1.

The most basic inner product space is the Euclidean space $(\mathbb{C}^n, \cdot)$ with the dot (scalar) product defined as $\langle x, y\rangle = x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n}$

For $\mathbb{C}^3$: $\langle u, v\rangle = u_1\overline{v_1} + u_2\overline{v_2} + u_3\overline{v_3}$

Let $v_1 = (1,\,0,\,i), \quad v_2 = (1,\,1,\,2+i)$

For $w_1 = v_1 = (1,\,0,\,i)$,

$$\|w_1\|^2 = \langle w_1, w_1\rangle = \langle (1,\,0,\,i), (1,\,0,\,i)\rangle = 1\cdot\overline{1} + 0\cdot\overline{0} + i\cdot\overline{i} = 1 + 0 + i(-i) = 1 + 0 + 1 = 2$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2}\, w_1$$

$$\langle v_2, w_1\rangle = \langle (1,\,1,\,2+i), (1,\,0,\,i)\rangle = 1\cdot\overline{1} + 1\cdot\overline{0} + (2+i)\overline{i} = 1 + 0 + (2+i)(-i) = 1 - 2i - i^2 = 1 - 2i + 1 = 2 - 2i$$

$$\begin{aligned}
w_2 &= (1,\,1,\,2+i) - \frac{2-2i}{2}\,(1,\,0,\,i)\\
&= (1,\,1,\,2+i) - (1-i)\,(1,\,0,\,i)\\
&= (1,\,1,\,2+i) - \big((1-i)\cdot 1,\ (1-i)\cdot 0,\ (1-i)\cdot i\big)\\
&= (1,\,1,\,2+i) - (1-i,\,0,\,i-i^2)\\
&= (1,\,1,\,2+i) - (1-i,\,0,\,i+1)\\
&= \big(1-(1-i),\ 1-0,\ (2+i)-(i+1)\big)\\
&= (i,\,1,\,1)
\end{aligned}$$

$$\|w_2\|^2 = \langle w_2, w_2\rangle = \langle (i,\,1,\,1), (i,\,1,\,1)\rangle = i\cdot\overline{i} + 1\cdot\overline{1} + 1\cdot\overline{1} = (i)(-i) + 1 + 1 = 1 + 1 + 1 = 3$$

$$e_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}}(1,\,0,\,i)$$

$$e_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{3}}(i,\,1,\,1)$$

$\therefore$ An orthonormal basis for the subspace of $\mathbb{C}^3$ is $\left\{\frac{1}{\sqrt{2}}(1,\,0,\,i),\ \frac{1}{\sqrt{3}}(i,\,1,\,1)\right\}$

2.

a.

$$\langle v, w\rangle = \langle v, w\rangle_1 + \langle v, w\rangle_2$$

Let $f : V \times V \to \mathbb{C}$, for all $u, v, w \in V$ and $c \in \mathbb{C}$.

$$\begin{aligned}
\langle u+v, w\rangle &= \langle u+v, w\rangle_1 + \langle u+v, w\rangle_2\\
&= (\langle u, w\rangle_1 + \langle v, w\rangle_1) + (\langle u, w\rangle_2 + \langle v, w\rangle_2)\\
&= (\langle u, w\rangle_1 + \langle u, w\rangle_2) + (\langle v, w\rangle_1 + \langle v, w\rangle_2)\\
&= \langle u, w\rangle + \langle v, w\rangle
\end{aligned}$$

$$
\begin{aligned}
\langle cu, w\rangle &= \langle cu, w\rangle_1 + \langle cu, w\rangle_2 \\
&= c\langle u, w\rangle_1 + c\langle u, w\rangle_2 \\
&= c(\langle u, w\rangle_1 + \langle u, w\rangle_2) \\
&= c\langle u, w\rangle
\end{aligned}
$$

$$
\begin{aligned}
\overline{\langle w, v\rangle} &= \overline{\langle w, v\rangle_1 + \langle w, v\rangle_2} \\
&= \overline{\langle w, v\rangle_1} + \overline{\langle w, v\rangle_2} \\
&= \langle v, w\rangle_1 + \langle v, w\rangle_2 \\
&= \langle v, w\rangle.
\end{aligned}
$$

For any $v \in V$, $\langle v, v\rangle = \langle v, v\rangle_1 + \langle v, v\rangle_2$

Since each $\langle \cdot, \cdot\rangle_i$ is an inner product, we know that

$$\langle v, v\rangle_i \geq 0 \quad \text{for all } v \in V, \quad \text{and} \quad \langle v, v\rangle_i = 0 \iff v = 0$$

If $\langle v, v\rangle = 0$ then both $\langle v, v\rangle_1 = 0$ and $\langle v, v\rangle_2 = 0$

$\langle v, v\rangle_i = 0 \Rightarrow v = 0$. Hence, $\langle v, v\rangle = 0 \Rightarrow v = 0$. Equivalently $\langle v, v\rangle > 0$ for $v \neq 0$.

For $v \in V$, each $\langle v, v\rangle_i \geq 0$, so $\langle v, v\rangle = \langle v, v\rangle_1 + \langle v, v\rangle_2 \geq 0$.

$\therefore$ $\langle v, w\rangle$ satisfies linearity, conjugate symmetry, and positive definiteness and hence it is an inner product on V.

b.

$$\langle v, w\rangle = \langle v, w\rangle_1 - \langle v, w\rangle_2$$

Let $f : V \times V \to \mathbb{C}$, for all $u, v, w \in V$ and $c \in \mathbb{C}$.

$$
\begin{aligned}
\langle u + v, w\rangle &= \langle u + v, w\rangle_1 - \langle u + v, w\rangle_2 \\
&= (\langle u, w\rangle_1 + \langle v, w\rangle_1) - (\langle u, w\rangle_2 + \langle v, w\rangle_2) \\
&= (\langle u, w\rangle_1 - \langle u, w\rangle_2) + (\langle v, w\rangle_1 - \langle v, w\rangle_2) \\
&= \langle u, w\rangle + \langle v, w\rangle.
\end{aligned}
$$

$$
\begin{aligned}
\langle cu, w\rangle &= \langle cu, w\rangle_1 - \langle cu, w\rangle_2 \\
&= c\langle u, w\rangle_1 - c\langle u, w\rangle_2 \\
&= c(\langle u, w\rangle_1 - \langle u, w\rangle_2) \\
&= c\langle u, w\rangle
\end{aligned}
$$

$$
\begin{aligned}
\overline{\langle w, v\rangle} &= \overline{\langle w, v\rangle_1 - \langle w, v\rangle_2} \\
&= \overline{\langle w, v\rangle_1} - \overline{\langle w, v\rangle_2} \\
&= \langle v, w\rangle_1 - \langle v, w\rangle_2 \\
&= \langle v, w\rangle
\end{aligned}
$$

Thus, linearity and conjugate symmetry hold.

For any $v \in V$, $\langle v, v\rangle = \langle v, v\rangle_1 - \langle v, v\rangle_2$

Suppose $\langle \cdot, \cdot \rangle_2 = \langle \cdot, \cdot \rangle_1$. Then, for any $v \neq 0$, if $\langle v, v \rangle_2 = \langle v, v \rangle_1$, we have

$$\langle v, v \rangle = \langle v, v \rangle_1 - \langle v, v \rangle_1$$

If $\langle v, v \rangle = 0$ then $\langle v, v \rangle_1 > 0$ for $v \neq 0$, the definiteness property ($\langle v, v \rangle = 0 \Rightarrow v = 0$) fails.

Suppose $\langle \cdot, \cdot \rangle_2 = 2\langle \cdot, \cdot \rangle_1$. Then, for any $v \neq 0$, if $\langle v, v \rangle_2 = 2\langle v, v \rangle_1$, we have

$$\langle v, v \rangle = \langle v, v \rangle_1 - 2\langle v, v \rangle_1 = -\langle v, v \rangle_1$$

As $\langle v, v \rangle < 0$, the nonnegativity property fails.

$\therefore$ $\langle v, w \rangle$ fails the definiteness and positivity conditions, and hence is not an inner product on V.

3.

Cauchy–Schwarz Inequality $\quad |\langle x, y \rangle| \leq \|x\| \, \|y\|$

$f(x) = x, \qquad g(x) = x^2, \qquad h(x) = x^3$

$$\begin{aligned} \langle f, g \rangle &= \int_0^1 f(x)\, g(x)\, dx \\ &= \int_0^1 x(x^2)\, dx \\ &= \int_0^1 (x^3)\, dx = \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \langle f, f \rangle &= \int_0^1 f(x)\, f(x)\, dx \\ &= \int_0^1 x(x)\, dx \\ &= \int_0^1 (x^2)\, dx = \frac{1}{3} \end{aligned}$$

$$\|f\| = \sqrt{\frac{1}{3}}$$

$$\begin{aligned} \langle g, g \rangle &= \int_0^1 g(x)\, g(x)\, dx \\ &= \int_0^1 (x^2)(x^2)\, dx \\ &= \int_0^1 (x^4)\, dx = \frac{1}{5} \end{aligned}$$

$$\|g\| = \sqrt{\frac{1}{5}}$$

$$|\langle f, g \rangle| = \frac{1}{4} = 0.25 \leq \|f\| \, \|g\| = \sqrt{\frac{1}{3}}\sqrt{\frac{1}{5}} = \sqrt{\frac{1}{15}} \approx 0.258$$

$\therefore$ Cauchy–Schwarz Inequality holds for f and g

$$\|f + g\|^2 = \langle f + g, f + g \rangle = \|f\|^2 + 2\langle f, g \rangle + \|g\|^2 = \frac{1}{3} + 2 \cdot \frac{1}{4} + \frac{1}{5} = \frac{31}{30} = \frac{15.5}{15}$$

$$(\|f\| + \|g\|)^2 = \left(\sqrt{\tfrac{1}{3}} + \sqrt{\tfrac{1}{5}}\right)^2 = \frac{1}{3} + \frac{1}{5} + 2\sqrt{\frac{1}{15}} = \frac{8}{15} + \frac{2}{\sqrt{15}} = \frac{8 + 2\sqrt{15}}{15} = \frac{15.745}{15}$$

As $15.5 < 15.745$ we have $\|f + g\|^2 < (\|f\| + \|g\|)^2$

$\therefore$ $\|f + g\| < \|f\| + \|g\|$, so the triangle inequality holds for f and g

4.

a.

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 4 \\ 0 \\ 5 \\ -2 \end{bmatrix}$$

For $w_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix}$,

$$\|w_1\|^2 = v_1 \cdot v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} = 1 + 0 + 4 + 4 = 9$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$$

$$\langle v_2, w_1 \rangle = v_2 \cdot w_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} = 0 + 0 + 2 - 2 = 0$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 = v_2 - 0 = v_2$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2$$

$$\langle v_3, w_1 \rangle = v_3 \cdot w_1 = \begin{bmatrix} 4 \\ 0 \\ 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} = 4 + 0 + 10 + 4 = 18$$

$$\langle v_3, w_2 \rangle = v_3 \cdot w_2 = \begin{bmatrix} 4 \\ 0 \\ 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0 + 0 + 5 - 2 = 3$$

$$\|w_2\|^2 = v_2 \cdot v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 3$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2 = v_3 - 2w_1 - 1w_2 = \begin{bmatrix} 4 \\ 0 \\ 5 \\ -2 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\|w_3\|^2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} = 6$$

$$e_1 = \frac{w_1}{\|w_1\|} = \frac{1}{3}\begin{bmatrix}1\\0\\2\\-2\end{bmatrix}, \qquad e_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{3}}\begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \qquad e_3 = \frac{w_3}{\|w_3\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}2\\-1\\0\\1\end{bmatrix}$$

$\therefore$ An orthonormal basis of $V = \operatorname{span}(v_1, v_2, v_3)$ is $\left\{\frac{1}{3}(1,0,2,-2),\ \frac{1}{\sqrt{3}}(0,1,1,1),\ \frac{1}{\sqrt{6}}(2,-1,0,1)\right\}$

b.

The orthogonal complement $W^\perp$ of $W = \operatorname{span}\{v_1, v_2\} \subset \mathbb{R}^4$ is the null space of the matrix formed by the basis vectors of W being $v_1 = \begin{bmatrix}1\\0\\2\\-2\end{bmatrix}$ and $v_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$

$$W^\perp = \operatorname{Null}(v_1, v_2)$$

$$W^\perp = \operatorname{Null}\begin{bmatrix}1 & 0 & 2 & -2\\0 & 1 & 1 & 1\end{bmatrix}$$

Let $x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$

Then, the condition for orthogonality is

$$v_1 \cdot x = 0 \quad \text{and} \quad v_2 \cdot x = 0$$

From $v_1 \cdot x = 0$:

$$x_1 + 2x_3 - 2x_4 = 0 \Rightarrow x_1 = -2x_3 + 2x_4$$

From $v_2 \cdot x = 0$:

$$x_2 + x_3 + x_4 = 0 \Rightarrow x_2 = -x_3 - x_4$$

Let $x_3 = s$ and $x_4 = t$. Then

$$x = \begin{bmatrix}-2s+2t\\-s-t\\s\\t\end{bmatrix} = s\begin{bmatrix}-2\\-1\\1\\0\end{bmatrix} + t\begin{bmatrix}2\\-1\\0\\1\end{bmatrix}$$

$$\therefore \quad W^\perp = \left\{\begin{bmatrix}-2\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\0\\1\end{bmatrix}\right\}$$

c.

$$y = y_1 + y_2 \implies y_2 = y - y_1 \in V^\perp,$$

where $y_1 = \operatorname{proj}_V(y) \in V$ and $y_2 \perp V$

If e_1, e_2, e_3 form an orthonormal basis of a subspace W of space $V = \operatorname{span}(v_1, v_2, v_3)$, then the orthogonal projection of $y \in V$ onto W is:

$$\operatorname{proj}_W(y) = \langle y, e_1\rangle e_1 + \langle y, e_2\rangle e_2 + \langle y, e_3\rangle e_3$$

In this case, where $W = V = \text{span}(v_1, v_2, v_3)$,

$$\text{proj}_V(y) = \langle y, e_1\rangle e_1 + \langle y, e_2\rangle e_2 + \langle y, e_3\rangle e_3$$

$$\langle y, e_1\rangle = (9,9,9,9)\cdot\left(\tfrac{1}{3},\ 0,\ \tfrac{2}{3},\ -\tfrac{2}{3}\right) = 9\cdot\tfrac{1}{3} + 9\cdot 0 + 9\cdot\tfrac{2}{3} + 9\cdot\left(-\tfrac{2}{3}\right) = 3 + 0 + 6 - 6 = 3$$

$$\langle y, e_1\rangle e_1 = 3\cdot\frac{1}{3}(1,0,2,-2) = (1,0,2,-2)$$

$$\langle y, e_2\rangle = (9,9,9,9)\cdot\left(0,\ \tfrac{1}{\sqrt{3}},\ \tfrac{1}{\sqrt{3}},\ \tfrac{1}{\sqrt{3}}\right) = 0 + \frac{9}{\sqrt{3}} + \frac{9}{\sqrt{3}} + \frac{9}{\sqrt{3}} = \frac{27}{\sqrt{3}} = 9\sqrt{3}$$

$$\langle y, e_2\rangle e_2 = 9\sqrt{3}\cdot\frac{1}{\sqrt{3}}(0,1,1,1) = 9\,(0,1,1,1) = (0,9,9,9)$$

$$\langle y, e_3\rangle = (9,9,9,9)\cdot\left(\tfrac{2}{\sqrt{6}},\ -\tfrac{1}{\sqrt{6}},\ 0,\ \tfrac{1}{\sqrt{6}}\right) = \frac{18 - 9 + 0 + 9}{\sqrt{6}} = \frac{18}{\sqrt{6}} = 3\sqrt{6}$$

$$\langle y, e_3\rangle e_3 = 3\sqrt{6}\cdot\frac{1}{\sqrt{6}}(2,-1,0,1) = 3\,(2,-1,0,1) = (6,-3,0,3)$$

$$\text{proj}_V(y) = \langle y, e_1\rangle e_1 + \langle y, e_2\rangle e_2 + \langle y, e_3\rangle e_3 = (1,0,2,-2) + (0,9,9,9) + (6,-3,0,3) = (7,6,11,10)$$

$$\therefore\quad y_1 = \text{proj}_V(y) = (7,6,11,10),\qquad y_2 = y - y_1 = (2,3,-2,-1)$$

The orthogonal complement $V^\perp$ of $V = \text{span}\{v_1, v_2, v_3\} \subset \mathbb{R}^4$ is the null space of the matrix formed by the basis vectors:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix},\quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},\quad v_3 = \begin{bmatrix} 4 \\ 0 \\ 5 \\ -2 \end{bmatrix}$$

Therefore, the condition for orthogonality is:

$$v_i \cdot y_2 = 0 \quad \text{for all } i = 1,2,3.$$

$$y_2 \cdot v_1 = (2,3,-2,-1)\cdot(1,0,2,-2) = 2 + 0 - 4 + 2 = 0,$$

$$y_2 \cdot v_2 = (2,3,-2,-1)\cdot(0,1,1,1) = 0 + 3 - 2 - 1 = 0,$$

$$y_2 \cdot v_3 = (2,3,-2,-1)\cdot(4,0,5,-2) = 8 + 0 - 10 + 2 = 0.$$

$$\therefore\quad y_2 \perp v_1,\ y_2 \perp v_2,\ y_2 \perp v_3,\quad \text{and hence } y_2 \in V^\perp$$

5.

$$\langle A, B\rangle = \text{tr}(AB^\dagger) = \text{tr}(A\overline{B}^T)$$

a.

Let $v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

For $w_1 = v_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\|w_1\|^2 = \langle w_1, w_1\rangle = \text{tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^\dagger\right)$$

$$v_1 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \quad v_1^T = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \qquad v_1^\dagger = \overline{v_1^T} = \overline{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}^T = \begin{bmatrix}\overline{1} & \overline{0}\\\overline{0} & \overline{1}\end{bmatrix}^T = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\|w_1\|^2 = \langle w_1, w_1\rangle = \operatorname{tr}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}^\dagger\right) = \operatorname{tr}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = 1 + 1 = 2$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2}\, w_1$$

$$\langle v_2, w_1\rangle = \operatorname{tr}\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}^\dagger\right) = \operatorname{tr}\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) = 1$$

$$\begin{aligned} w_2 &= v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2}\, w_1 \\ &= \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 - \frac{1}{2} & 0\\0 & 0 - \frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & 0\\0 & -\frac{1}{2}\end{bmatrix}\end{aligned}$$

$$\begin{aligned} \langle w_1, w_2\rangle &= \operatorname{tr}\left(w_1\, w_2^\dagger\right) \\ &= \operatorname{tr}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\\0 & -\frac{1}{2}\end{bmatrix}^\dagger\right) = \operatorname{tr}\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\\0 & -\frac{1}{2}\end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix}1\cdot\frac{1}{2} + 0\cdot 0 & 1\cdot 0 + 0\cdot(-\frac{1}{2})\\0\cdot\frac{1}{2} + 1\cdot 0 & 0\cdot 0 + 1\cdot(-\frac{1}{2})\end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix}\frac{1}{2} & 0\\0 & -\frac{1}{2}\end{bmatrix}\right) = 0\end{aligned}$$

$$\therefore\ As\langle w_1, w_2\rangle = 0, \quad \text{so} \ \left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}\frac{1}{2} & 0\\0 & -\frac{1}{2}\end{bmatrix}\right\} \text{ is an orthogonal basis for the subspace } W$$

b.

$$W = \operatorname{span}\{w_1, w_2\}, \quad w_1 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \qquad w_2 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

Let

$$A = \begin{bmatrix}a & b\\c & d\end{bmatrix} \in M_{2\times 2}(\mathbb{C})$$

The orthogonal complement $W^\perp$ is the null space of the linear map

$$L : M_{2\times 2}(\mathbb{C}) \to \mathbb{C}^2, \quad L(A) = \big(\langle A, w_1\rangle, \langle A, w_2\rangle\big) = \big(\operatorname{tr}(Aw_1^\dagger), \operatorname{tr}(Aw_2^\dagger)\big)$$

Since $A \in W^\perp$ if $L(A) = (0, 0)$, we must have

$$\operatorname{tr}(Aw_1^\dagger) = 0 \quad \text{and} \quad \operatorname{tr}(Aw_2^\dagger) = 0$$

$$\langle A, w_1\rangle = \operatorname{tr}\left(A\, w_1^\dagger\right) = \operatorname{tr}\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = a + d = 0$$

$$\langle A, w_2\rangle = \operatorname{tr}\left(A\, w_2^\dagger\right) = \operatorname{tr}\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix}a & 0\\c & 0\end{bmatrix}\right) = a = 0$$

Hence, $a = 0$ and $d = 0$. Therefore,

$$W^\perp = \left\{\begin{bmatrix}0 & b\\c & 0\end{bmatrix} : b, c \in \mathbb{C}\right\} = \operatorname{span}\left\{\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right\}$$

6.

A linear map $H : V \to W$ is Hermitian if $H^\dagger = H$.

Let $S : V \to W \quad \text{and} \quad T : V \to W$ be linear maps between finite-dimensional inner product spaces V and W. Assuming S and T are Hermitian, so that $S^\dagger = S \quad \text{and} \quad T^\dagger = T$.

Assume ST is Hermitian. Then $(ST)^\dagger = ST$

By the properties of the adjoint, $(ST)^\dagger = T^\dagger S^\dagger$.

Since S and T are Hermitian ($S^\dagger = S$, $T^\dagger = T$), it follows that $ST = (ST)^\dagger = T^\dagger S^\dagger = TS$. Hence, S and T commute.

Conversely, assume $ST = TS$. Then $(ST)^\dagger = T^\dagger S^\dagger$.

Using $S^\dagger = S$ and $T^\dagger = T$, we have$(ST)^\dagger = TS = ST$. Thus, ST equals its adjoint and is Hermitian.

$$\therefore \quad ST \text{ is Hermitian if and only if } ST = TS.$$