9}{4} - \frac{5}{4} = 1 = T(1)$$

For $f = x$:

$$\langle x, g\rangle = \int_{-1}^{1} x \cdot \left(\frac{9}{8} - \frac{15}{8}x^2\right) dx = \frac{9}{8}\int_{-1}^{1} x\, dx - \frac{15}{8}\int_{-1}^{1} x^3\, dx = 0 = T(x)$$

For $f = x^2$:

$$\langle x^2, g\rangle = \int_{-1}^{1} x^2 \cdot \left(\frac{9}{8} - \frac{15}{8}x^2\right) dx = \frac{9}{8} \cdot \frac{2}{3} - \frac{15}{8} \cdot \frac{2}{5} = \frac{3}{4} - \frac{3}{4} = 0 = T(x^2)$$

Checking for any $f \in V$ has the form $f(x) = a + bx + cx^2$. By linearity of the inner product:

$$\begin{aligned}
\langle f, g\rangle &= \int_{-1}^{1} (a + bx + cx^2) g(x)\, dx \\
&= a\int_{-1}^{1} g(x)\, dx + b\int_{-1}^{1} x g(x)\, dx + c\int_{-1}^{1} x^2 g(x)\, dx \\
&= a \cdot 1 + b \cdot 0 + c \cdot 0 = a = f(0) = T(f)
\end{aligned}$$

b.

For $f(x) = x^2$ and $g(x) = \frac{9}{8} - \frac{15}{8}x^2$:

$$f(0) = 0$$

$$\begin{aligned}
\langle f, g\rangle &= \int_{-1}^{1} x^2\left(\frac{9}{8} - \frac{15}{8}x^2\right) dx \\
&= \frac{9}{8}\int_{-1}^{1} x^2\, dx - \frac{15}{8}\int_{-1}^{1} x^4\, dx \\
&= \frac{9}{8} \cdot \frac{2}{3} - \frac{15}{8} \cdot \frac{2}{5} \\
&= \frac{3}{4} - \frac{3}{4} \\
&= 0
\end{aligned}$$

$$\therefore T(f) = f(0) = \langle f, g\rangle$$

2.

a.

$V = \mathbb{C}^{n\times n}$ and $I : V \to V$ is the identity map on V

For any linear map $T : V \to W$ between inner product spaces, the adjoint $T^\dagger : W \to V$ is defined such that

$$\langle T(v), w\rangle_W = \langle v, T^\dagger(w)\rangle_V \quad \text{for all } v \in V, w \in W$$

For the identity map $I : V \to V$, the adjoint property gives

$$\langle I(v), w\rangle = \langle v, I^\dagger(w)\rangle \quad \text{for all } v, w \in V$$

Since I is the identity map, $I(v) = v$, so

$$\langle v, w\rangle = \langle v, I^\dagger(w)\rangle \quad \text{for all } v, w \in V$$

By the uniqueness of the adjoint, we have $I^\dagger(w) = w$ for all $w \in V$. Since this is in the same form as $I(v) = v$ for all $v \in V$, we conclude that

$$I^\dagger = I$$

b.

$P \in V$ is an invertible matrix.

$$\begin{aligned}
PP^{-1} &= I \\
\Rightarrow (PP^{-1})^\dagger &= I^\dagger \quad \text{(taking adjoint of both sides)} \\
\Rightarrow (P^{-1})^\dagger P^\dagger &= I \quad \text{(using } (ST)^\dagger = T^\dagger S^\dagger \text{ and } I^\dagger = I\text{)} \\
\Rightarrow (P^{-1})^\dagger P^\dagger (P^\dagger)^{-1} &= I(P^\dagger)^{-1} \quad \text{(multiplying both sides by } (P^\dagger)^{-1}\text{)} \\
\Rightarrow (P^{-1})^\dagger I &= (P^\dagger)^{-1} \quad \text{(since } P^\dagger(P^\dagger)^{-1} = I\text{)} \\
\Rightarrow (P^{-1})^\dagger &= (P^\dagger)^{-1}
\end{aligned}$$

c.

$T : V \to V$ is defined by $T(A) = P^{-1}AP$

$\langle A, B\rangle = \mathrm{tr}(AB^\dagger)$

By the definition of the adjoint:

$$\langle T(B), A\rangle = \langle B, T^\dagger(A)\rangle$$

We have to prove:

$$T^\dagger(A) = (P^\dagger)^{-1}AP^\dagger$$

Therefore we need to verify that for all $B \in V$:

$$\langle T(B), A\rangle = \langle B, (P^\dagger)^{-1}AP^\dagger\rangle$$

$$\begin{aligned}
\langle T(B), A\rangle &= \langle P^{-1}BP, A\rangle \quad \text{(as } T(B) = P^{-1}BP\text{)} \\
&= \mathrm{tr}\big((P^{-1}BP)A^\dagger\big) \\
&= \mathrm{tr}\big(A^\dagger(P^{-1}BP)\big) \quad \text{(using } \mathrm{tr}(XY) = \mathrm{tr}(YX)\text{)} \\
&= \mathrm{tr}\big(B(PA^\dagger P^{-1})\big) \\
&= \mathrm{tr}\Big(B\big((P^\dagger)^{-1}AP^\dagger\big)^\dagger\Big) \quad \text{(since } PA^\dagger P^{-1} = \big((P^\dagger)^{-1}AP^\dagger\big)^\dagger \text{ using } T = (T^\dagger)^\dagger\text{)} \\
&= \langle B, (P^\dagger)^{-1}AP^\dagger\rangle
\end{aligned}$$

Since this equality holds for all $B \in V$, by the definition of the adjoint operator it can be concluded:

$$T^\dagger(A) = (P^\dagger)^{-1}AP^\dagger$$

3.

$$
\begin{aligned}
L(\theta) &= \|X\theta - Y\|^2 \\
&= (X\theta - Y)^\top (X\theta - Y) \\
&= \theta^\top X^\top X\theta - 2(X^\top Y)^\top \theta + Y^\top Y
\end{aligned}
$$

$$
\begin{aligned}
&\nabla_\theta(b^\top \theta) = b \\
&\nabla_\theta(\theta^\top A\theta) = (A^\top + A)\theta
\end{aligned}
$$

If A is symmetric, i.e., $A^\top = A$, then

$$\nabla_\theta(\theta^\top A\theta) = 2A\theta$$

$$
\begin{aligned}
&\nabla_\theta L(\theta) \\
&= \nabla_\theta(\theta^\top X^\top X\theta - 2(X^\top Y)^\top \theta + Y^\top Y) \\
&= \nabla_\theta(\theta^\top X^\top X\theta) + \nabla_\theta(-2(X^\top Y)^\top \theta) + \nabla_\theta(Y^\top Y) \\
&= 2X^\top X\theta - 2X^\top Y + 0 = 2X^\top X\theta - 2X^\top Y
\end{aligned}
$$

$$
\begin{aligned}
H(L) &= \nabla_\theta^2 L \\
&= \frac{\partial}{\partial \theta}(2X^\top X\theta - 2X^\top Y) \\
&= 2X^\top X
\end{aligned}
$$

4.

Fourier Series Approximation of $f(x) = x^2$ up to $k = 2$

The Fourier series is given by:

$$f(x) = \frac{1}{\sqrt{2}}a_0 + \sum_{k=1}^{\infty} a_k \cos(k\pi x) + \sum_{k=1}^{\infty} b_k \sin(k\pi x)$$

The Fourier coefficients are:

$$
\begin{aligned}
a_0 &= \frac{1}{\sqrt{2}} \int_{-1}^{1} f(x)\, dx \\
a_k &= \int_{-1}^{1} f(x) \cos(k\pi x)\, dx \\
b_k &= \int_{-1}^{1} f(x) \sin(k\pi x)\, dx
\end{aligned}
$$

For the approximation up to $k = 2$, we compute a_0, a_1, a_2, b_1, b_2.
For $f(x) = x^2$:

$$a_0 = \frac{1}{\sqrt{2}} \int_{-1}^{1} x^2\, dx = \frac{1}{\sqrt{2}} \cdot \frac{2}{3} = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}$$

$$
\begin{aligned}
a_1 &= \int_{-1}^{1} x^2 \cos(\pi x)\, dx \\
&= \left[\frac{x^2 \sin(\pi x)}{\pi}\right]_{-1}^{1} + \int_{-1}^{1} \frac{2x \sin(\pi x)}{\pi}\, dx \quad \text{(integration by parts with } u = x^2\text{)} \\
&= 0 + \frac{2}{\pi}\left[\frac{-x\cos(\pi x)}{\pi}\right]_{-1}^{1} - \frac{2}{\pi^2}\int_{-1}^{1} \cos(\pi x)\, dx \\
&= \frac{2}{\pi}\left(-\frac{\cos(\pi)}{\pi} + \frac{-\cos(-\pi)}{\pi}\right) - 0 \\
&= \frac{2}{\pi}\left(\frac{1}{\pi} - \frac{1}{\pi}\right) = -\frac{4}{\pi^2}
\end{aligned}
$$

$$
\begin{aligned}
a_2 &= \int_{-1}^{1} x^2 \cos(2\pi x)\, dx \\
&= \left[\frac{x^2 \sin(2\pi x)}{2\pi}\right]_{-1}^{1} + \int_{-1}^{1} \frac{2x \sin(2\pi x)}{2\pi}\, dx \quad \text{(integration by parts with } u = x^2\text{)} \\
&= 0 + \frac{1}{\pi}\left[\frac{-x\cos(2\pi x)}{2\pi}\right]_{-1}^{1} - \frac{1}{2\pi^2}\int_{-1}^{1} \cos(2\pi x)\, dx \\
&= \frac{1}{2\pi^2}\left[-x\cos(2\pi x)\right]_{-1}^{1} - 0 \\
&= \frac{1}{2\pi^2}\left[-1\cdot\cos(2\pi) - (-1)\cdot\cos(-2\pi)\right] \\
&= \frac{1}{2\pi^2}\left[-1 - (-1)\right] = \frac{1}{2\pi^2}\cdot 2 = \frac{1}{\pi^2}
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= \int_{-1}^{1} x^2 \sin(\pi x)\, dx \\
&= \left[-\frac{x^2 \cos(\pi x)}{\pi}\right]_{-1}^{1} + \int_{-1}^{1} \frac{2x \cos(\pi x)}{\pi}\, dx \quad \text{(integration by parts)} \\
&= \left(-\frac{\cos(\pi)}{\pi} - \frac{\cos(-\pi)}{\pi}\right) + \frac{2}{\pi}\left[\frac{x\sin(\pi x)}{\pi} + \frac{\cos(\pi x)}{\pi^2}\right]_{-1}^{1} \\
&= \left(\frac{1}{\pi} + \frac{1}{\pi}\right) + \frac{2}{\pi}(0 - 0) = 0
\end{aligned}
$$

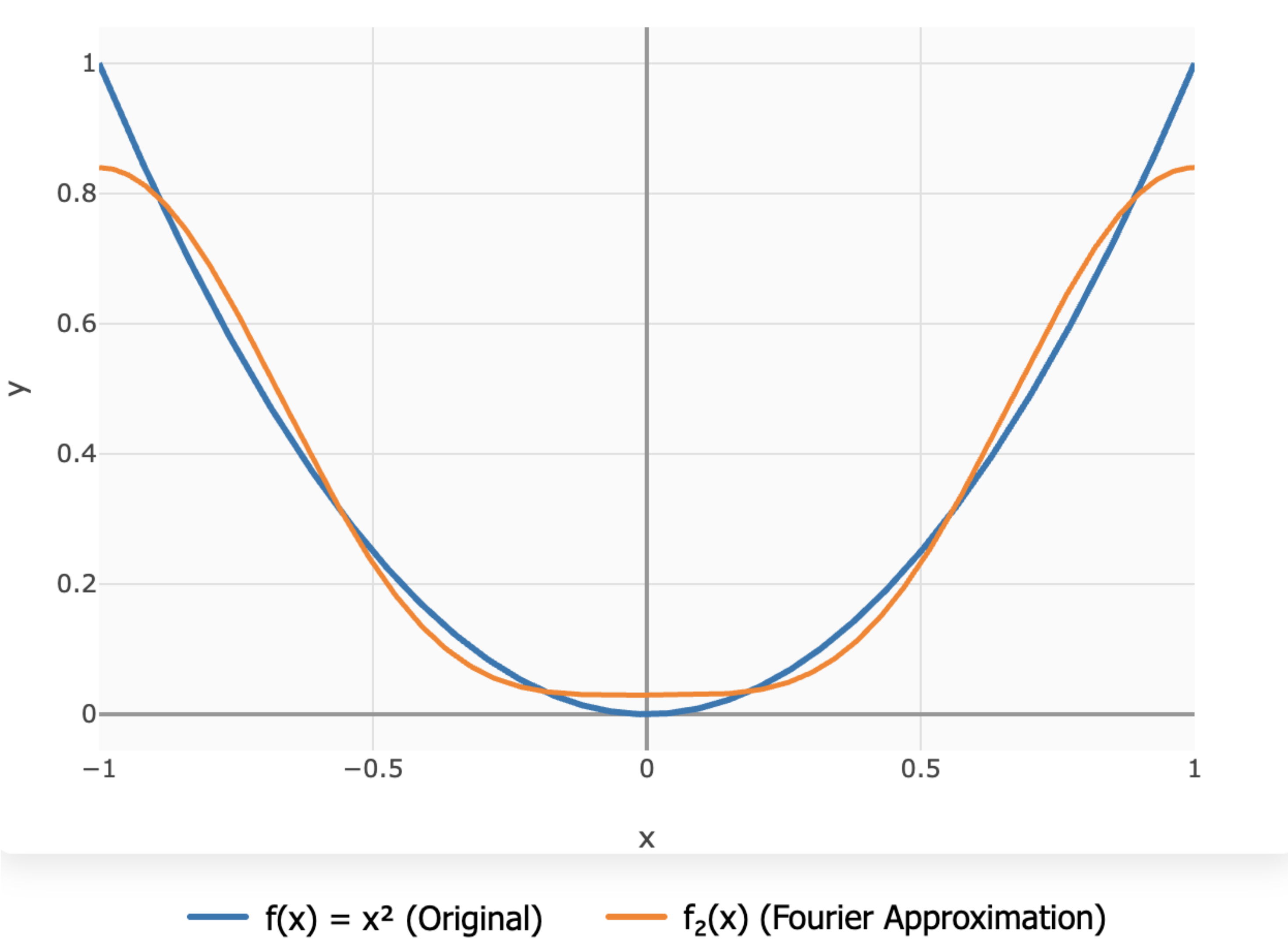
$$
b_2 = \int_{-1}^{1} x^2 \sin(2\pi x)\, dx = 0 \quad \text{(since } x^2 \text{ is even and } \sin(2\pi x) \text{ is odd)}
$$

The best approximation up to $k = 2$ is:

$$
\begin{aligned}
f_2(x) &= \frac{1}{\sqrt{2}}a_0 + a_1\cos(\pi x) + a_2\cos(2\pi x) \\
&= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{3} - \frac{4}{\pi^2}\cos(\pi x) + \frac{1}{\pi^2}\cos(2\pi x) \\
&= \frac{1}{3} - \frac{4}{\pi^2}\cos(\pi x) + \frac{1}{\pi^2}\cos(2\pi x)
\end{aligned}
$$

Therefore:

$$
f_2(x) = \frac{1}{3} - \frac{4}{\pi^2}\cos(\pi x) + \frac{1}{\pi^2}\cos(2\pi x)
$$



5.

As we need to find the best approximation in the form $\begin{bmatrix} a & a \\ 0 & a \end{bmatrix}$, we define the basis matrix:

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Since the approximation has a single parameter a^*, we use the normal equation to solve:

$$a^* = \frac{\langle Y, M \rangle}{\langle M, M \rangle}$$

where $Y = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ is the matrix to be approximated.

$$\begin{aligned}
\langle Y, M\rangle &= \operatorname{tr}(YM^\top) \\
&= \operatorname{tr}\left(\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\right) \\
&= \operatorname{tr}\left(\begin{bmatrix}3 & 2\\ 7 & 3\end{bmatrix}\right) \\
&= 3 + 3 = 6
\end{aligned}$$

$$\begin{aligned}
\langle M, M\rangle &= \operatorname{tr}(MM^\top) \\
&= \operatorname{tr}\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\right) \\
&= \operatorname{tr}\left(\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\right) \\
&= 2 + 1 = 3
\end{aligned}$$

$$\begin{aligned}
a^* &= \frac{\langle Y, M\rangle}{\langle M, M\rangle} \\
&= \frac{6}{3} = 2
\end{aligned}$$

The best approximation of $Y = \begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$ in the given form is:

$$a^* M = 2\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}2 & 2\\ 0 & 2\end{bmatrix}$$