Optimization - B. Math, 3rd Semester
Assignment 1 — Odd Semester 2021-2022

Due date: October 11, 2021 (by 11:59 pm)
Note: Each question is worth 10 points, and subparts are worth equal points. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only** after you have understood it.

1. (a) Let $K \subseteq \mathbb{R}^n$ be a convex set, with $\vec{x}_1, \ldots, \vec{x}_k \in K$, and let $\theta_1, \ldots, \theta_k \in \mathbb{R}$ satisfy $\theta_i \geq 0, \theta_1 + \cdots + \theta_k = 1$. Show that $\theta_1\vec{x}_1 + \cdots + \theta_k\vec{x}_k \in K$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.)

   (b) Show that a set is convex if and only its intersection with any line is convex. Show that a set is affine if and only if its intersection with any line is affine.

2. Let $S \subseteq \mathbb{R}^n$.

   (a) Prove that $conv(S) := \{\sum_{i=1}^k \theta_i\vec{x}_i : k \in \mathbb{N}, \theta_i \geq 0, \sum_{i=1}^k \theta_i = 1, \vec{x}_i \in S\}$ is a convex set. (This defines the convex hull of S.)

   (b) Prove that $conic(S) := \{\sum_{i=1}^k \theta_i\vec{x}_i : k \in \mathbb{N}, \theta_i \geq 0, \vec{x}_i \in S\}$ is a convex cone. (This defines the positive hull of S.)

3. Show that the *hyperbolic set* $\{(x_1, x_2) \in \mathbb{R}^2_+ : x_1x_2 \geq 1\}$ is convex. As a generalization, show that $\{(x_1, \ldots, x_n) \in R^n_+ : \prod_{i=1}^n x_i \geq 1\}$ is convex.

4. (*When does one halfspace contain another?*) For a non-zero vector $\vec{a} \in \mathbb{R}^n$ and a scalar $b \in \mathbb{R}$, recall the definition of the closed half-space $H^-_{\vec{a},b} := \{\vec{x} \in \mathbb{R}^n : \vec{a}^T\vec{x} \leq b\}$.

   (a) Find conditions in terms of non-zero vectors $\vec{a}_1, \vec{a}_2 \in \mathbb{R}^n$ and scalars $b_1, b_2, \in \mathbb{R}$ such that
   $$H^-_{\vec{a}_1,b_1} \subseteq H^-_{\vec{a}_2,b_2}.$$

   (b) Find Find conditions in terms of non-zero vectors $\vec{a}_1, \vec{a}_2 \in \mathbb{R}^n$ and scalars $b_1, b_2, \in \mathbb{R}$ such that
   $$H^-_{\vec{a}_1,b_1} = H^-_{\vec{a}_2,b_2}.$$

5. (*Voronoi description of half-space*)

   Let $\vec{a}, \vec{b}$ be distinct points in $\mathbb{R}^n$. Show that the set of all points closer (in Euclidean norm) to $\vec{a}$ than $\vec{b}$, that is, $\{\vec{x} : \|\vec{x} - \vec{a}\|_2 \leq \|\vec{x} - \vec{b}\|_2\}$, is a half-space. Describe it explicitly as an inequality of the form $\vec{c}^T\vec{x} \leq d$. (Including a picture for visualization is encouraged.(

6. (*Voronoi sets and polyhedral decomposition*) Let $\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_k \in \mathbb{R}^n$. Consider the set of points that are closer (in Euclidean norm) to $\vec{x}_0$ than the other $\vec{x}_i$, that is,

$$V = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\|_2 \leq \|\vec{x} - \vec{x}_i\|_2, i = 1, 2, \ldots, k\}.$$

   (a) Show that V is a polyhedron by expressing V in the form $V = \{\vec{x} \in \mathbb{R}^n : A\vec{x} \precsim \vec{b}\}$.

   (b) Conversely, given a polyhedron with nonempty interior, show how to find $\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_k$ so that the polyhedron is the Voronoi region of of $\vec{x}_0$ with respect to $\vec{x}_1, \ldots, \vec{x}_k$.

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