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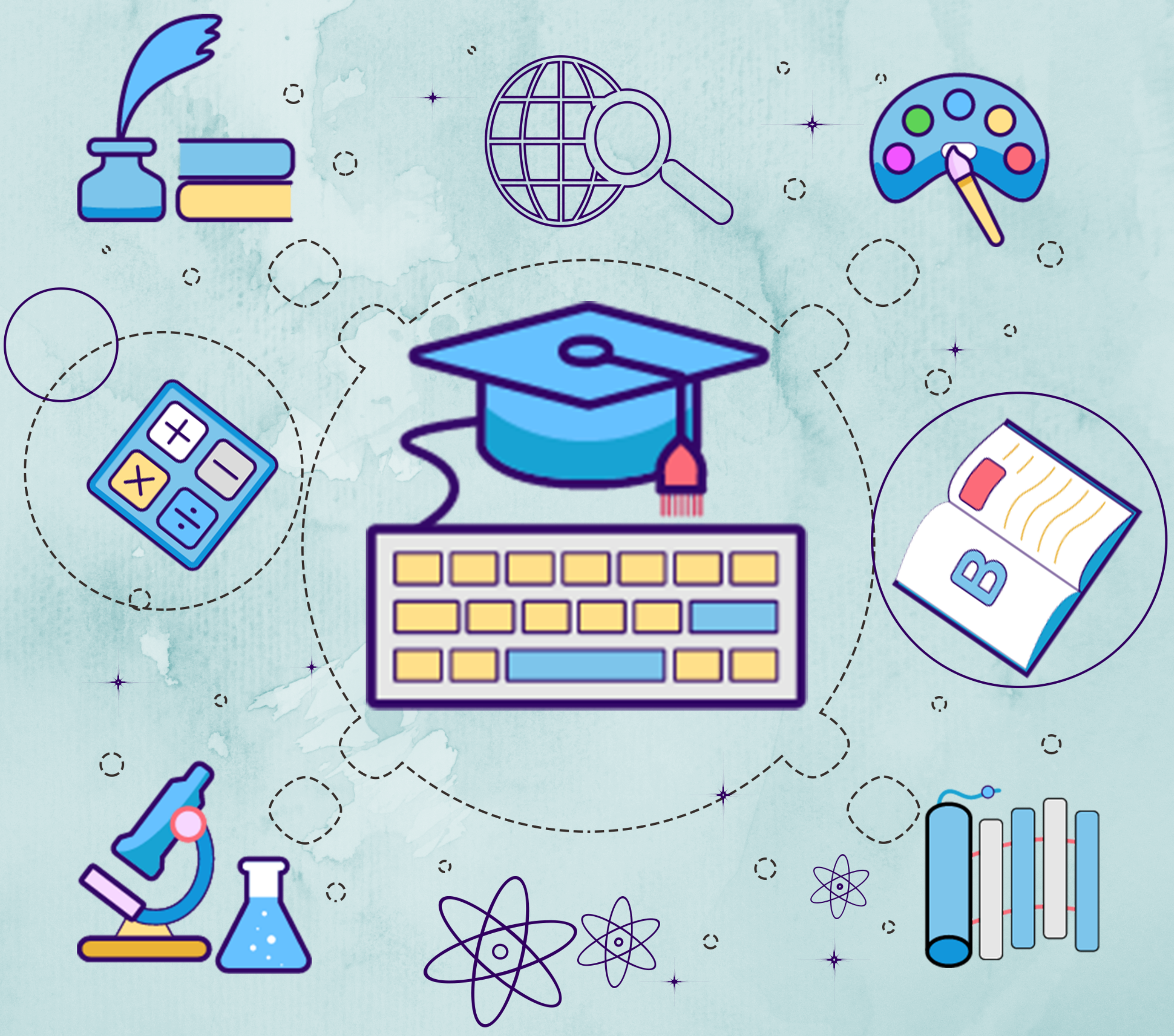

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GENERATING FUNCTIONS & RECCURENCE RELATIONS:

THE FIRST ORDER LINEAR RECCURENCE RELATION:

An equation that express a_n, viz, the general term of the the sequence $\{a_n\}$ in terms of one or more previous terms of the sequence, namely, $a_0, a_1, a_2 \ldots\ldots a_{n-1}$ for all integers n with $n \geq n_0$, where n_0 is an non negative integer is called a recurrence relation for $\{a_n\}$.

For ex:-

$$a_{n+1} = 3a_n$$

Let us consider the g.p 4, 12, 36, 108, ... the common ratio of which is 3. If $\{a_n\}$ represents this infinite sequence, we see that $\frac{a_{n+1}}{a_n} = 3$

ie $a_{n+1} = 3a_n$, $n \geq 0$ is the recurrence relation corresponding to the geometric sequence $\{a_n\}$

Also, the sequence 5, 15, 45, 135, ... also satisfies the above recurrence relation.

If the terms of the sequence satisfy a recurrence relation, then the sequence is called a solution of the recurrence relation.

Now, a recurrence relation $a_{n+1} = 3a_n$, $n \geq 0$, may represent a unique sequence, we should know, one of the terms of the sequence, say $a_0 = 4$,

If $a_0 = 4$, then the recurrence relation represents the sequence $4, 12, 36, 108, \ldots$

The value $a_0 = 4$ is called Initial Condition.

If $a_0 = 4$, then from the recurrence relation, we get,

$a_1 = 3(4)$, $a_2 = 3^2(4), \ldots\ldots$

In general when $n \geq 0$, $\boxed{a_n = 4 \cdot 3^n}$. $[a_n = a_0 3^n]$ This is called the general solution of the recurrence relation.

eg:-

Consider the Fibonacci sequence,

$0, 1, 1, 2, 3, 5, 8, 13 \ldots$

which can be represented by a relation

$$F_{n+2} = F_{n+1} + F_n \quad \text{where } n \geq 0 \text{ \& } F_0 = 0, F_1 = 1.$$

Q. Solve the recurrence relation $a_n = 7a_{n-1}$, where $n \geq 1$ and $a_2 = 98$.

This is just an alternative form of the relation,

$a_{n+1} = 7a_n$, , $a_2 = 98$

~~Here soln is~~ So, the solution is $a_n = a_0(7^n)$

a. Since $a_2 = 98 = a_0(7^2)$

$\Rightarrow a_0 = 2.$

$\therefore$ Solution is $a_n = a_0(7^n)$

ie $a_n = 2(7^n)$

Q. Find a recurrence relation with initial condition that uniquely determines each of the following sequences that begin with the given terms.

(a) 3, 7, 11, 15, 19, ...

(b) 8, $\frac{24}{7}$, $\frac{72}{49}$, $\frac{216}{343}$, ...

Ans. (a) Here first term is $a_0 = 3$, and increases by 4,

So, the recurrence relation is

$$a_n = a_{n-1} + 4, \quad \text{for } n > 1$$

(b) Here $a_0 = 8$ and each term get multiplied by $3/7$

So, the recurrence relation is

$$a_n = 3/7 \, a_{n-1} \quad \text{for } n > 1$$

DEFINITION :

A recurrence relation of the form,

$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots\cdots + c_k a_{n-k} = f(n)$ is called linear recurrence relation of degree k with constant coefficients, where $c_0, c_1, c_2 \cdots c_k$ are real numbers.

The degree or order of the recurrence relation is said to be k.

In otherwords, the degree is the difference between the greatest and least subscripts of the members of the sequence occuring in the recurrence relation.

If $f(n)=0$, the recurrence relation is said to be homogeneous, otherwise it is said to be non-homogeneous

⇒ Homogeneous Solution of recurrence relation: (degree 2)

Consider the recurrence relation,

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} = 0 \longrightarrow (1)$$

Let $a_n = r^n$ be a solution of (1)

Then, (1) becomes,

$$c_0 r^n + c_1 r^{n-1} + c_2 r^{n-2} = 0$$

ie

$$c_0 r^2 + c_1 r + c_2 = 0 \longrightarrow (2)$$

(2) is a quadratic equation in r, which is called characteristic equation, whose roots r_1 and r_2 are called characteristic roots of the recurrence relation.

Depending on the nature of the roots r_1 and r_2, we get, three different forms of the soln.

Case (i)

"r_1 and r_2 are real and distinct."

Then the solution of the recurrence relation is

$a_n = k_1 r_1^n + k_2 r_2^n$, where k_1 and k_2 are arbitrary constants determined by Initial Conditions

Case (ii) "r_1 and r_2 are real and equal"

The solution is $a_n = (k_1 + k_2 n) r^n$, where $r_1 = r_2 = r$

Case (iii) "r_1 and r_2 are Complex Conjugate"

Then the solution is

$$a_n = r^n (k_1 \cos n\theta + k_2 \sin n\theta).$$

Problems:

Q. Solve the recurrence relation, $a_n - 5a_{n-1} + 6a_{n-2} = 0$

put $a_n = r^n$

$r^n - 5r^{n-1} + 6r^{n-2}$

degree $= 2 \implies r^2 - 5r + 6 = 0$

$\therefore$ ~~r~~ $r = 2, 3$ ie $r_1 = 2$, and $r_2 = 3$

roots are real and distinct.

$\therefore$ the solution is

$$a_n^{(h)} = K_1 r_1^n + k_2 r_2^n$$

$$\text{ie } a_n^{(h)} = k_1 2^n + k_2 3^n$$

Q. Solve the recurrence relation $a_n - 2a_{n-1} = 0$

$\therefore$ put $a_n = r^n$

$$r^n - 2r^{n-1} = 0 \quad n = \cancel{0}, 1$$

$$r - 2 = 0$$

$$\Rightarrow r = 2$$

$\therefore$ solution is $a_n = k_1 r_1^n$

ie $\underline{a_n = k_1 2^n}$

Q. $a_n - \cancel{4a_{n-1}} + \cancel{4a_{n-2}}$

Q. $a_n - 4a_{n-1} + 4a_{n-2} = 0$ with $a_0 = 1$, $a_1 = 2$.

put $a_n = r^n$

$$r^n - 4r^{n-1} + 4r^{n-2} = 0 \quad , n = \cancel{n-2-n} = n-(n-2) = 2$$

$$r^2 - 4r + 4 = 0$$

$$r = 2, 2$$

$\therefore$ the solution is $a_n^{(h)} = \cancel{k_1 r_1^n} (k_1 + k_2 n) r^n$

ie $a_n^{(h)} = (K_1 + K_2 n) 2^n$

$\underline{a_0 = 1}$

$$(K_1 + K_2 \times 0) 2^0 = 1$$

$$K_1 = 1$$

$\underline{a_1 = 2}$

$$(K_1 + K_2 \times 1) 2^1 = 2$$

$$(K_1 + K_2) 2 = 2$$

$$2K_1 + 2K_2 = 2$$

$$2 \times 1 + 2K_2 = 2$$

$$2K_2 = 0$$

$$K_2 = 0$$

$\therefore$ we get $K_1 = 1$ & $K_2 = 0$

$\therefore$ the solution becomes $a_n^{(h)} = (1 + 0 \times n) 2^n$

$$= 2^n$$

ie $a_n^{(h)} = 2^n$

Q) $a_n = 4(a_{n-1} - a_{n-2})$

$a_n = 1$ for $n=0$ & $n=1$

$a_n = r^n$

$\Rightarrow a_n - 4a_{n-1} + 4a_{n-2} = 0$

$r^n - 4r^{n-1} + 4r^{n-2} = 0$

$r^2 - 4r + 4 = 0$

$r = 2, 2$

$a_n^{(h)} = (k_1 + k_2 n)2^n$

$a_n = 1$ for $n = 0$

$\Rightarrow 1 = (k_1 + k_2 \times 0)2^0$

$1 = k_1$

$a_n = 1$ for $n = 1$

$1 = (k_1 + k_2 \times 1)2^1$

$= 1 = 2k_1 + 2k_2$

$1 = 2 + 2k_2$

$2k_2 = 1 - 2$

$k_2 = -1/2$

$a_n = (1 + {}^{-}1/2)n)2^n$

$a_n = (1 - 1/2\, n)2^n$

Q ~~A~~ n.31'

Q. Solve the recurrence relation $F_{n+2} = F_{n+1} + F_n$, where $n \geq 0$ and $F_0 = 0$ and $F_1 = 1$ (Fibonacci Sequence).

A. $F_{n+2} = F_{n+1} + F_n$ $\qquad\qquad \left(\begin{array}{l} a_n = r^n; \\ F_n = r^n; \end{array} \Big/ r \neq 0 \right)$

$F_{n+2} - F_{n+1} - F_n = 0$

put $F_n = r^n \Rightarrow r^{n+2} - r^{n+1} - r^n = 0$

$\Rightarrow r^4 - r^3 - r^2 = 0$ $\qquad$ degree $= n+2-n$
$= 2$

$r^2(r^2 - r - 1) = 0$

$r^2 - r - 1 = 0$

$\therefore r =$ The characteristic roots are $r = \dfrac{1 \pm \sqrt{5}}{2}$

ie $r_1 = \dfrac{1+\sqrt{5}}{2}$ and $r_2 = \dfrac{1-\sqrt{5}}{2}$

$\therefore$ the solun is $F_n = k_1 r_1^n + k_2 r_2^n$

ie $F_n = k_1 \left(\dfrac{1+\sqrt{5}}{2}\right)^n + k_2 \left[\dfrac{1-\sqrt{5}}{2}\right]^n$

To find the values of k_1 and k_2,

$\underline{\underline{F_0 = 0}}$

$k_1 \left(\dfrac{1+\sqrt{5}}{2}\right)^0 + k_2 \left[\dfrac{1-\sqrt{5}}{2}\right]^0 = 0$

$k_1 + k_2 = 0 \Rightarrow$ $\longrightarrow (1)$

$-k_1 = k_2$

$F_1 = 1$

$k_1 \left[\dfrac{1+\sqrt{5}}{2}\right]^1 + k_2 \left[\dfrac{1-\sqrt{5}}{2}\right]^1 = 1$

$$k_1\left(\frac{1+\sqrt{5}}{2}\right)-k_1\left(\frac{1-\sqrt{5}}{2}\right)=1$$

$$\frac{k_1}{2}\left[1+\sqrt{5}-(1-\sqrt{5})\right]=1$$

$$k_1\left[1+\sqrt{5}-1+\sqrt{5}\right]=2$$

$$k_1\left(2\sqrt{5}\right)=2$$

$$k_1=\frac{\cancel{2}}{\cancel{2}\sqrt{5}}=\underline{\underline{1/\sqrt{5}}}$$

$\therefore$ Since $k_2=-k_1$

$\therefore k_2=-1/\sqrt{5}$

$\therefore$ the solun is $\underline{\underline{\dfrac{1}{\sqrt{5}}\left[\dfrac{1+\sqrt{5}}{2}\right]^n-\dfrac{1}{\sqrt{5}}\left[\dfrac{1-\sqrt{5}}{2}\right]^n}}$

Q. Solve the recurrence relation $a_n=2(a_{n-1}-a_{n-2})$, where $n\geq 2$ and $a_0=1$ and $a_1=2$. ?

A. put $a_n=r^n$

$$a_n=2a_{n-1}-2a_{n-2}$$

$$a_n-2a_{n-1}+2a_{n-2}=0$$

$$r^n-2r^{n-1}+2r^{n-2}=0$$

$$r^2-2r+2=0$$

roots are $r=1\pm i$

ie $r_1=1+i$, and $r_2=1-i$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \ n \geq 0$$

$z = x+iy$, $z = r(\cos\theta + i\sin\theta)$

where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$

Here $r_1 = 1+i$ and $r_2 = 1-i$

$r = \sqrt{1+1} = \sqrt{2}$ $\qquad$ $r = \sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1}(y/x) = \pi/4$ $\qquad$ $\theta = \tan^{-1}(-1/1) = -\pi/4$

$r_1 = 1+i = \sqrt{2}\left(\cos(\pi/4) + i\sin(\pi/4)\right)$

$r_2 = 1-i = \sqrt{2}\left(\cos(-\pi/4) + i\sin(-\pi/4)\right)$

$= \sqrt{2}\left(\cos(\pi/4) - i\sin(\pi/4)\right).$

$\therefore$ the soln is $a_n = (\sqrt{2})^n\left[k_1 \cos(n\pi/4) + k_2 \sin(n\pi/4)\right]$

Here, $a_0 = 1$

$(\sqrt{2})^0\left[k_1 \cos 0 + k_2 \sin 0\right) = 1$

$\Rightarrow k_1 = 1$

$a_1 = 2$

$(\sqrt{2})^1\left[k_1 \cos(\pi/4) + k_2 \sin(\pi/4)\right] = 2$

$(\sqrt{2})\left[\cos(\pi/4) + k_2 \sin(\pi/4)\right) = 2$

$\sqrt{2}\left[\frac{1}{\sqrt{2}} + \frac{k_2}{\sqrt{2}}\right] = 2$

$\sqrt{2}\left[\frac{k_2+1}{\sqrt{2}}\right] = 2 \Rightarrow k_2 + 1 = 2 \Rightarrow k_2 = 1$

$\therefore$ Solun is

$$a_n = (\sqrt{2})^n \left[\cos(n \pi/4) + \sin(n(\pi/4))\right], \ n \geq 0$$

~~PART-IV~~

NON-HOMOGENEOUS SOLUTIONS :

The following table gives certain forms of $f(n)$ and the forms of the Corresponding particular solution,

Form of $f(n)$	Form of $a_n^{(P)}$ to be assumed.
C, a constant	A, a constant
n	$A_0 n + A_1$
n^2	$A_0 n^2 + A_1 n + A_2$
$n^t, t \in Z^+$	$A_0 n^t + A_1 n^{t-1} + \dots\dots + A_n$
$r^n, r \in R$	$A r^n$
$n^t r^n$	$r^n (A_0 n^t + A_1 n^{t-1} + \dots\dots + A_n$
$\sin \theta n$	$A \sin \theta n + B \cos \theta n$
$\cos \theta n$	$A \sin \theta n + B \cos \theta n$
$r^n \sin \theta n$	$r^n (A \sin \theta n + B \cos \theta n)$
$r^n \cos \theta n$	$r^n (A \sin \theta n + B \cos \theta n)$

Q) find the particular Solun for the recurrence relation

$$a_n - 3a_{n-1} + 2a_{n-2} = 3^n \quad , \quad n \geq 2$$

$$f(n) = 3^n$$

$$a_n^{(P)} = A\,3^n$$

$$A\,3^n - 3A\,3^{n-1} + 2A\,3^{n-2} = 3^n$$

$$A\,3^2 - 3A\,3^1 + 2A\,3^0 = 3^2$$

$$9A - 9A + 2A = 9$$

$$A = 9/2$$

$\therefore$ particular Solun is $a_n^{(P)} = 9/2\,3^n$

$$= \frac{3^2 \cdot 3^n}{2}$$

$$= \frac{3^{n+2}}{2}$$

Q) find the particular Solun of the

$$a_n + a_{n-1} = 2n\,3^n$$

$$f(n) = 2n\,3^n$$

$$a_n^{(P)} = 3^n(A_0 n + A_1)$$

Q) find the particular Soln of

$$a_n + a_{n-1} = 2n3^n,$$

Here $f(n) = 2n3^n$

$$r^n + r^{n-1} = 0$$

$$- 3^n(A_0 n + A_1)$$

$$3^n(A_0 n + A_1) + 3^{n-1}(A_0(n-1) + A_1) = 2(n)3^n$$

$$= \cancel{3^n A_0 n + A_1 3^n + 3^{n-1} A_0 (n-1) + 3^{n-1} A_1} = 2n3^n$$

$$= 3^n A_0 n + A_1 3^n + 3^n \cdot 3^{-1}(A_0(n-1) + A_1) = 2n3^n$$

$$\cancel{3^n}\left(A_0 n + A_1 + \tfrac{1}{3}(A_0(n-1) + A_1)\right) = 2n\cancel{3^n}$$

$$\therefore A_0 n + A_1 + \tfrac{1}{3}(A_0 n - A_0 + A_1) = 2n$$

$$3A_0 n + 3A_1 + A_0 n - A_0 + A_1 = 2n$$

$$\therefore 4A_0 n + 4A_1 - A_0 = 2n$$

$$4A_0 = 2$$

$$A_0 = 2/4 = 1/2$$

$$4A_1 - A_0 = 0$$

$$4A_1 = A_0$$

$$4A_1 = 1/2$$

$$A_1 = 1/8$$

$\therefore$ particular Solun is

$3^n\left(\frac{1}{2}n+\frac{1}{10}\right)$

Q. $a_{n+2} - 8a_{n+1} + 16a_n = 8(5^n)$

$r^{n+2} - 8r^{n+1} + 16r^n = 0$

$r^4 - 8r^3 + 16r^2 = 0$

$r^2(r^2 - 8r + 16) = 0$

$r^2 - 8r + 16 = 0 \implies r = 4, 4$

$\therefore$ Solun $a_n^{(h)} = (k_1 + k_2 n)4^n$

$a_n^{(p)} = A5^n$

Substitute $a_n^{(p)}$ in the given reccurence relation

$A5^{n+2} - 8A5^{n+1} + 16A5^n = 8(5^n)$

$A5^n \cdot 5^2 - 8A5^n \cdot 5 + 16A5^n = 8 5^n$

$\cancel{5^n}(A5^2 - 8A \cdot 5 + 16A) = 8 \cancel{5^n}$

$A5^2 - 8A \cdot 5 + 16A = 8$

$25A - 40A + 16 = 8$

$-15A + 16 = 8$

$-15A = -8$

$A = 8/15$

$\therefore$ Solun $a_n^{(p)} = A5^n = \frac{8}{15} \cdot 5^n$

$\therefore$ total Solun is $a_n^{(h)} + a_n^{(p)} = (k_1 + k_2 n)4^n + \frac{8}{15} * 5^n$

Generating function :

Let $a_0, a_1, a_2, \ldots\ldots$ be a sequence of real numbers. The function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots = \sum_{i=0}^{\infty} a_i x^i$$

is called the generating function for the given sequence.

Eg:—

For any $n \in \mathbb{Z}^+$

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots\ldots + {}^nC_n x^n.$$

So, $(1+x)^n$ is the generating function for the sequence

${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots {}^nC_n, 0, 0, 0 \ldots\ldots$

Eg:—

For $n \in \mathbb{Z}^+$,

$$(1-x^{n+1}) = (1-x)(1+x+x^2+x^3+\ldots\ldots+x^n)$$

So,

$$\frac{1-x^{n+1}}{1-x} = 1+x+x^2+\ldots\ldots+x^n$$

and $\frac{(1-x^{n+1})}{(1-x)}$ is the generating function for the sequence $1, 1, 1, \ldots\ldots 1, 0, 0, 0 \ldots\ldots$ where the first $n+1$ terms are 1.

eg:— $\frac{1}{1-x} = 1+x+x^2+x^3+\ldots\ldots = \sum_{i=0}^{\infty} x^i$

$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}(1-x)^{-1} = (-1)(1-x)^{-2}(-1)$

$= \frac{1}{(1-x)^2}$

ie $\frac{d}{dx}(1+x+x^2+\ldots) = 1+2x+3x^2+4x^3+\ldots$

ie $\frac{1}{(1-x)^2}$ is the generating function for the sequence

1, 2, 3, 4, ...

while $\frac{x}{(1-x)^2} = 0+1x+2x^2+3x^3+4x^4+\ldots$

is the generating function for the sequence 0, 1, 2, 3, ...

eg:— $\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = \frac{d}{dx}(0+x+2x^2+3x^3+4x^4+\ldots)$

$\frac{(1-x)^2 \times 1 - x(2(1-x)\times -1)}{(1-x)^4} = 1+\cancel{4}\,2\cdot 2x+3\cdot 3x^2+\ldots$

$\frac{(1-x)^2 - x(2-2x)\times -1}{(1-x)^4} = 1+2^2x+3^2x^2+\ldots$

$\frac{1-2x+x^2+2x-2x^2}{(1-x)^4} = 1+2^2x+3^2x^2+\ldots$

$$\frac{1-x^2}{(1-x)^4} = 1+2^2x+3^2x^2+\cdots\cdots$$

$$\therefore \frac{(1+x)(1-x)}{(1-x)^4} = 1+2^2x+3^2x^2+\cdots\cdots$$

$$= \frac{(1+x)}{(1-x)^3} = 1+2^2x+3^2x^2+\cdots\cdots$$

$\therefore \dfrac{1+x}{(1-x)^3}$ is the generating function for the sequence

$1, 2^2, 3^2, \ldots\ldots$

eg:- we know, $\dfrac{1}{1-x} = 1+x+x^2+x^3+\cdots\cdots$

put $x = 2y$

$$\frac{1}{1-2y} = 1+2y+2^2y^2+2^3y^3+\cdots\cdots$$

ie $\dfrac{1}{1-2y}$ is the generating function for the sequence

$2^0, 2^1, 2^2, 2^3, \ldots\ldots$

If $x = 3y$

$$\frac{1}{1-3y} = 1+3y+3^2y^2+3^3y^3+\cdots\cdots$$

ie $\dfrac{1}{1-3y}$ is the generating function for the sequence

$3^0, 3^1, 3^2, 3^3, \ldots\ldots$

If $x = ky$

then in general, $\frac{1}{1-ky} = 1 + ky + k^2y^2 + k^3y^3 + \cdots$

ie $\frac{1}{1-ky}$ is the generating function for the Sequence

$k^0, k^1, k^2, k^3, k^4, \ldots\ldots$

eg:- we know, $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots$

multiply by 2 on both sides, we get,

$$\frac{2}{1-x} = 2 + 2x + 2x^2 + 2x^3 + \ldots\ldots$$

ie $\frac{2}{1-x}$ is the generating function for the Sequence

$2, 2, 2, 2, \ldots\ldots$

generally $\frac{a}{1-x} = a + ax + ax^2 + ax^3 + \ldots\ldots$

ie $\frac{a}{1-x}$ is the generating function for the Sequence

$a, a, a, a, \ldots\ldots$

Q. Find the generating function for the sequence
3, 7, 11, 15, 19, 23, ...

A. Here $a_0 = 3, \; a_1 = 7, \; a_2 = 11, \; a_3 = 15, \; \ldots\ldots$

Let $A = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots$ be the sequence

ie $A = 3 + 7x + 11x^2 + 15x^3 + \ldots\ldots \longrightarrow (1)$

$xA = 3x + 7x^2 + 11x^3 + 15x^4 + \ldots\ldots \longrightarrow (2)$

$(1) - (2) \Longrightarrow$ x

$$(A - xA) = 3 + 4x + 4x^2 + 4x^3 + \ldots\ldots$$

$$(1-x)A = 3 + 4x + 4x^2 + 4x^3 + \ldots\ldots$$

$$= 3 + 4x\left[1 + x + x^2 + \ldots\ldots\right]$$

$$(1-x)A = 3 + 4x\left[\frac{1}{1-x}\right]$$

$$= 3 + \frac{4x}{(1-x)}$$

$$\Rightarrow A = \frac{3}{(1-x)} + \frac{4x}{(1-x)^2}$$

It is the generating function for the sequence,
3, 7, 11, 15, 19, 23, ...

Q. Find the generating function for the sequence
0, 2, 6, 12, 20, 30, 42, ... ?

Here, we know,

$$a_0 = 0, = 0^2 + 0$$

$$a_1 = 2, = 1^2 + 1$$

$$a_2 = 6, = 2^2 + 2$$

$$a_3 = 12 = 3^2 + 3$$

$$a_4 = 20 = 4^2 + 4$$

In general $a_n = n^2 + n$

we know, in previous examples,

$\frac{x(x+1)}{(1-x)^3}$ is the generating function for the sequence,

$0^2, 1^2, 2^2, 3^2, 4^2, \ldots\ldots$ and,

$\frac{x}{(1-x)^2}$ is the generating function for the sequence,

$0, 1, 2, 3, 4, 5, \ldots\ldots$

$\therefore$ from these two examples,

$$\frac{x(x+1)}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x(x+1)}{(1-x)^3} + \frac{x(1-x)}{(1-x)^3}$$

$$= \frac{x(x+1) + x(1-x)}{(1-x)^3}$$

$$= \frac{x^2 + x + x - x^2}{(1-x)^3} = \frac{2x}{(1-x)^3}$$

$\therefore \frac{2x}{(1-x)^3}$ is the generating function for the sequence.

$0, 2, 6, 12, 20, 30, \ldots\ldots\ldots$

Result *

we know, $(1+x)^n = nC_0 + nC_1 x + nC_2 x^2 + \dots\dots + nC_n x^n$

$$nC_r = \frac{n!}{r!(n-r)!}$$

$$= \frac{n(n-1)(n-2)(n-3)\dots\dots(n-r+1)(n-r)\dots\dots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\dots\dots 3\cdot 2\cdot 1 \times r!}$$

$$= \frac{n(n-1)(n-2)\dots\dots\dots(n-(r-1))}{r!}$$

$n = -n$

$$-nC_r = \frac{(-n)(-n-1)(-n-2)\dots\dots(-n-(r-1))}{r!}$$

$$= \frac{(-1)^r n(n+1)(n+2)\dots\dots(n+(r-1))}{r!}$$

$$= \frac{(-1)^r 1\cdot 2\cdot 3\cdot \dots\dots(n-1)\, n(n+1)(n+2)\dots\dots(n+(r-1))}{1\cdot 2\cdot 3\dots\dots(n-1)\; r!}$$

$$= \frac{(-1)^r (n+(r-1))!}{(n-1)!\, r!}$$

nC_r $\boxed{nC_r = \frac{n!}{r!(n-r)!}}$

$$= (-1)^r \, {}^{n+r-1}C_r$$

$$= (-1)^r \binom{n+r-1}{r}$$

Q. Find the Maclaurin Series expansion for $(1+x)^{-n}$

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots\cdots + {}^nC_n x^n + \cdots\cdots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots\cdots + x^n + \cdots\cdots$$

$$(1+x)^{-n} = 1 + (-nx) + \frac{(-n)(-n-1)}{2!}x^2 + \frac{(-n)(-n-1)(-n-2)}{3!}x^3 + \cdots\cdots + x^n + \cdots$$

$$= 1 + \sum_{r=1}^{\infty} \binom{n+r-1}{r} x^r$$

$$= \underline{\underline{\sum_{r=0}^{\infty} (-1)^r \binom{n+(r-1)}{r} x^r}}$$

$$(1+x)^{-n} = {}^{-n}C_0 + {}^{(-n)}C_1 x + {}^{-n}C_2 x^2 + \cdots\cdots$$

$$= \sum_{r=0}^{\infty} {}^{-n}C_r x^r$$

ie $\frac{1}{(1+x)^n}$ generate ${}^{-n}C_0, {}^{-n}C_1, \cdots\cdots$

Q. Find the Coefficient of x^5 in $(1-2x)^{-7}$.

A. $(1-2x)^{-7}$,

we know, $(1+x)^{-n} = \sum_{r=0}^{\infty} {}^{-n}C_r x^r$

$n=7, \quad r=5,$

$$(1+(-2x))^{-7} = \sum_{r=0}^{\infty} {}^{-7}C_r (-2x)^r$$

$$= \sum \left({}^{-7}C_r (-2)^r\right) x^r$$

coefficient of x^5 is $-7C_5 (-2)^5$

$= -7C_5 (-32)$

$= (-1)^5 \binom{7+5-1}{5} (-32)$

$= 32 \binom{11}{5}$

$= 32 \times \frac{11!}{5!(6!)} = 14784$

The Exponential generating function :—

For a Sequence $a_0, a_1, a_2, a_3, \ldots\ldots$ of real numbers.. $f(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \ldots\ldots = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$

is called exponential generating function for the given sequence.

Example:

The maclaurin Series expansion for the function e^x, we find,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

So, e^x is the exponential generating function for the Sequence $1, 1, 1, 1, \ldots\ldots$.

* — <u>Example</u>

Consider the maclaurian series expansion of e^x & e^{-x}

$$e^x = 1 + x + x^2/2! + x^3/3! + \cdots$$

$$e^{-x} = 1 - x + x^2/2! - x^3/3! + \cdots$$

Adding these series together, we find that,

$$e^x + e^{-x} = 2 + 2x^2/2! + 2x^3/3! + 2\frac{x^4}{4!} + \cdots$$

$$= 2\left(1 + x^2/2! + x^3/3! + \cdots\right)$$

or

$$\frac{e^x + e^{-x}}{2} = 1 + x^2/2! + x^3/3! + \cdots$$

substracting e^{-x} from e^x

$$\frac{e^x - e^{-x}}{2} = x + x^3/3! + x^5/5! + \cdots$$

Q Determine the sequences generated by each of the following exponential generating functions.

(1) $f(x) = 5e^{5x}$

A. we know $e^x = 1 + x + x^2/2! + x^3/3! + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

So, $f(x) = 5e^{5x}$

$$= 5 \sum_{n=0}^{\infty} \frac{(5x)^n}{n!}$$

$$= 5\left[1 + 5x + \frac{5^2 x^2}{2!} + \frac{5^3 x^3}{3!} + \cdots\right.$$

ie $5 + 5^2 x + 5^3 \frac{x^2}{2!} + 5^4 \frac{x^3}{3!} + \cdots\cdots$

$\therefore$ $f(x)$ is the exponential generating function for the Sequence $5, 5^2, 5^3, 5^4, \cdots\cdots$

(2) $f(x) = 7e^{8x} - 4e^{3x}$

A. $f(x) = 7e^{8x} - 4e^{3x}$

$$= 7 \sum_{n=0}^{\infty} \frac{(8x)^n}{n!} - 4 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!}$$

$\therefore$ the Sequence is $7(8)^n - 4(3^n)$ with $n = 0, 1, 2, 3, \cdots\cdots$

ie $3, 44, 412, 3476, \cdots\cdots$

(3) $f(x) = 2e^x + 3x^2$.

A Here $f(x) = 2e^x + 3x^2$

$$= 2 \sum_{n=0}^{\infty} \frac{x^n}{n!} + 3x^2$$

$$= \left(2\frac{x^0}{0!} + 3x^2\right) + 2\frac{x}{1!} + 3x^2 + 2\frac{x^2}{2!} + 3x^2 + 2\frac{x^3}{3!} + 3x^2 + \cdots\cdots$$

$$= (2 + 3x^2) + (2x + 3x^2) + \left((2+3)\frac{x^2}{2!}\right) + \left(2\frac{x^3}{3!} + 3x^2\right) + \cdots$$

$\therefore$ So, the Sequence is $2, 2, (2+3), 2, 2, 2, \cdots\cdots$

(4) $f(x) = e^{3x} - 28x^3 + 6x^2 + 9x$

A

ie $= \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} - 28x^3 + 6x^2 + 9x$

$= \left(\frac{(3x)^0}{0!} - 28x^3 + 6x^2 + 9x \right) + \left(\frac{3x}{1!} - 28x^3 + 6x^2 + 9x \right) + \left(\frac{3^2 x^2}{2!} - 28x^3 + 6x^2 + 9x \right)$

$+ \left(\frac{3^3 x^3}{3!} - 28x^3 + 6x^2 + 9x \right) + \cdots\cdots$

$\therefore$ the Sequence is $3^0, 3^1 + 9, 3^2 - 6, 3^3 - 28, 3^4, 3^5, 3^6, \ldots$

ie $1, 12, 3, -1, 3^4, 3^5, 3^6 \ldots$

THE METHOD OF GENERATING FUNCTIONS:

Q. Solve the relation $a_n - 3a_{n-1} = n$, $n \geq 1$, $a_0 = 1$

A. This relation represents an infinite set of equations,

$$(n=1), \quad a_1 - 3a_0 = 1$$
$$(n=2), \quad a_2 - 3a_1 = 2$$
$$\vdots \qquad \vdots \qquad \vdots$$

multiplying first of these by x, the Second by x^2, and So on, we

$$(n=1) \quad a_1 x^1 - 3a_0 x^1 = 1x^1$$
$$(n=2) \quad a_2 x^2 - 3a_1 x^2 = 2x^2$$
$$\vdots \qquad \vdots \qquad \vdots$$

~~Summing up,~~

Adding this set of equations,

$$\sum_{n=1}^{\infty} a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} n x^n \longrightarrow (1)$$

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for the Sequence $a_0, a_1, a_2, \ldots\ldots$ Then (1) can be written as

$$(f(x) - a_0) - 3x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = \sum_{n=1}^{\infty} n x^n \left(= \sum_{n=0}^{\infty} n x^n\right) \rightarrow (2)$$

Since $\sum_{n=1}^{\infty} a_{n-1} x^{n-1} = \sum_{n=0}^{\infty} a_n x^n = f(x)$ & $a_0 = 1$, then

L.H.S of eqn (1) becomes $(f(x) - 1) - 3x f(x)$

In previous examples, we know,

$\frac{x}{(1-x)^2}$ is the generating function for the Sequence

$0, 1, 2, 3, \ldots\ldots$

ie $\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \ldots\ldots \infty$, so

$(f(x)-1) - 3x f(x) = \frac{x}{(1-x)^2}$ and

ie $f(x)\left[1-3x\right] - 1 = \frac{x}{(1-x)^2}$

$f(x)\left[1-3x\right] = \frac{x}{(1-x)^2} + 1$

ie $f(x) = \frac{1}{(1-3x)} + \frac{x}{(1-x)^2(1-3x)}$

using partial fraction

$$\frac{x}{(1-x)^2(1-3x)} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{(1-3x)}$$

$$x = A(1-x)(1-3x) + B(1-3x) + C(1-x)^2$$

From the following

put $x = 1$, and

$1 = B(-2) \Rightarrow B = -1/2$

$x = 1/3 \Rightarrow 1/3 = C(2/3)^2 \Rightarrow C = 3/4$

$x = 0 \Rightarrow 0 = A + B + C \Rightarrow A = -(B+C)$

$= -1/4$

Therefore

$$f(x) = \frac{1}{(1-3x)} + \frac{(-1/4)}{(1-x)} + \frac{(-1/2)}{(1-x)^2} + \frac{(3/4)}{(1-3x)}$$

$$= \frac{7/4}{(1-3x)} + \frac{(-1/4)}{(1-x)} + \frac{-1/2}{(1-x)^2}$$

we find a_n by determining the Co-efficient of x^n in each of these summands.

(a) $\frac{(7/4)}{(1-3x)} = 7/4 \left[\frac{1}{1-3x}\right]$

$= 7/4 \left[1 + (3x) + (3x)^2 + \ldots\ldots\right]$

and the Coefficient of x^n is $(7/4)(3^n)$.

(b) $\frac{(-1/4)}{(1-x)} = -1/4 \left[\frac{1}{1-x}\right] = -1/4 \left[1 + x + x^2 + \ldots\ldots\right]$

Here the Coefficient of x^n is $-1/4$.

(c) $\frac{(-1/2)}{(1-x)^2} = -1/2 \left[(1-x)^{-2}\right]$

$= -1/2 \left[\binom{-2}{0} + \binom{-2}{1} x + \binom{-2}{2} x^2 + \ldots\ldots\right]$

Here, the Coefficient of x^n is $-1/2 \binom{-2}{n} (-1)^n$

$= (-1/2)(-1)^n \binom{2+n-1}{n} (-1)^n$

$= -1/2 (n+1)$

Therefore

Therefore $a_n = (7/4)3^n - (1/2)n$

$= (7/4)\,3^n - 1/4 - 1/2(n+1)$

$= (7/4)\,3^n - 1/2\,n - (3/4)$