Optimization - B. Math, 3rd Semester
Assignment 4 — Odd Semester 2021-2022

Due date: November 27, 2021 (by 11:59 pm)
Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

1. (10 points) Solve the following LP problem using the simplex method:

$$\text{minimize } \; x_1 - 2x_2$$

subject to

$$\begin{aligned} 2x_1 + 3x_3 &= 1 \\ 3x_1 + 2x_2 - x_3 &= 5 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Use the two-phase simplex method (the first phase identifies an initial basis) and Bland's rule (for a choice of the entering and exiting basis which ensures termination of algorithm in finitely many steps.)

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2. (20 points) Consider the following LP problem:

$$\text{maximize } \; 3x_1 + 2x_2$$

subject to

$$2x_1 + x_2 \leq 4 \tag{1}$$
$$-2x_1 + x_2 \leq 2 \tag{2}$$
$$x_1 - x_2 \leq 1 \tag{3}$$
$$x_1, x_2 \geq 0 \tag{4}$$

Answer the questions below with proper justification.

(a) (4 points) Determine the bases associated to all the vertices of the feasible polyhedron.

(b) (4 points) Start the simplex algorithm with the basis corresponding to the slack variables with x_1 as the first entering variable. Specify the sequences of bases visited by the simplex algorithm to reach the optimal solution.

(c) (4 points) Determine the value of the reduced costs relative to the basic solutions associated to the following vertices of the feasible polyhedron:

a) (eq. 1) $\cap$ (eq. 2)

b) (eq. 1) $\cap$ (eq. 3)

where (eq. i) is the equality obtained by tightness at inequality i.

(d) (4 points) For which values of the RHS coefficient b_1 in constraint (1) is the feasible set non-empty? For what values of b_1 does the optimal basis not change? (In the LPP above, $b_1 = 4$.)

(e) (4 points) For which values of the objective function coefficient c_1 is there more than one optimal solution? (In the above LPP, $c_1 = 3$.)

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3. (10 points) Consider the simplex method applied to an LP in standard form (the feasible set is of the form $A\vec{x} = \vec{b}, \vec{0} \precsim \vec{x}$, where A is an $m \times n$ matrix of full-rank, and $\vec{b} \in \mathbb{R}^m$.) For each of the following statements that follow, give either a proof or a counterexample:
   (a) (2 points) An iteration of the simplex method may move the feasible solution by a positive distance while leaving the cost unchanged.
   (b) (2 points) A variable that has just left the basis cannot reenter in the very next iteration.
   (c) (2 points) A variable that just entered the basis cannot leave in the very next iteration.
   (d) (2 points) If there is a nondegenerate optimal basis, then there exists a unique optimal basis.
   (e) (2 points) If $\vec{x}$ is an optimal solution found by the simplex method, no more than m of its components can be positive, where m is the number of equality constraints.

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4. (10 points) What is the dual of the following LP problems?

   (a) (3 points)

$$\text{minimize } 3x_1 + 5x_2 - x_3$$

subject to

$$x_1 - x_2 + x_3 \leq 3 \tag{5}$$

$$2x_1 - 3x_2 \leq 4 \tag{6}$$

$$x_1, x_2, x_3 \geq 0 \tag{7}$$

   (b) (3 points)

$$\text{minimize } x_1 - x_2 - x_3$$

subject to

$$-3x_1 - x_2 + x_3 \leq 3 \tag{8}$$

$$2x_1 - 3x_2 - 2x_3 \geq 4 \tag{9}$$

$$x_1 - x_3 = 2 \tag{10}$$

$$x_1 \geq 0 \tag{11}$$

$$x_2 \geq 0 \tag{12}$$

   (c) (4 points)

$$\text{maximize } x_1 - x_2 - 2x_3 + 3$$

subject to

$$-3x_1 - x_2 + x_3 \leq 3 \tag{13}$$

$$2x_1 - 3x_2 \geq 4x_3 \tag{14}$$

$$x_1 - x_3 = x_2 \tag{15}$$

$$x_1 \geq 0 \tag{16}$$

$$x_2 \leq 0 \tag{17}$$

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5. (10 points) The primal simplex method works by visiting a sequence of basic feasible solutions converging to the optimal basic feasible solution; in other words, it maintains feasibility while aiming for optimality. Application of the simplex algorithm to the DUAL problem and inferring the optimal solution to the PRIMAL problem from the optimal solution to the DUAL problem is called the dual simplex method. In other words, the dual simplex method maintains optimality whilst working towards feasibility (with respect to the PRIMAL problem).

   Solve the following LP problem using the dual simplex method:

$$\text{minimize } \; 3x_1 + 4x_2 + 5x_3 + 10$$

   subject to

$$2x_1 + 2x_2 + x_3 \geq 6 \tag{18}$$

$$x_1 + 2x_2 + 3x_3 \geq 5 \tag{19}$$

$$x_1, x_2, x_3 \geq 0 \tag{20}$$

   What are the advantages in comparison with the primal simplex method?

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