1.

For a vector

$$v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n,$$

$$\ell_1 \text{ norm:} \quad \|v\|_1 = |v_1| + |v_2| + \cdots + |v_n|.$$

$$\ell_2 \text{ norm:} \quad \|v\|_2 = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

$$\ell_\infty \text{ norm:} \quad \|v\|_\infty = \max_{1 \le i \le n} |v_i|.$$

a.

The square of the ℓ_2 norm is

$$\|v\|_2^2 = v_1^2 + v_2^2 + \cdots + v_n^2 = \sum_{i=1}^{n} |v_i|^2.$$

By definition of the ℓ_∞ norm,

$$\|v\|_\infty = \max_{1 \le i \le n} |v_i|.$$

Therefore, for every component $i = 1, 2, \ldots, n$,

$$|v_i| \le \|v\|_\infty.$$

Multiplying both sides by $|v_i|$ gives

$$|v_i|^2 = |v_i| \cdot |v_i| \le |v_i| \cdot \|v\|_\infty.$$

Summing this inequality over all components $i = 1, 2, \ldots, n$:

$$\begin{aligned}
\|v\|_2^2 &= \sum_{i=1}^{n} |v_i|^2 \\
&\le \sum_{i=1}^{n} |v_i| \cdot \|v\|_\infty \\
&= \|v\|_\infty \sum_{i=1}^{n} |v_i| \\
&= \|v\|_\infty \|v\|_1 \\
&= \|v\|_1 \|v\|_\infty
\end{aligned}$$

Therefore, $\|v\|_2^2 \le \|v\|_1 \|v\|_\infty$.

b.

The square of the ℓ_2 norm is

$$\|v\|_2^2 = \sum_{i=1}^{n} |v_i|^2.$$

By definition of the ℓ_∞ norm,

$$\|v\|_\infty = \max_{1\le i\le n} |v_i|.$$

Therefore, for every component $i = 1, 2, \dots, n$,

$$|v_i| \le \|v\|_\infty.$$

Squaring both sides gives

$$|v_i|^2 \le \|v\|_\infty^2.$$

Summing this inequality over all components $i = 1, 2, \dots, n$:

$$\begin{aligned} \|v\|_2^2 &= \sum_{i=1}^{n} |v_i|^2 \\ &\le \sum_{i=1}^{n} \|v\|_\infty^2 \\ &= n\|v\|_\infty^2 \end{aligned}$$

Therefore, $\|v\|_2^2 \le n\|v\|_\infty^2$.

2.

$$B_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

General Form of the tridiagonal matrix:

$$B_n = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2 \end{bmatrix}$$

For a square matrix $A = [a_{ij}]$, the cofactor of a_{ij} is

$$A_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is the determinant of the submatrix obtained by deleting the ith row and jth column. The determinant expands along the ith row as

$$|A| = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} M_{ij}.$$

As the first row of B_n has only two nonzero entries $a_{11} = 2$ and $a_{12} = -1$:

$$\begin{aligned}\det(B_n) &= (-1)^{1+1}a_{11}M_{11} + (-1)^{1+2}a_{12}M_{12} \\ &= (1)(2)M_{11} + (-1)^3(-1)M_{12} \\ &= 2M_{11} + M_{12}\end{aligned}$$

For the cofactor A_{11}:

$$A_{11} = (-1)^{1+1}M_{11} = M_{11}$$

Deleting the first row and first column from B_n results in the $(n-1) \times (n-1)$ submatrix B_{n-1}. Thus,

$$M_{11} = \det(B_{n-1})$$

For the cofactor A_{12}:

$$A_{12} = (-1)^{1+2}M_{12} = -M_{12}.$$

Deleting the first row and second column from B_n results in a matrix whose first row is $[-1, 0, 0, \ldots, 0]$ and the remaining $(n-2) \times (n-2)$ block is B_{n-2}. Expanding along this first row gives

$$M_{12} = (-1) \cdot \det(B_{n-2}) = -\det(B_{n-2})$$

Therefore:

$$\begin{aligned}\det(B_n) &= 2M_{11} + M_{12} \\ &= 2\det(B_{n-1}) + (-\det(B_{n-2})) \\ &= 2\det(B_{n-1}) - \det(B_{n-2})\end{aligned}$$

The formula relating $\det(B_n)$ to $\det(B_{n-1})$ and $\det(B_{n-2})$ is $\det(B_n) = 2\det(B_{n-1}) - \det(B_{n-2})$.

$$B_1 = [2], \quad \det(B_1) = 2.$$

$$B_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \det(B_2) = 2 \cdot 2 - (-1) \cdot (-1) = 3.$$

$$\det(B_n) = 2\det(B_{n-1}) - \det(B_{n-2})$$

$$\det(B_3) = 2\det(B_{3-1}) - \det(B_{3-2}) = 2\det(B_2) - \det(B_1) = 2 \cdot 3 - 2 = 4$$

$$\det(B_4) = 2\det(B_{4-1}) - \det(B_{4-2}) = 2\det(B_3) - \det(B_2) = 2 \cdot 4 - 3 = 5$$

$$\det(B_5) = 2\det(B_{5-1}) - \det(B_{5-2}) = 2\det(B_4) - \det(B_3) = 2 \cdot 5 - 4 = 6$$

From the computed sequence,

$$\det(B_1) = 2, \quad \det(B_2) = 3, \quad \det(B_3) = 4, \quad \det(B_4) = 5, \ \ldots$$

it follows that

$$\det(B_n) = n + 1.$$

$$2\det(B_{n-1}) - \det(B_{n-2}) = 2\big((n-1)+1\big) - \big((n-2)+1\big) = 2n - (n-1) = n+1,$$

so the pattern holds for all n.

3.

The derivative operator

$$T : P_2(\mathbb{R}) \to P_2(\mathbb{R}), \quad T(p) = p',$$

As $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$,

$$P_2(\mathbb{R}) = \text{span}\{1, x, x^2\}.$$

Applying T to each basis element:

$$T(1) = 0, \quad T(x) = 1, \quad T(x^2) = 2x.$$

$$T(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

First column: $[0, 0, 0]^T$

$$T(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

Second column: $[1, 0, 0]^T$

$$T(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$$

Third column: $[0, 2, 0]^T$

Therefore, the matrix of T in the standard basis is

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\det(\lambda I - A) = 0.$$

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -2 \\ 0 & 0 & \lambda \end{bmatrix}.$$

$$\det(\lambda I - T) = \det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -2 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda^3.$$

$$\therefore \quad \det(\lambda I - T) = \lambda^3 = 0.$$

Therefore, the only eigenvalue is $\lambda = 0$.

The algebraic multiplicity of an eigenvalue λ is the number of times λ appears as a root of the characteristic polynomial. Since

$$p(\lambda) = \lambda^3 = (\lambda - 0)^3,$$

the eigenvalue $\lambda = 0$ appears three times as a root.

$$\therefore \text{ the algebraic multiplicity of } \lambda = 0 \text{ is } 3.$$

To find the eigenspace corresponding to $\lambda = 0$, solve

$$(T - 0I)p = 0 \quad \Longrightarrow \quad T(p) = 0.$$

Since $T(p) = p'$, this means

$$p'(x) = 0.$$

Integrating both sides gives

$$p(x) = c,$$

where c is a constant. Thus, the eigenspace corresponding to $\lambda = 0$ is

$$E_0 = \text{span}\{1\}.$$

The geometric multiplicity of an eigenvalue $\lambda = 0$ is the dimension of its eigenspace:

$$\dim(E_0) = 1.$$

Since the algebraic multiplicity is 3 and the geometric multiplicity is 1, they are not equal.

$$\therefore T \text{ is not diagonalizable.}$$

4.

Suppose v is an eigenvector of T with eigenvalue λ, so that

$$T(v) = \lambda v.$$

Applying the given condition $T^2 = T$:

$$T^2(v) = T(T(v)).$$

Substituting $T(v) = \lambda v$ gives

$$T^2(v) = T(\lambda v) = \lambda T(v).$$

By the linearity of T (as $T(cv) = cT(v) = c(\lambda v) = \lambda(cv)$):

$$T^2(v) = \lambda T(v) = \lambda(\lambda v) = \lambda^2 v.$$

Thus, there are two expressions for $T^2(v)$:

$$T^2(v) = T(v) \quad \text{(from the given condition)}$$

$$T^2(v) = \lambda^2 v \quad \text{(from substitution)}$$

Equating these,

$$\lambda^2 v = T(v)$$

$$\Rightarrow \lambda^2 v = \lambda v \quad (\text{as } T(v) = \lambda v)$$

$$\Rightarrow (\lambda^2 - \lambda)v = 0$$

$$\Rightarrow \lambda(\lambda - 1)v = 0$$

As eigenvectors are nonzero (so $v \neq 0$), it follows that

$$\lambda(\lambda - 1) = 0.$$

$$\therefore\ \lambda = 0 \text{ or } \lambda = 1.$$

Therefore, for any linear operator T satisfying $T^2 = T$, the only possible eigenvalues of T are 0 and 1.

5.

A linear operator $T : V \to V$ (or a matrix A) is diagonalizable if it has a basis of eigenvectors. Equivalently, there exists an invertible matrix Q such that

$$A = QDQ^{-1},$$

where Q contains the eigenvectors of A as columns, and D is a diagonal matrix whose entries are the corresponding eigenvalues. For $A + I$:

$$\begin{aligned} A + I &= QDQ^{-1} + I \\ &= QDQ^{-1} + QIQ^{-1}. \end{aligned}$$

Since $QI = Q$ (multiplying any matrix by the identity leaves it unchanged), and $QQ^{-1} = I$ by definition of the inverse, it follows that $QIQ^{-1} = QQ^{-1} = I$.

$$\begin{aligned} A + I &= QDQ^{-1} + QIQ^{-1} \\ &= Q(D + I)Q^{-1}. \end{aligned}$$

This is a diagonalization form. Therefore, $A+I$ is diagonalizable with the same eigenvectors Q, and the corresponding eigenvalues are each shifted by 1:

$$A + I = Q(D + I)Q^{-1}, \qquad \lambda_i(A + I) = \lambda_i(A) + 1.$$

Thus, $A + I$ has the same eigenvectors as A, and its eigenvalues are increased by 1.

6.

Let A be a real matrix, and suppose $Av = \lambda v$ where v may have complex entries and λ may be complex.

Taking the complex conjugate of both sides gives

$$\overline{Av} = \overline{\lambda v}$$

Conjugating the LHS:

For the LHS $\overline{Av}$. The ith component of the product Av is

$$(Av)_i = \sum_j a_{ij} v_j.$$

Taking the complex conjugate entry-wise gives

$$\overline{(Av)_i} = \overline{\sum_j a_{ij} v_j}.$$

Since complex conjugation distributes over both addition and multiplication, that is,

$$\overline{a+b} = \overline{a} + \overline{b} \quad \text{and} \quad \overline{ab} = \overline{a}\,\overline{b} \quad \text{for any complex numbers } a, b \in \mathbb{C},$$

these properties can be applied term by term in the summation:

$$\overline{\sum_j a_{ij} v_j} = \sum_j \overline{a_{ij} v_j} = \sum_j \overline{a_{ij}}\,\overline{v_j}.$$

Therefore, entry-wise,

$$\overline{(Av)_i} = \sum_j \overline{a_{ij}}\,\overline{v_j}.$$

Conjugating the RHS:

For the RHS $\overline{\lambda v}$. The ith component of λv is

$$(\lambda v)_i = \lambda v_i.$$

Taking the complex conjugate entry-wise gives

$$\overline{(\lambda v)_i} = \overline{\lambda v_i}.$$

Since complex conjugation is a distributive operation over multiplication, that is,

$$\overline{ab} = \overline{a}\,\overline{b} \quad \text{for any complex numbers } a, b \in \mathbb{C},$$

this property can be applied with $a = \lambda$ and $b = v_i$ to obtain

$$\overline{\lambda v_i} = \overline{\lambda}\,\overline{v_i}.$$

Therefore, entry-wise,

$$(\overline{\lambda v})_i = \overline{\lambda}\,\overline{v_i}.$$

From $\overline{Av} = \overline{\lambda v}$, equate corresponding components:

$$\sum_j \overline{a_{ij}}\,\overline{v_j} = \overline{\lambda}\,\overline{v_i}, \quad \text{for all } i.$$

Since A is real, $\overline{a_{ij}} = a_{ij}$ for all i, j. Therefore:

$$\sum_j a_{ij}\,\overline{v_j} = \overline{\lambda}\,\overline{v_i}, \quad \text{for all } i.$$

This is exactly the component form of the matrix equation

$$A\,\overline{v} = \overline{\lambda}\,\overline{v}.$$

7.

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}, \quad n > 1.$$

a.

Let A be the $n \times n$ matrix with all entries equal to 1, and let $v \neq 0$ be an eigenvector of A with eigenvalue λ. Then

$$Av = \lambda v$$

Let

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

As every entry of A equals 1, each row of A is identical:

$$\mathrm{Row}_1(A) = \mathrm{Row}_2(A) = \cdots = \mathrm{Row}_n(A) = [\,1\;1\;\cdots\;1\,].$$

When A multiplies v, each component of Av is the dot product of this row with v:

$$(Av)_i = [\,1\;1\;\cdots\;1\,] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = 1 \cdot v_1 + 1 \cdot v_2 + \cdots + 1 \cdot v_n = v_1 + v_2 + \cdots + v_n, \quad \text{for all } i = 1, 2, \ldots, n.$$

Therefore, every entry of Av equals the same sum of all entries of v:

$$Av = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} (v_1 + v_2 + \cdots + v_n).$$

If $\lambda = 0$, the eigenvalue equation $Av = \lambda v$ reduces to

$$Av = 0 \quad \Longrightarrow \quad v_1 + v_2 + \cdots + v_n = 0.$$

Therefore, all nonzero vectors v whose components sum to 0 are eigenvectors corresponding to $\lambda = 0$. The 0-eigenspace is

$$E_0 = \{\, v \in \mathbb{R}^n : v_1 + v_2 + \cdots + v_n = 0 \,\}.$$

b.

Case 1: $\lambda = 0$ From part (a), the eigenspace is

$$E_0 = \{\, v \in \mathbb{R}^n : v_1 + v_2 + \cdots + v_n = 0 \,\}.$$

Case 2: $\lambda \neq 0$ From the eigenvalue equation $Av = \lambda v$ and the result from part (a):

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} (v_1 + v_2 + \cdots + v_n) = \lambda \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Equating the ith components on both sides:

$$v_1 + v_2 + \cdots + v_n = \lambda v_i, \quad \text{for all } i = 1, 2, \ldots, n.$$

Since the left-hand side $(v_1 + v_2 + \cdots + v_n)$ is the same for every i, all values λv_i must also be equal:

$$\lambda v_1 = \lambda v_2 = \cdots = \lambda v_n.$$

Since $\lambda \neq 0$, dividing by λ gives

$$v_1 = v_2 = \cdots = v_n.$$

Therefore, each component of v is equal. Let $v_1 = v_2 = \cdots = v_n = c$ for some constant c. Then:

$$(Av)_i = v_1 + v_2 + \cdots + v_n = nc, \quad \text{for all } i.$$

From the eigenvalue equation $(Av)_i = \lambda v_i$:

$$nc = \lambda c.$$

Since v is an eigenvector, $v \neq 0$, so $c \neq 0$. Dividing by c gives

$$\lambda = n.$$

Therefore, every eigenvector corresponding to $\lambda \neq 0$ must have all components equal and the eigenvalue must be $\lambda = n$. The eigenspace associated with $\lambda = n$ is

$$E_n = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\}.$$

The matrix A has exactly two eigenvalues:

- $\lambda = 0$ with eigenspace $E_0 = \{\, v \in \mathbb{R}^n : v_1 + v_2 + \cdots + v_n = 0 \,\}$
- $\lambda = n$ with eigenspace $E_n = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\}$

8.

$$A = \begin{bmatrix} -3 & 4 & 4 \\ -5 & 9 & 5 \\ -7 & 4 & 8 \end{bmatrix}$$

a.

For

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$Av_1 = \begin{bmatrix} -3 & 4 & 4 \\ -5 & 9 & 5 \\ -7 & 4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} (-3)(1) + 4(0) + 4(1) \\ (-5)(1) + 9(0) + 5(1) \\ (-7)(1) + 4(0) + 8(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = v_1.$$

Therefore, $\lambda_1 = 1$.

For

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

$$Av_2 = \begin{bmatrix} -3 & 4 & 4 \\ -5 & 9 & 5 \\ -7 & 4 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3(0) + 4(1) + 4(-1) \\ -5(0) + 9(1) + 5(-1) \\ -7(0) + 4(1) + 8(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix} = 4 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 4v_2.$$

Therefore, $\lambda_2 = 4$.

For

$$v_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

$$Av_3 = \begin{bmatrix} -3 & 4 & 4 \\ -5 & 9 & 5 \\ -7 & 4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3(1) + 4(2) + 4(1) \\ -5(1) + 9(2) + 5(1) \\ -7(1) + 4(2) + 8(1) \end{bmatrix} = \begin{bmatrix} 9 \\ 18 \\ 9 \end{bmatrix} = 9 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 9v_3.$$

Therefore, $\lambda_3 = 9$.

The matrix P is formed from the eigenvectors as columns and D is the diagonal matrix of corresponding eigenvalues:

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Thus,

$$A = PDP^{-1}.$$

b.

Given $M^2 = A$ and $N = P^{-1}MP$, and using the diagonalization relation $A = PDP^{-1}$:

$$\begin{aligned} N^2 &= (P^{-1}MP)(P^{-1}MP) \\ &= P^{-1}M(PP^{-1})MP \quad \text{(by associativity of matrix multiplication)} \\ &= P^{-1}MIMP \quad (\text{since } PP^{-1} = I) \\ &= P^{-1}M^2P \end{aligned}$$

Since $M^2 = A$:

$$N^2 = P^{-1}AP$$

Using the diagonalization relation $A = PDP^{-1}$:

$$\begin{aligned} N^2 &= P^{-1}(PDP^{-1})P \\ &= (P^{-1}P)D(P^{-1}P) \\ &= IDI \quad (\text{since } P^{-1}P = I) \\ &= D \end{aligned}$$

Therefore, $N^2 = D$.

c.

Given $M^2 = A$, $N = P^{-1}MP$, $A = PDP^{-1}$, and $N^2 = D$.

Proving $ND = DN$ Multiplying both sides of $N^2 = D$ on the left by N gives

$$N(N^2) = ND$$

$$\Rightarrow N^3 = ND$$

Multiplying both sides of $N^2 = D$ on the right by N gives

$$(N^2)N = DN$$

$$\Rightarrow N^3 = DN$$

Since both expressions equal N^3, it follows that

$$ND = DN.$$

Thus, N commutes with D.

Proving N is diagonal Since D is a diagonal matrix with distinct eigenvalues and N commutes with D, the matrix N must be diagonal. Any matrix commuting with a diagonal matrix having distinct diagonal entries must itself be diagonal.

Since $N^2 = D$ and both are diagonal, if

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},$$

then N must have the form

$$N = \begin{bmatrix} n_{11} & 0 & 0 \\ 0 & n_{22} & 0 \\ 0 & 0 & n_{33} \end{bmatrix},$$

where $n_{ii}^2 = \lambda_i$ for each i. Therefore,

$$n_{ii} = \pm\sqrt{\lambda_i}.$$

Therefore, $ND = DN$ and N is diagonal.

d.

Given $M^2 = A$, $N = P^{-1}MP$, $A = PDP^{-1}$, and $N^2 = D$.

Counting possible matrices N: Since N is diagonal with $N^2 = D$, each diagonal entry of N satisfies

$$n_{ii} = \pm\sqrt{d_i},$$

where d_i is the ith diagonal entry of D.

For the diagonal matrix D, there are two choices (positive or negative square root) for each of the three diagonal entries. Therefore, the total number of possible matrices N is $2^3 = 8$.

Constructing M from N: From $N = P^{-1}MP$, solving for M gives

$$M = PNP^{-1}.$$

To obtain a particular M, choose one specific N. For example, taking all positive square roots:

$$N = \begin{bmatrix} \sqrt{1} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{9} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

then

$$M = PNP^{-1}.$$

Verification:

$$M^2 = (PNP^{-1})(PNP^{-1}) = PN^2P^{-1} = PDP^{-1} = A.$$

Therefore, the chosen M satisfies $M^2 = A$. Since there are 8 possible choices for N, there are 8 matrices M satisfying $M^2 = A$.