Midterm Exam - Optimization
B. Math. II

12-13 November, 2021

1. Duration of the exam is 24 hours.
2. The maximum number of points you can score in the exam is 80.
3. You may refer to your notes or other textbooks used in the course to solve the questions.
4. You must justify all calculations by referring to the appropriate theorems/results discussed in the above sources.
5. You are **not** permitted to discuss the solutions of the exam questions with anyone during the exam.

Name: ______________________________

Roll Number: ______________________________

1. (LU decomposition) Let $\mathbb{K}$ be a field. All matrices considered below lie in $M_n(\mathbb{K})$.
   (a) (4 points) Show that the inverse of an invertible upper-triangular (lower-triangular, respectively) matrix in is an upper-triangular (lower-triangular, respectively) matrix.
   (b) (3 points) Let A be an invertible matrix. Show that there are permutation matrices P, Q, a lower-triangular matrix L and an upper-triangular matrix U such that $A = PLUQ$. (**Hint:** Interprete Gaussian elimination appropriately)
   (c) (3 points) Let A be an invertible matrix which can be decomposed as the product of a lower-triangular matrix and an upper-triangular matrix. Show that there is a lower-triangular matrix L with 1's on the diagonal and an upper-triangular matrix U such that $A = LU$, and the choice of such L and U is unique.

Total for Question 1: 10

2. Consider the polyhedron $\mathcal{P}$ in $\mathbb{R}^3$ given by:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

   (a) (3 points) Find all extreme points of $\mathcal{P}$, with justification.

   (b) (3 points) Describe all edges (1-dimensional faces) of $\mathcal{P}$ in terms of their extreme points, with justification.

   (c) (4 points) Describe all facets (2-dimensional faces) of $\mathcal{P}$ in terms of their extreme points, with justification.

   (d) (5 points) Compute the surface area of $\mathcal{P}$.

   (**Hint:** Recall that the extreme point of a face of a polyhedron is an extreme point of the polyhedron.)

Total for Question 2: 15

3. Consider the following linear programming problem:

$$\text{Minimize } x_1 + \alpha x_2 + x_3$$

   subject to

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &\geq 0 \\ -x_1 + x_3 &\geq 0 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

   (a) (5 points) Solve the above LPP for $\alpha = -1$.

   (b) (10 points) Which values of α lead to an unbounded LPP?

Total for Question 3: 15

4. For an LP problem in standard form, we arrived at the following simplex tableau:

3	1	0	γ	?
2	0	0	-1	1
α	2	1	β	0

   (a) (4 points) What is the value of the missing entry? Why?

   (b) (4 points) What is the current basic feasible solution and the corresponding cost?

   (c) (4 points) Give specific values of α, β, γ (if they exist) such that the LP is unbounded.

   (d) (6 points) Find (with justification) necessary and sufficient conditions on α, β, γ for the above tableau to be optimal, and there to be multiple optimal solutions.

(e) (7 points) Assume that the basis associated with this tableau is optimal.Suppose also that c_1 (the first coefficient of the cost) in the original problem is replaced by $c_1+\varepsilon$. Give upper and lower bounds on ε on so that this basis remains optimal (for the updated LPP with the updated cost vector).

Total for Question 4: 25

5. A pharmaceutical laboratory manufactures four drugs $1, 2, 3, 4$ using three basic API's (active pharmaceutical ingredients) A, B, C. The company decides on the quantity of each drug (x_1, x_2, x_3, x_4) to be produced (to maximize profit) based on the following LPP:

$$\text{Maximize } 16x_1 + 14x_2 + 15x_3 + 50x_4$$

subject to

$$\begin{aligned} 2x_1 + 2x_2 + 5x_3 + 16x_4 &\le 800 \quad \text{(constraint on API A availability)} \\ 3x_1 + 2x_2 + 2x_3 + 5x_4 &\le 1000 \quad \text{(constraint on API B availability)} \\ 2x_1 + 1.2x_2 + x_3 + 4x_4 &\le 680 \quad \text{(constraint on API C availability)} \\ x_1, x_2, x_3, x_4 &\ge 0. \end{aligned}$$

The company solves the problem using the simplex method, and obtains the following optimal tableau:

-6000	0	0	-14	-40	-5	-2	0
200	0	1	5.5	19	1.5	-1	0
200	1	0	-3	-11	-1	1	0
40	0	0	0.4	3.2	0.2	-0.8	1

(a) (3 points) What is the optimal solution, and the optimal solution value? Is the optimal solution unique?

(b) (1 point) In words, what is the optimal strategy?

(c) (3 points) By at least how much should the profit of drug 3 increase so that it is used in an optimal solution?

(d) (5 points) What should the minimum profit of drug 2 be, so that the company continues to produce it?

(e) (3 points) Find the range on API B, so that the current basis remains optimal.

Total for Question 5: 15