1.

a.

$p(\lambda) = (\lambda - 2)^3 = 0$, the only eigenvalue is $\lambda = 2$ and the algebraic multiplicity of $\lambda = 2$ is 3.

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 4 & 5 & 2 \end{bmatrix}$$

To find the 2-eigenspace of A, we solve $(A - 2I)\mathbf{x} = 0$:

$$A - \lambda I = A - 2I = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 4 & 5 & 2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 4 & 5 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 4 & 5 & 0 & 0 \end{array}\right]$$

Taking pivot $a_{11} = 1$

- Multiplier for R_2: $m_{21} = \frac{-1}{1} = -1$. Operation: $R_2 \leftarrow R_2 - (-1)R_1 = R_2 + R_1$.
- Multiplier for R_3: $m_{31} = \frac{4}{1} = 4$. Operation: $R_3 \leftarrow R_3 - 4R_1$.

Applying the operations:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1+1 & -1+1 & 0+0 & 0 \\ 4-4(1) & 5-4(1) & 0-4(0) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

As pivot a_{22} is 0, we swap R_2 and R_3

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore,

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 &= 0 \end{aligned}$$

Let x_3 be t where $t \in \mathbb{R}$.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$E_2(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

To find the generalized 2-eigenspace of A, we solve $(A-2I)^3\mathbf{x}=0$:

$$(A-\lambda I)^3 = (A-2I)^3 = \left(\begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 4 & 5 & 2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^3 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 4 & 5 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since every row is zero, there are no additional constraints on $\mathbf{x}$. All components x_1, x_2, x_3 are free. Let $x_1 = r, \quad x_2 = s, \quad x_3 = t, \quad \text{where } r, s, t \in \mathbb{R}$ Then

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$G_2(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Given the matrix B:

$$B = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 3 & 2 \end{bmatrix}$$

To find the 2-eigenspace of B, we solve $(B-2I)\mathbf{x}=0$:

$$B - \lambda I = B - 2I = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 3 & 2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 3 & 3 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 3 & 3 & 0 & 0 \end{array} \right]$$

Taking pivot $a_{11} = 1$

- Multiplier for R_2: $m_{21} = \frac{-1}{1} = -1$. Operation: $R_2 \leftarrow R_2 - (-1)R_1 = R_2 + R_1$.
- Multiplier for R_3: $m_{31} = \frac{3}{1} = 3$. Operation: $R_3 \leftarrow R_3 - 3R_1$.

Applying the operations:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1+1 & -1+1 & 0+0 & 0 \\ 3-3(1) & 3-3(1) & 0-3(0) & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

As pivot a_{22} is 0, and the entry below it in the same column is also 0, no row swap is possible.

Therefore,

$$x_1 + x_2 = 0$$

Let $x_2 = s$ and $x_3 = t$, where $s, t \in \mathbb{R}$.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ s \\ t \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} -s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$E_2(B) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

To find the generalized 2-eigenspace of B, we solve $(B - 2I)^3\mathbf{x} = 0$:

$$(B - \lambda I)^3 = (B - 2I)^3 = \left(\begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 3 & 2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^3 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 3 & 3 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since every row is zero, there are no additional constraints on $\mathbf{x}$. All components x_1, x_2, x_3 are free. Let $x_1 = r, \quad x_2 = s, \quad x_3 = t, \quad \text{where } r, s, t \in \mathbb{R}$ Let $x_1 = r$, $x_2 = s$, and $x_3 = t$, where $r, s, t \in \mathbb{R}$.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$G_2(B) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

b.

For matrix A, $S = A - 2I$

$$E_2(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad G_2(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

- $\dim(\ker S) = \dim(E_2(A)) = 1$
- $\dim(\ker S^2) = 2$
- $\dim(\ker S^3) = 3 = \dim(G_2(A)) = 3$

The exact number of Jordan blocks of each size is given by

$$\#(\text{blocks of size exactly } d) = \left[\dim(\ker S^d) - \dim(\ker S^{d-1})\right] - \left[\dim(\ker S^{d+1}) - \dim(\ker S^d)\right]$$

$$\text{Number of blocks of size } 1 = (1 - 0) - (2 - 1) = 0$$

$$\text{Number of blocks of size } 2 = (2 - 1) - (3 - 2) = 0$$

$$\text{Number of blocks of size } 3 = (3 - 2) - (3 - 3) = 1$$

Therefore, there is exactly 1 Jordan block of size 3×3. All 3 generalized eigenvectors belong to a single Jordan chain of length 3.

$$J_A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

This requires finding a Jordan chain of length 3, $\{v_1, v_2, v_3\}$, such that $Sv_1 = 0$, $Sv_2 = v_1$ and $Sv_3 = v_2$.

We solve $(A - 2I)v_2 = v_1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + x_2 = 0$$
$$x_2 = 1$$

$$x_1 + 1 = 0 \implies x_1 = -1$$

Let x_3 be t, where $t \in \mathbb{R}$.

$$v_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

We solve $(A - 2I)v_3 = v_2$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 \\ 4 & 5 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + x_2 = -1$$
$$x_2 = 4$$

$$x_1 + 4 = -1 \implies x_1 = -5$$

Let x_3 be t, where $t \in \mathbb{R}$.

$$\mathbf{v}_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \\ t \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} -5 \\ 4 \\ 0 \end{bmatrix}$$

The Jordan basis:

$$\mathcal{B} = \{v_3, v_2, v_1\} = \left\{ \begin{bmatrix} -5 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$Q = \begin{bmatrix} -5 & -1 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These matrices satisfy: $J = Q^{-1}AQ$.

For matrix B, let $S = B - 2I$.

$$E_2(B) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad G_2(B) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- $\dim(\ker S) = \dim(E_2(B)) = 2$
- $\dim(\ker S^2) = 3$
- $\dim(\ker S^3) = 3$

The exact number of Jordan blocks of each size is given by

$$\#(\text{blocks of size d}) = [\dim(\ker S^d) - \dim(\ker S^{d-1})] - [\dim(\ker S^{d+1}) - \dim(\ker S^d)].$$

$$\text{number of blocks of size } 1 = (2-0) - (3-2) = 1$$
$$\text{number of blocks of size } 2 = (3-2) - (3-3) = 1$$
$$\text{number of blocks of size } 3 = (3-3) - (3-3) = 0$$

Therefore, there are 2 Jordan blocks: one 2×2 block and one 1×1 block.

$$J_B = \begin{bmatrix} \boxed{\begin{matrix} \lambda & 1 \\ 0 & \lambda \end{matrix}} & 0 \\ 0 & \boxed{\lambda} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & 0 \\ 0 & \boxed{2} \end{bmatrix}$$

This requires finding two Jordan chains: A chain of length 2, $\{v_1, v_2\}$, satisfying $Sv_2 = v_1$ and $Sv_1 = 0$ and an independent eigenvector v_3 satisfying $Sv_3 = 0$.

From $E_2(B) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, we choose $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

We Solve $(B - 2I)v_2 = v_1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 \\ 3 & 3 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + x_2 = -1 \quad \Rightarrow \quad x_1 = -x_2 - 1.$$

Let $x_2 = s,\ x_3 = t,\ s, t \in \mathbb{R}$.

$$v_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$v_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

The Jordan basis:

$$\mathcal{B} = \{v_3,\ v_2,\ v_1\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$Q = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

These matrices satisfy: $J = Q^{-1}BQ$.

2.

a.

$$p(\lambda) = \lambda^3(\lambda - 1)^2.$$

The roots of the characteristic polynomial are the eigenvalues of A:

$$p(\lambda) = \lambda^3(\lambda - 1)^2 = 0 \quad \Rightarrow \quad \lambda_1 = 0, \quad \lambda_2 = 1$$

The algebraic multiplicity of each eigenvalue is the exponent of its factor in $p(\lambda)$:

$$m_a(0) = 3, \qquad m_a(1) = 2$$

The geometric multiplicity of each eigenvalue satisfies

$$1 \le m_g(\lambda) \le m_a(\lambda),$$

For $\lambda = 0$:

$$m_a(0) = 3 \quad \Rightarrow \quad m_g(0) \in \{1, 2, 3\}$$

The possible dimensions of the eigenspace E_0 are $\{1, 2, 3\}$

For $\lambda = 1$:

$$m_a(1) = 2 \quad \Rightarrow \quad m_g(1) \in \{1, 2\}$$

The possible dimensions of the eigenspace E_1 are $\{1, 2\}$

The total dimension of all eigenspaces cannot be greater compared to the size of the matrix:

$$m_g(0) + m_g(1) \le 5$$

b.

Eigenvalue $\lambda = 0$ (partitions (3) $(2,1)$ $(1,1,1)$) and eigenvalue $\lambda = 1$ (partitions (2) $(1,1)$).

$$\begin{bmatrix} \boxed{\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}} & 0 \\ 0 & \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}} \end{bmatrix} \qquad \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}} & 0 \\ 0 & \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \end{bmatrix} \qquad \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & 0 & 0 \\ 0 & \boxed{0} & 0 \\ 0 & 0 & \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}} \end{bmatrix}$$

$$\begin{bmatrix} \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & 0 & 0 \\ 0 & \boxed{0} & 0 \\ 0 & 0 & \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \end{bmatrix} \qquad \begin{bmatrix} \boxed{0} & 0 & 0 & 0 & 0 \\ 0 & \boxed{0} & 0 & 0 & 0 \\ 0 & 0 & \boxed{0} & 0 & 0 \\ 0 & 0 & 0 & \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}} \end{bmatrix} \qquad \begin{bmatrix} \boxed{0} & 0 & 0 & 0 & 0 \\ 0 & \boxed{0} & 0 & 0 & 0 \\ 0 & 0 & \boxed{0} & 0 & 0 \\ 0 & 0 & 0 & \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \end{bmatrix}$$

c.

Given $\dim \ker(A) = 2$ and $\dim \ker(A - I) = 2$.

According to the characteristic polynomial $p(\lambda) = \lambda^3(\lambda - 1)^2$, the eigenvalues are $\lambda = 0$ with $m_a(0) = 3$ and $\lambda = 1$ with $m_a(1) = 2$.

By definition,

$$m_g(\lambda) = \dim \ker(A - \lambda I).$$

Thus,

$$m_g(0) = \dim\ker(A) = 2, \qquad m_g(1) = \dim\ker(A - I) = 2$$

The number of Jordan blocks for an eigenvalue λ equals $\dim\ker(A - \lambda I)$. Hence:

$$\lambda = 0: \text{ 2 Jordan blocks, with total size 3,}$$
$$\lambda = 1: \text{ 2 Jordan blocks, with total size 2.}$$

For $\lambda = 0$, the only configuration summing to 3 with 2 blocks is one 2×2 block and one 1×1 block:

$$J_2(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad J_1(0) = [0].$$

For $\lambda = 1$, two 1×1 blocks sum to $m_a(1) = 2$:

$$J_1(1) = [1], \qquad J_1'(1) = [1].$$

Hence, the Jordan form is

$$J = \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & 0 & 0 & 0 \\ 0 & \boxed{[0]} & 0 & 0 \\ 0 & 0 & \boxed{[1]} & 0 \\ 0 & 0 & 0 & \boxed{[1]} \end{bmatrix},$$

3.

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 2 & -7 & -1 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix},$$

According to characteristic polynomial $p(\lambda) = (\lambda - 1)^3(\lambda - 2)$:

$$\lambda = 1 \text{ with } m_a(1) = 3, \qquad \lambda = 2 \text{ with } m_a(2) = 1$$

For $\lambda = 1$:

$$A - I = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 2 & -7 & -1 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 2 & 1 & -7 & -1 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & -2 & -1 \end{bmatrix}$$

As $a_{11} = 0$, swap R_1 and R_2

$$\begin{bmatrix} 2 & 1 & -7 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & -2 & -1 \end{bmatrix}$$

Taking pivot $a_{11} = 2$.

- Multiplier for R_4: $m_{41} = \frac{2}{2} = 1$. Operation: $R_4 \leftarrow R_4 - 1 \cdot R_1$.

Applying the operation:

$$\begin{bmatrix} 2 & 1 & -7 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 2-1(2) & 1-1(1) & -2-1(-7) & -1-1(-1) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -7 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

As the entry $a_{22} = 0$, we cannot use it as a pivot. We move to the next column and take the first non-zero entry in that row as the next pivot. Taking pivot $a_{23} = 2$.

- Multiplier for R_3: $m_{33} = \frac{1}{2}$. Operation: $R_3 \leftarrow R_3 - \frac{1}{2} \cdot R_2$.
- Multiplier for R_4: $m_{43} = \frac{5}{2}$. Operation: $R_4 \leftarrow R_4 - \frac{5}{2} \cdot R_2$.

Applying the operations:

$$\begin{bmatrix} 2 & 1 & -7 & -1 \\ 0 & 0 & 2 & 0 \\ 0-\frac{1}{2}(0) & 0-\frac{1}{2}(0) & 1-\frac{1}{2}(2) & 0-\frac{1}{2}(0) \\ 0-\frac{5}{2}(0) & 0-\frac{5}{2}(0) & 5-\frac{5}{2}(2) & 0-\frac{5}{2}(0) \end{bmatrix} = \begin{bmatrix} 2 & 1 & -7 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We solve $(A - I)\mathbf{x} = 0$:

$$\begin{aligned} 2x_1 + x_2 - 7x_3 - x_4 &= 0 \\ 2x_3 &= 0 \end{aligned}$$

Let $x_2 = s$ and $x_4 = t$.

$$2x_3 = 0 \implies x_3 = 0$$

$$\begin{aligned} 2x_1 + x_2 - 7(0) - x_4 &= 0 \\ \Rightarrow 2x_1 + x_2 - x_4 &= 0 \end{aligned}$$

$$2x_1 + s - t = 0 \implies 2x_1 = t - s \implies x_1 = -\frac{1}{2}s + \frac{1}{2}t$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s + \frac{1}{2}t \\ s \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, $\dim\ker(A - I) = 2 \Rightarrow m_g(\lambda = 1) = 2$

For $\lambda = 2$:

$$A - 2I = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 2 & -7 & -1 \\ 0 & 0 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 & 0 \\ 2 & 0 & -7 & -1 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & -2 & -2 \end{bmatrix}$$

Taking pivot $a_{11} = -1$.

- Multiplier for R_2: $m_{21} = \frac{2}{-1} = -2$. Operation: $R_2 \leftarrow R_2 - (-2)R_1 = R_2 + 2R_1$.
- Multiplier for R_4: $m_{41} = \frac{2}{-1} = -2$. Operation: $R_4 \leftarrow R_4 - (-2)R_1 = R_4 + 2R_1$.

Applying the operations:

$$\begin{bmatrix} -1 & 0 & 2 & 0 \\ 2+2(-1) & 0+2(0) & -7+2(2) & -1+2(0) \\ 0 & 0 & 0 & 0 \\ 2+2(-1) & 1+2(0) & -2+2(2) & -2+2(0) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

As $a_{22} = 0$, we swap R_2 and R_4.

$$\begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & -1 \end{bmatrix}$$

As $a_{33} = 0$, we swap R_3 and R_4.

$$\begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We solve $(A - 2I)\mathbf{x} = 0$:

$$\begin{aligned} -x_1 + 2x_3 &= 0 \\ x_2 + 2x_3 - 2x_4 &= 0 \\ -3x_3 - x_4 &= 0 \end{aligned}$$

Let $x_4 = t$.

$$-3x_3 - x_4 = 0 \implies -3x_3 = t \implies x_3 = -\frac{1}{3}t$$

$$x_2 + 2\left(-\frac{1}{3}t\right) - 2(t) = 0 \implies x_2 - \frac{2}{3}t - 2t = 0 \implies x_2 = \frac{8}{3}t$$

$$-x_1 + 2\left(-\frac{1}{3}t\right) = 0 \implies -x_1 = \frac{2}{3}t \implies x_1 = -\frac{2}{3}t$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2/3t \\ 8/3t \\ -1/3t \\ t \end{bmatrix} = t\begin{bmatrix} -2/3 \\ 8/3 \\ -1/3 \\ 1 \end{bmatrix}$$

Therefore, $\dim\ker(A - 2I) = 1 \Rightarrow m_g(\lambda = 1) = 1$.

Summary of multiplicities:

$$\lambda = 1: \quad m_a(1) = 3,\ m_g(1) = 2, \qquad \lambda = 2: \quad m_a(2) = 1,\ m_g(2) = 1.$$

- For $\lambda = 1$, there are two Jordan blocks whose sizes sum to 3: one 2×2 block and one 1×1 block.
- For $\lambda = 2$, there is one 1×1 block.

$$J = \begin{bmatrix} \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}} & 0 & 0 \\ 0 & \boxed{[1]} & 0 \\ 0 & 0 & \boxed{[2]} \end{bmatrix}$$

Chain lengths:

$$\begin{aligned} \lambda = 1: &\quad \text{two chains of lengths 2 and 1} \\ \lambda = 2: &\quad \text{one chain of length 1} \end{aligned}$$

4.

a.

If all eigenvectors of a matrix A are multiples of a single vector v, then A cannot have two distinct eigenvalues. Eigenvectors associated with distinct eigenvalues are linearly independent, which would result in at least two independent eigenvectors. Hence, A must have a single eigenvalue λ.

Because all eigenvectors of A are multiples of a single vector:

There is only 1 eigenvalue λ.

The geometric multiplicity (number of independent eigenvectors) is 1.

The algebraic multiplicity (size of the block) is n.

$$J = \begin{bmatrix} \boxed{\lambda} & 1 & 0 & \cdots & 0 \\ 0 & \boxed{\lambda} & 1 & \cdots & 0 \\ 0 & 0 & \boxed{\lambda} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & \boxed{\lambda} \end{bmatrix}$$

b.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

From the diagonal entries it follows that the only eigenvalue is $\lambda = 0$ with algebraic multiplicity 4.

We solve $Ax = 0$:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \qquad Ax = \begin{bmatrix} 0 \\ x_1 \\ 2x_2 \\ 3x_3 \end{bmatrix}.$$

$$x_1 = 0, \quad 2x_2 = 0 \Rightarrow x_2 = 0, \quad 3x_3 = 0 \Rightarrow x_3 = 0,$$

so x_4 is free. Therefore, $\ker A = \operatorname{span}\{e_4\}, \quad \dim\ker A = 1$.

We solve $A^2x = 0$:

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix} \qquad A^2x = \begin{bmatrix} 0 \\ 0 \\ 2x_1 \\ 6x_2 \end{bmatrix}.$$

$$2x_1 = 0 \Rightarrow x_1 = 0, \qquad 6x_2 = 0 \Rightarrow x_2 = 0,$$

so x_3, x_4 are free. Therefore, $\ker A^2 = \operatorname{span}\{e_3, e_4\}, \quad \dim\ker A^2 = 2$.

We solve $A^3x = 0$:

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix} \qquad A^3x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6x_1 \end{bmatrix}.$$

$$6x_1 = 0 \Rightarrow x_1 = 0,$$

so x_2, x_3, x_4 are free. Therefore, $\ker A^3 = \operatorname{span}\{e_2, e_3, e_4\}, \quad \dim\ker A^3 = 3$.

We solve $A^4x = 0$:

$$A^4 = 0,$$

so x_1, x_2, x_3, x_4 are free. Therefore, $\ker A^4 = \operatorname{span}\{e_1, e_2, e_3, e_4\}, \quad \dim\ker A^4 = 4$.

Therefore the number of blocks of size $\geq d$ is

$$\#\{\text{blocks of size } \geq d\} = \dim\ker(A^d) - \dim\ker(A^{d-1}) = \begin{cases} 1, & d = 1, 2, 3, 4, \\ 0, & d \geq 5. \end{cases}$$

Equivalently, the exact number of blocks of size d is

$$\#\big(\text{blocks of size exactly } d\big) = \big[\dim\ker(A^d) - \dim\ker(A^{d-1})\big] - \big[\dim\ker(A^{d+1}) - \dim\ker(A^d)\big],$$

which results in 1 block of size 4

Jordan Form:

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$S = A - \lambda I = A$. Building a length-4 chain v_0, v_1, v_2, v_3 with $Sv_0 = 0$, $Sv_1 = v_0$, $Sv_2 = v_1$ and $Sv_3 = v_2$. Since $\ker A = \operatorname{span}\{e_4\}$, we choose $v_0 = e_4$.

Solve $Sv_1 = v_0$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \implies x_1 = 0,\ x_2 = 0,\ 3x_3 = 1 \Rightarrow x_3 = \tfrac{1}{3},$$

so $v_1 = \frac{1}{3}e_3$

Solve $Sv_2 = v_1$:

$$\begin{bmatrix} 0 \\ x_1 \\ 2x_2 \\ 3x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} \\ 0 \end{bmatrix} \implies x_1 = 0,\ 2x_2 = \tfrac{1}{3} \Rightarrow x_2 = \tfrac{1}{6},\ x_3 = 0,$$

so $v_2 = \frac{1}{6}e_2$

Solve $Sv_3 = v_2$:

$$\begin{bmatrix} 0 \\ x_1 \\ 2x_2 \\ 3x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{6} \\ 0 \\ 0 \end{bmatrix} \implies x_1 = \tfrac{1}{6},\ x_2 = 0,\ x_3 = 0,$$

so $v_3 = \frac{1}{6}e_1$

Jordan basis:

$$\mathcal{B} = \{v_3, v_2, v_1, v_0\} = \left\{\tfrac{1}{6}e_1,\ \tfrac{1}{6}e_2,\ \tfrac{1}{3}e_3,\ e_4\right\}$$

5.

a.

An $n \times n$ matrix A is nilpotent if $A^k = 0$ for some integer $k \geq 1$.

Let (λ, x) be an eigenpair of A, so that $Ax = \lambda x$.

$$A^k x = \lambda^k x.$$

Since $A^k = 0$, we have

$$0 = A^k x = \lambda^k x.$$

Because $x \neq 0$, it follows that

$$\lambda^k = 0 \quad \Rightarrow \quad \lambda = 0.$$

Hence, every eigenvalue of a nilpotent matrix is 0.

Since all eigenvalues are 0,

$$\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i = 0, \qquad \det(A) = \prod_{i=1}^{n} \lambda_i = 0.$$

Therefore,

$$\operatorname{tr}(A) = 0, \qquad \det(A) = 0.$$

b.

The characteristic polynomial of A is

$$p_A(t) = \det(tI - A) = \prod_{i=1}^{n} (t - \lambda_i).$$

Substituting $\lambda_i = 0$ for all i,

$$p_A(t) = \prod_{i=1}^{n} (t - 0) = t^n.$$

According to the Cayley–Hamilton theorem, if $p(x)$ is the characteristic polynomial of a matrix A, then

$$p(A) = 0.$$

With $p(t) = t^n$,

$$p(A) = A^n = 0.$$

6.

a.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

The characteristic polynomial is $p_A(t) = \det(tI - A)$

$$tI - A = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} t - a & -b \\ -c & t - d \end{bmatrix}$$

$$\det(tI - A) = (t - a)(t - d) - (-b)(-c) = (t - a)(t - d) - bc = (t - a)(t - d) - bc = t^2 - (a + d)t + (ad - bc)$$

Since

$$\operatorname{tr}(A) = a + d, \qquad \det(A) = ad - bc,$$

we have

$$p_A(t) = t^2 - (\operatorname{tr} A)t + \det(A)$$

b.

For any 2×2 matrix M, its characteristic polynomial is

$$p(t) = t^2 - (\operatorname{tr} M)t + \det(M).$$

For AB,

$$p(t) = t^2 - (\operatorname{tr}(AB))t + \det(AB).$$

According to the Cayley–Hamilton theorem, if $p(x)$ is the characteristic polynomial of a matrix A, then

$$p(A) = 0.$$

Therefore,

$$p(AB) = (AB)^2 - (\operatorname{tr}(AB))(AB) + (\det AB)I = 0.$$

Given that $(AB)^2 = 0$, the matrix AB is nilpotent. All eigenvalues of a nilpotent matrix are 0, so

$$\operatorname{tr}(AB) = 0, \qquad \det(AB) = 0.$$

$p(AB)$ simplifies to 0.

Using the identities

$$\operatorname{tr}(AB) = \operatorname{tr}(BA), \qquad \det(AB) = \det(BA),$$

it follows that

$$\operatorname{tr}(BA) = 0, \qquad \det(BA) = 0$$

The characteristic polynomial of BA is

$$p(t) = t^2 - (\operatorname{tr}(BA))t + \det(BA) = t^2$$

Applying Cayley–Hamilton theorem and $\operatorname{tr}(BA) = 0$ and $\det(BA) = 0$:

$$p(BA) = (BA)^2 = 0$$

Therefore,

$$(AB)^2 = 0 \implies (BA)^2 = 0$$

7.

a.

A is an invertible $n \times n$ matrix with the characteristic polynomial

$$p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

The characteristic polynomial is defined as

$$p(t) = \det(tI - A).$$

To find a_0, we can evaluate the polynomial at 0:

$$a_0 = p(0) = \det(0 \cdot I - A) = \det(-A)$$

Using the properties of determinants, we know that for an $n \times n$ matrix, $\det(-A) = (-1)^n \det(A)$. Therefore,

$$a_0 = (-1)^n \det(A)$$

We are given that A is an invertible matrix. A a matrix is invertible if its determinant is nonzero. Thus, $\det(A) \neq 0$.

Since a_0 is the product of $(-1)^n$ (which is either 1 or -1) and a non-zero determinant, we can conclude that $a_0 \neq 0$

(b)

The characteristic polynomial is:

$$p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

According to the Cayley–Hamilton theorem, if $p(x)$ is the characteristic polynomial of a matrix A, then

$$p(A) = 0$$

Therefore,

$$p(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A = -a_0 I$$

Multiplying both sides by $-\frac{1}{a_0}A^{-1}$ (since A is invertible):

$$-\frac{1}{a_0}A^{-1}\big(A^n + a_{n-1}A^{n-1} + \cdots + a_1 A\big) = I$$

$$-\frac{1}{a_0}\big(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I\big) = A^{-1}$$

Let $B = -\frac{1}{a_0}\big(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2 A + a_1 I\big)$

$$B = A^{-1}$$

Therefore

$$AB = I$$