KeralaNotes

KTU STUDY MATERIALS

DISCRETE MATHEMATICAL STRUCTURES

MAT 203

Module 2

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FUNDAMENTALS OF LOGIC

Logic is the discipline that deals with the methods of reasoning. One of the aims of logic is to provide us rules by which, we can determine whether a particular reasoning or argument is valid.

Rules of logic are used to provide proofs of theorems in mathematics. In this chapter we shall introduce certain logical symbols using which we shall state and apply rules of valid inference, and hence understand how to construct correct mathematical arguments.

PROPOSITIONS :

A declarative sentence (or assertion) which is true or false, but not both is called a preposition (or statement).

Lower case letters such as p, q, r are used to denote prepositions.

For, example,

consider, the following sentences,

1. New Delhi is the Capital city of India
2. How beautiful is Rose
3. $2+2=3$
4. what time is it ?
5. $x+y=z$

In the given statements, (2), (4) are ~~not~~ obviously not propositions as they are not declarative in nature.

(1) and (3) are propositions. but 5 is not, since (1) is true, (3) is false, and (5) is neither true nor false as the values of x, y, z are not assigned.

If a proposition is true, we say that the truth value of that proposition is true., denoted by 'T' or 1.

If a proposition is false, we say that, the truth value is false and denoted by 'F' or 0.

CONNECTIVES:

DEFINITION:

when p and q are any two prepositions, the preposition 'p and q' denoted by $p \wedge q$ called Conjunction of p and q is defined as the Compound preposition, that is true, when both p and q are true, and is false otherwise.

A truth table that displays the relationships between the truth values of Sub-prepositions and that of Compound preposition constructed for them.

Truth table for '$p \wedge q$'

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

DEFINITION:

when p and q are any two prepositions, the preposition 'p or q' denoted by $p \vee q$ is called disjunction of p and q is defined as the compound preposition, that is false, when both p and q are false, and is true otherwise.

The truth table for disjunction is

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

DEFINITION :

Given any proposition p, another preposition formed by writing "It is not the case that" or "It is false that" before p or by inserting the word 'not' suitable in p is called the ~~exp~~ the negation of p and denoted by $\neg p$. / $\sim p$ / p'.

If p is true, then $\neg p$ is false
If p is false, then $\neg p$ is true.

p	$\neg p$
T	F
F	T

Example :

If p is the statement, "New Delhi is in India", then $\neg p$ is any of the following statements,

(1) It is not the case that "New Delhi is in India"

(2) It is false that, New Delhi ~~In~~ is India

(3) NewDelhi is not in India.

DEFINITION:

If p and q are propositions, the Compound proposition, "if p, then q" that is denoted by $p \rightarrow q$ is called a conditional proposition., which is False when p is true and q is false. and true otherwise.

In this Conditional proposition, p is called hypothesis or premise & q is called Conclusion or consequence.

The truth table for $p \rightarrow q$ is

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

DEFINITION:

If p and q are propositions, the Compound proposition "p if and only if q" that is denoted by $p \leftrightarrow q$ is called biconditional proposition. which is true, when p and q have the same truth values. and is false otherwise.

Alternatively, $p \leftrightarrow q$ is also expressed as p iff q and "p is necessary and sufficient for q"

Alternatively, $p \leftrightarrow q$ is also expressed as "p iff q" and "p is necessary and sufficient for q"

P	q	$P \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

TUATOLOGY & CONTRADICTION

A Compound proposition $P = P(P_1, P_2 \cdots P_n)$ where $P_1, P_2, \ldots P_n$ are variables (elemental propositions), is called a tautology if it is true for every truth assignment for $P_1, P_2 \ldots P_n$.

P is called a Contradiction, if it is false for every truth assignment for $P_1, P_2 \ldots P_n$.

Example:

$P \vee \neg P$ is a tautology, where as $P \wedge \neg P$ is a contradiction. as seen from the table.

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	F	T	F
F	T	T	F

EQUIVALANCE OF PROPOSITIONS:

Two Compound propositions $A(P_1, P_2 \ldots P_n)$ and $B(P_1, P_2 \ldots P_n)$ are said to be logically equivalent or simply, equivalent, if they have identical truth values tables,

The equivalence of A two propositions, A and B is denoted as $A \Longleftrightarrow B$ or $A \equiv B$

Example:

Consider the truth table of $\sim(P \vee q)$ and $\sim P \wedge \sim q$ final coloums of truth table are Identical.

Hence $\neg(P \vee q) \equiv \sim P \wedge \sim q$

Table: 1

P	q	$P \vee q$	$\sim(P \vee q)$	$\sim P$	$\sim q$	$\sim P \wedge \sim q$
T	T	T	F	F	F	F
T	F	~~F~~	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

NOTE:

The biconditional proposition, $A \longleftrightarrow B$ is true, whenever both A and B have the same truth value,

$A \longleftrightarrow B$ is a tautology when A and B are equivalent.

Conversely, $A \equiv B$, when $A \leftrightarrow B$ is a tautology.

For example: $(p \rightarrow q) \equiv (\neg p \vee q)$ since $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$ is a tautology., (displayed in table).

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$	$(p \rightarrow q) \leftrightarrow \neg p \vee q$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

ORDER FOR LOGICAL CONNECTIVES:

* The negation operator has precedence over all other logical operators.

  ie $\neg p \wedge q$ means $(\neg p) \wedge q$ not $\{\neg(p \wedge q)$

* The conjunction operator has precedence over the disjunction operator.

  ie $p \wedge q \vee r$ means $(p \wedge q) \vee r$

  but not $p \wedge (q \vee r)$

* The Conditional & biconditional operators, $\rightarrow$ & $\leftrightarrow$,

  Here $\rightarrow$ has precedence over $\leftrightarrow$

DUALITY LAW: (Imp)

The dual of a Compound proposition, that Contains only the logical operators $\vee$, $\wedge$ and $\sim$ is the proposition obtained by replacing each $\vee$ by $\wedge$ & each $\wedge$ by $\vee$, each T by F and each F by T, where T and F are special variables, representing Compound propositions that are tautologies and Contradictions respectively,

The dual of a proposition A is denoted by A^*.

DUALITY THEOREM: (DUALITY PRINCIPLE)

If $A(P_1, P_2, \ldots\ldots P_n) \equiv B(P_1 . P_2 . \ldots . P_n)$

where A and B are Compound propositions, then,

$A^*(P_1, P_2, \ldots\ldots P_n) \equiv B^*(P_1\ P_2 \ldots\ldots P_n)$.

Proof:

In above table (Table: 1), we have proved that

$$\left.\begin{aligned} \sim(p \vee q) &\equiv \sim p \wedge \sim q \\ \text{or} \quad p \vee q &\equiv \sim(\sim p \wedge \sim q) \end{aligned}\right\} \longrightarrow (1)$$

Similarly, we can prove that

$$p \wedge q \equiv \sim(\sim p \vee \sim q) \longrightarrow (2)$$

using ① & ②, we can show that,

$$\neg A(P_1, P_2, \dots P_n) \equiv A^*(\neg P_1, \neg P_2, \dots \neg P_n) \rightarrow ③$$

Equation ③ means that, the negation of a proposition is equivalent to its dual in which every variable (is replaced by it's negation.

③ follows that

$$A(P_1, P_2 \dots P_n) \equiv \neg A^*(\neg P_1, \neg P_2, \dots \neg P_n) \rightarrow ④$$

Now, Since $A(P_1 P_2 \dots P_n) \equiv B(P_1 P_2 \dots P_n)$

we have $A(P_1 P_2 \dots P_n) \longleftrightarrow B(P_1 P_2 \dots P_n)$ is a tautology.

$\therefore A(\neg P_1, \neg P_2 \dots \neg P_n) \longleftrightarrow B(\neg P_1 \neg P_2 \dots \neg P_n)$

is also a tautology. $\longrightarrow$ ⑤

using ④ & ⑤, we get,

$\neg A^*(P_1 P_2 \dots P_n) \longleftrightarrow \neg B^*(P_1 P_2 \dots P_n)$ is a tautology.

$\therefore A^* \longleftrightarrow B^*$ is a tautology

$\therefore A^* \equiv B^*$

⇒ LAWS OF ALGEBRA OF PROPOSITIONS :

	* Name of law	primal form	Dual form :
1.	Idempotent law	$p \vee p \equiv p$	$p \wedge p \equiv p$
2.	Identity law	$p \vee F \equiv p$	$p \wedge T \equiv p$
3.	Dominant law	$p \vee T \equiv T$	$p \wedge F \equiv F$
4.	complement law	$p \vee \neg p \equiv T$	$p \wedge \neg p \equiv F$
5.	commutative law	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
6.	Associative law	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
7.	Distributive law	$p \vee (q \wedge r) \equiv (p \vee q) \wedge p(\vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (q \wedge r)$
8.	Absorption law	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
9.	Demorgan law	$\neg (p \vee q) \equiv \neg p \wedge \neg q$	$\neg (p \wedge q) \equiv \neg p \vee \neg q$

→ Problems:

Construct a truth table for each of the following Compound propositions.

(a) $(p \vee q) \longrightarrow (p \wedge q)$

(b) $(p \rightarrow q) \longrightarrow (q \rightarrow p)$

(c) $(q \longrightarrow \neg p) \longleftrightarrow (p \longleftrightarrow q)$

(d) $(p \longleftrightarrow q) \longleftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$

(a) $(p \vee q) \longrightarrow (p \wedge q)$

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \longrightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

(b) $(p \rightarrow q) \longrightarrow (q \rightarrow p)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \rightarrow (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	F	T	T

(c) $(q \longrightarrow \sim p) \longleftrightarrow (p \longleftrightarrow q)$

p	q	$\sim p$	$q \rightarrow \sim p$	$p \leftrightarrow q$	$(q \rightarrow \sim p) \leftrightarrow p \leftrightarrow q$.
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

Q. Construct the truth table for each of the compound propositions given as follows?

(a) $(p \rightarrow (q \rightarrow r)) \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)$

(b) $\neg(p \vee (q \wedge r)) \longleftrightarrow ((p \vee q) \wedge (p \rightarrow r))$

(c) $(\neg p \leftrightarrow \neg q) \longleftrightarrow (q \leftrightarrow r)$

(d) $(p \rightarrow (q \rightarrow s)) \wedge (\neg r \vee p) \wedge q$

(e) $(p \rightarrow q) \rightarrow r) \rightarrow s$

A. (a) Truth table for $(p \rightarrow (q \rightarrow r)) \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)$

p	q	r	$p \rightarrow q$	$p \rightarrow r$	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$(p \rightarrow q) \rightarrow (p \rightarrow r)$	$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Q. Construct truth table for the following compound propositions.

① $\neg(p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \rightarrow r))$ $\quad \neg a \leftrightarrow b$

p	q	r	$q \wedge r$	$p \vee (q \wedge r) \equiv a$	$\neg a$	$p \vee q$	$p \rightarrow r$	$(p \vee q) \wedge (p \rightarrow r) \equiv b$	$\neg a \leftrightarrow b$
T	T	T	T	T	F	T	T	T	F
T	T	F	F	T	F	T	F	F	T
T	F	T	F	T	F	T	T	T	F
T	F	F	F	T	F	T	F	F	T
F	T	T	T	T	F	T	T	T	F
F	T	F	F	F	T	T	T	T	T
F	F	T	F	F	T	F	T	F	F
F	F	F	F	F	T	F	T	F	T

② $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$

p	q	r	$\neg p$	$\neg q$	$\neg p \leftrightarrow \neg q \equiv a$	$q \leftrightarrow r \equiv b$	$a \leftrightarrow b$
T	T	T	F	F	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	T	F	F	T
T	F	F	F	T	F	T	F
F	T	T	T	F	F	T	F
F	T	F	T	F	F	F	T
F	F	T	T	T	T	F	F
F	F	F	T	T	T	T	T

(3) Truth table for $((p \rightarrow q) \rightarrow r) \rightarrow s$

p	q	r	s	$p \rightarrow q$	$(p \rightarrow q) \rightarrow r$	$((p \rightarrow q) \rightarrow r) \rightarrow s$
T	T	T	T	T	T	T
T	T	T	F	T	T	F
T	T	F	T	T	F	T
T	T	F	F	T	F	T
T	F	T	T	F	T	T
T	F	T	F	F	T	F
T	F	F	T	F	T	T
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	T	T	F	T	T	F
F	T	F	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	T	T
F	F	T	F	T	T	F
F	F	F	T	T	F	T
F	F	F	F	T	F	T

1. Determine which of the following compounds propositions are tautologies and which of them are contradictions using truth tables?

(a) $(\neg q \wedge (p \rightarrow q)) \rightarrow p$

(b) $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow p \rightarrow r)$

(c) $\neg (q \rightarrow r) \wedge r \wedge (p \rightarrow q)$.

(d) $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow r$

A. (a) Truth table for $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \wedge (p \rightarrow q)$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

⇒ Equivalences Involving Conditionals:

1. $p \rightarrow q \equiv \neg p \vee q$
2. $p \rightarrow q \equiv \neg q \rightarrow \neg p$
3. $p \vee q \equiv \neg p \rightarrow q$

4. $p \vee q \equiv \neg p \rightarrow q$

5. $\neg(p \rightarrow q) \equiv p \wedge \neg q$

6. $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

7. $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

8. $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

9. $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Equivalences Involving biconditionals:

1. $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

2. $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

3. $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

4. $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$.

Q. With out using truth table's prove the following ?

Q. (i) $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \equiv p \wedge q$.

(ii) $p \longrightarrow (q \rightarrow \neg p) \equiv \neg p \longrightarrow (p \longrightarrow q)$

(iii) $\neg (p \longleftrightarrow q) \equiv (p \vee q) \wedge \neg(p \vee q) \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

(Ans)

Ans. (i) $(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \equiv (\neg p \vee q) \wedge (p \wedge p) \wedge q$

(by associative law)

$\equiv (\neg p \vee q) \wedge (p \wedge q)$ by Idempotent law.

$\equiv (p \wedge q) \wedge (\neg p \vee q)$ by Commutative law

$\equiv ((p \wedge q) \wedge \neg p) \vee ((p \wedge q) \wedge q)$ by distributive law.

$\equiv (\neg p \wedge (p \wedge q)) \vee ((p \wedge q) \wedge q)$ by Commutative law.

$\equiv ((\neg p \wedge p) \wedge q) \vee (p \wedge (q \wedge q))$ by associative law

$\equiv (F \wedge q) \vee (p \wedge q)$ by Complement & Idempotent law

$\equiv F \vee (p \wedge q)$ by dominent law.

$\equiv p \wedge q$, by dominent law.

(ii) $p \to (q \to p) \equiv \neg p \vee (q \to p)$ {using the results in equivalence conditions)
$\equiv \neg p \vee (\neg q \vee p)$

$\equiv \neg q \vee (p \vee \neg p)$ by Commutative & associative laws.

$\equiv \neg q \vee T$ by Complement law.

$\equiv T$, by dominent law. $\to$ (1)

$\neg p \to (p \to q) \equiv \cancel{\neg p} \vee (\cancel{q \to p})$
$\equiv p \vee (p \to q)$ {using results in equivalence condition.
$\equiv p \vee (\neg p \vee q)$

$\equiv (p \vee \neg p) \vee q$ by Associative law.

$\equiv T \vee q$ by Complement law.

$\equiv T$ by dominent law. $\to$ (2)

from (1) & (2) result follows.

(iii) $\neg(p \leftrightarrow q) \equiv \neg((p \to q) \wedge (q \to p))$ {by using the result in equivalence involving biconditionals).

$\equiv \neg((\neg p \vee q) \wedge (\neg q \vee p))$ {using the results in equivalence using conditionals)

$\equiv \neg[((\neg p \vee q) \wedge \neg q) \vee ((\neg p \vee q) \wedge p)]$ by distributive law

$\equiv \neg[((\neg p \wedge \neg q) \vee (q \wedge \neg q)) \vee ((\neg p \wedge p) \vee (q \wedge p))]$

by distributive law.

$\equiv \neg[((\neg p \wedge \neg q) \vee F) \vee ((F \vee (q \wedge p))]$ by Complement law.

$\equiv \neg[(\neg p \wedge \neg q) \vee (q \wedge p)]$ by Identity law.

$\equiv \neg[\neg(p \vee q) \vee (q \wedge p)]$ by demorgans law.

$\equiv (p \vee q) \wedge \neg(q \wedge p)$ by demorgan's law. $\longrightarrow$ (1)

$\equiv (p \vee q) \wedge (\neg q \vee \neg p)$ by demorgans law.

$\equiv ((p \vee q) \wedge \neg q)) \vee ((p \vee q) \wedge \neg p)$ by distributive law

$\equiv ((p \wedge \neg q) \vee (q \wedge \neg q)) \vee (p \wedge \neg p) \vee (q \wedge \neg p))$ by distributive law

$\equiv ((p \wedge \neg q) \vee F) \vee (F \vee (q \wedge \neg p))$ by Complement law.

$\equiv (p \wedge \neg q) \vee (q \wedge \neg p)$, by Identity law.

$\equiv \cancel{(p \wedge \neg q) \vee (q \wedge \neg p)}$

$\equiv (p \wedge \neg q) \vee (\neg p \wedge q)$ by Commutative. $\longrightarrow$ (2)

from (1) & (2) result follows.

Q. with out Constructing truth table, prove the following

① $\neg p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \vee r)$

$\{ p \vee q \equiv \neg p \rightarrow q \}$

$\neg p \rightarrow (q \rightarrow r) \equiv p \vee (q \rightarrow r)$ {3^{rd} result in conditionals}.

$\equiv p \vee (\neg q \vee r)$ { 1^{st} result }

$\equiv (p \vee \neg q) \vee r$, by associative.

$\equiv (\neg q \vee p) \vee r$, by Commutative.

$\equiv \neg q \vee (p \vee r)$, by associative.

$\equiv q \rightarrow (p \vee r)$ {by 3^{rd} result in conditionals}

② $p \rightarrow (q \rightarrow r) \equiv p \rightarrow (\neg q \vee r) \equiv (p \wedge q) \rightarrow r$

$p \rightarrow (q \rightarrow r) \equiv p \rightarrow (\neg q \vee r)$ { by result in 3^{rd} conditionals

$\equiv \neg p \vee (\neg q \vee r)$ {by result in Conditionals}

$\equiv (\neg p \vee \neg q) \vee r$, by associative law.

$\equiv \neg (p \wedge q) \vee r$ by demorgans law

$\equiv p \wedge q \rightarrow r$ by result in Conditionals.

Q. prove the following equivalences by proving the equalences of the duals?

① $\neg ((\neg p \wedge q) \vee (\neg p \wedge \neg q)) \vee (p \wedge q) \equiv p$

The dual of the given equivalence is

$\cancel{\neg((\neg p \vee q)}$

$\neg((\neg p \vee q) \wedge (\neg p \vee \neg q)) \wedge (p \vee q) \equiv p$

Now, let us prove the dual equivalence.

L.H.S $= \neg(\neg p \vee (q \wedge \neg q)) \wedge (p \vee q)$, by distributive law.

$\equiv \neg(\neg p \vee (F)) \wedge (p \vee q)$ by Complement law

$\equiv \neg(\neg p) \wedge (p \vee q)$, by Identity law

$\equiv p \wedge (p \vee q)$

$\equiv p$, by absorption law.

(2) $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$

ie $\neg(p \vee q) \vee r \equiv (\neg p \vee r) \wedge (\neg q \vee r)$

Dual of this equivalence is

$\neg(p \wedge q) \wedge r \equiv (\neg p \wedge r) \vee (\neg q \wedge r)$

L.H.S $\equiv (\neg p \vee \neg q) \wedge r$ by De,morgan's law.

$\equiv (\neg p \wedge r) \vee (\neg q \wedge r)$ by distributive law.

$\equiv$ R.H.S

CONVERSE, INVERSE, & CONTRAPOSITIVE OF A IMPLICATION :

* The Converse of the implication $p \to q$ is $q \to p$
* The inverse of the implication $p \to q$ is $\sim p \to \sim q$.
* The Contrapositive of the implication $p \to q$ is $\sim q \to \sim p$.

Example :

(1) Let $p \to q$: If ABCD is a parallogram, then $AB = CD$.

Here p : ABCD is a parallogram

q : $AB = CD$.

- Converse : $q \to p$: If $AB = CD$, then ABCD is a parallogram.
- Inverse : $\sim p \to \sim q$: If ABCD is not a parallogram then $AB \neq CD$.
- contrapositive : $\sim q \to \sim p$: If $AB \neq CD$, then ABCD is not a parallogram.

(2) Eg: p.

$p \rightarrow q$: If Deepthi learns discrete mathematics. then she will find a good job.

Here p: Deepthi learns discrete mathematics.

q: she will find a good job.

- Converse : $q \rightarrow p$: If Deepthi will find a good job, then she learn Discrete mathematics.

Inverse : $\neg p \rightarrow \neg q$: If Deepthi does not learn mathematics, then she will not find a good job.

Contra-positive : $\neg q \rightarrow \neg p$: If Deepthi will not find a good job, then she does not learn Discrete mathematics.

TAUTOLOGICAL IMPLICATION:

A Compound proposition $A(P_1, P_2 \ldots\ldots P_n)$ is said to be "tautologically imply" or simply "imply" the compound proposition $B(P_1, P_2, \ldots\ldots P_n)$, if B is true, whenever A is true. or equivalently $A \rightarrow B$ is a tautology.

This is denoted as $A \Rightarrow B$.

read as "A implies B"

Example :

$$p \Rightarrow p \vee q$$

Here $p \vee q$ is true whenever p is true, so $p \rightarrow p \vee q$ is a tautology.

P	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Example:

$(p \rightarrow q) \Rightarrow (\neg q \rightarrow \neg p)$ from the following truth table:

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$	$(p \to q) \to (\neg q \to \neg p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Some important implications:

1. $p \wedge q \Rightarrow p$
2. $p \wedge q \Rightarrow q$
3. $p \Rightarrow p \vee q$
4. $q \Rightarrow p \to q$
5. $q \Rightarrow p \to q$
6. $\neg(p \to q) \Rightarrow p$
7. $\neg(p \to q) \Rightarrow \neg q$
8. $p \wedge (p \to q) \Rightarrow q$
9. $\neg q \wedge (p \to q) \Rightarrow \neg p$
10. $\neg p \wedge (p \vee q) \Rightarrow q$
11. $(p \to q) \wedge (q \to r) \Rightarrow p \to r$
12. $(p \vee q) \wedge (p \to r) \wedge (q \to r) \Rightarrow r$.

* NOTE:

If $A \Rightarrow B$ & $B \Rightarrow A$, then $A \equiv B$.

Q. prove the following implication by using truth tables.

(i) $p \rightarrow (p \rightarrow r) \Rightarrow (p \rightarrow q) \rightarrow (p \rightarrow r)$

p	q	r	$p \rightarrow r$ $\equiv a$	$p \rightarrow (p \rightarrow r)$ $\equiv b$	$p \rightarrow q$ $\equiv c$	$c \rightarrow a$ $\equiv d$	$b \rightarrow d$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Q. prove the following implications, with out using truth tables?

(1) $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \Rightarrow r$

$(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \rightarrow r$

$\equiv (p \vee q) \wedge ((p \vee q) \rightarrow r) \rightarrow r$

{ by the 6^{th} result in Conditionals,
ie $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$ }

$\equiv (p \vee q) \wedge (\neg(p \vee q) \vee r) \rightarrow r$

$\equiv (F \vee (p \vee q) \wedge r) \rightarrow r$

$\equiv ((p \vee q) \wedge r) \rightarrow r$

$\equiv \neg((p \vee q) \wedge r) \vee r$

$\equiv \neg((p \wedge r) \vee (q \wedge r)) \vee r$

$\equiv (\neg(p \wedge r) \wedge \neg(q \wedge r)) \vee r$

$\equiv (\neg(p \wedge r) \vee r) \wedge (\neg(q \wedge r) \vee r)$

$\equiv ((\neg p \vee \neg r) \vee r) \wedge ((\neg q \vee \neg r) \vee r)$

$\equiv (\neg p \vee T) \wedge (\neg q \vee T)$

$\equiv T \wedge T$

$\equiv T$

Q. prove the implication with out using truth table.

$$((p \vee \neg p) \rightarrow q) \rightarrow ((p \vee \neg p) \rightarrow r) \Rightarrow q \rightarrow r$$

A) $$[((p \vee \neg p) \rightarrow q) \rightarrow ((p \vee \neg p) \rightarrow r)] \rightarrow (q \rightarrow r)$$

$$\equiv [(T \rightarrow q) \rightarrow (T \rightarrow r)] \rightarrow (q \rightarrow r)$$

$$\equiv [(\neg T \vee q) \rightarrow (\neg T \vee r)] \rightarrow (q \rightarrow r)$$

$$\equiv [(F \vee q) \rightarrow (F \vee r)] \rightarrow (q \rightarrow r)$$

$$\equiv [(q \rightarrow r)] \rightarrow (q \rightarrow r)$$

$$\equiv T$$

THEORY OF INFERENCE

Inference theory is concerned with the inferring of a conclusion from certain hypothesis or basic assumptions, called premises by applying certain principles called rules of inference.

Any conclusion which is arrived at by following this rules is called valid argument.

RULES OF INFERENCE

Here we state two basic rules of Inference called rules P and T

Rule P : A premise may be introduced at any step in the derivation

Rule T : A formula S may be Introduced in the derivation.

Rules of Inferences :

* Example :

If it is raining (p), I will need an umberlla (q) — $p \to q$

It is raining (p) — p

therefore I will need an umberla — $\therefore q$

Modus ponens

Rules   Name of the rules

1. $(p \wedge q) \rightarrow p$ $(p \wedge q \Rightarrow p)$
   $(p \wedge q) \rightarrow q$ $(p \wedge q \Rightarrow q)$ $\rightarrow$ Simplification.

2. $p \rightarrow (p \vee q)$
   $q \rightarrow (p \vee q)$ $\rightarrow$ Addition.

3. $((p) \wedge (q)) \rightarrow (p \wedge q)$ $\rightarrow$ conjunction.

4. $[p \wedge (p \rightarrow q)] \rightarrow q$ $\rightarrow$ Modus ponens.

5. $[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$ $\rightarrow$ Modus tollens.

6. $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ $\rightarrow$ Hypothetical Syllogism.

7. $[(p \vee q) \wedge \neg p] \rightarrow q$ $\rightarrow$ disjunctive Syllogism

8. $[(p \vee q) \wedge (\neg p \vee q)] \rightarrow (q \vee r)$ $\rightarrow$ Resolution

9. $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$ – Dilemma

* example :

If it rains heavily, then travelling will be difficult. If students arrive on time, then travelling was not difficult. They arrived on time. Therefore it did not rain heavily.

Let the statements be defined as follows,

p : It rains heavily

q : Travelling is difficult

r : Students arrived on time.

Now, we have to show that premises $p \rightarrow q$, $r \rightarrow \neg q$ & r lead to the conclusion $\neg p$.

The form of argument gives as follows.

Step No :	Statement	Reason
1.	$p \rightarrow q$	Rule : P
2.	$\neg q \rightarrow \neg p$	contrapositive (Rule T)
3.	$r \rightarrow \neg q$	Rule P
4.	$r \rightarrow \neg p$	T, step 2, 3, hypothetical Syllogism
5.	r	Rule P

6.	$\neg p$	~~MT~~ steps
		4, 5 modus ponens.

Q. Show that $(t \wedge s)$ can be derived from the premises $p \to q$, $q \to \neg r$, r, $p \vee (t \wedge s)$.

A.

Step No :	Statement	Reason.
1.	$p \to q$	P
2.	$q \to \neg r$	P
3.	$p \to \neg r$	T, (1, 2 & hypothetical syllogism).
4.	$r \to \neg p$	T, (3 & $p \to q \equiv \neg q \to \neg p$)
5.	r	P
6.	$\neg p$	T, 4, 5 & modus ponens.
7.	$p \vee (t \wedge s)$	P
8.	$t \wedge s$	T, 6, 7 & Disjunctive syllogism.

Q. Show that, $a \vee b$ follows logically from the primises, $p \vee q$, $(p \vee q) \to \neg r$, $\neg r \to (s \wedge \neg t)$ & $(s \wedge \neg t) \to (a \vee b)$.

A.

Step: No	Statement	Reason :
1.	$(p \vee q) \to \neg r$	P
2.	$\neg r \to (s \wedge \neg t)$	P

3.	$(p \vee q) \rightarrow (s \wedge \neg t)$	T, 1, 2 & hypothetical syllogism.
4	$p \vee q$	P
5.	$s \wedge \neg t$	T, 3, 4 & Modus ponens
6.	$(s \wedge \neg t) \rightarrow (a \vee b)$	P
7.	$a \vee b$	T, 5, 6 & modus ponens.

Q. Show that $(a \rightarrow b) \wedge (a \rightarrow c), \neg(b \wedge c), (d \vee a) \Rightarrow d$

Step No:	Statement	Reason
1.	$(a \rightarrow b) \wedge (a \rightarrow c)$	P
2.	$a \rightarrow b$	T, 1 & simplication
3	$a \rightarrow c$	T, 1 & Simplication
4.	$\neg b \rightarrow \neg a$	T, 2 & Contrapositive
5.	$\neg c \rightarrow \neg a$	T, 3 & contrapositive
6.	$(\neg b \vee \neg c) \rightarrow \neg a$	T, 4 & 5
7.	$\neg(b \wedge c) \rightarrow \neg a$	T, & Demorgans law.
8.	$\neg(b \wedge c)$	P
9.	$\neg a$	T, 7 & 8 & Modus ponens.
10.	$d \vee a$	P

$\cancel{(d\vee a)\wedge\neg a}$

11.	$(d \vee a) \wedge \neg a$	T, 9, 10, & Conjunction
12.	$(d \wedge \neg a) \vee (a \wedge \neg a)$	T, 11 & distributive.
13.	$(d \wedge \neg a) \vee F$	T, 12 & negation law
14.	$(d \wedge \neg a)$	T, 13, identity law
15.	$\therefore d$	T, 14 & Simplification.

⇒ TRUTH TABLE TECHNIQUE :-

* example:

(i) $H_1 : \neg p$, $H_2 : p \vee q$, $C : q$

(ii) $H_1 : p \rightarrow q$, $H_2 : q$, $C : p$

p	q	$\neg p$	$p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	T

ans.
* (i) H_1 and H_2 are true only in third row, in which case C is also true. Hence (i) is valid.

(ii) H_1 and H_2 are true in the first & third rows, but C is not true in the third row, Hence (ii) is not a valid conclusion.

Q. Find whether the Conclusion C follows from the premises H_1, H_2, H_3 in the following cases, using truth table technique.

(i) $H_1 : \neg p$, $H_2 : p \vee q$, $C : p \wedge q$.

INCONSISTENT PREMISES:

A Set of premises (formulas) $H_1, H_2, \ldots H_n$ is said to be inconsistent, if their Conjunction implies a Contradiction.

ie $H_1 \wedge H_2 \wedge \ldots \wedge H_n \Rightarrow R \wedge \neg R$ for some formula R.

A set of premises is said to be Consistent, if it is not inconsistent.

⇒ ## INDIRECT METHOD OF PROOF:

The notion of inconsistency is used to derive a proof at times, this procedure is called indirect method of proof or proof by contradiction

ie To show that a conclusion C follows from the premises $H_1, H_2, \ldots H_n$ by this method, we assume that C is false and include $\neg C$ as an additional premise. If the new set of premises is inconsistent leading lead to a Contradiction, then the assumption $\neg C$ is true does not hold good. Hence C is true whenever $H_1 \wedge H_2 \wedge \ldots \wedge H_n$ is true.

Thus C follows from $H_1, H_2, \dots H_n$.

* Example :

we prove the premises $\neg q$, $p \rightarrow q$ result in Conclusion $\neg p$ by Indirect method

we now include $\neg\neg p$ as an additional premise The argument form is given below.

Step No :	Statement	Reason
1	$\neg\neg p$	P
2.	p	T double negation
3.	$p \rightarrow q$	P
4.	$\neg q \rightarrow \neg p$	T, contrapositive
5.	$\neg q$	P
6.	$\neg p$	T, Modus ponens 4, 5
7.	$p \wedge \neg p$	T conjunction 2, 6

Thus the inclusion of $\neg C$ leads to the contradiction

Q. Show that b can be derived from the premises $a \rightarrow b$, $c \rightarrow b$, $d \rightarrow (a \vee c)$, d by the Indirect method.

A. Let us include $\neg b$ as an additional premise & prove a contradiction.

Step No.	Statement	Reason
1.	$a \rightarrow b$	P
2.	$c \rightarrow b$	P
3.	$(a \vee c) \rightarrow b$	T, 1,2 & equivalence condition
4.	$d \rightarrow (a \vee c)$	P
5.	$d \rightarrow b$	T, 3,4 & hypothetical syllogism
6.	d	P
7.	b	T, 5, 6, modus ponens
8.	$\neg b$	P (additional)
9.	$b \wedge \neg b$	T, 7,8 conjunction
10.	F	T, 9 negation law

Q. Use indirect method to show that, $r \to \neg q$, $r \vee s$, $s \to \neg q$, $p \to q \Rightarrow \neg p$.

A. To use the Indirect method, we will include, $\neg\neg p \equiv p$ as an additional premise and prove a contradiction.

Step No:	Statement	Reason
1.	p	P
2.	$p \to q$	P
3.	q	T, 1, 2 modus ponens
4.	$r \to \neg q$	P
5.	$s \to \neg q$	P
6.	$(r \vee s) \to q$	T, 4, 5 equivalence condition
7.	$r \vee s$	P
8.	$\neg q$	T, 6, 7 modus ponens
9.	$q \wedge \neg q$	T, 3, 8, conjunction
10.	F	T, 9 negation law.

Q. Using indirect method of proof, derive $p \rightarrow \neg s$ from the premises $p \rightarrow (q \vee r)$, $q \rightarrow \neg p$, $s \rightarrow \neg r$, p.

A. Let us include $\neg(p \rightarrow \neg s)$ as an additional premise and prove a contradiction.

Now, $\neg(p \rightarrow \neg s) \equiv \neg(\neg p \vee \neg s) = p \wedge s$.

Hence the additional premise to be introduced may be taken as $p \wedge s$.

Step No	Statement	Reason.
1.	$p \rightarrow (q \vee r)$	P
2.	p	P
3.	$q \vee r$	T, 1,2 modus ponens.
4.	$p \wedge s$	P (additional)
5.	s	T, 4, simplification
6.	$s \rightarrow \neg r$	P
7.	$\neg r$	T, 5, 6 modus ponens
8.	q	T, 3, 7 disjunctive syllogism
9.	$q \rightarrow \neg p$	P
10.	$\neg p$	T, 8, 7, modus ponens
11.	$p \wedge \neg p$	T, 2, 10 Conjunction

12. | 12. | F | T, 11 & negation law.

Q. prove that the premises $p \rightarrow q$, $q \rightarrow r$, $s \rightarrow \neg r$ and $p \wedge s$ are inconsistent.

A. If we derive a contradiction, by using the given premises, it means that they are inconsistent.

Step No:	Statement	Reason.
1.	$p \rightarrow q$	P
2.	$q \rightarrow r$	P
3.	$p \rightarrow r$	T, 1, 2, & hypothetical syllogism.
4.	$s \rightarrow \neg r$	P
5.	$r \rightarrow \neg s$	T, 4 & contrapositive
6.	$q \rightarrow \neg s$	T, 2, 5 hypothetical syllogism
7.	$\neg q \vee \neg s$	T, 6 & equivalence of (6) $(p \rightarrow q \equiv \neg p \vee q)$
8.	$\neg(q \wedge s)$	T, 7 & Demorgan's law.
9.	$q \wedge s$	P
10.	$(q \wedge s) \wedge \neg(q \wedge s)$	T, 8, 9 & conjunction
11.	F	T, 10, negation law.

Q. Prove that the premises $a \rightarrow (b \rightarrow c)$, $d \rightarrow (b \wedge \neg c)$, and $(a \wedge d)$ are inconsistent?

A.

Step No	Statement	Reason
1.	$a \wedge d$	P
2.	a	T, 1 & Simplification
3.	d	T, 1, Simplification
4.	$a \rightarrow (b \rightarrow c)$	P
5.	$b \rightarrow c$	T, 2, 4 modus ponens
6.	$\neg b \vee c$	T, 5 & equivalence of (5)
7.	$d \rightarrow (b \wedge \neg c)$	P
8.	$\neg (b \wedge \neg c) \rightarrow \neg d$	T, 7 & Contrapositive
9.	$\neg b \vee c \rightarrow \neg d$	T, 8 & equivalence
10.	$\neg d$	T, 6, 9 & modus ponens.
11.	$d \wedge \neg d$	T, 3, 10 & conjunction.
12.	F	T, 11 & negation law.

Q. Construct an argument to show that the following premises imply the Conclusion "it rained".

"If it does not rain or if there is no traffic dislocation, then the sports day will be held & the Cultural programme will go on". "If sports day is held, the trophy will be awarded" and "the trophy was not awarded."

A. Let us symbolise the statement as follows.

p : It rains

q : There is a traffic dislocation

r : Sports day will be held

s : cultural programme will go on

t : The trophy will be awarded.

Then, we have to prove that,

$$\neg p \vee \neg q \rightarrow r \wedge s, \quad r \rightarrow t, \quad \neg t \implies p$$

Step No:	Statement	Reason.
1.	$\neg p \vee \neg q \rightarrow r \wedge s$	P
2.		

2. $(\neg p \rightarrow (r \wedge s)) \wedge (\neg q \rightarrow (r \wedge s))$ T, 1,

equalence in conditionals,

$[(p \rightarrow r) \wedge (q \rightarrow r)$
$\equiv (p \vee q) \rightarrow r$

3. $\neg(r \wedge s) \longrightarrow \neg p$ T, 2 & contrapositive

4. $r \longrightarrow t$ P

5. $\neg t \longrightarrow \neg r$ T, 4, contrapositive

6. $\neg t$ P

7. $\neg r$ T, 5, 6, moduspones

8. $\neg r \vee \neg s$ T, 7 & addition

9. $\neg(r \wedge s)$ T, 8 & Demorgans law.

10. p T, 3, 9 & moduspones

Q. Show that, the following set of premises is inconsistent.

- If Rama gets his degree, he will go for a Job.
- If he goes for a Job, he will get married soon
- If he goes for heigher Study, he will not get married

. Ruma gets his degree and goes for heigher study.

9. .

PREDICATE LOGIC

Consider the statements "$x > 10$", "$x = y + 5$" & "$x + y = z$". These statements are neither true, nor false, when the values of the variables are not specified.

The statement $x > 10$, has two parts, the first part, the variable 'x', the second part "is greater than 10." is called predicate.

It is denoted by $P(x)$, where P denote <u>predicate</u> "is greater than 10" and 'x' is <u>the variable</u>.

Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value. For example,

the truth value of $P(15) \{ \equiv 15 > 10$ is T, truth value of $P(5) \equiv \{5 > 10\}$ are F.

The statements "$x = y + 5$" & $x + y = z$ will be denoted by $p(x, y)$ & $p(x, y, z)$ respectively.

QUANTIFIERS

$p(x)$ is true for all values of x in the universe of discourse (a particular domain) and is denoted by the notatation $(x)\,p(x)$ or $\forall x\, p(x)$. The preposition $(x)\,p(x)$ or $\forall x\, p(x)$ is read as "for all x, $P(x)$" or "for every x, $P(x)$". The symbol $\forall$ is called the <u>universal quantifier</u>.

⇒ **Examples:**

⇒ * If $P(x) \equiv \{(-x)^2 = x^2\}$ where universe consist of all integers, then the truth value of $\forall x\,((-x)^2 = x^2)$ is T.

⇒ * If $Q(x) \equiv$ "$2x > x$" where the universe consists of all real numbers, then the truth value of $\forall x\, Q(x)$ is F

⇒ * If $P(x) \equiv$ "$x^2 < 10$" where the universe consist of the positive integers 1, 2, 3, and 4. then $\forall x\, P(x) = P(1) \wedge P(2) \wedge P(3) \wedge P(4)$ and so, the truth value of $\forall x\, P(x) = T \wedge T \wedge T \wedge F = F$

Existential Quantifier:

The existential quantification of $P(x)$ is "There exist at least one x (or an x) such that $P(x)$ is true" and is denoted by $\exists x\, P(x)$

The proposition $\exists x\, P(x)$

The symbol $\exists$ is called existential quantifier.

The proposition $\exists x\, p(x)$ is read as "For some x, $P(x)$"

Example:

* when $P(x)$ denotes the propositional function "$x > 3$", the truth value of $\exists x\, P(x)$ is T when the universe of discourse consists of all real numbers, since $x > 3$ is true for $x = 4$.

NEGATION OF A QUANTIFIED EXPRESSION :

If $P(x)$ is the statement "x has studied Computer programming" then $\forall x\, P(x)$ means that, "every student (in the class) has studied Computer programming"

The negation of this statement is

"It is not the case that every student in the class has studied Computer programming".

or equivalently,

"There is a student in the class who has not studied Computer programming". which is denoted by $\exists x \neg P(x)$.

Thus, we see that $\neg \forall x\, P(x) \equiv \exists x \neg P(x)$.

IIIly,

$$\neg \exists x\, P(x) \equiv \forall x \neg P(x)$$

Q. Express each of the following statements using mathematical & logical operations, predicates and quantifiers, where the universe of discourse consist of all Computer Science students / mathematical Courses.

(a) Every Computer Science student need a course in mathematics.

(b) There is a Student in this class who owns a personal Computer.

(c) Every Student in this class who has taken, at least one mathematical Course.

A.

(a) Let $M(x) \equiv$ 'x needs a Course in mathematics' where the universe of discourse Consists of all Computer Science Students

Then $\forall x\, M(x)$

(b) Let $P(x) \equiv$ 'x owns a personal computer' where the universe Consists of all Students in this class.

Then $\exists x\, P(x)$.

(c) Let $Q(x, y) \equiv$ 'x has taken y' where the universe of x Consists of all Students in this class and that of y Consists of all mathematics Courses.

Then $\forall x\, \exists y\, Q(x, y)$.

Q. Express the negation of the following statements using quantifiers?

(a) If the teacher is absent, then some students do not keep quiet.

(b) All the students keep quiet & the teacher is present.

A. (a) Let T represent the presence of the teacher, and $Q(x)$ represent "x keeps quiet."

Then the given statement is

$$\neg T \longrightarrow \exists x\, \neg Q(x) \equiv \neg T \longrightarrow \neg \forall x\, Q(x)$$
$$\equiv T \vee \neg \forall x\, Q(x)$$

$\therefore$ Negation of the given statement is

$$\neg (T \vee \neg \forall x\, Q(x))$$
$$\equiv \neg T \wedge \forall x\, Q(x)$$

ie the teacher is absent ~~all~~ and all the students keep quiet.

(b)

(b) The given statement is

$$\forall x\, Q(x) \wedge T$$

$\therefore$ negation of the given statement is

$$\neg(\forall x\, Q(x) \wedge T) \equiv \neg \forall x\, Q(x) \vee \neg T$$

$$\equiv \exists x\, \neg Q(x) \vee \neg T$$

ie Some students donot keep quiet or the teacher is absent.