Optimization - B. Math, 3rd Semester

Assignment 5 — Odd Semester 2021-2022

Due date: December 13, 2021 (by 11:59 pm)

Note: Total number of points is 60. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

1. (10 points) For each of the two problems below, check the optimality of the proposed solution (using the relevant corollary of the complementary slackness theorem):

   (a) (5 points)

   $$\text{maximize } 7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5$$

   subject to

   $$\begin{aligned} x_1 + 3x_2 + 5x_3 - 2x_4 + 2x_5 &\leq 4, \\ 4x_1 + 2x_2 - 2x_3 + x_4 + x_5 &\leq 3, \\ 2x_1 + 4x_2 + 4x_3 - 2x_4 + 5x_5 &\leq 5, \\ 3x_1 + x_2 + 2x_3 - x_4 + 2x_5 &\leq 1, \\ x_1, x_2, x_3, x_4, x_5 &\geq 0. \end{aligned}$$

   Proposed optimal solution: $x_1 = 0, x_2 = \frac{4}{3}, x_3 = \frac{2}{3}, x_4 = \frac{5}{3}, x_5 = 0.$

   (b) (5 points)

   $$\text{maximize } 4x_1 + 5x_2 + x_3 + 3x_4 - 5x_5 + 8x_6$$

   subject to

   $$\begin{aligned} x_1 - 4x_3 + 3x_4 + x_5 + x_6 &\leq 1 \\ 5x_1 + 3x_2 + x_3 - 5x_5 + 3x_6 &\leq 4 \\ 4x_1 + 5x_2 - 3x_3 + 3x_4 - 4x_5 + x_6 &\leq 4 \\ -x_2 + 2x_4 + x_5 - 5x_6 &\leq 5 \\ -2x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_6 &\leq 7 \\ 2x_1 - 3x_2 + 2x_3 - x_4 + 4x_5 + 5x_6 &\leq 5 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

   Proposed optimal solution: $x_1 = 0, x_2 = 0, x_3 = \frac{2}{3}, x_4 = \frac{5}{2}, x_5 = \frac{7}{2}, x_6 = \frac{1}{2}.$

Soumyashant Nayak: soumyashant@isibang.ac.in

2. (12 points) (Matrix games)

   Every matrix $A = (a_{ij})$ defines the following game. In each round, the *row player* selects one of the rows $i = 1, 2, \ldots, m$, and the *column player* selects one of the columns $j = 1, 2, \ldots, n$, the resulting *payoff* to the row player is a_{ij}; that is to say, the column player gives the row player a_{ij} monetary units. (If a_{ij} is negative, then it is the row player who pays $|a_{ij}|$ to the column player.)

   A row r of the $m \times n$ payoff matrix A is said to *dominate* a row s if $a_{rj} \geq a_{sj}$ for all $1 \leq j \leq n$. Similarly, a column r of the $m \times n$ payoff matrix is said to *dominate* a column s if $a_{ir} \geq a_{is}$ for all $1 \leq i \leq m$.

   (a) (5 points) If a row r is dominated by another row, show that the row player has at least one optimal (mixed) strategy $\vec{x}$ in which $x_r = 0$. In particular, if row r is deleted from the payoff matrix, then the value of the game does not change.

   (b) (4 points) If a column s dominates another column, show that the column player has at least one optimal (mixed) strategy $\vec{y}$ in which $y_s = 0$. In particular, if column s is deleted from the payoff matrix, then the value of the game does not change.

   (c) (3 points) Use the above facts to reduce the following payoff matrix to size 2×2.

$$\begin{bmatrix} -2 & 3 & 0 & -6 & -3 \\ 0 & -4 & 9 & 2 & 1 \\ 6 & -2 & 7 & 4 & 5 \\ 7 & -3 & 8 & 3 & 2 \end{bmatrix}$$

3. (8 points) We use the matrix game as defined in question 2. (The payoff matrix determines payment by column player to row player.) In the game with the payoff matrix:

$$\begin{bmatrix} 3 & 2 & 0 & -1 & 5 & -2 \\ -2 & -3 & 2 & 4 & 0 & 4 \\ 5 & -3 & 4 & 0 & 4 & 7 \\ 1 & 3 & 3 & 2 & -6 & 5 \end{bmatrix}$$

   the row player's mixed strategy $[\frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}]$ is optimal. Describe all the optimal strategies of the column player.

Soumyashant Nayak: soumyashant@isibang.ac.in

4. (10 points) For each of the following functions, determine whether it is convex, concave or neither (with proper justification):

   (a) (2 points) $f(x) = e^x - 1$ on $\mathbb{R}$;

   (b) (2 points) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{>0}$;

   (c) (2 points) $f(x_1, x_2) = \frac{1}{x_1 x_2}$ on $\mathbb{R}^2_{>0}$;

   (d) (2 points) $f(x_1, x_2) = \frac{x_1^2}{x_2}$ on $\mathbb{R} \times \mathbb{R}_{>0}$;

   (e) (2 points) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $\mathbb{R}^2_{>0}$.

5. (12 points) Let $f : \Omega \to \mathbb{R}$ be a convex function on the non-empty convex domain $\Omega \subseteq \mathbb{R}^n$.

   (a) (5 points) Show that every local minimum of f is a global minimum. In other words, if $x_0 \in \Omega$ is such that for some $\varepsilon > 0$, $f(x_0) = \min\{f(y) : y \in B(x_0, \varepsilon) \cap \Omega\}$, then $f(x_0) = \min\{f(y) : y \in \Omega\}$.

   (b) (5 points) Show that f is continuous in the interior of Ω.

   (c) (2 points) Is f necessarily continuous on the boundary of Ω? Prove or provide counterexample.

6. (8 points) (a) (4 points) Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a convex function, and Ω be a non-empty convex domain of $\mathbb{R}^m$. For $\vec{x} \in \mathbb{R}^n$, we define

$$g(\vec{x}) = \inf_{\vec{y} \in \Omega} f(\vec{x}, \vec{y}).$$

   Show that if $g(\vec{x}) > -\infty$ for some $\vec{x} \in \mathbb{R}^n$, then $g(\vec{x}) > -\infty$ for all $\vec{x} \in \mathbb{R}^n$. (In class, we have shown that $g : \mathbb{R}^n \to \mathbb{R}$ is a convex function.)

   (b) (2 points) Let $A, B, C \in M_n(\mathbb{R})$ such that A, C are symmetric, and C is invertible. Show that the quadratic function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$f(\vec{x}, \vec{y}) = \langle \vec{x}, A\vec{x} \rangle + 2\langle \vec{x}, B\vec{y} \rangle + \langle \vec{y}, C\vec{y} \rangle$$

   is a convex function on $\mathbb{R}^n \times \mathbb{R}^n$ if and only if the $2n \times 2n$ matrix $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ is positive-semidefinite.

   (c) (2 points) With notation as in part (b), show that $g(\vec{x}) = \inf_{\vec{y} \in \mathbb{R}^n} f(\vec{x}, \vec{y})$ is given by:

$$g(\vec{x}) = \left\langle \vec{x}, (A - BC^{-1}B^T)\vec{x} \right\rangle.$$

Soumyashant Nayak: soumyashant@isibang.ac.in