Optimization - B. Math, 3rd Semester

Assignment 3 — Odd Semester 2021-2022

Due date: November 11, 2021 (by 11:59 pm)

Note: Each question is worth 10 points, and subparts are worth equal points. Plagiarism is prohibited. But after sustained effort, if you cannot find a solution, you may discuss with others and write the solution in your own words **only after** you have understood it.

1. Consider the following optimization problem:

$$\text{minimize } \; x_1 + |2x_2 + 5|$$

   subject to

$$|x_1 + 1| + |x_2 - 1| \leq 2.$$

   Reformulate this as a linear programming problem. Recast the LP in standard form and canonical form.

2. (a) Describe a process, with justification, to obtain an optimal solution of a linear programming problem given an optimal solution of its standard form (canonical form, respectively).

   (b) Show that a linear programming problem is bounded if and only if the corresponding linear program in standard form is bounded.

3. (a) Consider a polyhedron $P := \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{b}, \vec{x} \geq \vec{0}\}$ in standard form. Show that when P is non-empty, we may assume A is of full rank, without loss of generality.

   (b) Suppose that $\{\vec{x} \in \mathbb{R}^n : A_1\vec{x} \leq \vec{b}_1\}$ and $\{\vec{x} \in \mathbb{R}^n : A_2\vec{x} \leq \vec{b}_2\}$ are two representations of the same (non-empty) polyhedron in $\mathbb{R}^n$. Show that A_1 and A_2 must have the same rank.

4. Suppose $P = \{\vec{x} \in \mathbb{R}^n : A\vec{x} \leq \vec{b}\}$ is a polyhedron. Show that $\vec{v} \in \mathbb{R}^n$ belongs to a k-dimensional face of P if and only if there are $n - k$ linearly independent constraints among the constraints $A\vec{x} \leq \vec{b}$ that are tight at $\vec{v}$. (A face of P is said to have dimension k if its affine hull has dimension k.)

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5. Consider the standard-form polyhedron $\{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{b}, \vec{x} \geq \vec{0}\}$, where A is a $m \times n$ matrix of full rank and $\vec{b} \in \mathbb{R}^m$..
   (a) Consider a degenerate basic solution (that is, it has more than $n - m$ zeroes). Is it true that it corresponds to two or more distinct bases? Prove or give a counterexample.
   (b) Suppose that a basic solution $\vec{v}$ is degenerate. Is it true that there is an adjacent/neighbouring basic solution $\vec{w}$ (that is, there are $n - 1$ linearly independent constraints that are both tight at $\vec{v}$ and $\vec{w}$) which is degenerate? Prove or give a counterexample.

6. Consider the standard-form polyhedron $P := \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{b}, \vec{x} \geq \vec{0}\}$, where A is a $m \times n$ matrix of full rank and $\vec{b} \in \mathbb{R}^m$. For each one of the following statements, state whether it is true (provide justification) or false (provide counterexample).
   (a) If $n = m + 1$, then P has at most two basic feasible solutions.
   (b) The set of all optimal solutions is bounded.
   (c) At every optimal solution, no more than m variables can be positive.
   (d) If there are several optimal solutions, then there exist at least two basic feasible solutions that are optimal.
   (e) Consider the problem of minimizing $max\{\langle\vec{c}, \vec{x}\rangle, \langle\vec{d}, \vec{x}\rangle\}$ over the set P. If this problem has an optimal solution, it must have an optimal solution which is an extreme point of P.

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