KTU STUDY MATERIALS

DISCRETE MATHEMATICAL STRUCTURES

MAT 203

Module 3

MODULE : 3

RELATIONS & FUNCTIONS

* Cartesian product / cross product :

The Cartesian product of two sets is the set of all possible ordered pairs whose 1st element is from the first set and second element is from the second set.

ie the Cartesian product is the direct product of the elements of two sets.

ie If A and B are any two ~~sets~~, non empty sets, then the Cartesian product is defined as

$$A \times B = \{(a,b) / a \in A \text{ and } b \in B\}.$$

The Cartesian product of two sets A and B, denoted by $A \times B$ is defined as the collection of all ordered pairs (a,b) s.t a is an element of A and b is an element of B.

ex:- (1)

Let $A = \{1,2,3\}$, and $B = \{a,b,c\}$

Then $A \times B = \{(1,a), (1,b) (1,c), (2,a) (2,b), (2,c)$

$(3,a), (3,b), (3,c)\}.$

eg: — 2

Let $A = \{Book, pen, pencil\}$ and $B = \{1, 2\}$

$A \times B = \{(Book, 1), (Book, 2), (pen, 1), (pen, 2), (pencil, 1), (pencil, 2)\}$.

eg:- Suppose there are n sets, $A_1, A_2, \ldots A_n$.

Then its Cartesian product is

$A_1 \times A_2 \times \ldots \times A_n = \{(x_1, x_2 \ldots x_n) : x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n\}$.

RELATIONS

A binary relation R from a set A to B (written as ARB or $R(A,B)$) is a subset of the Cartesian product $A \times B$

* A binary relation R on a single set A is a subset of $A \times A$

* For two distinct sets, A and B having cardinalities m and n respectively, then the maximum cardinality of a relation R from A to B is mn.

Domain and Range

If there are two sets A and B and a relation R have ordered pair (x, y), then

* The domain of R is the set $\{x \mid (x,y) \in R$ for some $y \in B\}$

* The range of R is the set $\{y \mid (x,y) \in R$ for some x in $A\}$

ex:-

let $A = \{1, 2, 9\}$ and $B = \{1, 3, 7\}$

$A \times B = \{(1,1), (1,3) (1,7), (2,1) (2,3) (2,7) (9,1), (9,3) (9,7)\}$.

* A relation R is $(=)$ "equal to"

$R = \{(1,1)\}$

* A relation R is $(<)$ 'less than'

$R = \{(1,3), (1,7) (2,3), (2,7)\}$

* A relation R is $(>)$ 'greater than'

$R = \{(2,1) (9,1), (9,3), (9,7)\}$

* Types of Relations

* Empty Relation (void relation)

The empty relation between sets A and B or on A is the empty set Φ

* Full relation (universal relation)

The full relation between sets A and B is the set $A \times B$

* Identity relation

The identity relation on a set X is the $\{(x,x) / x \in X\}$

* Inverse Relation

The inverse relation (R') of a Relation R is defined as $R' = \{(b,a) / (a,b) \in R\}$

ex:- if $R = \{(1,2) (2,3)\}$, then $R' = \{(2,1), (3,2)\}$

FUNCTIONS :

A relation from a set X to a set Y is said to be a function if for every element in X has a unique element in Y. denoted by $f: X \longrightarrow Y$.

If $y = f(x)$, Here y is the Image of 'x' under 'f'.

Here X is the domain of f

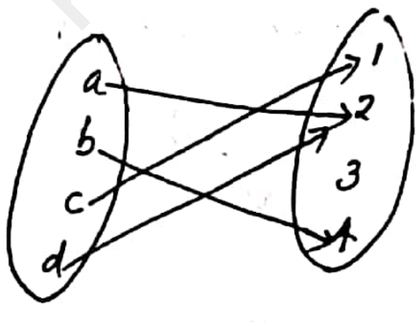
Y is the Co-domain of f

& The set of Images of all elements of X is called Range of 'f' i.e. ($R(f)$ or R_f).

ex: $y = f(x)$

$f(a) = f(d) = 2$

$a \neq d$



$f(a) = 2$

$f(b) = 4$

$f(c) = 1$

$f(d) = 2$

D_f, Domain of $f = \{a, b, c, d\}$

Co-domain of $f = \{1, 2, 3, 4\}$

R_f, range of $f = \{1, 2, 4\}$

ONE-ONE Functions / INJECTIVE FUNCTIONS:

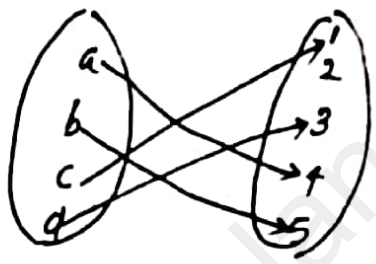
A function $f: X \to Y$ is called one to one function (1-1) if distinct elements of X mapped in to distinct elements of Y.

In other words,

f is 1-1, iff $f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2$ or

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

ex:- $f: X \to Y$



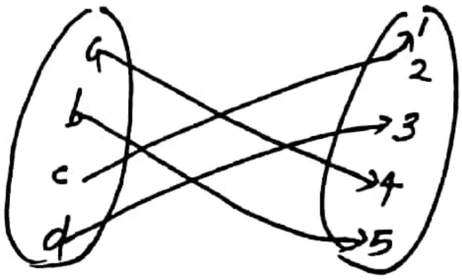
It is 1-1

ON-TO FUNCTION / SURJECTIVE FUNCTION:

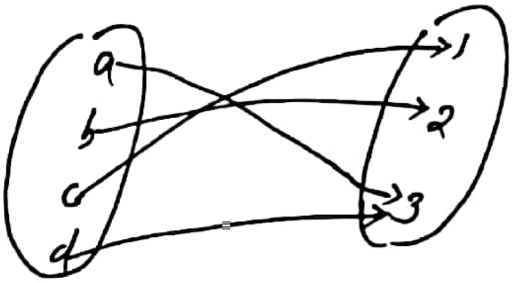
A function $f: X \to Y$ is said to be on-to if Range of $f = Y$.

Otherwise it is called in-to function.

ie a function is on-to iff every element $y \in Y$ there is an element $x \in X$ $\therefore$ $f(x) = y$.



not -on-to

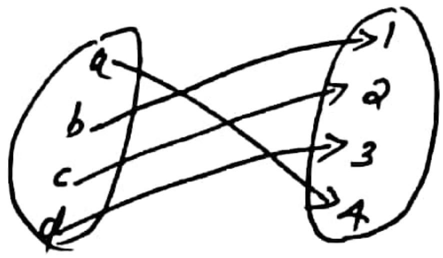


is -on-to.

BIJECTIVE FUNCTION : / ~~on~~ 1-1 & ON-TO- function

A function $f: X \rightarrow Y$ is called bijective /1-1 & on-to, if it is both 1-1 & on-to.

* If $f: X \rightarrow Y$ is bijective, then X and Y have the same no: of elements.



is bijective.

COMPOSTION OF FUNCTIONS :

If $f: A \rightarrow B$ & $g: B \rightarrow C$ then the composition of f and g is a new function from $A \rightarrow C$ denoted by gof, is d-given by,

$$gof(x) = g(f(x)) \quad \forall \ x \in A.$$

Problems:

* Determine, whether or not each of the following relations is a function with domain $\{1, 2, 3, 4\}$. If any relation is not a function, explain why?

(a) $R_1 = \{(1,1), (2,1), (3,1) (4,1) (3,3)\}$

(b) $R_2 = \{(1,2), (2,3), (4,2)\}$

(c) $R_3 = \{(1,1), (2,1), (3,1), (4,1)\}$

(d) $R_4 = \{(1,4), (2,3), (3,2), (4,1)\}$

Ans. (a) R_1 is not a function. Since there are 2 pairs $(3,1)$ & $(3,3)$, which means that, the image of the element 3 is not unique.

(b) R_2 is not a function, Since there is no - Image for the element 3.

(c) R_3 is a function. even though the images of 1, 2, 3, 4 of the domain are one and the same element 1.

(d) R_4 is a function.

⇒ Determine whether the following relation is a function. If it is a function find it's range.

(a) $R_1 = \{(x,y) / x, y \in Z, y = x^2 + 7\}$ which is a relation from Z to Z.

(b) $R_2 = \{(x,y) / x,y \in R,\ y^2 = x\}$ which is relation from $R \xrightarrow{to} R$.

Ans. (a) R_1 is a function, since for each $x \in Z$ there is a unique y, given by $y = x^2 + 7$

$$Range = \{7, 8, 11, 16, 23, \ldots\}$$

(b) R_2 is not a function, since for a given 'x', the value of y, given by y is not unique. as $y = \pm\sqrt{x}$

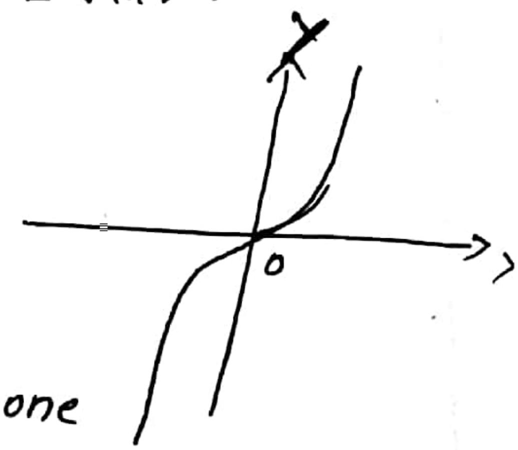
Q. Determine whether each of the following functions is an injection and / or a surjection?

(a) $f: R \longrightarrow R$ defined by, $f(x) = 3x^3 + x$

(b) $f: Z^+ \longrightarrow Z^+$ defined by $f(x) = x^2 + 2$

A. Let us consider the graph of $y = f(x)$,

$y = 3x^3 + x$



From the graph,

It is obvious that, for a given real value of 'x', there is only one real value y, as a line drawn parallel to Y axis intersects the graph at only one point, Hence the function is injective.

iii Since it is Surjective,

Since, (If we draw a line parallel to x-axis, intersects the graph at a point.)

(b) $f(x) = x^2+2$

$\therefore f(x_1) = f(x_2) \Rightarrow x_1^2+2 = x_2^2+2$

$\Rightarrow x_1^2 - x_2^2 = 0$

$\Rightarrow (x_1+x_2)(x_1-x_2) = 0$

$\Rightarrow x_1 - x_2 = 0$ ($\because x_1+x_2 \neq 0$ since $x_1, x_2 \in$.

$\Rightarrow x_1 = x_2$

ie $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$

$\therefore f(x) = x^2+2$ is ~~surjective~~. 1-1 or injective.

when $y = f(x)$ ie $y = x^2+2$, $x^2 = y-2$

$\therefore$ when $y=1$, x does not exist

Also, when $y=4$, $x = \pm\sqrt{2} \notin Z^+$

Hence $f(x)$ is not Surjective. / on-to.

Q $f: R \rightarrow R$, & $g: R \rightarrow R$, where R is the set of all real numbers find fog and gof, where $f(x) = x^2 - 2$, & $g(x) = x + 4$.

State they are injective, surjective or bijective?

A. $f(x) = x^2 - 2$, $g(x) = x + 4$

$$fog = fog(x) = f(g(x))$$
$$= f(x+4)$$
$$= (x+4)^2 - 2$$
$$= x^2 + 8x + 16 - 2$$

$$gof - gof(x) = g(f(x))$$
$$= g(x^2 - 2)$$
$$= (x^2 - 2) + 4$$
$$= \underline{\underline{x^2 + 2}}$$

* $f(x) = x^2 - 2$

$$f(x_1) = f(x_2) \Rightarrow x_1^2 - 2 = x_2^2 - 2$$
$$\Rightarrow x_1^2 = x_2^2$$
$$\Rightarrow x_1 \neq x_2$$

$\therefore$ It is not 1-1

$y = f(x)$, $y = x^2 + 2$, we have $x^2 = y - 2$

when $y = 1$, x doesn't exist

$\therefore$ It is not on-to

when $y = 1$

$x^2 = 1 - 2$

$x = \pm\sqrt{-1}$

$= \pm i \notin R$

Q. Consider f, g, h are functions on integers, $f(n) = n^2$, $g(n) = n+1$, $h(n) = n-1$, Determine,
(i) $f \circ g \circ h$ ② $g \circ f \circ h$, ③ $h \circ f \circ g$.

A.
$$f \circ g \circ h = f(g \circ h) = f(g(h(n)))$$
$$= f(g(n-1))$$
$$= f(n-1+1)$$
$$= f(n) = \underline{\underline{n^2}}$$

$$g \circ f \circ h = g(f \circ h) = g(f(h(n)))$$
$$= g(f(n-1))$$
$$= g((n-1)^2)$$
$$= (n-1)^2 + 1$$
$$= n^2 - 2n + 1 + 1$$
$$= \underline{\underline{n^2 - 2n + 2}}$$

$$h \circ f \circ g = h(f \circ g) = h(f(g(n)))$$
$$= h(f(n+1))$$
$$= h((n+1)^2)$$
$$= (n+1)^2 - 1$$
$$= n^2 + 2n + 1 - 1$$
$$= \underline{\underline{n^2 + 2n}}$$

PROPERTIES OF RELATIONS:

- **REFLEXIVE**

A relation R on a set A is said to be reflexive, if aRa for every $a \in A$.

ex:— if R is a relation on $A = \{1, 2, 3\}$ defined by, $(a,b) \in R$ if $a \leq b$, where $a, b \in A$.

then $R = \{(1,1)\ (1,2)\ (1,3)\ (2,2)\ (2,3)\ (3,3)\}$

Now, R is reflexive, since each elements of A is related to itself.

- **SYMMETRIC**

A relation R on a set A is said to be symmetric, if whenever aRb, then bRa

ie whenever $(a,b) \in R$ then $(b,a) \in R$.

* A relation R on a set A is not symmetric, if there exist $a, b \in A$ & $(a,b) \in R$ but $(b,a) \notin R$.

- ~~Anti-Symmetric~~
- **ANTI-SYMMETRIC:**

A relation R on a set A is said to be Anti-symmetric if $(a,b) \in R$ & $(b,a) \in R$ then $a = b$.

ie If $\exists\ a, b \in A$ & (a,b) & $(b,a) \in R$ but $a \neq b$, then R is not symmetric.

ex:— ~~The relation $\nleq$ on the set of integer's ($\mathbb{Z}$)~~

~~is not symmetric.~~

~~$4 \nleq 5$ but 5~~

TRANSITIVE :

A relation R on a set A is said to be transitive, if aRb & bRc then aRc

ie $(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$

ie If $\exists$ $a,b,c \in A$ $\ni$ (a,b) & $(b,c) \in R$ but $(a,c) \notin R$, then R is not transitive.

Problems:

1. Determine whether the relation R on the set of all integers is reflexive, symmetric, anti-symmetric / transitive, where a R b if and only if

(a) aRb iff $a \neq b$

(b) aRb iff $ab \geq 0$

(c) aRb iff $ab \geq 1$

(d) aRb iff $a \equiv b \pmod 7$ ($a-b$ is a multiple of 7)

(e) aRb, iff $a = b^2$.

Ans: (a) aRb iff $a \neq b$

$a \neq a$ is not true

Hence R is not reflexive.

$a \neq b \Rightarrow b \neq a$ ∴ R is symmetric

$a \neq b$, & $b \neq c$ $\Rightarrow a \neq c$, ∴ R is transitive.

(b) a $a \cdot a \geq 0$

$a^2 \geq 0$ ∴ R is reflexive.

$ab \geq 0 \Rightarrow ba \geq 0$ ∴ R is symmetric

R is not transitive.

Since $ab \geq 0$ & $bc \geq 0$ does not imply $ac \geq 0$

i.e. consider $(2,0)$ and $(0,-3)$ belong to R

but $2 \times -3 < 0$.

$\therefore$ R is not transitive.

(c) $a^2 \geq 1$ need not be true, Since a may be zero,

$\therefore$ R is not reflexive.

$ab \geq 1 \Longrightarrow ba \geq 1$ $\therefore$ R is Symmetric

if $ab \geq 1$ & $bc \geq 1 \Longrightarrow$ all of $a, b, c > 0$ or < 0

If all of $a, b, c > 0$, least a = least b = least c $= 1$

$\therefore$ $ac \geq 1$

If all of $a, b, c < 0$, greatest a = greatest b = greatest c $= -1$

$\therefore$ $ac \geq 1$ Hence R is transitive.

$\therefore$ R is Symmetric & transitive.

(d) $a \equiv b \pmod 7$

$a - b$ is a multiple of 7

Since $(a - a)$ is a multiple of 7

$\therefore$ R is reflexive

when $(a-b)$ is a multiple of 7
$(b-a)$ is also a multiple of 7

$\therefore$ R is symmetric.

when $(a-b)$ and $(b-c)$ are multiple of 7,
then $(a-b)+(b-c) = a-c$ is also a multiple of 7,

$\therefore$ R is transitive.

$\therefore$ R is reflexive, symmetric & transitive.

Q. R is a relation defined on the set $\{0,1,2,3\}$. check it is reflexive, symmetric and transitive.

(1) $R_1 = \{(0,0),(1,1),(2,2)(3,3)\}$

R_1 is reflexive and symmetric & transitive.

(2) $R_2 = \{(0,0),(0,2),(2,0)(2,2)(2,3)(3,2)(3,3)(1,1)\}$

R_2 is reflexive, R_2 is symmetric

but not transitive, since $(3,2)$ & $(2,0) \in R_2$,

but $(3,0)$ does not belong to R_2.

EQUALENCE RELATION:

A relation R on a Set A is equalence relation, if R is reflexive, symmetric & transitive

ie a relation R is said to be equalence relation if

(a) $a R a \ \forall a \in A$ (reflex

(b) if aRb then bRa (symme

(c) if aRb & bRc then aRc (transi

PARTIAL ORDER RELATIONS:

A relation R on a Set A is called partial order relation, if R is reflexive, antisymmetric & transitive.

ie R is a partial order relation on a set A, if it has the following properties

(a) $a R a \ \forall a \in A$

(b) $a R b$ & $b R a \Longrightarrow a = b$

(c) aRb & $bRc \Longrightarrow a R c$

PARTIALLY ORDERED SET / POSET :

A Set 'A' to together with a partial order relation R is called a partially ordered set or poset.

Example:

the greater than or equal to $(\geq)$ relation is a partial order relation on the set of integers $(\mathbb{Z})$

Since,

(a) $a \geq a$ for every integer a, ie $\geq$ is reflexive

(b) $a \geq b$ & $b \geq a \Rightarrow a = b$ ie $\geq$ is antisymmetric

(c) $a \geq b$ & $b \geq c \Rightarrow a \geq c$ ie $\geq$ is transitive.

$\therefore$ Thus $(\mathbb{Z}, \geq)$ is a poset.

Q. check whether the following relation is an equivalence relation?

(a) R is a relation on the set of integers such that aRb iff $a \equiv b \pmod{m}$, where m is a positive integer greater than 1.

(b) R is a relation on the set of real numbers such that aRb iff $(a-b)$ is an integer.

Ans. (a) • $(a-a)$ is a multiple of m

∴ $a \equiv a \pmod m$ ∴ It is reflexive.

• when $(a-b)$ is a multiple of m, then $(b-a)$ is also a multiple of m

ie $a \equiv b \pmod m \Rightarrow b \equiv a \pmod m$

∴ It is symmetric

• when $a-b = k_1 m$ & $b-c = k_2 m$

we get $a-b+b-c = a-c = (k_1+k_2)m$

ie when $a \equiv b \pmod m$ & $b \equiv c \pmod m$

$\rightarrow a \equiv c \pmod m$

∴ R_2 is transitive.

∴ R is an equivalence relation

(b) $(a-a)$ is an integer ∴ R_3 is reflexive.

when $(a-b)$ is an integer then $(b-a)$ is an integer

∴ It is symmetric.

when $(a-b)$ & $(b-c)$ are integers,

clearly $(a-c)$ is also an integer, (by addition)

∴ R_3 is transitive.

∴ R is an equivalence relation.

Q. If R is a relation on the set of positive integers, Such that . $(a,b) \in R$ iff a^2+b is even. prove that R is an equivalence relation?

A. $a^2+a = a(a+1) = \text{even}$ Since a and $a+1$ are consecutive positive integers.

$\therefore (a,a) \in R$.

Hence R is reflexive.

when a^2+b is even, a and b must be both even or both odd

In either case, b^2+a is even.

$\therefore (a,b) \in R \Rightarrow (b,a) \in R$

Hence R is Symmetric.

when a,b,c are even, a^2+b and b^2+c are even.

Also a^2+c is even.

when a,b,c are odd, a^2+b & b^2+c are even

Also a^2+c is even

Then $(a,b) \in R$ & $(b,c) \in R \Rightarrow (a,c) \in R$.

$\therefore R$ is transitive.

Hence R is an equivalence relation.

Q. If R is a relation on the set of integers, such that $(a,b) \in R$ iff $3a+4b=7n$ for some integers 'n'. prove that R is an equivalence relation?

A. $3a+4a = 7a$ when a is an integer.

$\therefore (a,a) \in R$, ie R is reflexive.

<u>Symmetry</u>

$(a,b) \in R$

$$3b+4a = 7a+7b-(3a+4b)$$
$$= 7(a+b) - 7n$$
$$= 7(a+b-n), \text{ where } (a+b-n)$$
is an integer,

$\therefore (b,a) \in R$

ie R is symmetric

• Let $(a,b) \in R$ & $(b,c) \in R$

ie $3a+4b = 7m \longrightarrow (1)$

$3b+4c = 7n \longrightarrow (2)$

Adding (1) & (2) $\Rightarrow$ $3a+4c+7b = 7m+7n$

$3a+4c = 7(m+n-b)$

where $(m+n-b)$ is an integer.

$\therefore$ $(a,c) \in R$.

ie R is transitive.

$\therefore$ R is an equivalence relation.

$\Rightarrow$ **EQUALANCE CLASS:**

EQUIVALANCE CLASSES

If R is an equalence relation on the set A the set of all elements of A that are related to an element a of A is called the equalence class of a denoted by [a]

$$[a] = \{x \mid (a, x) \in R\}$$

ex:- $A = \{1, 2, 3\}$ R is a relation on A

$R = \{(1,1)\ (1,2)\ (2,1)\ (2,2)\ (3,3)\}$ is a equalence relation ∵ reflexive, symmetric, transitive

$[1] = \{1, 2\}$

$[2] = \{1, 2\}$

$[3] = \{3\}$

Imp

THEOREM

If R is an equivalence relation on non-empty set A and if a and $b \in A$ are arbitrary, then

(i) $a \in [a] \ \forall a \in A$

(ii) $[a] = [b]$ iff $(a,b) \in R$

(iii) if $[a] \cap [b] \neq \phi$ then $[a] = [b]$

PROOF

① Since R is reflexive ($\because$ equivalence relation)

$(a, a) \in R \ \forall a \in A$

Hence $a \in [a]$

② Let us assume $(a,b) \in R \ \forall a \in A$ or $aRb \to ①$

Let $x \in [b]$ hence $(b,x) \in R$ or $bRx \to ②$

from ① and ②,

it follows that, aRx or $(a,x) \in R$ ($\because R$ is transitive)

$\therefore x \in [a]$

Thus $x \in [b] \Rightarrow x \in [a]$

$\therefore [b] \subseteq [a] \longrightarrow (3)$

Thus ~~$x \in [b]$~~ $\Rightarrow$

Let $y \in [a]$, then $aRy \longrightarrow (4)$

From (1) we have bRa, Since R is Symmetric $\rightarrow (5)$

From (5) and (4), we get bRy ; Since R is transitive

$\therefore y \in [b]$

Thus $y \in [a] \Rightarrow y \in [b]$ $\therefore [a] \subseteq [b]$

from (3) and (6) we get $[a] = [b]$

Conversely, let, $[a] = [b]$

Now, $b \in [b]$ by (i)

ie $b \in [a]$ or $(a, b) \in R$.

(3) Since $[a] \cap [b] \neq \phi$ $\exists$ an element $x \in A$ $\ni$

$x \in [a] \cap [b]$

Hence $x \in [a]$ and $x \in [b]$

ie xRa and xRb

or aRx and xRb

$\therefore aRb$, Since R transitive

Hence by (ii) $[a] = [b]$

Equivalently, if $[a] \neq [b]$, then $[a] \cap [b] = \phi$

DEFINITION / PARTITION

If S is a non-empty set, a collection of disjoint non empty subsets of S, whose union is S is called partition of

i.e. the collection subsets A_i is a partition of S if

(i) $A_i \neq \phi$ for each i

(ii) $A_i \cap A_j = \phi$ $\forall$ for $i \neq j$

(iii) $\bigcup_i A_i = S$ where $\bigcup_i A_i$ represents the union of the subsets A_i $\forall i$

ex :- $S = \{1, 2, 3, 4, 5, 6\}$

$[\{1, 3, 5\}, \{2, 4\}]$ is not a partition since the union of the subsets is not S as the element 6 is missing.

$[\{1,3\}, \{3,5\}, \{2,4,6\}]$ is not a partition

Since $\{1,3\}$ and $\{3,5\}$ are not disjoint

- $[\{1,2,3\}, \{4,5\} \{6\}]$ is a partition

(a, b) is a multiple of

Representation of Relations using graph:

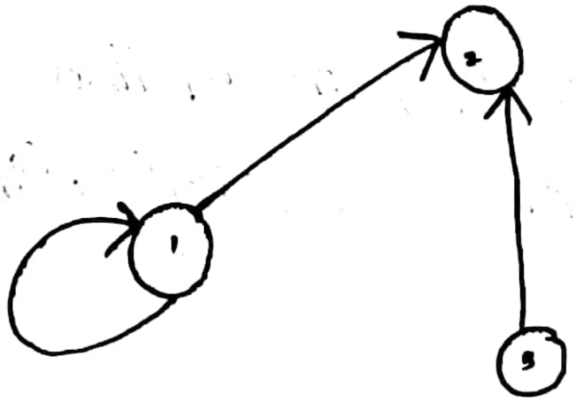
A relation can be represented using a directed graph.

The number of vertices in the graph is equal to the number of elements in the set from which the relation has been defined.

For each ordered pair (x, y) in the Relation R there will be a directed edge 'from the vertex x to vertex y'. If there is an ordered pair (x, x), there will be self-loop on vertex x.

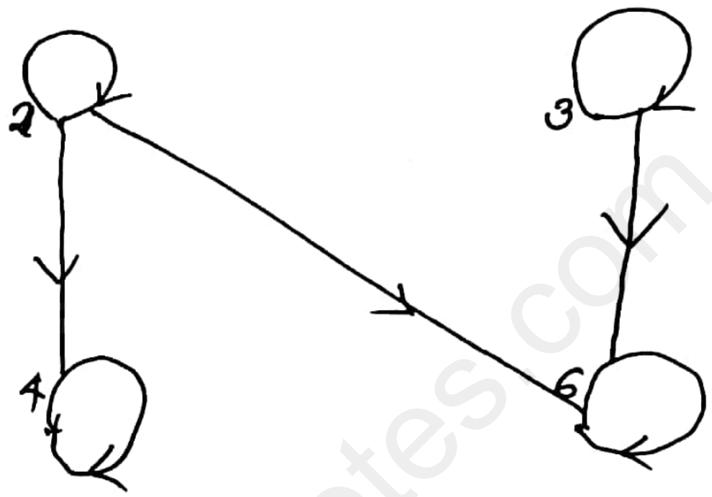
* ex:— Suppose there is a relation $R = \{(1,1), (1,2), (3,2)\}$ on the set $A = \{1, 2, 3\}$

It can be represented by the following graph.



Q. If $A = \{2, 3, 4, 6\}$ and R is defined by aRb if a divides b. Draw the digraph of R.

A. $R = \{(2,2)\ (3,3),\ (4,4)\ (6,6),\ (2,4)\ (2,6)\ (3,6)\}$



→ Note:

The digraph of R^{-1}, the inverse of R has exactly the same edges of the digraph of R, but the directions of the edges are reversed.

→ ~~Pd~~ Properties of digraph:

(*) A relation R is reflexive if

* Irreflexive Relations

A relation R on a Set A is irreflexive, if for every $a \in A$, $(a, a) \notin R$

ie if there is no $a \in A$, Such that aRa.

ex:—

If $A = \{1, 2, 3\}$

$R = \{(1,2)(2,3)(1,3)\}$ is irreflexive.

→ Properties of digraph:

(*) A relation R is reflexive, if and only if there is a loop at every vertex of the digraph of the relation R, So, that, every ordered pair of the form (a, a) occurs in R. If no vertex has a loop R is irreflexive.

(*) A relation R is Symmetric if and only if for every edge between distinct vertices in its digraph there is an edge in the opposite direction So, that (b, a) is in R, whenever (a, b) is in R

(*) A relation R is ~~Sym.~~ anti-Symmetric if and only if there are never two edges in opposite directions between distinct vertices.

(*) A relation R is transitive, if and only if whenever there is an edge from a vertex a to b, and from b to c, then there is an edge from a to c.

HASSE DIAGRAM FOR PARTIAL ORDERINGS

The simplified form of the digraph of a partial ordering on a finite set contains sufficient information about the partial ordering is called Hasse digram.

(*) Since the partial ordering is a reflexive relation it's digraph has loops at all vertices. We need not to show these loops since they must be present

(*) Since all the partial ordering is transitive, we need not to show, those edges that must be present due to transitivity.

ex: if (1,2) & (2,3) are edges the (1,3) will also be an edge due to transitivity.

this edge need not to show in the corresponding Hasse diagram

(*) If we assume that all edges are directed upward, we need not to show the directions of the edges.

Thus, Hasse digraph representing a partial ordering can be obtained from its digraph, by removing all loops, by removing all edges that are present due to transitivity and by drawing each edge without arrow so that initial vertex is below it's terminal vertex.

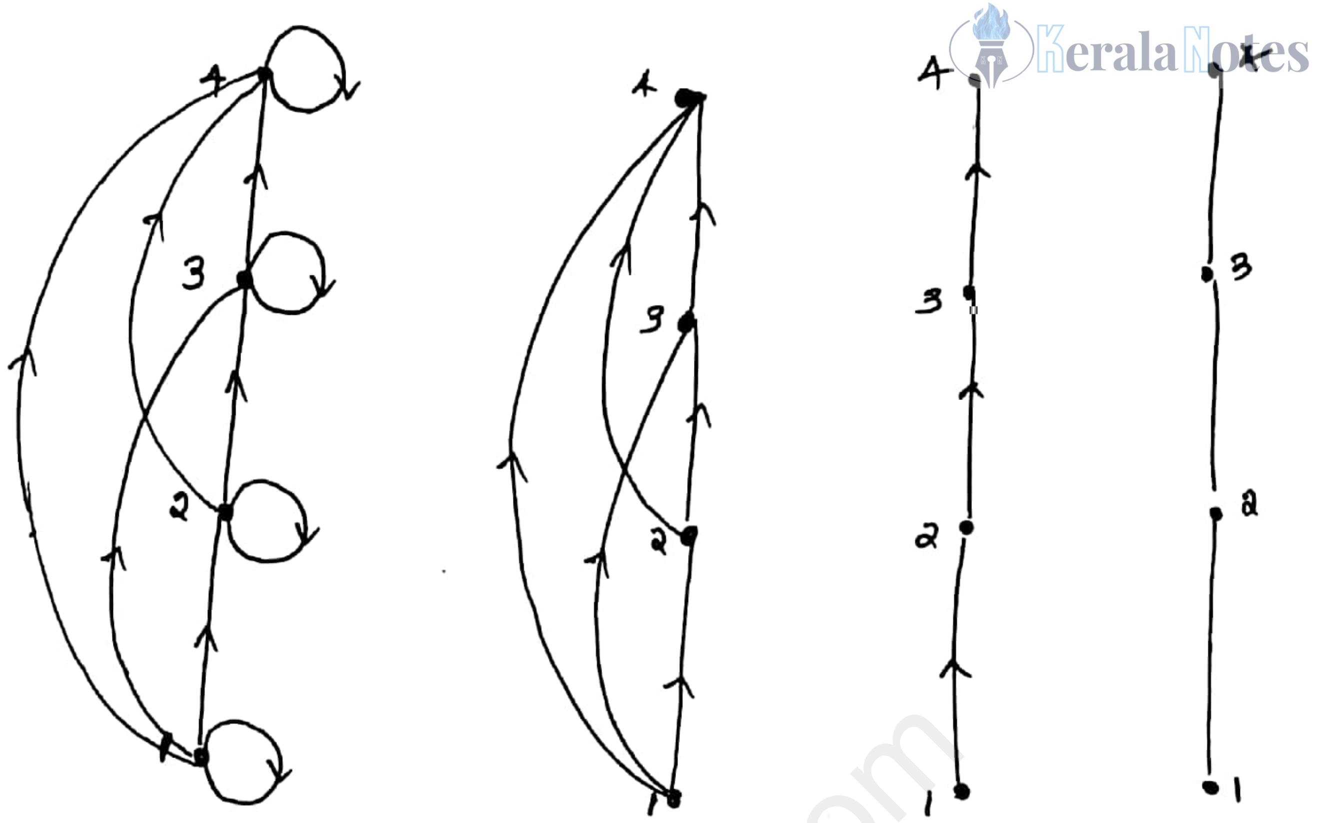
→Q Q Construct a Hasse digraph for the partial ordering $\{(a,b) / a \preceq b\}$ on the set $\{1,2,3,4\}$ starting from its digraph?

A. $\{1,2,3,4\}$

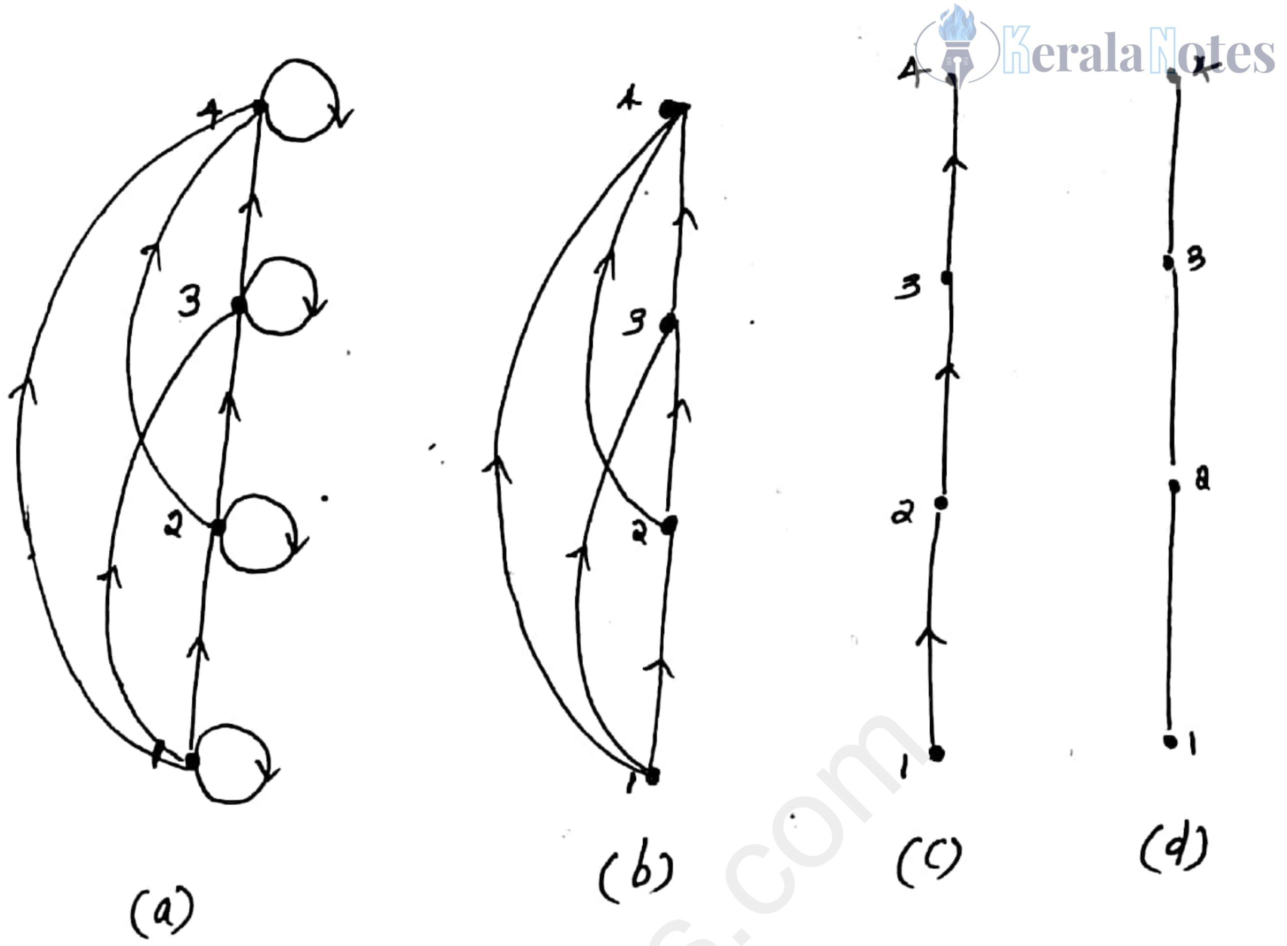
$R = \{(a,b) / a \preceq b\}$

$R = \{(1,1)\ (2,2)\ (3,3)\ (4,4),\ (1,2)\ (1,3)$

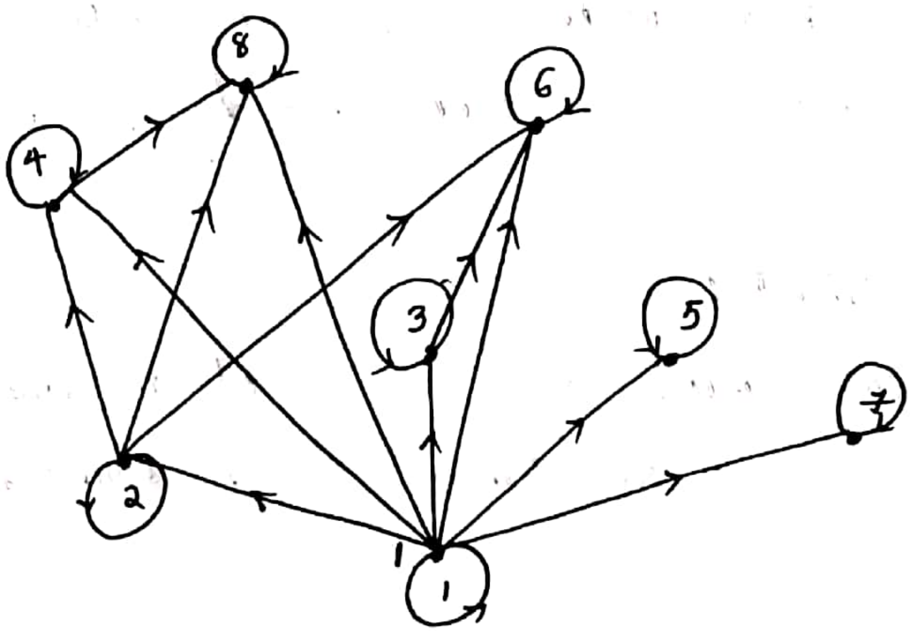
$(1,4),\ (2,3),\ (2,4)\ (3,4)\}$

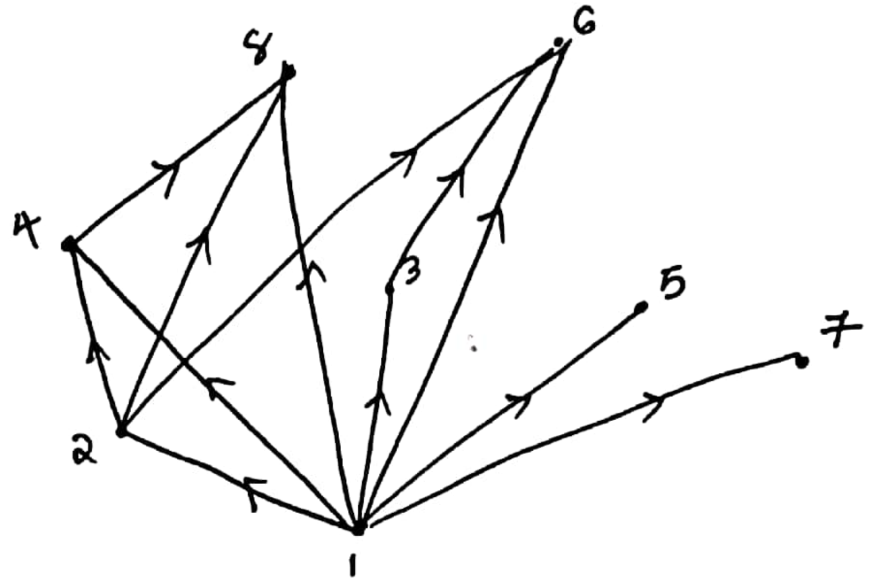


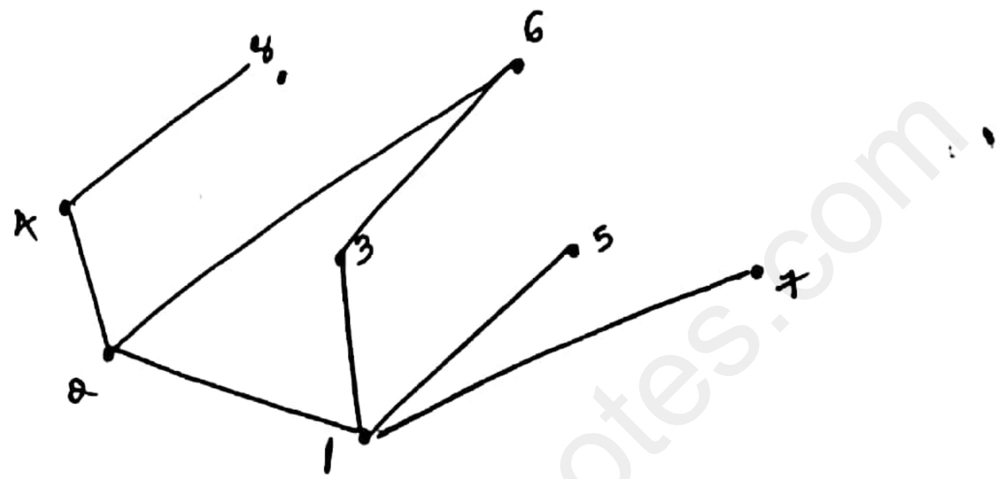
(a) (b) (c) (d)



Draw the digraph representing the partial ordering $\{(a,b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Reduce to it to Hasse diagram representing the given partial ordering?







DEFINITIONS : [RELATED TO POSETS].

MAXIMAL MEMBER :

When $\{P, \leq\}$ is a poset, an element $a \in P$ is called maximal member of P, if there is no element $b \in P$ such that $a < b$. [a strictly precedes b].

MINIMAL MEMBER :

An element $a \in P$ is called minimal member of P, if there is no element $b \in P$ such that $b < a$

GREATEST MEMBER:

If there exist an element $a \in P$ such that $b \preceq a$ for all $b \in P$, then a is called the greatest member of the poset $\{P, \preceq\}$.

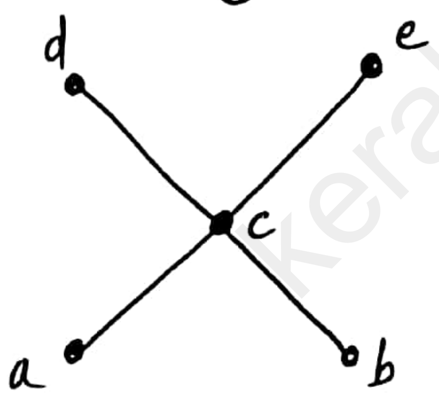
LEAST MEMBER:

If there exists an element $a \in P$ such that, $a \preceq b$ for all $b \in P$, then a is called the least member of the poset $\{P, \preceq\}$.

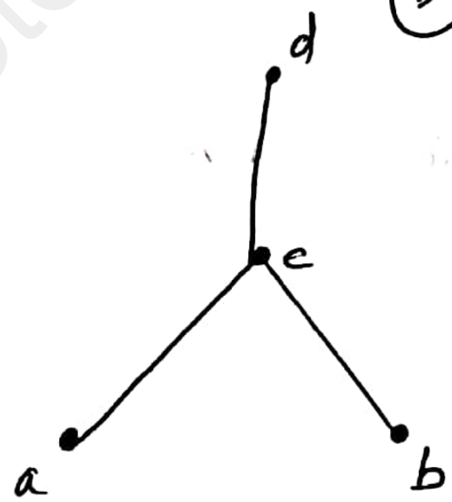
Example:

Let us consider the Hasse diagrams of four posets given below.

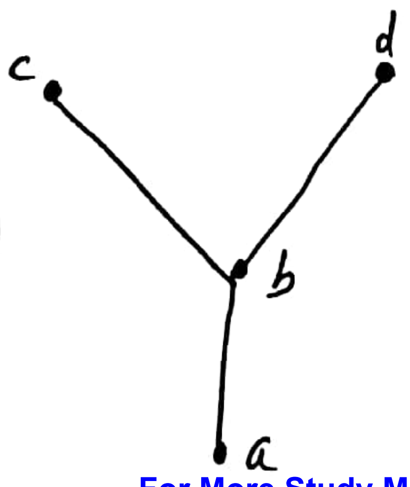
(a)



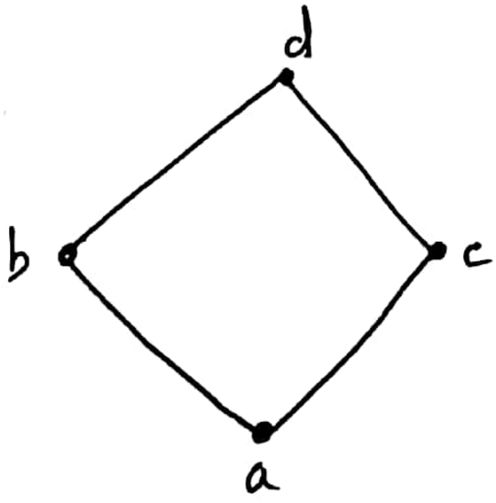
(b)



(c)



(d)

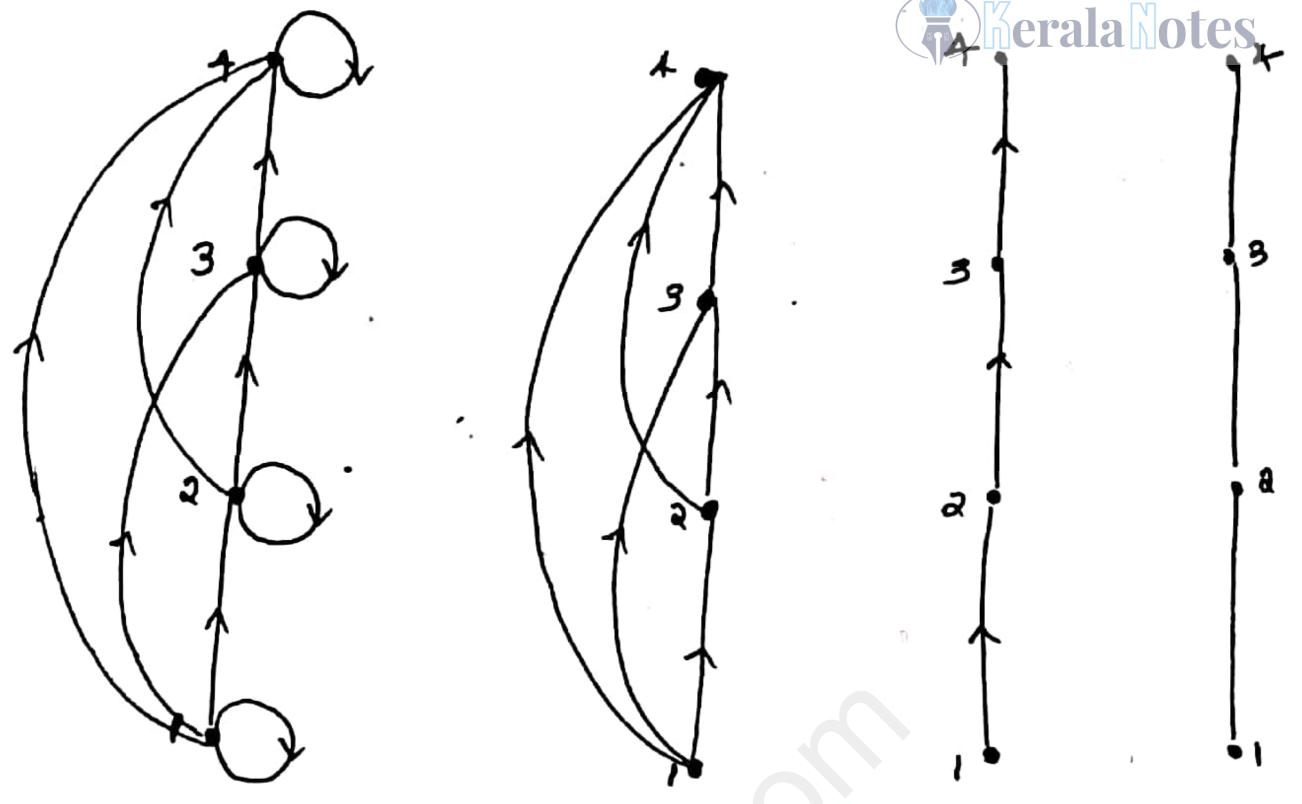


* In figu (a) => a and b are minimal elements and d and e are maximal elements. , but the poset has neither greatest nor the least element.

* In fig (b) => a and b are minimal elements and d is the maximal element (also the only maximal element). There is no least element.

* In fig (c) => a is the least element (only the minimal element) & c and d are the maximal elements. There is no greatest element.

* In figu (d) => a is the least element & d is the greatest element.



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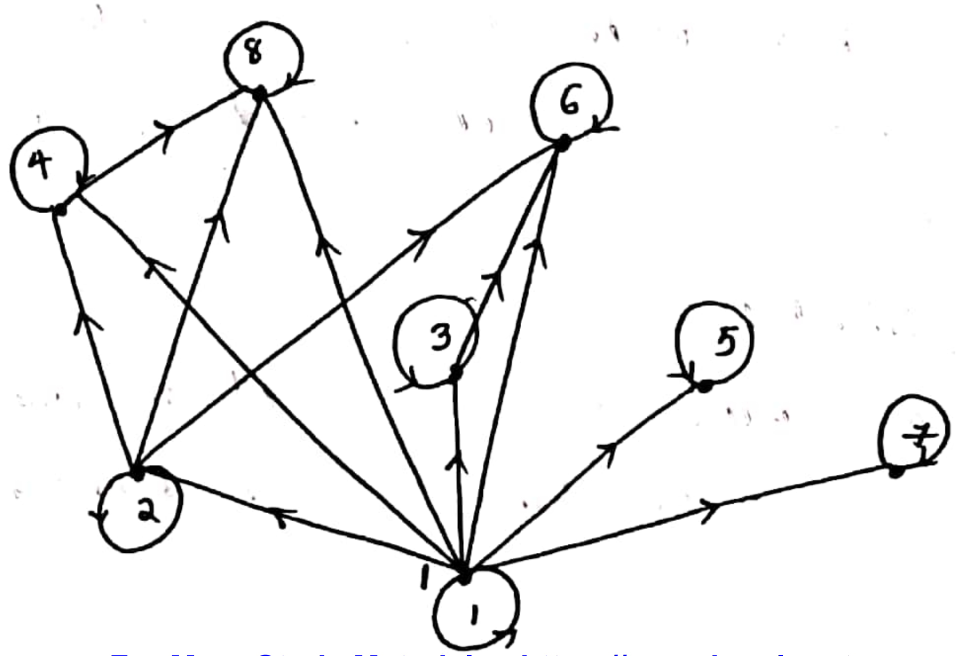

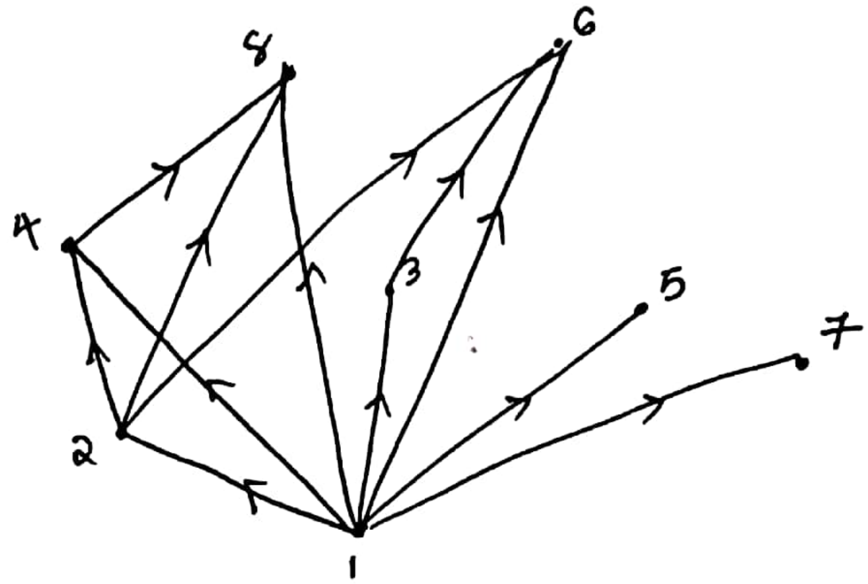

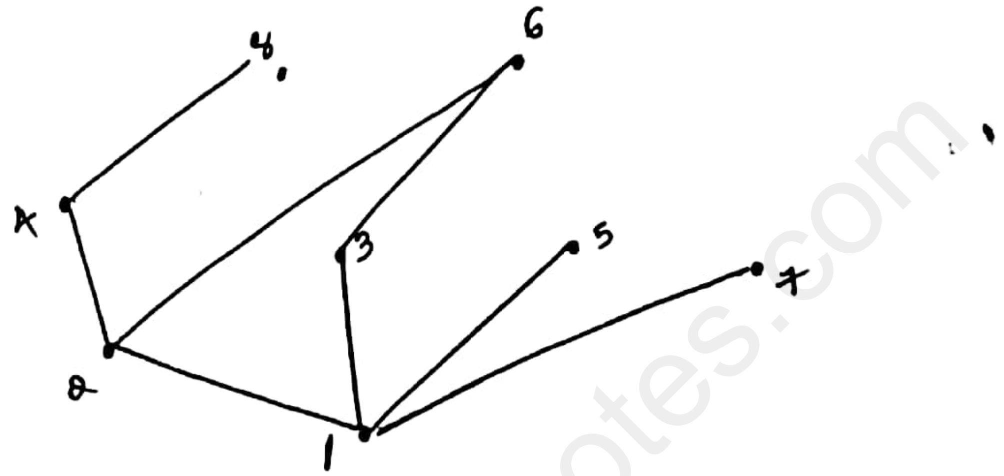

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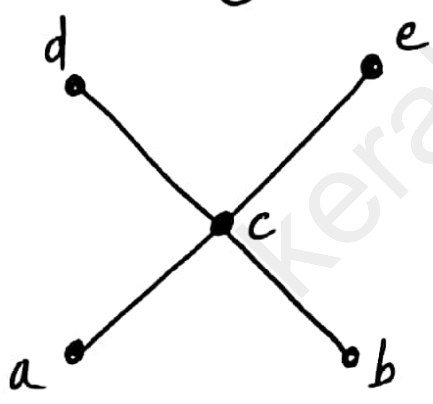
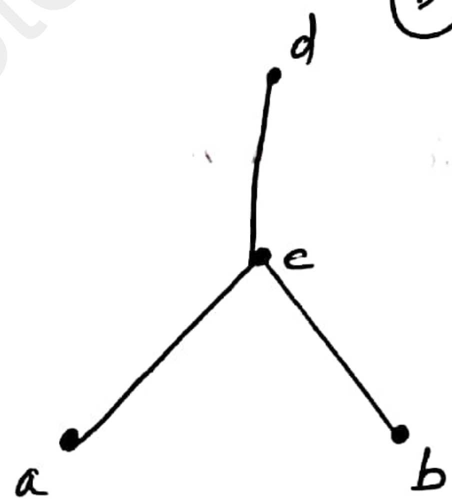
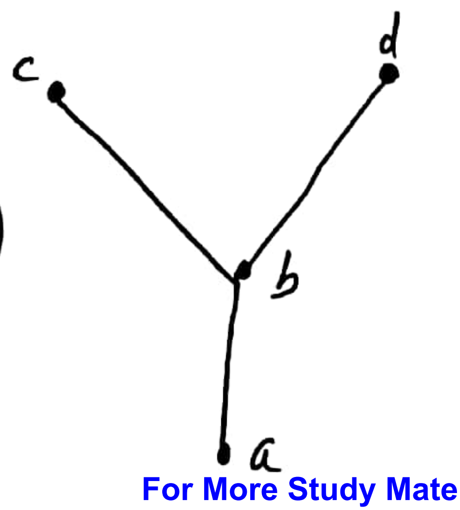


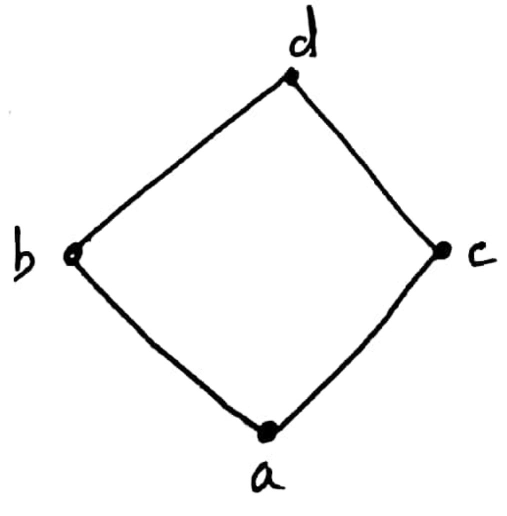
(b)



(c)



(d)



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* In figu (d) => a is the least element & d is the greatest element.

DEFINITIONS :

when A is a subset of a poset $\{P, \preceq\}$ and if u is an element of P such that $a \preceq u$ for all elements $a \in A$, then u is called an <u>upper bound</u> of A.

Similiarly if l is an element of P such that $l \preceq a$ for all elements $a \in A$, then l is called a <u>lower bound</u> of A.

Note :-

The upper & lower bounds of a subset of a poset are not necessarily unique.

DEFINITION :

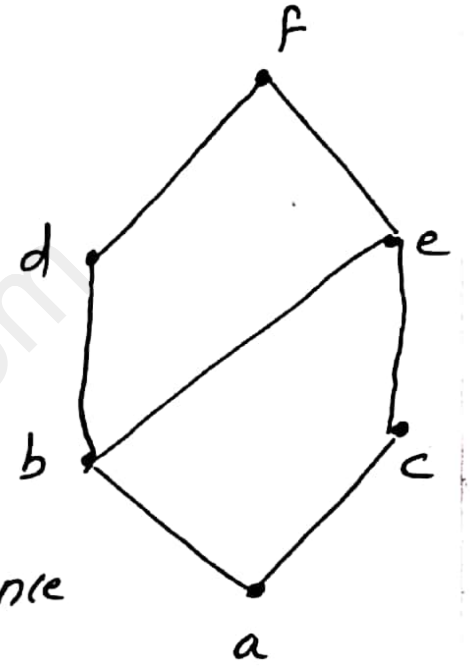
The element 'x' is called <u>least upper bound</u> (LUB) <u>or supremum</u> of the subset A of a poset $\{P, \preceq\}$, if x is an upperbound that is less than every other upper bound of A.

Similiarly the element 'y' is called <u>greatest lower bound</u> (GLB) or <u>infimum</u> of the subset A of a poset $\{P, \preceq\}$ if y is a lower bound that is greater than every other lower bound of A.

Note:-

The LUB & GLB, of a subset of a poset if they exist, are unique.

example:-



Let us Consider the poset with the Hasse diagram

The upper bounds of a subset {a, b, c} are e & f.

[∵ d is not an upper bound, Since c is not related to d].

and LUB of {a, b, c} is e.

The lower bounds of a subset ~~{a, b, c}~~ {d, e} are a and b and GLB of {d, e} is b. [c is not a lower bound, Since c is not related to d].

LATTICES:

A partially ordered set $\{L, \leq\}$ in which every pair of elements has a least upper bound and a greatest lower bound is called a lattice.

The LUB (supremum) of a subset $\{a,b\} \subseteq L$ is denoted by $a \vee b$ [or $a \oplus b$ or $a+b$ or $a \cup b$] and is called the join or sum of a and b.

The GLB (infemum) of a subset $\{a,b\} \subseteq L$ is denoted by $a \wedge b$ [or $a * b$ or $a \cdot b$ or $a \cap b$] is called meet or product of a and b.

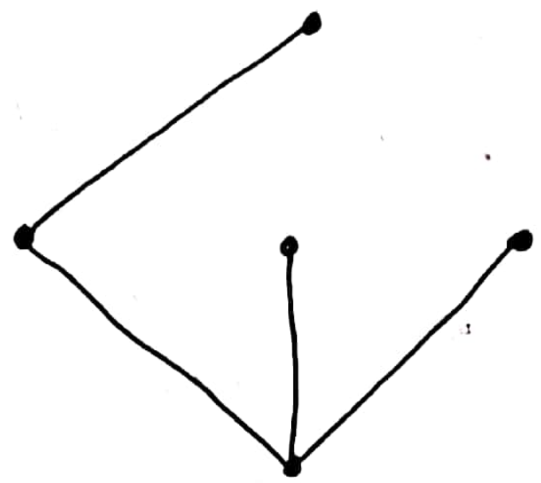
example :-

Consider the poset $(\{1,2,3,4,8,16\}, /)$ where / means "divisor of." The Hasse diagram is given by,

It is a lattice, since the LUB of any two elements of this poset is obviously the larger of them. and GLB of any two elements is the smaller of them.

Hence this poset is a lattice.

ex:- consider the poset $(\{1, 2, 3, 4, 5\}, |)$ whose diagram is given by



The LUB of the pairs $(2, 3)$ and $(3, 5)$ do not exist and hence do not have LUB.

Hence this poset is not a lattice.

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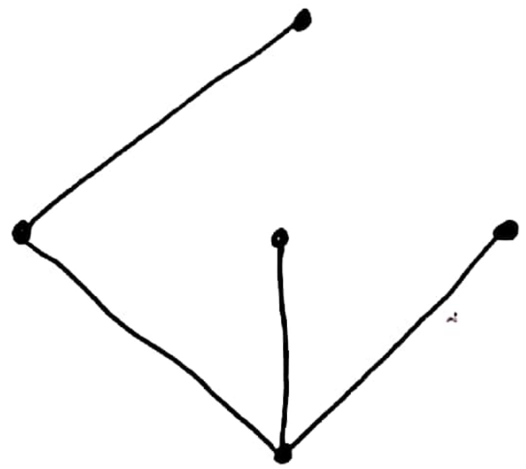


The LUB of the pairs (2, 3) and (3, 5) do not exist and hence do not have LUB.

Hence this poset is not a lattice.

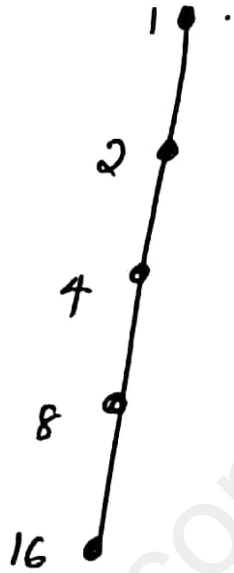
DUALITY PRINCIPLE:-

When $\leq$ is a partial ordering relation on the set S, the converse $\geq$ is also a partial ordering relation on S.

For ex:-

if $\leq$ denote "divisor of", then $\geq$ denotes "multiple of".

For ex:— the Hasse diagram of the poset $(\{1, 2, 4, 8, 16\}, \text{'division of'} \leq)$ is, then, Hasse diagram of the poset $(\{1, 2, 4, 8, 16\}, \text{'multiple of'} \geq)$ is



From this example, it is obvious that LUB of a set A with respect to $\leq$ is the same as GLB of the set A with respect to $\geq$, and vice versa.

i.e LUB & GLB are interchanged.

Duality principle:

In the case of lattices, if $\{L, \leq\}$ is a lattice, so also is $\{L, \geq\}$. Also the operations of join & meet on $\{L, \leq\}$ become the operations of meet and join respectively on $\{L, \geq\}$.

It is known as Duality principle.

The lattices $\{L, \leq\}$ & $\{L, \geq\}$ are called duals of each other.

PROPERTIES OF LATTICES:

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Property : 1

If $\{L, \preceq\}$ is a lattice, then for any $a, b, c \in L$,

(i) $L_1 : a \vee a = a$ or $(L_1)' : a \wedge a = a$ (Idempotency)

(ii) $L_2 : a \vee b = b \vee a$ or $(L_2)' : a \wedge b = b \wedge a$ (Commutative)

(iii) $L_3 : a \vee (b \vee c) = (a \vee b) \vee c$ or $(L_3)' : a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (Associative)

(iv) $L_4 : a \vee (a \wedge b) = a$ or $(L_4)' : a \wedge (a \vee b) = a$ (Absorption).

Proof:

(i) $a \vee a = LUB$
$(a, a) = LUB$
$= a$ Hence L_1 follows,

(ii) $a \vee b = LUB$
$(a, b) = LUB$
$(b, a) = b \vee a$ $\{\because LUB\ (a, b)$ is unique$\}$.

Hence L_2 follows,

(iii), Since, $(a \vee b) \vee c$ is the LUB of $\{(a \vee b), c\}$

we have, $a \vee b \preceq (a \vee b) \vee c \longrightarrow (1)$

$c \preceq (a \vee b) \vee c \longrightarrow (2)$

or

and

Since, $a \vee b$ is the LUB of (a, b), we have

$$a \preceq a \vee b \longrightarrow (3) \quad \&$$
$$b \preceq a \vee b \longrightarrow (4)$$

from (1) & (3) $\Rightarrow a \preceq (a \vee b) \vee c \longrightarrow (5)$ — by transitivity

from (1) & (4) $\longrightarrow b \preceq (a \vee b) \vee c \longrightarrow (6)$ — by transitivity.

from (2) & (6) $\Rightarrow b \vee c \preceq (a \vee b) \vee c \longrightarrow (7)$ by definition of join

from (5) & (7) $\Rightarrow a \vee (b \vee c) \preceq (a \vee b) \vee c \longrightarrow (8)$ by definition of join

Similarly, $a \preceq a \vee (b \vee c) \longrightarrow (9)$

$b \preceq b \vee c \preceq a \vee (b \vee c) \longrightarrow (10)$

& $c \preceq b \vee c \preceq a \vee (b \vee c) \longrightarrow (11)$

from (9) & (10), $a \vee b \preceq a \vee (b \vee c) \longrightarrow (12)$

from (11) & (12) $(a \vee b) \vee c \preceq a \vee (b \vee c) \longrightarrow (13)$

from (8) & (13). by anti-symmetry of $\preceq$, we get,

$$a \vee (b \vee c) = (a \vee b) \vee c$$

Hence, L_3 follows.

(iv) Since, $a \wedge b$ is the GLB of (a, b), we have,

$$a \wedge b \preceq a \longrightarrow (1)$$

Also $$a \preceq a \longrightarrow (2)$$

from (1) & (2), $a \vee (a \wedge b) \preceq a \longrightarrow (3)$

Also $a \preceq a \vee (a \wedge b) \longrightarrow (4)$

By definition of LUB.

$\therefore$ from (3) & (4), by anti-symmetry,

we get $a \vee (a \wedge b) = a$

Hence L_4 follows.

Property : 2

If $\{L, \preceq\}$ is a lattice, in which $\vee$ and $\wedge$ denote the operations of join & meet respectively, then for $a, b, \in L$, $a \preceq b \Longleftrightarrow a \vee b = b \Longleftrightarrow a \wedge b = a$

In other words,

(i) $a \vee b = b$ if and only if $a \preceq b$

(ii) $a \wedge b = a$ if and only if $a \preceq b$

(iii) $a \wedge b = a$ if and only if $a \vee b = b$.

Proof

Let $a \preceq b$

Now $b \preceq b$ (by reflexivity)

$\therefore a \vee b \preceq b \longrightarrow (1)$

Since $a \vee b$ is the LUB of (a, b)

$b \preceq a \vee b \longrightarrow (2)$

from (1) & (2) we get $a \vee b = b. \longrightarrow (3)$

Conversely let,

$$a \vee b = b$$

Since $a \vee b$ is the $LUB(a,b)$

$$a \leq a \vee b$$

ie $a \leq b$ (by assumption) $\longrightarrow$ (4)

from (3) & (4) result (i) follows.

* Result (ii) can be proved in a way similar to (i) &
from (i) & (ii) result (iii) follows.

Property : 3 (Isotonic property).

If $\{L, \leq\}$ is a lattice, then for any $a, b, c, \in L$, the following properties holds.

If $b \leq c$, then (i) $a \vee b \leq a \vee c$ &
(ii) $a \wedge b \leq a \wedge c$.

Proof

Since $b \leq c$, $b \vee c = c$ (by property: 2 (i).)

Also, $a \vee a = a$ by Idempotent property.

Now, $a \vee c = (a \vee a) \vee (b \vee c)$ by above steps
$= a \vee (a \vee b) \vee c$, by associativity
$= a \vee (b \vee a) \vee c$, by Commutativity
$= (a \vee b) \vee (a \vee c)$, by associativity

This is the form $x \vee y = y$, $\therefore x \leq y$ by property 2 (i) ie $a \vee b \le a \vee c$, which is the required result (i)

Similarly, result (ii) can be proved.

* Property : 4 (Distributive, Inequalities)

If $\{L, \le\}$ is a lattice, then for any $a, b, c \in L$,

(i) $a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c)$

(ii) $a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$.

Proof

Since $a \wedge b$ is the GLB (a, b), $a \wedge b \le a \longrightarrow (1)$

Also, $a \wedge b \le b \le b \vee c \longrightarrow (2)$

Since $b \vee c$ is the LUB of b & c

From (1) & (2), we have $a \wedge b$ is the lower bound of $\{a, b \vee c\}$

$\therefore a \wedge b \le a \wedge (b \vee c)$

Similarly $a \wedge c \le a$

& $a \wedge c \le c \le b \vee c$

~~$a \wedge c \le a \le$~~

$\therefore$ a. $a \wedge c \le a \wedge (b \vee c) \longrightarrow (4)$

From (3) & (4), we get,

$(a \wedge b) \vee (a \wedge c) \le a \wedge (b \vee c)$

ie $a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c)$, which is the result (i)

Result (ii) follows, by the principle of duality.

* Property : 5 : (Modular Inequality):

If $\{L, \leq\}$ is a lattice, then for any $a, b, c \in L$, $a \leq c \iff a \vee (b \wedge c) \leq (a \vee b) \wedge c$.

Proof :-

Since $a \leq c$, $a \vee c = c \longrightarrow$ (1) {∵ by property 2 (i)}

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c) \longrightarrow$ (2) {∵ by property-4 (ii)}

ie $a \vee (b \wedge c) \leq (a \vee b) \wedge c \longrightarrow$ (3) {by (1)}

Now,

Conversely assume, $a \vee (b \wedge c) \leq (a \vee b) \wedge c$

∴ $a \leq a \vee (b \wedge c) \leq (a \vee b) \wedge c \leq c$ – by definition of LUB and GLB

ie $a \leq c \longrightarrow$ (4)

from (3) & (4), we get,

$a \leq c \iff a \vee (b \wedge c) \leq (a \vee b) \wedge c$.

SUBLATTICES:

A non empty subset M of a Lattice. $\{L, \vee, \wedge\}$ is called a subset lattice of L, iff M is closed under both operations $\vee$ and $\wedge$.

ie if $a, b \in M$ then $a \vee b$ & $a \wedge b$ also $\in M$.

" From the definition, the sublattice itself is a lattice with respect to $\vee$ and $\wedge$.

example:—

if aRb whenever a divides b, where $a, b \in Z^+$ (the set of all positive integers) then $\{Z^+, R\}$ is a lattice in which $a \vee b$ = LCM of (a, b) (least common multiple) and

$a \wedge b$ = GCD of (a, b). (greatest common divisor).

If $\{S_n, R\}$ is the lattice of divisors of any positive integer n, then $\{S_n, R\}$ is a sublattice of $\{Z^+, R\}$.

SOME SPECIAL LATTICES:

BOUNDED LATTICE

A lattice L is said to have a lower bound denoted by 0, if $0 \leq a$ for all $a \in L$. Similarly L is said to have, an upper bound denoted by 1, if $a \leq 1$ for all $a \in L$.

Then the lattice L is said to be <u>bounded lattice</u> if it has both a lower bound 0 and an upper bound 1.

ie For any $a \in L$,

$$a \vee 1 = 1 \quad \& \quad a \wedge 1 = a \quad \&$$

$$a \vee 0 = a \quad \& \quad a \wedge 0 = 0$$

DISTRIBUTIVE LATTICE

A Lattice, $\{L, \vee, \wedge\}$ is called distributive lattice, if for any elements $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \&$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

In other words, if the operations $\vee$ and $\wedge$ distribute over each other in a lattice, it is said to be distributive lattice.

* **COMPLETE LATTICE**

A lattice $\{L, \vee, \wedge\}$ is said to be complete lattice if every non-empty subset of L has a least upperbound & greatest lower bound.

* **COMPLEMENT OF A LATTICE**

If $\{L, \vee, \wedge, 0, 1\}$ is a bounded lattice, and $a \in L$, then an element $b \in L$ is called a complement of a, if

$$a \vee b = 1 \quad \& \quad a \wedge b = 0$$

Since $0 \vee 1 = 1$ & $0 \wedge 1 = 0$

So, 0 and 1 are complements of each other.