



Lab Manual
for
Linear Algebra
by
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Cover: my Chocolate Lab, Suzy.

Preface

WARNING! This is an incomplete draft, no doubt riddled with errors.

This collection supplements the text *Linear Algebra*¹ with explorations to help students solidify and extend their understanding of the subject, using the mathematical software Sage.²

A major goal of any undergraduate Mathematics program is to move students toward a higher-level, more abstract, grasp of the subject. For instance, Calculus classes work on elaborate computations while later courses spend more effort on concepts and proofs, focussing less on the details of calculations.

The text *Linear Algebra* fits into this development process. Its presentation works to bring students to a deeper understanding, but it does so by expecting that for them at this point a good bit of calculation helps the process. Naturally it uses examples and practice problems that are small-sized and have manageable numbers: an assignment to by-hand multiply a pair of three by three matrices of small integers will build intuition, whereas asking students to do that same question with twenty by twenty matrices of ten decimal place numbers would be badgering.

However, an instructor can be concerned that this misses a chance to make the point that Linear Algebra is widely applied, or to develop students's understanding through explorations that are not hindered by the mechanics of paper and pencil. Mathematical software can mitigate these concerns by extending the reach of what is reasonable to bigger systems, harder numbers, and longer computations. For instance, an advantage of learning how to handle larger jobs is that they are more like the ones that appear when students apply Linear Algebra to other areas. Another advantage is that students see new ideas such as runtime growth measures.

Well then, why not teach straight from the computer?

Our goal is to develop a higher-level understanding of the material so we want to keep the focus on vector spaces and linear maps. Our exposition takes computation to be a tool to develop that understanding, not the main object.

Some instructors will find that their students are best served by keeping a tight focus on the core material, and leaving aside altogether the work in this manual. Other instructors have students who will benefit from the increased reach that the software provides. This manual's existence, and status as a separate book, gives teachers the freedom to make the choice that suits their class.

¹The text's home page <http://joshua.smcvt.edu/linearalgebra> has the PDF, the ancillary materials, and the L^AT_EX source.

²See <http://www.sagemath.org> for the software and documentation.

Why Sage?

In *Open Source Mathematical Software*[[Joyner and Stein, 2007](#)]¹ the authors argue that for Mathematics the best way forward is to use software that is Open Source.

Suppose Jane is a well-known mathematician who announces she has proved a theorem. We probably will believe her, but she knows that she will be required to produce a proof if requested. However, suppose now Jane says a theorem is true based partly on the results of software. The closest we can reasonably hope to get to a rigorous proof (without new ideas) is the open inspection and ability to use all the computer code on which the result depends. If the program is proprietary, this is not possible. We have every right to be distrustful, not only due to a vague distrust of computers but because even the best programmers regularly make mistakes.

If one reads the proof of Jane's theorem in hopes of extending her ideas or applying them in a new context, it is limiting to not have access to the inner workings of the software on which Jane's result builds.

Professionals choose their tools by balancing many factors but this argument is persuasive. We use Sage because it is very capable so students can learn a great deal from it, and because it is Free² and Open Source.³

This manual

This is Free. Get the latest version from <http://joshua.smcvt.edu/linearalgebra>. Also see that page for the license details and for the \LaTeX source. I am glad to get feedback, especially from instructors who have class-tested the material. My contact information is on the same page.

The computer output included here was generated automatically (I have automatically edited some lines). This is my Sage.

```
1 sage: version()
2 'Sage Version 5.2, Release
```

Acknowledgements

I am glad for this chance to thank the Sage Development Team. In particular, without [[Sage Development Team, 2012b](#)] this work would not have happened. I am glad also for the chance to mention [[Beezer, 2011](#)] as an inspiration.

We emphasize practice.

–[Shunryu Suzuki \[2006\]](#)

*[A]n orderly presentation is not necessarily bad
but by itself may be insufficient.*

–[Ron Brandt \[1998\]](#)

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¹See <http://www.ams.org/notices/200710/tex071001279p.pdf> for the full text. ²The Free Software Foundation page <http://www.gnu.org/philosophy/free-sw.html> gives background and a definition. ³See <http://opensource.org/osd.html> for a definition.

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Python and Sage

To work through the Linear Algebra in this manual you need to be acquainted with how to run Sage. Sage uses the computer language Python so we'll start with that.

Python

Python is a popular computer language, often used for scripting, that is appealing for its simple style and powerful libraries. The significance for us of 'scripting' is that Sage uses it in this way, as a glue to bring together separate parts.

Python is Free. If your operating system doesn't provide it then go to the home page www.python.org and follow the download and installation instructions. Also at that site is Python's excellent tutorial. That tutorial is thorough; here you will see only enough Python to get started.

Comment. There is a new version, Python 3, with some differences. Here we stick to the older version because that is what Sage uses.

Basics Start Python, for instance by entering `python` at a command line. You'll get a couple of lines of information followed by three greater-than characters.

```
1 >>>
```

This is a prompt. It lets you experiment: if you type Python code and *(Enter)* then the system will read your code, evaluate it, and print the result. We will see below how to write and run whole programs but for now we will experiment. You can always leave the prompt with *(Ctrl)-D*.

Try entering these expressions (double star is exponentiation).

```
1 >>> 2 - (-1)
2 3
3 >>> 1 + 2 * 3
4 7
5 >>> 2 ** 3
6 8
```

Part of Python's appeal is that simple things tend to be simple to do. Here is how you print something to the screen.

```
1 >>> print 1, "plus", 2, "equals", 3
2 1 plus 2 equals 3
```

Often you can debug just by putting in commands to print things, and having a straightforward print operator helps with that.

As in any other computer language, variables give you a place to keep values.

```
1 >>> i = 1
2 >>> i + 1
3 2
```

In some programming languages you must declare the ‘type’ of a variable before you use it; for instance you would have to declare that *i* is an integer before you could set *i* = 1. In contrast, Python deduces the type of a variable based on what you do to it—above we assigned 1 to *i* so Python figured that it must be an integer. Further, we can change how we use the variable and Python will go along; here we change what is in *x* from an integer to a string.

```
1 >>> x = 1
2 >>> x
3 1
4 >>> x = 'a'
5 >>> x
6 'a'
```

Python complains by raising an *error*. Here we are trying to combine a string and an integer.

```
1 >>> 'a'+1
2 Traceback (most recent call last):
3   File "<stdin>", line 1, in <module>
4 TypeError: cannot concatenate 'str' and 'int' objects
```

The error message’s bottom line is the useful one.

The hash mark # makes the rest of a line a comment.

```
1 >>> t = 2.2
2 >>> d = (0.5) * 9.8 * (t**2) # d in meters
3 >>> d
4 23.716000000000005
```

(Comments are more useful in a program than at the prompt.) Programmers often comment an entire line by starting that line with a hash.

As in the listing above, we can represent real numbers and even complex numbers.

```
1 >>> 5.774 * 3
2 17.322
3 >>> (3+2j) - (1-4j)
4 (2+6j)
```

Notice that Python uses ‘*j*’ for the square root of -1 , not the ‘*i*’ traditional in mathematics.

The examples above show addition, subtraction, multiplication, and exponentiation. Division is a bit awkward. Python was originally designed to have the division bar / mean real number division when at least one of the numbers is real. However between two integers the division bar was taken to mean a quotient, as in “2 goes into 5 with quotient 2 and remainder 1.”

```
1 >>> 5.2 / 2.0
2 2.6
3 >>> 5.2 / 2
4 2.6
5 >>> 5 / 2
6 2
```

This was a mistake and one of the changes in Python 3 is that the quotient operation will be `//` while the single-slash operator will be real division in all cases. In Python 2 the simplest thing is to make sure that at least one number in a division is real.

```

1 >>> x = 5
2 >>> y = 2
3 >>> (1.0*x) / y
4 2.5

```

Incidentally, do the remainder operation (sometimes called ‘modulus’) with a percent character: `5 % 2` returns 1.

Variables can also represent truth values; these are *Booleans*.

```

1 >>> yankees_stink = True
2 >>> yankees_stink
3 True

```

You need the initial capital: `True` or `False`, not `true` or `false`.

Above we saw a string consisting of text between single quotes. You can use either single quotes or double quotes, as long as you use the same at both ends of the string. Here `x` and `y` are double-quoted, which makes sense because they contain apostrophes.

```

1 >>> x = "I'm Popeye the sailor man"
2 >>> y = "I yam what I yam and that's all what I yam"
3 >>> x + ', ' + y
4 "I'm Popeye the sailor man, I yam what I yam and that's all what I yam"

```

The `+` operation concatenates strings. Inside a double-quoted string you can use slash-*n* `\n` to get a newline.

A string marked by three sets of quotes can contain line breaks.

```

1 >>> """ THE ROAD TO WISDOM
2 ...
3 ... The road to wisdom?
4 ... -- Well, it's plain
5 ... and simple to express:
6 ... Err
7 ... and err
8 ... and err again
9 ... but less
10 ... and less
11 ... and less. --Piet Hein"""

```

The triple dots at the start of each line is a prompt that Python’s read-eval-print loop gives when what you have typed is not complete. One use for triple-quoted strings is as documentation in a program; we’ll see that below.

A Python *dictionary* is a finite function, that is, it is a finite set of ordered pairs `(key, value)` subject to the restriction that no key can be repeated.

```

1 >>> english_words = {'one': 1, 'two': 2, 'three': 3}
2 >>> english_words['one']
3 1
4 >>> english_words['four'] = 4

```

Dictionaries are a simple database. But the fact that a dictionary is a set means that the elements come in no apparently-sensible order.

```
1 >>> english_words
2 {'four': 4, 'three': 3, 'two': 2, 'one': 1}
```

(Don't be mislead by this example, they do not always just come in the reverse of the order in which you entered them.) If you assign to an existing key then that will replace the previous value.

```
1 >>> english_words['one'] = 5
2 >>> english_words
3 {'four': 4, 'three': 3, 'two': 2, 'one': 5}
```

Dictionaries are central to Python, in part because looking up values in a dictionary is very fast.

While dictionaries are unordered, a Python *list* is ordered.

```
1 >>> a=['alpha', 'beta', 'gamma']
2 >>> b=[]
3 >>> c=['delta']
4 >>> a
5 ['alpha', 'beta', 'gamma']
6 >>> a+b+c
7 ['alpha', 'beta', 'gamma', 'delta']
```

Get an element from a list by specifying its index, its place in the list, inside square brackets. Lists are zero-offset indexed, that is, the initial element of the list is numbered 0. You can count from the back by using negative indices.

```
1 >>> a[0]
2 'alpha'
3 >>> a[1]
4 'beta'
5 >>> a[-1]
6 'gamma'
```

Also, specifying two indices separated by a colon will get a *slice* of the list.

```
1 >>> a[1:3]
2 ['beta', 'gamma']
3 >>> a[1:2]
4 ['beta']
```

You can add to a list.

```
1 >>> c.append('epsilon')
2 >>> c
3 ['delta', 'epsilon']
```

Lists can contain anything, including other lists.

```
1 >>> x = 4
2 >>> a = ['alpha', ['beta', x]]
3 >>> a
4 ['alpha', ['beta', 4]]
```

The function `range` returns a list of numbers.

```

1 >>> range(10)
2 [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
3 >>> range(1,10)
4 [1, 2, 3, 4, 5, 6, 7, 8, 9]

```

Observe that by default `range` starts at 0, which is convenient because lists are zero-indexed. Observe also that 9 is the highest number in the sequence given by `range(10)`. This makes `range(10)+range(10,20)` give the same list as `range(20)`.

One of the most common things you'll do with a list is to run through it performing some action on each entry. Python has a shortcut for this called *list comprehension*.

```

1 >>> a = [2**i for i in range(4)]
2 >>> a
3 [1, 2, 4, 8]
4 >>> [i-1 for i in a]
5 [0, 1, 3, 7]

```

This is a special form of a loop; we'll see more on loops below.

A *tuple* is like a list in that it is ordered.

```

1 >>> a = (0, 1, 2)
2 >>> a
3 (0, 1, 2)
4 >>> a[0]
5 0

```

However, unlike a list a tuple cannot change.

```

1 >>> a[0] = 3
2 Traceback (most recent call last):
3   File "<stdin>", line 1, in <module>
4 TypeError: 'tuple' object does not support item assignment

```

That is an advantage sometimes: because tuples cannot change they can be keys in dictionaries, while list cannot.

```

1 >>> a = ['Jim', 2138]
2 >>> b = ('Jim', 2138)
3 >>> d = {a: 1}
4 Traceback (most recent call last):
5   File "<stdin>", line 1, in <module>
6 TypeError: unhashable type: 'list'
7 >>> d = {b: 1}
8 >>> d
9 {('Jim', 2138): 1}

```

Python has a special value `None` for you to use where there is no sensible value. For instance, if your program keeps track of a person's address and includes a variable `appt_no` then `None` is the right value for that variable when the person does not live in an apartment.

Flow of control Python supports the traditional ways of affecting the order of statement execution, with a twist.

```

1 >>> x = 4

```

```

2 >>> if (x == 0):
3 ...     y = 1
4 ... else:
5 ...     y = 0
6 ...
7 >>> y
8 0

```

The twist is that while many languages use braces or some other syntax to mark a block of code, Python uses indentation. (We shall always indent with four spaces.) Here, Python executes the single-line block `y = 1` if `x` equals 0, otherwise Python sets `y` to 0.

Notice also that double equals `==` means “is equal to.” We have already seen that single equals is the assignment operation so that `x = 4` means “`x` is assigned the value 4.”

Python has two variants on the `if` statement. The first has only one branch

```

1 >>> x = 4
2 >>> y = 0
3 >>> if (x == 0):
4 ...     y = 1
5 ...
6 >>> y
7 0

```

while the second has more than two branches.

```

1 >>> x = 2
2 >>> if (x == 0):
3 ...     y = 1
4 ... elif (x == 1):
5 ...     y = 0
6 ... else:
7 ...     y = -1
8 ...
9 >>> y
10 -1

```

A specialty of computers is iteration, looping through the same steps.

```

1 >>> for i in range(5):
2 ...     print i, "squared is", i**2
3 ...
4 0 squared is 0
5 1 squared is 1
6 2 squared is 4
7 3 squared is 9
8 4 squared is 16

```

A `for` loop often involves a `range`.

```

1 >>> x=[4,0,3,0]
2 >>> for i in range(len(x)):
3 ...     if (x[i] == 0):
4 ...         print "item",i,"is zero"
5 ...     else:

```

```

6     ...         print "item", i, "is nonzero"
7 ...
8 item 0 is nonzero
9 item 1 is zero
10 item 2 is nonzero
11 item 3 is zero

```

We could instead have written `for c in x:` since the `for` loop can iterate over any sequence, not just a sequence of integers.

A `for` loop is designed to execute a certain number of times. The natural way to write a loop that will run an uncertain number of times is `while`.

```

1 >>> n = 27
2 >>> i = 0
3 >>> while (n != 1):
4     ...     if (n%2 == 0):
5     ...         n = n / 2
6     ...     else:
7     ...         n = 3*n + 1
8     ...     i = i + 1
9     ...     print "i=", i
10 ...
11 i = 1
12 i = 2
13 i = 3

```

(This listing is incomplete; it takes 111 steps to finish.)¹ Note that “not equal” is `!=`.

The `break` command gets you out of a loop right away.

```

1 >>> for i in range(10):
2     ...     if (i == 3):
3     ...         break
4     ...     print "i is", i
5 ...
6 i is 0
7 i is 1
8 i is 2

```

Functions A *function* is a group of statements that executes when it is called, and can return values to the caller. Here is a naive version of the quadratic formula.

```

1 >>> def quad_formula(a, b, c):
2     ...     discriminant = (b**2 - 4*a*c)**(0.5)
3     ...     r1=(-1*b+discriminant) / (2.0*a)
4     ...     r2=(-1*b-discriminant) / (2.0*a)
5     ...     return (r1, r2)
6 ...
7 >>> quad_formula(1,0,-9)
8 (3.0, -3.0)
9 >>> quad_formula(1,2,1)
10 (-1.0, -1.0)

```

¹The *Collatz conjecture* is that for any starting n this loop will terminate, but this is not known.

(One way that it is naive is that it doesn't handle complex roots gracefully.) Functions organize code into blocks that may be run a number of different times or which may belong together conceptually. In a Python program the great majority of code is in functions.

At the end of the `def` line, in parentheses, are the function's *parameters*. These can take values passed in by the caller. Functions can have *optional parameters* that have a default value.

```

1 >>> def hello(name="Jim"):
2 ...     print "Hello, ", name
3 ...
4 >>> hello("Fred")
5 Hello, Fred
6 >>> hello()
7 Hello, Jim

```

Sage uses this aspect of Python a great deal.

Functions can contain multiple `return` statements. They always return something; if a function never executes a `return` then it will return the value `None`.

Objects and modules In Mathematics, the real numbers is a set associated with some operations such as addition and multiplication. Python is *object-oriented*, which means that we can similarly bundle together data and actions.

```

1 >>> class person(object):
2 ...     def __init__(self, name, age):
3 ...         self.name = name
4 ...         self.age = age
5 ...     def hello(self):
6 ...         print "Hello", self.name
7 ...
8 >>> a=person("Jim", 53)
9 >>> a.hello()
10 Hello Jim
11 >>> a.age
12 53

```

You work with objects by using the *dot notation*: to get the age data bundled with `a` you write `a.age`, and to call the `hello` function bundled with `a` you write `a.hello()` (a function bundled in this way is called a *method*).

You won't be writing your own classes in this lab manual but you will be using ones from the extensive libraries of code that others have written, including the code for Sage. For instance, Python has a library, or *module*, for math.

```

1 >>> import math
2 >>> math.pi
3 3.141592653589793
4 >>> math.factorial(4)
5 24
6 >>> math.cos(math.pi)
7 -1.0

```

The `import` statement gets the module and makes its contents available.

Another library is for random numbers.

```
1 >>> import random
2 >>> while (random.randint(1,10) != 1):
3 ...     print "wrong"
4 ...
5 wrong
6 wrong
```

Programs The read-eval-print loop is great for small experiments but for more than four or five lines you want to put your work in a separate file and run it as a standalone program.

To write the code, use a text editor; one example is Emacs¹. You should try to use an editor with support for Python such as automatic indentation, and syntax highlighting, where the editor colors your code to make it easier to read.

Here is one example. Start your editor, open a file called `test.py`, and enter these lines. Note the triple-quoted documentation string at the top of the file; good practice is to include this documentation in everything you write.

```
1 # test.py
2 """test
3
4 A test program for Python.
5 """
6
7 import datetime
8
9 current = datetime.datetime.now() # get a datetime object
10 print "the month number is", current.month
```

Run it under Python (for instance, from the command line run `python test.py`) and you should see output like `the month number is 9`.

Here is a small game (it has some Python constructs that you haven't seen but that are straightforward).

```
1 # guessing_game.py
2 """guessing_game
3
4 A toy game for demonstration.
5 """
6
7 import random
8 CHOICE = random.randint(1,10)
9
10 def test_guess(guess):
11     """Decide if the guess is correct and print a message.
12     """
13     if (guess < CHOICE):
14         print " Sorry, your guess is too low"
15         return False
16     elif (guess > CHOICE):
17         print " Sorry, your guess is too high"
18         return False
```

¹It may come with your operating system or see <http://www.gnu.org/software/emacs>.

```

18     print " You are right!"
19     return True
20
21 flag = False
22 while (not flag):
23     guess = int(raw_input("Guess an integer between 1 and 10: "))
24     flag = test_guess(guess)

```

Here is the output.

```

1 $ python guessing_game.py
2 Guess an integer between 1 and 10: 5
3 Sorry, your guess is too low
4 Guess an integer between 1 and 10: 8
5 Sorry, your guess is too high
6 Guess an integer between 1 and 10: 6
7 Sorry, your guess is too low
8 Guess an integer between 1 and 10: 7
9 You are right!

```

As above, note the triple-quoted documentation strings both for the file as a whole and for the function. Go to the directory containing `guessing_game.py` and start Python. At the prompt type in `import guessing_game`. You will play through a round of the game (there is a way to avoid this but it doesn't matter here) and then type `help("guessing_game")`. You will see documentation that includes these lines.

```

1 DESCRIPTION
2     A toy game for demonstration.
3
4 FUNCTIONS
5     test_guess(guess)
6         Decide if the guess is correct and print a message.

```

Obviously, Python got this from the file's documentation strings. In Python, and also in Sage, good practice is to always include documentation that is accessible with the `help` command.

Sage

Learning what Sage can do is the object of much of this book so this is a very brief walk-through of preliminaries. See [Sage Development Team, 2012a] for a more broad-based introduction.

First, if your system does not already supply it then install Sage by following the directions at www.sagemath.org.

Command line Sage's command line is like Python's but adapted to mathematical work. First start Sage, for instance, enter `sage` into a command line window. You get some initial text and then a prompt (leave the prompt by typing `exit` and `<Enter>`.)

```

1 sage:
Experiment with some expressions.

```

```

1 sage: 2**3

```

```

2 8
3 sage: 2^3
4 8
5 sage: 3*1 + 4*2
6 11
7 sage: 5 == 3+3
8 False
9 sage: sin(pi/3)
10 1/2*sqrt(3)

```

The second expression shows that Sage provides a convenient shortcut for exponentiation. The fourth shows that Sage sometimes returns exact results rather than an approximation. You can still get the approximation.

```

1 sage: sin(pi/3).numerical_approx()
2 0.866025403784439
3 sage: sin(pi/3).n()
4 0.866025403784439

```

The function `n()` is an abbreviation for `numerical_approx()`.

Script You can group Sage commands together in a file. This way you can test the commands, and also reuse them without having to retype.

Create a file with the extension `.sage`, such as `sage_try.sage`. Enter this function and save the file.

```

1 def normal_curve(upper_limit):
2     """Approximate area under the Normal curve from 0 to upper_limit.
3     """
4     stdev = 1.0
5     mu = 0.0
6     area=numerical_integral((1/sqrt(2*pi)) * e^(-0.5*(x)^2)),
7                             0, upper_limit)
8     print "area is", area[0]

```

Bring in the commands in with a `load` command.

```

1 sage: load "sage_try.sage"
2 sage: normal_curve(1.0)
3 area is 0.341344746069

```

Notebook Sage also offers a browser-based interface, where you can set up worksheets to run alone or with other people, where you can easily view plots integrated with the text, and many other nice features.

From the Sage prompt run `notebook()` and work through the tutorial.

Gauss's Method

Sage can solve linear systems in a number of ways. Here you will enter systems of linear equations, both as equations and in the matrix form. You will use those to solve the systems.

Systems

To enter a system of equations you must first enter single equations, so you must enter variables. We have seen one kind of variable in giving commands like these.

```
1 sage: x = 3
2 sage: 7*x
3 21
```

Here x is the name of a location to hold values. For variables in equations we want something different; in the equation $C = 2\pi \cdot r$ the two variables do not necessarily have fixed values, nor are they associated with a location in the computer's memory. To illustrate the difference enter this.

```
1 sage: y
```

The response is `NameError: name 'y' is not defined` because Sage defaults to expecting that y is the name of a location that holds a value. Instead, you need to declare to the system that something is a symbolic variable.

```
1 sage: var('x,y')
2 sage: x
3 x
```

With that, a system of equations is a list.

```
1 sage: var('x,y,z')
2 (x, y, z)
3 sage: eqns = [x-y+2*z == 4, 2*x+2*y == 12, x-4*z==5]
4 sage: solve(eqns, x,y,z)
5 [[x == 5, y == 1, z == 0]]
```

Note that you must write double equals `==` instead of the assignment operator `=` and also that you must write `2*x` instead of `2x`. Either mistake will give you `SyntaxError: invalid syntax`.

You can put a parameter in the right side and solve for the variables in terms of the parameter.

```
1 sage: var('x,y,z,a')
2 (x, y, z, a)
3 sage: eqns = [x-y+2*z == a, 2*x+2*y == 12, x-4*z==5]
```

```

4 sage: solve(eqns, x,y,z)
5 [[x == 2/5*a + 17/5, y == -2/5*a + 13/5, z == 1/10*a - 2/5]]

```

Matrices The *solve* routine is general-purpose but for the special case of linear systems matrix notation is the best tool.

Most of your matrices have entries that are fractions. Sage uses ‘QQ’ for the rational numbers, ‘RR’ for the reals, ‘CC’ for the complex numbers, and ‘ZZ’ for the integers. Also, you enter a row of the matrix as a list, and you enter the entire matrix as a list of rows.

```

1 sage: M = matrix(QQ, [[1, 2, 3], [4, 5, 6], [7, 8, 9]])
2 sage: M
3 [1 2 3]
4 [4 5 6]
5 [7 8 9]
6 sage: M[1,2]
7 6
8 sage: M.nrows()
9 3
10 sage: M.ncols()
11 3

```

Sage lists are zero-indexed, so $M[1,2]$ asks for the entry in the second row and third column.

Enter a vector in much the same way.

```

1 sage: v = vector(QQ, [2/3, -1/3, 1/2])
2 sage: v
3 (2/3, -1/3, 1/2)
4 sage: v[1]
5 -1/3

```

This is a column vector; it appears with rounded brackets so that it looks different than a one-row matrix. Augment the matrix with this vector.

```

1 sage: M_prime = M.augment(v)
2 sage: M_prime
3 [ 1 2 3 10]
4 [ 4 5 6 11]
5 [ 7 8 9 12]

```

Have Sage remember the distinction between the two parts of M' .

```

1 sage: M_prime = M.augment(v, subdivise=True)
2 sage: M_prime
3 [ 1 2 3|10]
4 [ 4 5 6|11]
5 [ 7 8 9|12]

```

Row operations Computers are good for jobs that are tedious and error-prone. Row operations are both.

First enter an example matrix.

```

1 sage: M = matrix(QQ, [[0, 2, 1], [2, 0, 4], [2, -1/2, 3]])

```

```

2 sage: v = vector(QQ, [2, 1, -1/2])
3 sage: M_prime = M.augment(v, subdivide=True)
4 sage: M_prime
5 [ 0 2 1| 2]
6 [ 2 0 4| 1]
7 [ 2 -1/2 3|-1/2]

```

Swap the top rows (remember that row indices start at zero).

```

1 sage: M_prime.swap_rows(0,1)
2 sage: M_prime
3 [ 2 0 4| 1]
4 [ 0 2 1| 2]
5 [ 2 -1/2 3|-1/2]

```

Rescale the top row.

```

1 sage: M_prime.rescale_row(0, 1/2)
2 sage: M_prime
3 [ 1 0 2| 1/2]
4 [ 0 2 1| 2]
5 [ 2 -1/2 3|-1/2]

```

Get a new bottom row by adding -2 times the top to the current bottom row.

```

1 sage: M_prime.add_multiple_of_row(2,0,-2)
2 sage: M_prime
3 [ 1 0 2| 1/2]
4 [ 0 2 1| 2]
5 [ 0 -1/2 -1|-3/2]

```

Finish by finding echelon form.

```

1 sage: M_prime.add_multiple_of_row(2,1,1/4)
2 sage: M_prime
3 [ 1 0 2| 1/2]
4 [ 0 2 1| 2]
5 [ 0 0 -3/4|-1]

```

(By the way, Sage would have given us echelon form in a single operation had we run the command `M_prime.echelon_form()`.)

Now by-hand back substitution gives the solution, or we can use `solve`.

```

1 sage: var('x,y,z')
2 sage: eqns=[-3/4*z == -1, 2*y+z == 2, x+2*z == 1/2]
3 sage: solve(eqns, x, y, z)
4 [[x == (-13/6), y == (1/3), z == (4/3)]]

```

The operations `swap_rows`, `rescale_rows`, and `add_multiple_of_row` changed the matrix M' . Sage has related commands that return a changed matrix but leave the starting matrix unchanged.

```

1 sage: M = matrix(QQ, [[1/2, 1, -1], [1, -2, 0], [2, -1, 1]])
2 sage: v = vector(QQ, [0, 1, -2])
3 sage: M_prime = M.augment(v, subdivide=True)
4 sage: M_prime

```

```

5 [1/2    1   -1|   0]
6 [ 1   -2    0|   1]
7 [ 2   -1    1|  -2]
8 sage: N = M_prime.with_rescaled_row(0, 2)
9 sage: M_prime
10 [1/2    1   -1|   0]
11 [ 1   -2    0|   1]
12 [ 2   -1    1|  -2]
13 sage: N
14 [ 1   2   -2|   0]
15 [ 1   -2    0|   1]
16 [ 2   -1    1|  -2]
```

Here, M' is unchanged by the routine, while N is the returned changed matrix. The other two routines of this kind are `with_swapped_rows` and `with_added_multiple_of_row`.

Nonsingular and singular systems Resorting to `solve` after going through the row operations is artless. Sage will give reduced row echelon form straight from the augmented matrix.

```

1 sage: M_prime.rref()
2 [ 1     0     0| -13/6]
3 [ 0     1     0|  1/3]
4 [ 0     0     1|  4/3]
```

In that example the matrix M on the left is nonsingular because it is square and because Gauss's Method leads to echelon forms where every one of M 's columns has a leading variable. The next example starts with a different square matrix, a singular matrix, and consequently leads to echelon form systems where there are columns on the left that do not have a leading variable.

```

1 sage: M = matrix(QQ, [[1, 1, 1], [1, 2, 3], [2, 3, 4]])
2 sage: v = vector(QQ, [0, 1, 1])
3 sage: M_prime = M.augment(v, subdivide=True)
4 sage: M_prime
5 [1 1 1|0]
6 [1 2 3|1]
7 [2 3 4|1]
8 sage: M_prime.rref()
9 [ 1  0 -1|-1]
10 [ 0  1  2| 1]
11 [ 0  0  0| 0]
```

Recall that the singular case has two subcases. The first happens above: because in echelon form every row that is all zeros on the left has an entry on the right that is also zero, the system has infinitely many solutions. In contrast, with the same starting matrix the example below has a row that is zeros on the left but is nonzero on the right and so the system has no solution.

```

1 sage: v = vector(QQ, [0, 1, 2])
2 sage: M_prime = M.augment(v, subdivide=True)
3 sage: M_prime.rref()
4 [ 1  0 -1|  0]
5 [ 0  1  2|  0]
6 [ 0  0  0|  1]
```

The difference between the subcases has to do with the relationships among the rows of M and the relationships among the rows of the vector. In both cases, the relationship among the rows of the matrix M is that the first two rows add to the third. In the first case the vector has the same relationship while in the second it does not.

The easy way to ensure that a zero row in the matrix on the left is associated with a zero entry in the vector on the right is to make the vector have all zeros, that is, to consider the homogeneous system associated with M .

```

1 sage: v = zero_vector(QQ, 3)
2 sage: v
3 (0, 0, 0)
4 sage: M = matrix(QQ, [[1, 1, 1], [1, 2, 3], [2, 3, 4]])
5 sage: M_prime = M.augment(v, subdivide=True)
6 sage: M_prime
7 [1 1 1|0]
8 [1 2 3|0]
9 [2 3 4|0]
10 sage: M_prime.rref()
11 [ 1  0 -1| 0]
12 [ 0  1  2| 0]
13 [ 0  0  0| 0]
```

You can get the numbers of the columns having leading entries with the pivots method (there is a complementary nonpivots).

```

1 sage: v = vector(QQ, [0, 1, 2])
2 sage: M_prime
3 [1 1 1|0]
4 [1 2 3|1]
5 [2 3 4|2]
6 sage: M_prime.pivots()
7 (0, 1, 3)
8 sage: M_prime.rref()
9 [ 1  0 -1| 0]
10 [ 0  1  2| 0]
11 [ 0  0  0| 1]
```

Because column 2 is not in the list of pivots we know that the system is singular before we see it in echelon form.

We can use this observation to write a routine that decides if a square matrix is nonsingular.

```

1 sage: def check_nonsingular(mat):
2 ....:     if not(mat.is_square()):
3 ....:         print "ERROR: mat must be square"
4 ....:         return
5 ....:     p = mat.pivots()
6 ....:     for col in range(mat.ncols()):
7 ....:         if not(col in p):
8 ....:             print "nonsingular"
9 ....:             break
10 ....:
11 sage: N = Matrix(QQ, [[1, 2, 3], [4, 5, 6], [7, 8, 9]])
12 sage: check_nonsingular(N)
```

```

13 nonsingular
14 sage: N = Matrix(QQ, [[1, 0, 0], [0, 1, 0], [0, 0, 1]])
15 sage: check_nonsingular(N)

```

Actually, Sage matrices already have a method `is_singular` but this illustrates how you can write routines to extend Sage.

Parametrization You can use `solve` to give the solution set of a system with infinitely many solutions. Start with the square matrix of coefficients from above, where the top two rows add to the bottom row, and adjoin a vector satisfying the same relationship to get a system with infinitely many solutions. Then convert that to a system of equations.

```

1 sage: M = matrix(QQ, [[1, 1, 1], [1, 2, 3], [2, 3, 4]])
2 sage: v = vector(QQ, [1, 0, 1])
3 sage: M_prime = M.augment(v, subdivide=True)
4 sage: M_prime
5 [1 1 1|1]
6 [1 2 3|0]
7 [2 3 4|1]
8 sage: eqns = [x+y+z == 1, x+2*y+3*z == 0, 2*x+3*y+4*z == 1]
9 sage: solve(eqns, x, y)
10 [[x == z + 2, y == -2*z - 1]]
11 sage: solve(eqns, x, y, z)
12 [[x == r1 + 2, y == -2*r1 - 1, z == r1]]

```

The first of the two `solve` calls asks Sage to solve only for x and y and so the solution is in terms of z . In the second call Sage produces a parameter of its own.

Automation

We finish by showing two routines to automate the by-hand row reductions of the kind that the text has in the homework. These use the matrix capabilities of Sage to both describe and perform the row operations that bring a matrix to echelon form or to reduced echelon form.

Loading and running The source file of the script is below, at the end. First here are a few sample calls. Start Sage in the directory containing the file `gauss_method.sage`.

```

1 sage: load "gauss_method.sage"
2 sage: M = matrix(QQ, [[1/2, 1, 4], [2, 4, -1], [1, 2, 0]])
3 sage: v = vector(QQ, [-2, 5, 4])
4 sage: M_prime = M.augment(v, subdivide=True)
5 sage: gauss_method(M_prime)
6 [1/2 1 4| -2]
7 [ 2 4 -1| 5]
8 [ 1 2 0| 4]
9 take -4 times row 1 plus row 2
10 take -2 times row 1 plus row 3
11 [1/2 1 4| -2]
12 [ 0 0 -17| 13]
13 [ 0 0 -8| 8]

```

```

14 take -8/17 times row 2 plus row 3
15 [ 1/2      1      4|     -2]
16 [ 0       0     -17|    13]
17 [ 0       0      0|32/17]

```

The remaining examples skip the extra steps to make an augmented matrix.

```

1 sage: M1 = matrix(QQ, [[2, 0, 1, 3], [-1, 1/2, 3, 1], [0, 1, 7, 5]])
2 sage: gauss_method(M1)
3 [ 2   0   1   3]
4 [ -1 1/2   3   1]
5 [ 0   1   7   5]
6 take 1/2 times row 1 plus row 2
7 [ 2   0   1   3]
8 [ 0 1/2 7/2 5/2]
9 [ 0   1   7   5]
10 take -2 times row 2 plus row 3
11 [ 2   0   1   3]
12 [ 0 1/2 7/2 5/2]
13 [ 0   0   0   0]

```

The script has a routine to go all the way to reduced echelon form.

```

1 sage: M2 = matrix(QQ,
2 . . .:           [[1, 2, 3, 4], [1, 2, 3, 4],
3 . . .:                     [2, 4, -1, 5], [1, 2, 0, 4]])
4 sage: gauss_jordan(M2)
5 [ 1   2   3   4]
6 [ 1   2   3   4]
7 [ 2   4   -1   5]
8 [ 1   2   0   4]
9 take -1 times row 1 plus row 2
10 take -2 times row 1 plus row 3
11 take -1 times row 1 plus row 4
12 [ 1   2   3   4]
13 [ 0   0   0   0]
14 [ 0   0   -7  -3]
15 [ 0   0   -3   0]
16 swap row 2 with row 3
17 [ 1   2   3   4]
18 [ 0   0   -7  -3]
19 [ 0   0   0   0]
20 [ 0   0   -3   0]
21 take -3/7 times row 2 plus row 4
22 [ 1   2   3   4]
23 [ 0   0   -7  -3]
24 [ 0   0   0   0]
25 [ 0   0   0   9/7]
26 swap row 3 with row 4
27 [ 1   2   3   4]
28 [ 0   0   -7  -3]
29 [ 0   0   0   9/7]
30 [ 0   0   0   0]

```

```

31 take -1/7 times row 2
32 take 7/9 times row 3
33 [ 1 2 3 4]
34 [ 0 0 1 3/7]
35 [ 0 0 0 1]
36 [ 0 0 0 0]
37 take -4 times row 3 plus row 1
38 take -3/7 times row 3 plus row 2
39 [1 2 3 0]
40 [0 0 1 0]
41 [0 0 0 1]
42 [0 0 0 0]
43 take -3 times row 2 plus row 1
44 [1 2 0 0]
45 [0 0 1 0]
46 [0 0 0 1]
47 [0 0 0 0]

```

These are naive implementations of Gauss's Method that do not try to work under stressful conditions, such as very small leading entries — they are just for fun. But they do illustrate perfectly the point made in the Preface, since a person builds intuition by doing a reasonable number of reasonably hard Gauss's Method reductions by hand, and at that point automation can take over.

Code comment: lines 50 and 88 are too long for this page so they end with a slash \ to make Python continue on the next line.

```

1 # code gauss_method.sage
2 # Show Gauss's method and Gauss-Jordan reduction steps.
3 # 2012-Apr-20 Jim Hefferon Public Domain.
4
5 # Naive Gaussian reduction
6 def gauss_method(M, rescale_leading_entry=False):
7     """Describe the reduction to echelon form of the given matrix of
8     rationals.
9
10    M    matrix of rationals   e.g., M = matrix(QQ, [[...], [...], ...])
11    rescale_leading_entry=False  boolean  make leading entries to 1's
12
13    Returns: None.  Side effect: M is reduced.  Note that this is
14    echelon form, not reduced echelon form;, and that this routine
15    does not end the same way as does M.echelon_form().
16
17    """
18    num_rows=M.nrows()
19    num_cols=M.ncols()
20    print M
21
22    col = 0  # all cols before this are already done
23    for row in range(0, num_rows):
24        # ?Need to swap in a nonzero entry from below
25        while (col < num_cols
26               and M[row][col] == 0):
27            for i in M.nonzero_positions_in_column(col):

```

```
28         if i > row:
29             print " swap row",row+1,"with row",i+1
30         M.swap_rows(row,i)
31         print M
32         break
33     else:
34         col += 1
35
36     if col >= num_cols:
37         break
38
39     # Now guaranteed M[row][col] != 0
40     if (rescale_leading_entry
41         and M[row][col] != 1):
42         print " take",1/M[row][col],"times row",row+1
43         M.rescale_row(row,1/M[row][col])
44     print M
45     change_flag=False
46     for changed_row in range(row+1,num_rows):
47         if M[changed_row][col] != 0:
48             change_flag=True
49             factor=-1*M[changed_row][col]/M[row][col]
50             print " take",factor,"times row",row+1, \
51                   "plus row",changed_row+1
52             M.add_multiple_of_row(changed_row,row,factor)
53     if change_flag:
54         print M
55     col +=1
56
57 # Naive Gauss-Jordan reduction
58 def gauss_jordan(M):
59     """Describe the reduction to reduced echelon form of the
60     given matrix of rationals.
61
62     M matrix of rationals e.g., M = matrix(QQ, [[..], [..], ...])
63
64     Returns: None. Side effect: M is reduced.
65
66     """
67     gauss_method(M,rescale_leading_entry=False)
68     # Get list of leading entries [le in row 0, le in row1, ...]
69     pivot_list=M.pivots()
70     # Rescale leading entries
71     change_flag=False
72     for row in range(0,len(pivot_list)):
73         col=pivot_list[row]
74         if M[row][col] != 1:
75             change_flag=True
76             print " take",1/M[row][col],"times row",row+1
77             M.rescale_row(row,1/M[row][col])
78     if change_flag:
```

```
79     print M
80 # Pivot
81 for row in range(len(pivot_list)-1, -1, -1):
82     col=pivot_list[row]
83     change_flag=False
84     for changed_row in range(0, row):
85         if M[changed_row, col] != 0:
86             change_flag=True
87             factor=-1*M[changed_row][col]/M[row][col]
88             print " take", factor, "times row", \
89                   row+1, "plus row", changed_row+1
90             M.add_multiple_of_row(changed_row, row, factor)
91     if change_flag:
92         print M
```

Vector Spaces

Sage can operate with vector spaces, for example by finding a basis for a space.

Real n -spaces

Start by creating a real vector space \mathbb{R}^n .

```
1 sage: V = RR^3
2 sage: V
3 Vector space of dimension 3 over Real Field with 53 bits of precision
```

You can test for membership.

```
1 sage: v1 = vector(RR, [1, 2, 3])
2 sage: v1 in V
3 True
4 sage: v2 = vector(RR, [1, 2, 3, 4])
5 sage: v2 in V
6 False
```

Consider the plane through the origin in \mathbb{R}^3 described by the equation $x - 2y + 2z = 0$. It is a subspace of \mathbb{R}^3 and we can describe it as a span.

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x = 2y - 2z \right\} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} z \mid y, z \in \mathbb{R} \right\}$$

You can create that subspace.

```
1 sage: V = RR^3
2 sage: v1 = vector(RR, [2, 1, 0])
3 sage: v2 = vector(RR, [-2, 0, 1])
4 sage: W = V.span([v1, v2])
```

And you can check that it is the desired space by doing some membership tests.

```
1 sage: v3 = vector(RR, [0, 1, 1])
2 sage: v3 in W
3 True
4 sage: v4 = vector(RR, [1, 0, 0])
5 sage: v4 in W
6 False
```

For a closer look at what's happening here, create a subspace of \mathbb{R}^4 . (Some of the output in this chapter is edited to fit the page width.)

```

1 sage: V = RR^4
2 sage: V
3 Vector space of dimension 4 over Real Field with 53 bits of precision
4 sage: v1 = vector(RR, [2, 0, -1, 0])
5 sage: W = V.span([v1])
6 sage: W
7 Vector space of degree 4 and dimension 1 over Real Field with 53 bits
8     of precision
9 Basis matrix:
10 [ 1.00000000000000  0.00000000000000 -0.50000000000000
11   0.00000000000000]

```

Sage has identified the dimension of the subspace and found a basis containing one vector (it prefers the vector with a leading 1). Sage also mentions the “Real Field” because we could take scalars from other number systems—in the book’s final chapter the scalars are from the complex numbers—but we will here stick with real numbers. The “53 bits of precision” refers to the fact that Sage is using this computer’s built-in model of real numbers.¹

As earlier, the membership set relation works here.

```

1 sage: v2 = vector(RR, [2, 1, -1, 0])
2 sage: v2 in W
3 False
4 sage: v3 = vector(RR, [-4, 0, 2, 0])
5 sage: v3 in W
6 True

```

Basis Sage will give you a basis for your space.

```

1 sage: V=RR^2
2 sage: v=vector(RR, [1, -1])
3 sage: W = V.span([v])
4 sage: W.basis()
5 [
6 (1.00000000000000, -1.00000000000000)
7 ]

```

Here is another example.

```

1 sage: V=RR^3
2 sage: v1 = vector(RR, [1, -1, 0])
3 sage: v2 = vector(RR, [1, 1, 0])
4 sage: W = V.span([v1, v2])
5 sage: W.basis()
6 [
7 (1.00000000000000, 0.00000000000000, 0.00000000000000),
8 (0.00000000000000, 1.00000000000000, 0.00000000000000)
9 ]
10 sage: W.basis()

```

¹This computer uses IEEE 754 double-precision binary floating-point numbers; if you have programmed then you may know this number model as binary64.

```

11 [
12 (1.00000000000000, 0.00000000000000, 0.00000000000000),
13 (0.00000000000000, 1.00000000000000, 0.00000000000000)
14 ]

```

Adding a linearly dependent vector \vec{v}_3 to the spanning set doesn't change the space.

```

1 sage: v3 = vector(RR, [2, 3, 0])
2 sage: W_prime = V.span([v1, v2, v3])
3 sage: W_prime.basis()
4 [
5 (1.00000000000000, 0.00000000000000, 0.00000000000000),
6 (0.00000000000000, 1.00000000000000, 0.00000000000000)
7 ]

```

In the prior example, notice that Sage does not simply give you back the linearly independent vectors that you gave it. Instead, Sage takes the vectors from the spanning set as the rows of a matrix, brings that matrix to reduced echelon form, and reports the nonzero rows (transposed to column vectors and so printed with parentheses) as the members of the basis. Because each matrix has one and only one reduced echelon form, each vector subspace of real n-space has one and only one such basis; this is a *canonical basis* for the space.

Sage will show you that it keeps the canonical basis as defining the space.

```

1 sage: W
2 Vector space of degree 3 and dimension 2 over Real Field with 53 bits
3     of precision
4 Basis matrix:
5 [ 1.00000000000000 0.00000000000000 0.00000000000000]
6 [ 0.00000000000000 1.00000000000000 0.00000000000000]
7 sage: W_prime
8 Vector space of degree 3 and dimension 2 over Real Field with 53 bits
9     of precision
10 Basis matrix:
11 [ 1.00000000000000 0.00000000000000 0.00000000000000]
12 [ 0.00000000000000 1.00000000000000 0.00000000000000]

```

If you are quite keen on your own basis then Sage will accomodate you.

```

1 sage: V = RR^3
2 sage: v1 = vector(RR, [1, 2, 3])
3 sage: v2 = vector(RR, [2, 1, 3])
4 sage: W = V.span_of_basis([v1, v2])
5 sage: W.basis()
6 [
7 (1.00000000000000, 2.00000000000000, 3.00000000000000),
8 (2.00000000000000, 1.00000000000000, 3.00000000000000)
9 ]
10 sage: W
11 Vector space of degree 3 and dimension 2 over Real Field with 53 bits
12     of precision
13 User basis matrix:
14 [1.00000000000000 2.00000000000000 3.00000000000000]
15 [2.00000000000000 1.00000000000000 3.00000000000000]

```

Space equality You can test spaces for equality.

```

1 sage: V = RR^4
2 sage: v1 = vector(RR, [1, 0, 0, 0])
3 sage: v2 = vector(RR, [1, 1, 0, 0])
4 sage: W12 = V.span([v1, v2])
5 sage: v3 = vector(RR, [2, 1, 0, 0])
6 sage: W13 = V.span([v1, v3])

```

Now use set membership to check that the two spaces W_{12} and W_{13} are equal.

```

1 sage: v3 in W12
2 True
3 sage: v2 in W13
4 True

```

Then, since obviously $\vec{v}_1 \in W_{12}$ and $\vec{v}_1 \in W_{13}$, the two spans are equal.

You can also just ask Sage.

```

1 sage: W12 == W13
2 True

```

Your check that equality works would be half-hearted if you didn't have an unequal case.

```

1 sage: v4 = vector(RR, [1, 1, 1, 1])
2 sage: W14 = V.span([v1, v4])
3 sage: v2 in W14
4 False
5 sage: v3 in W14
6 False
7 sage: v4 in W12
8 False
9 sage: v4 in W13
10 False
11 sage: W12 == W14
12 False
13 sage: W13 == W14
14 False

```

This illustrates a point about algorithms. Sage could check for equality of two spans by checking whether every member of the first spanning set is in the second space and vice versa (since the two spanning sets are finite). But Sage does something different. For each space it maintains the canonical basis and it checks for equality of the two spaces just by checking that they have the same canonical bases. These two algorithms have the same external behavior, in that both report correctly whether the two spaces are equal, but the second is faster. Finding the fastest way to do jobs is an important research area of computing.

Space operations Sage finds the intersection of two spaces. Consider these members of \mathbb{R}^3 .

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

We form two spans, the xy -plane $W_{12} = [\vec{v}_1, \vec{v}_2]$ and the yz -plane $W_{23} = [\vec{v}_2, \vec{v}_3]$. The intersection of these two is the y -axis.

```

1 sage: V=RR^3
2 sage: v1 = vector(RR, [1, 0, 0])
3 sage: v2 = vector(RR, [0, 1, 0])
4 sage: W12 = V.span([v1, v2])
5 sage: W12.basis()
6 [
7 (1.00000000000000, 0.000000000000000, 0.000000000000000),
8 (0.000000000000000, 1.00000000000000, 0.000000000000000)
9 ]
10 sage: v3 = vector(RR, [0, 0, 2])
11 sage: W23 = V.span([v2, v3])
12 sage: W23.basis()
13 [
14 (0.000000000000000, 1.00000000000000, 0.000000000000000),
15 (0.000000000000000, 0.000000000000000, 1.000000000000000)
16 ]
17 sage: W = W12.intersection(W23)
18 sage: W.basis()
19 [
20 (0.000000000000000, 1.00000000000000, 0.000000000000000)
21 ]

```

Remember that the span of the empty set is the trivial space.

```

1 sage: v3 = vector(RR, [1, 1, 1])
2 sage: W3 = V.span([v3])
3 sage: W3.basis()
4 [
5 (1.00000000000000, 1.00000000000000, 1.00000000000000)
6 ]
7 sage: W4 = W12.intersection(W3)
8 sage: W4.basis()
9 [
10 ]
11 ]

```

Sage will also find the sum of spaces, the span of their union.

```

1 sage: W5 = W12 + W3
2 sage: W5.basis()
3 [
4 (1.00000000000000, 0.000000000000000, 0.000000000000000),
5 (0.000000000000000, 1.00000000000000, 0.000000000000000),
6 (0.000000000000000, 0.000000000000000, 1.000000000000000)
7 ]
8 sage: W5 == V
9 True

```

Other spaces

You can extend these computations to vector spaces that aren't a subspace of some \mathbb{R}^n . Just find a real space that matches the given one.

Consider this vector space set of quadratic polynomials (under the usual operations of polynomial addition and scalar multiplication).

$$\{a_2x^2 + a_1x + a_0 \mid a_2 - a_1 = a_0\} = \{(a_1 + a_0)x^2 + a_1x + a_0 \mid a_1, a_0 \in \mathbb{R}\}$$

It matches¹ this subspace of \mathbb{R}^3 .

$$\left\{ \begin{pmatrix} a_1 + a_0 \\ a_1 \\ a_0 \end{pmatrix} \mid a_1, a_0 \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} a_1 + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} a_0 \mid a_1, a_0 \in \mathbb{R} \right\}$$

```

1 sage: V=RR^3
2 sage: v1 = vector(RR, [1, 1, 0])
3 sage: v2 = vector(RR, [1, 0, 1])
4 sage: W = V.span([v1, v2])
5 sage: W.basis()
6 [
7 (1.00000000000000, 0.00000000000000, 1.00000000000000),
8 (0.00000000000000, 1.00000000000000, -1.00000000000000)
9 ]

```

Similarly you can represent this space of 2×2 matrices

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a - b + c = 0 \text{ and } b + d = 0 \right\}$$

by finding a real n -space just like it. Rewrite the given space

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a = -c - d \text{ and } b = -d \right\} = \left\{ \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} c + \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} d \mid c, d \in \mathbb{R} \right\}$$

and then here is a natural matching real space.

```

1 sage: V = RR^4
2 sage: v1 = vector(RR, [-1, 0, 1, 0])
3 sage: v2 = vector(RR, [-1, -1, 0, 1])
4 sage: W = V.span([v1, v2])
5 sage: W.basis()
6 [
7 (1.00000000000000, -0.00000000000000, -1.00000000000000,
8 -0.00000000000000),
9 (0.00000000000000, 1.00000000000000, 1.00000000000000,
10 -1.00000000000000)
11 ]

```

You could have gotten another matching space by going down the columns instead of across the rows, etc. But the important things about the space, such as its dimension, are unaffected by that choice.

¹The textbook's chapter on Maps Between Spaces makes "matches" precise.

Matrices

Matrix operations are mechanical, and are therefore perfectly suited for mechanizing.

Defining

To define a matrix you can use real number entries, or complex entries, or entries from other number systems such as the rationals.

```
1 sage: A = matrix(RR, [[1, 2], [3, 4]])
2 sage: A
3 [1.00000000000000  2.00000000000000]
4 [3.00000000000000  4.00000000000000]
5 sage: i = CC(i)
6 sage: A = matrix(CC, [[1+2*i, 3+4*i], [5+6*i, 7+8*i]])
7 sage: A
8 [1.00000000000000 + 2.00000000000000*I
9      3.00000000000000 + 4.00000000000000*I]
10 [5.00000000000000 + 6.00000000000000*I
11      7.00000000000000 + 8.00000000000000*I]
12 sage: A = matrix(QQ, [[1, 2], [3, 4]])
13 sage: A
14 [1 2]
15 [3 4]
```

(Sometimes, as here, the output is edited to fit the page. Note that before working with complex numbers we resetting i to be $\sqrt{-1}$. The letter i is used for many things so resetting is a good habit.) Unless we have a reason to do otherwise, we'll use rational numbers because the matrices easier to read.

The `matrix` constructor allows you to specify the number of rows and columns.

```
1 sage: B = matrix(QQ, 2, 3, [[1, 1, 1], [2, 2, 2]])
2 sage: B
3 [1 1 1]
4 [2 2 2]
```

If your specified size doesn't match the entries

```
1 sage: B = matrix(QQ, 3, 3, [[1, 1, 1], [2, 2, 2]])
```

then Sage's error says `Number of rows does not match up with specified number`. Until now we've let Sage figure out the matrix's number of rows and columns size from the entries but

a shortcut to get the zero matrix is to put the number zero in the place of the entries, and there you must say which size you want.

```

1 sage: B = matrix(QQ, 2, 3, 0)
2 sage: B
3 [0 0 0]
4 [0 0 0]
```

Another case is the shortcut to get an identity matrix.

```

1 sage: B = matrix(QQ, 2, 2, 1)
2 sage: B
3 [1 0]
4 [0 1]
```

The difference with this case is that `matrix(QQ, 3, 2, 1)` gives an error because an identity matrix must be square. Sage has another shortcut that can't lead to this error.

```

1 sage: I = identity_matrix(4)
2 sage: I
3 [1 0 0 0]
4 [0 1 0 0]
5 [0 0 1 0]
6 [0 0 0 1]
```

Sage has a wealth of methods on matrices. For instance, you can transpose the rows to columns or test if the matrix is *symmetric*, unchanged by transposition.

```

1 sage: A.transpose()
2 [1 3]
3 [2 4]
4 sage: A.is_symmetric()
5 False
```

Linear combinations

Addition and subtraction are natural.

```

1 sage: B = matrix(QQ, [[1, 1], [2, -2]])
2 sage: A+B
3 [2 3]
4 [5 2]
5 sage: A-B
6 [0 1]
7 [1 6]
8 sage: B-A
9 [ 0 -1]
10 [-1 -6]
```

Sage won't let you combine matrices with different sizes; this gives an error.

```

1 sage: C = matrix(QQ, [[0, 0, 2], [3, 2, 1]])
2 sage: A+C
```

The last line of the error says `TypeError: unsupported operand parent(s) for '+'` and describes the problem, albeit in somewhat technical terms, as that you can't add a 2×2 matrix to a 2×3 matrix.

Scalar multiplication is also natural so you have linear combinations.

```

1 sage: 3*A
2 [ 3   6]
3 [ 9  12]
4 sage: 3*A - 4*B
5 [-1   2]
6 [ 1  20]
```

Multiplication

Matrix-vector product Matrix-vector multiplication is just what you would guess.

```

1 sage: A = matrix(QQ, [[1, 3, 5, 9], [0, 2, 4, 6]])
2 sage: v = vector(QQ, [1, 2, 3, 4])
3 sage: A*v
4 (58, 40)
```

Here the 2×4 matrix A multiplies the 4×1 column vector \vec{v} , with the vector on the right side, as $A\vec{v}$. If you try this vector on the left as $v*A$ then Sage gives an error saying the sizes don't match.

You can make a vector to give a defined multiplication from the left side.

```

1 sage: w = vector(QQ, [3, 5])
2 sage: w*A
3 (3, 19, 35, 57)
```

In practice you see the vector on either side. We will have the habit of putting the vector on the right because that is what's in the book.

Matrix-matrix product If the sizes match then Sage will multiply the matrices in the obvious way. Here is the product of a 2×2 matrix A and a 2×3 matrix B .

```

1 sage: A = matrix(QQ, [[2, 1], [4, 3]])
2 sage: B = matrix(QQ, [[5, 6, 7], [8, 9, 10]])
3 sage: A*B
4 [18 21 24]
5 [44 51 58]
```

Trying $B*A$ gives the error `unsupported operand parent(s) for '*'`, meaning that the product operation in this order is undefined.

Same-sized square matrices have the product defined in either order.

```

1 sage: A = matrix(QQ, [[1, 2], [3, 4]])
2 sage: B = matrix(QQ, [[4, 5], [6, 7]])
3 sage: A*B
4 [16 19]
5 [36 43]
6 sage: B*A
```

```

7 [19  28]
8 [27  40]
```

Of course they are different; matrix multiplication is not commutative.

```

1 sage: A*B == B*A
2 False
```

It is very noncommutative, in that if you produce two $n \times n$ matrices at random then they almost surely don't commute.

```

1 sage: random_matrix(RR, 3, min=-1, max=1)
2 [ 0.316363590763196  0.401338054275907 -0.434566166245099]
3 [ 0.971906850779329 -0.189878350367539  0.0883946411446965]
4 [-0.892505949015383 -0.976541221718267 -0.278783066662399]
5 sage: random_matrix(RR, 3, min=-1, max=1)
6 [ 0.993363158808678  0.839878618006500 -0.431488448252715]
7 [-0.161186636590872  0.554717067849706 -0.414238110166975]
8 [-0.524416310634741 -0.897687685090845  0.829719717481328]
```

(Note the RR. We prefer real number entries here because `random_matrix` is more straightforward in this case than in the rational entry case.)

```

1 sage: number_commuting = 0
2 sage: for n in range(1000):
3 ....:     A = random_matrix(RR, 2, min=-1, max=1)
4 ....:     B = random_matrix(RR, 2, min=-1, max=1)
5 ....:     if (A*B == B*A):
6 ....:         number_commuting = number_commuting + 1
7 ....:
8 sage: print "number commuting of 1000=", number_commuting
9 number commuting of 1000= 0
```

Inverse Recall that if A is nonsingular then its *inverse* A^{-1} is the matrix such that $AA^{-1} = A^{-1}A$ is the identity matrix.

```

1 sage: A = matrix(QQ, [[1, 3, 1], [2, 1, 0], [4, -1, 0]])
2 sage: A.is_singular()
3 False
```

We have a formula for 2×2 matrix inverses but to compute inverses for larger matrices we write the original matrix next to the identity, and then do Gauss-Jordan reduction.

```

1 sage: I = identity_matrix(3)
2 sage: B = A.augment(I, subdivide=True)
3 sage: B
4 [ 1  3  1|  1  0  0]
5 [ 2  1  0|  0  1  0]
6 [ 4 -1  0|  0  0  1]
7 sage: C = B.rref()
8 sage: C
9 [ 1  0  0|  0  1/6  1/6]
10 [ 0  1  0|  0  2/3 -1/3]
11 [ 0  0  1|  1 -13/6 5/6]
```

The inverse is the resulting matrix on the right.

```

1 sage: A_inv = C.matrix_from_columns([3, 4, 5])
2 sage: A_inv
3 [ 0 1/6 1/6]
4 [ 0 2/3 -1/3]
5 [ 1 -13/6 5/6]
6 sage: A*A_inv
7 [1 0 0]
8 [0 1 0]
9 [0 0 1]
10 sage: A_inv*A
11 [1 0 0]
12 [0 1 0]
13 [0 0 1]
```

Since this is an operation that Sage users do all the time, there is a standalone command.

```

1 sage: A_inv = A.inverse()
2 sage: A_inv
3 [ 0 1/6 1/6]
4 [ 0 2/3 -1/3]
5 [ 1 -13/6 5/6]
```

One reason for finding the inverse is to make solving linear systems easier. Consider these.

$$\begin{array}{l} x + 3y + z = 4 \\ 2x + y = 4 \\ 4x - y = 4 \end{array} \quad \begin{array}{l} x + 3y + z = 2 \\ 2x + y = -1 \\ 4x - y = 5 \end{array} \quad \begin{array}{l} x + 3y + z = 1/2 \\ 2x + y = 0 \\ 4x - y = 12 \end{array}$$

They share the matrix of coefficients A but have different vectors on the right side. Having calculated the inverse of the matrix of coefficients, solving each system takes just a matrix-vector product.

```

1 sage: v1 = vector(QQ, [4, 4, 4])
2 sage: v2 = vector(QQ, [2, -1, 5])
3 sage: v3 = vector(QQ, [1/2, 0, 12])
4 sage: A_inv*v1
5 (4/3, 4/3, -4/3)
6 sage: A_inv*v2
7 (2/3, -7/3, 25/3)
8 sage: A_inv*v3
9 (2, -4, 21/2)
```

Run time

Since computers are fast and accurate they open up the possibility of solving problems that are quite large. Large linear algebra problems occur frequently in science and engineering. In this section we will suggest what limits there are to computers. (Here we will use matrices with real entries because they are common in applications.)

One of the limits on just how large a problem we can do is how quickly the computer program can give answers. Naturally computers take longer to perform operations on matrices that are larger but it may be that the time the program takes to compute the answer grows more quickly than does the size of the problem—for instance, when the size of the matrix doubles then the time to do the job more than doubles.

We will time the matrix inverse operation. This is an important operation; for instance, if we could do large matrix inverses quickly then we could quickly solve large linear systems, with just a matrix-vector product. We also use it because it makes a good illustration.

```

1 sage: A = matrix(RR, [[1, 3, 1], [2, 1, 0], [4, -1, 0]])
2 sage: A
3 [ 1.00000000000000  3.00000000000000  1.00000000000000]
4 [ 2.00000000000000  1.00000000000000  0.00000000000000]
5 [ 4.00000000000000 -1.00000000000000  0.00000000000000]
6 sage: A.is_singular()
7 False
8 sage: timeit('A.inverse()')
9 625 loops, best of 3: 124 µs per loop

```

It took 124 microseconds, which is 0.000124 seconds.¹ That's fast, but then A is only a 3×3 matrix.

To scale up the timing, you can find the inverse of a random matrix.

```

1 sage: timeit('random_matrix(RR, 3, min=-1, max=1).inverse()')
2 625 loops, best of 3: 186 µs per loop

```

This has the issue that we can't tell right away whether the time is spent generating the random matrix or finding the inverse. We also can't tell whether this command generates many random matrices and finds each's inverse, or generates one matrix and finds its inverse many times. This code is clearer.

```

1 sage: for size in [3, 10, 25, 50, 75, 100, 150, 200]:
2 ....:     print "size=", size
3 ....:     M = random_matrix(RR, size, min=-1, max=1)
4 ....:     timeit('M.inverse()')
5 ....:
6 size= 3
7 625 loops, best of 3: 125 µs per loop
8 size= 10
9 625 loops, best of 3: 940 µs per loop
10 size= 25
11 25 loops, best of 3: 12 ms per loop
12 size= 50
13 5 loops, best of 3: 92.4 ms per loop
14 size= 75
15 5 loops, best of 3: 308 ms per loop
16 size= 100
17 5 loops, best of 3: 727 ms per loop
18 size= 150
19 5 loops, best of 3: 2.45 s per loop
20 size= 200

```

¹Sage runs the command many times and comes up with a best guess about how long the operation ideally takes, because on any one time your computer may have been slowed down by a disk write or some other interruption.

```
21 5 loops, best of 3: 5.78 s per loop
```

Some of those times are in microseconds, some are in milliseconds, and some are in seconds. This table is consistently in seconds.

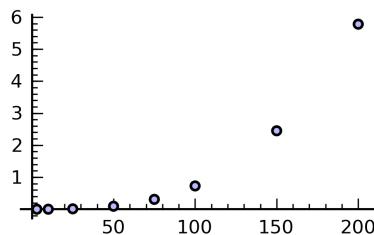
<i>size</i>	<i>seconds</i>
3	0.000125
10	0.000940
25	0.012
50	0.0924
75	0.308
100	0.727
150	2.45
200	5.78

The time grows faster than the size. For instance, in going from size 25 to size 50 the time more than doubles: $0.0924/0.012$ is 7.7. Similarly, increasing the size from 50 to 200 causes the time to increase by much more than a factor of four: $5.78/0.0924 \approx 62.55$.

To get a picture give Sage the data as a list of pairs.

```
1 sage: d = [(3, 0.000125), (10, 0.000940), (25, 0.012),
2 ....:      (50, 0.0924), (75, 0.308), (100, 0.727),
3 ....:      (150, 2.45), (200, 5.78)]
4 sage: g = scatter_plot(d)
5 sage: g.save("inverse_initial.png")
```

(If you enter `scatter_plot(d)` without saving it as `g` then Sage will pop up a window with the graphic.)¹



The graph dramatizes that the ratio time/size is not constant since the data clearly does not look like a line.

Here is some more data. The times are big enough that the computer had to run overnight.

```
1 sage: for size in [500, 750, 1000]:
2 ....:     print "size=", size
3 ....:     M = random_matrix(RR, size, min=-1, max=1)
4 ....:     timeit('M.inverse()')
5 ....:
6 size= 500
```

¹The graphics here use a few of Sage's options to make them better fit this page. For instance, the first graphic was generated with `g1 = scatter_plot(d, markersize=10, facecolor='#b9b9ff')`. It was saved with `g1.save("../inverse_initial.png", figsize=[2.25,1.5], axes_pad=0.05, fontsize=7, dpi=500)`.

```

7 5 loops, best of 3: 89.2 s per loop
8 size = 750
9 5 loops, best of 3: 299 s per loop
10 size = 1000
11 5 loops, best of 3: 705 s per loop

```

Again the table is neater.

<i>size</i>	<i>seconds</i>
500	89.2
750	299.
1000	705.

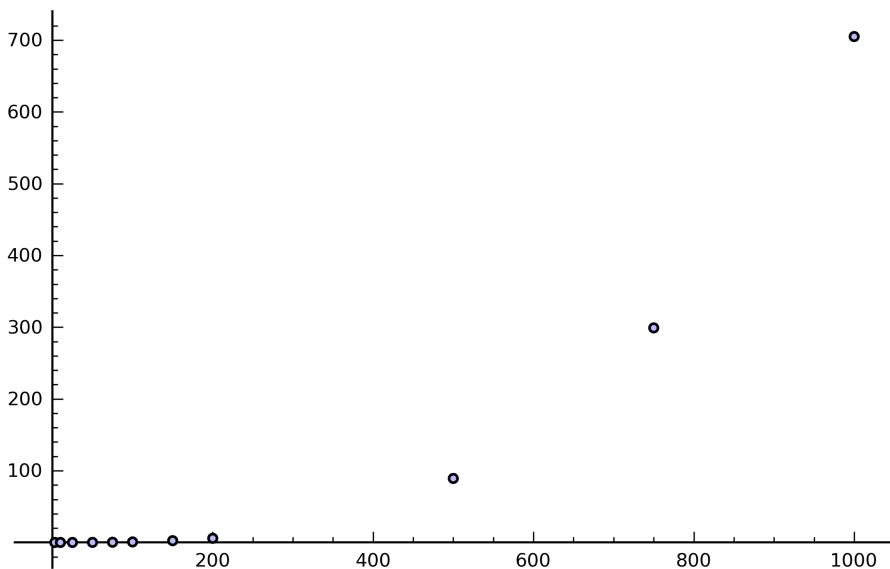
Get a graph by tacking the new data onto the existing data.

```

1 sage: d = d + [(500, 89.2), (750, 299), (1000, 705)]
2 sage: g = scatter_plot(d)
3 sage: g.save("inverse_full.png")

```

The result is this graphic.



Note that the two graphs have different scales. For one thing, if the vertical height for 6 was as large on the second graph as on the first then the data would go far off the top of the page.

So a practical limit to the size of a problem that we can solve with the matrix inverse operation comes from the fact that the graph above is not a line. Beyond some size the time required just gets too large.

Maps

We've used Sage to define vector spaces. Next we explore operations that we can do on vector spaces.

Left/right

Sage represents linear maps differently than the book does. Rather than quarrel with the tool, below we will do it Sage's way.

Consider the application of a linear map to a member of a vector space $t(\vec{v})$. For example, with this function $t: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ and this element of the domain

$$t\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{pmatrix} a+b \\ a-b \\ b \end{pmatrix} \quad \vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

the map application gives this.

$$t\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$$

To represent the map application we first fix bases. In this example we use the canonical bases $E_2 \subset \mathbb{R}^2$ and $E_3 \subset \mathbb{R}^3$. Then with respect to the bases the book finds a matrix $T = \text{Rep}_{E_2, E_3}(t)$ and a column vector $\vec{w} = \text{Rep}_{E_2}(\vec{v})$, and represents $t(\vec{v})$ with the product $T\vec{w}$.

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$$

That is, the book is write right: its notation puts the vector on the right of the matrix.

However, this choice is a matter of taste and many authors instead use a row vector that multiplies a matrix from the left. Sage is in this camp and represents the map application in this way.

$$(1 \ 3) \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} = (4 \ -2 \ 3)$$

Obviously the difference is cosmetic but can cause confusion. The translation is that, compared to the book's representation $T\vec{w}$, Sage prefers the transpose $(T\vec{w})^T = \vec{w}^T T^T$.

Defining

We will see two different ways to define a linear transformation.

Symbolically We first define a map that takes two inputs and returns three outputs.

```

1 sage: a, b = var('a, b')
2 sage: T_symbolic(a, b) = [a+b, a-b, b]
3 sage: T_symbolic
4 (a, b) |--> (a + b, a - b, b)

```

We have not yet defined a domain and codomain so this is not a function—instead it is a prototype for a function. Make an instance of a function by applying $T_{symbolic}$ on a particular domain and codomain.

```

1 sage: T_symbolic(a, b) = [a+b, a-b, b]
2 sage: T = linear_transformation(RR^2, RR^3, T_symbolic)
3 sage: T
4 Vector space morphism represented by the matrix:
5 [ 1.00000000000000  1.00000000000000  0.000000000000000]
6 [ 1.00000000000000 -1.00000000000000  1.00000000000000]
7 Do

```

Note the left/right issue again: Sage's matrix is the transpose of the matrix that the book would use.

Evaluating this function on a member of the domain gives a member of the codomain.

```

1 sage: v = vector(RR, [1, 3])
2 sage: T(v)
3 (4.00000000000000, -2.00000000000000, 3.00000000000000)

```

Sage can compute the interesting things about the transformation. Here it finds the null space and range space, using the equivalent terms *kernel* and *image*.

```

1 sage: T.kernel()
2 Vector space of degree 2 and dimension 0 over Real Field with 53 bits
3                                         of precision
4 Basis matrix:
5 []
6 sage: T.image()
7 Vector space of degree 3 and dimension 2 over Real Field with 53 bits
8                                         of precision
9 Basis matrix:
10 [ 1.00000000000000  0.00000000000000  0.500000000000000]
11 [ 0.00000000000000  1.00000000000000 -0.500000000000000]

```

The null space's basis is empty because it is the trivial subspace of the domain, with dimension 0. Therefore T is one-to-one.

The range space has a 2-vector basis. This agrees with the theorem that the dimension of the null space plus the dimension of the range space equals to the dimension of the domain.

For contrast consider a map that is not one-to-one.

```

1 sage: S_symbolic(a, b) = [a+2*b, a+2*b]
2 sage: S_symbolic

```

```

3 (a, b) |--> (a + 2*b, a + 2*b)
4 sage: S = linear_transformation(RR^2, RR^2, S_symbolic)
5 sage: S
6 Vector space morphism represented by the matrix:
7 [1.00000000000000 1.00000000000000]
8 [2.00000000000000 2.00000000000000]
9 Domain: Vector space of dimension 2 over Real Field with 53 bits of
10 precision
11 Codomain: Vector space of dimension 2 over Real Field with 53 bits of
12 precision

```

This map is not one-to-one since the input $(a, b) = (2, 0)$ gives the same result as $(a, b) = (0, 1)$.

```

1 sage: S.kernel()
2 Vector space of degree 2 and dimension 1 over Real Field with 53 bits
3 of precision
4 Basis matrix:
5 [ 1.00000000000000 -0.500000000000000]
6 sage: S.image()
7 Vector space of degree 2 and dimension 1 over Real Field with 53 bits
8 of precision
9 Basis matrix:
10 [1.00000000000000 1.00000000000000]

```

The null space has nonzero dimension, namely it has dimension 1, so Sage agrees that the map is not one-to-one.

Without looking at the range space we know that its dimension must be 1 because the dimensions of the null and range spaces add to the dimension of the domain. Again, Sage confirms our calculation.

Via matrices We can define a transformation by specifying the matrix representing its action.

```

1 sage:
2 sage: M = matrix(RR, [[1, 2], [3, 4], [5, 6]])
3 sage: m = linear_transformation(M)
4 sage: m
5 Vector space morphism represented by the matrix:
6 [1.00000000000000 2.00000000000000]
7 [3.00000000000000 4.00000000000000]
8 [5.00000000000000 6.00000000000000]
9 Do

```

Note again that Sage prefers the representation where the vector multiplies from the left.

```

1 sage: v = vector(RR, [7, 8, 9])
2 sage: m(v)
3 (76.0000000000000, 100.000000000000)

```

Sage has done this calculation.

$$(7 \ 8 \ 9) \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = (76 \ 100)$$

If you have a matrix intended for a vector-on-the-right calculation (as in the book) then Sage will make the necessary adaptation.

```

1 sage: N = matrix(RR, [[1, 3, 5], [2, 4, 6]])
2 sage: n = linear_transformation(N, side='right')
3 sage: n
4 Vector space morphism represented by the matrix:
5 [1.00000000000000 2.00000000000000]
6 [3.00000000000000 4.00000000000000]
7 [5.00000000000000 6.00000000000000]
8 Domain: Vector space of dimension 3 over Real Field with 53 bits of
9 precision
10 Codomain: Vector space of dimension 2 over Real Field with 53 bits of
11 precision

```

Although we gave it a `side='right'` option, the matrix that Sage shows by default is for `side='left'`.

Despite that we specified them differently, these two transformations are the same.

```

1 sage: m == n
2 True

```

We can ask the same questions of linear transformations created from matrices that we asked of linear transformations created from functions.

```

1 sage: m.kernel()
2 Vector space of degree 3 and dimension 1 over Real Field with 53 bits
3 of precision
4 Basis matrix:
5 [ 1.00000000000000 -2.00000000000000 1.00000000000000]

```

The null space of `m` is not the trivial subspace of \mathbb{R}^3 and so this function is not one-to-one. The domain has dimension 3 and the null space has dimension 1 and so the range space is a dimension 2 subspace of \mathbb{R}^2 .

```

1 sage: m.image()
2 Vector space of degree 2 and dimension 2 over Real Field with 53 bits
3 of precision
4 Basis matrix:
5 [ 1.00000000000000 0.00000000000000]
6 [0.00000000000000 1.00000000000000]
7 sage: m.image() == RR^2
8 True

```

Sage lets us have the matrix represent a transformation involving spaces with nonstandard bases.

```

1 sage: M = matrix(RR, [[1, 2], [3, 4]])
2 sage: domain_basis = [vector(RR, [1, -1]), vector(RR, [1, 1])]
3 sage: D = (RR^2).subspace_with_basis(domain_basis)
4 sage: codomain_basis = [vector(RR, [2, 0]), vector(RR, [0, 3])]
5 sage: C = (RR^2).subspace_with_basis(codomain_basis)
6 sage: m = linear_transformation(D, C, M)
7 sage: m(vector(RR, [1, 0]))
8 (4.00000000000000, 9.00000000000000)

```

Sage has calculated that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1/2) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (1/2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{so} \quad \text{Rep}_{\text{domain_basis}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

and then computed this.

$$\text{Rep}_{\text{codomain_basis}}(m(\vec{v})) = (1/2 \ 1/2) \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = (2 \ 3) \quad \text{so} \quad m(\vec{v}) = 2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$$

Operations

Consider the set of linear maps from some Fix some vector space domain D and codomain C and consider the set of all linear transformations between them. This set has some natural operations, including addition and scalar multiplication. Sage can work with those operations.

Addition Recall that matrix addition is defined so that the representation of the sum of two linear transformations is the matrix sum of the representatives. Sage can illustrate.

```

1 sage: M = matrix(RR, [[1, 2], [3, 4]])
2 sage: m = linear_transformation(RR^2, RR^2, M)
3 sage: m
4 Vector space morphism represented by the matrix:
5 [1.00000000000000 2.00000000000000]
6 [3.00000000000000 4.00000000000000]
7 Domain: Vector space of dimension 2 over Real Field with 53 bits of
8 precision
9 Codomain: Vector space of dimension 2 over Real Field with 53 bits of
10 precision
11 sage: N = matrix(RR, [[5, -1], [0, 7]])
12 sage: n = linear_transformation(RR^2, RR^2, N)
13 sage: n
14 Vector space morphism represented by the matrix:
15 [ 5.00000000000000 -1.00000000000000]
16 [ 0.00000000000000 7.00000000000000]
17 Domain: Vector space of dimension 2 over Real Field with 53 bits of
18 precision
19 Codomain: Vector space of dimension 2 over Real Field with 53 bits of
20 precision
21 sage: m+n
22 Vector space morphism represented by the matrix:
23 [ 6.00000000000000 1.00000000000000]
24 [ 3.00000000000000 11.00000000000000]
25 Domain: Vector space of dimension 2 over Real Field with 53 bits of
26 precision
27 Codomain: Vector space of dimension 2 over Real Field with 53 bits of
28 precision
29 sage: M+N
30 [ 6.00000000000000 1.00000000000000]
31 [ 3.00000000000000 11.00000000000000]
```

Similarly, linear map scalar multiplication is reflected by matrix scalar multiplication.

```

1 sage: m*3
2 Vector space morphism represented by the matrix:
3 [3.00000000000000 6.00000000000000]
4 [9.00000000000000 12.0000000000000]
5 Domain: Vector space of dimension 2 over Real Field with 53 bits of
6 precision
7 Codomain: Vector space of dimension 2 over Real Field with 53 bits of
precision

```

Composition The composition of linear maps gives rise to matrix multiplication. Sage uses the star $*$ to denote composition of linear maps.

```

1 sage: M = matrix(RR, [[1, 2], [3, 4]])
2 sage: m = linear_transformation(RR^2, RR^2, M)
3 sage: m
4 Vector space morphism represented by the matrix:
5 [1.00000000000000 2.00000000000000]
6 [3.00000000000000 4.00000000000000]
7 Domain: Vector space of dimension 2 over Real Field with 53 bits of
8 precision
9 Codomain: Vector space of dimension 2 over Real Field with 53 bits of
precision
10
11 sage: N = matrix(RR, [[5, -1], [0, 7]])
12 sage: n = linear_transformation(RR^2, RR^2, N)
13 sage: n
14 Vector space morphism represented by the matrix:
15 [ 5.00000000000000 -1.00000000000000]
16 [ 0.00000000000000 7.00000000000000]
17 Domain: Vector space of dimension 2 over Real Field with 53 bits of
18 precision
19 Codomain: Vector space of dimension 2 over Real Field with 53 bits of
precision
20
21 sage: M*N
22 [5.00000000000000 13.0000000000000]
23 [15.0000000000000 25.0000000000000]
24 sage: N*M
25 [2.00000000000000 6.00000000000000]
26 [21.0000000000000 28.0000000000000]
27 sage: m*n
28 Vector space morphism represented by the matrix:
29 [2.00000000000000 6.00000000000000]
30 [21.0000000000000 28.0000000000000]
31 Domain: Vector space of dimension 2 over Real Field with 53 bits of
32 precision
33 Codomain: Vector space of dimension 2 over Real Field with 53 bits of
precision
34

```

Note the left/right issue. Remember that $m \circ n$ is the map $\vec{v} \mapsto m(n(\vec{v}))$, so that n is applied first. Sage puts the representing vector on the left, so N must come left-most: $\vec{w} \cdot N \cdot M = \vec{w}(NM)$.

Singular Value Decomposition

Recall that a line through the origin in \mathbb{R}^n is $\{r \cdot \vec{v} \mid r \in \mathbb{R}\}$. One of the defining properties of a linear map is that $h(r \cdot \vec{v}) = r \cdot h(\vec{v})$. So the action of h on any line through the origin is determined by the action of h on any nonzero vector in that line.

For instance consider the line $y = 2x$ in the plane.

$$\{r \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid r \in \mathbb{R}\}$$

If $t: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is represented by the matrix

$$\text{Rep}_{E_2, E_2}(t) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

then here is the effect of t on one vector in the line.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \xrightarrow{t} \begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

And here is the effect of t on other members of the line.

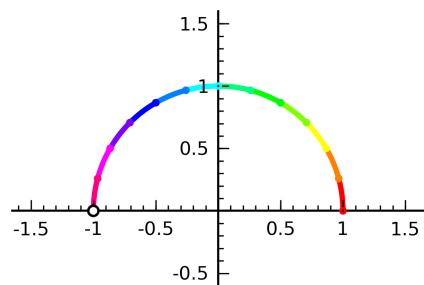
$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} \xrightarrow{t} \begin{pmatrix} 14 \\ 20 \end{pmatrix} \quad \begin{pmatrix} -3 \\ -6 \end{pmatrix} \xrightarrow{t} \begin{pmatrix} -21 \\ -30 \end{pmatrix} \quad \begin{pmatrix} r \\ 2r \end{pmatrix} \xrightarrow{t} \begin{pmatrix} 7r \\ 10r \end{pmatrix}$$

The map t 's action on the line is uniform.

Unit circle

The above means that one way to describe a transformation's action is to pick one nonzero element from each line through the origin and describe where the transformation maps those elements. An easy way to select one nonzero element from each line through the origin is to take the upper half unit circle.

$$U = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x = \cos(t), y = \sin(t), 0 \leq t < \pi \right\}$$



We produce the above graph by using a routine that draws the effect of an arbitrary matrix on the unit circle, and giving that routine the identity matrix. The colors will keep track the directionality in the set of before and after pictures below.

```

1 sage: load "plot_action.sage"
2 sage: p = plot_circle_action(1, 0, 0, 1) # identity matrix
3 sage: p.set_axes_range(-1.5, 1.5, -0.5, 1.5)
4 sage: p.save("sageoutput/plot_action10.png")

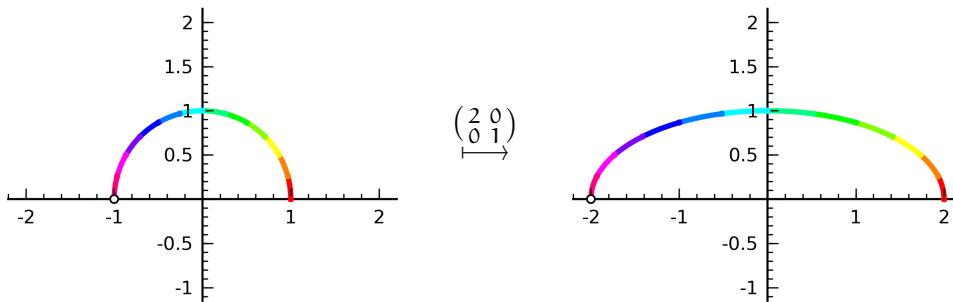
```

Here is the transformation that doubles the x components of all vectors.

```

1 sage: q.save("sageoutput/plot_action11.png")
2 sage: p.save("sageoutput/plot_action11a.png")

```



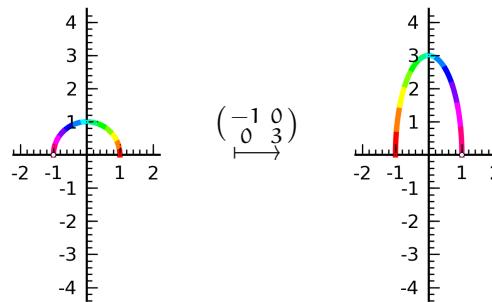
And here is the transformation that triples the y components and multiplies x components by -1 .

Note that it changes the orientation: the input circle moves counterclockwise from red to orange, then green, blue, indigo, and violet but the output does the opposite.

```

1 sage: q.save("sageoutput/plot_action12.png")
2 sage: p.save("sageoutput/plot_action12a.png")

```

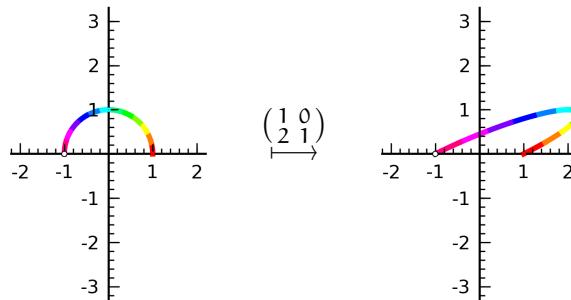


Here is a skew transformation.

```

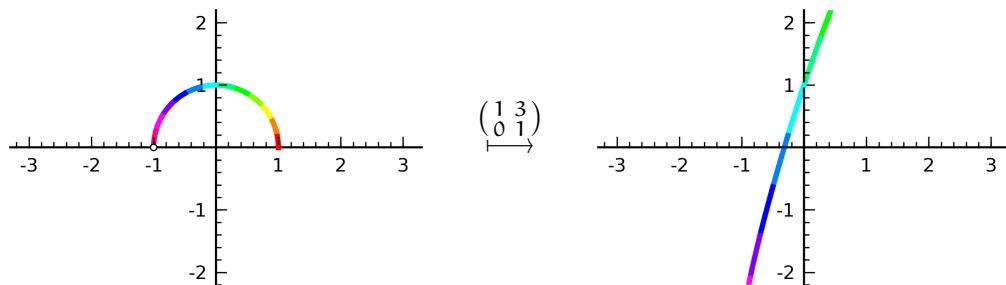
1 sage: q.save("sageoutput/plot_action13.png")
2 sage: p.save("sageoutput/plot_action13a.png")

```



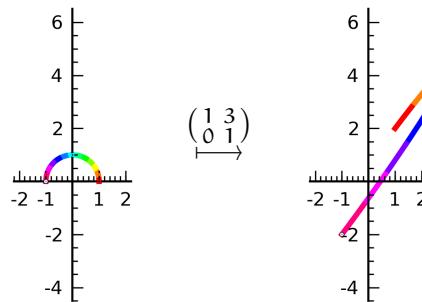
Here is a second skew transformation.

```
1 sage: q.save("sageoutput/plot_action14.png")
2 sage: p.save("sageoutput/plot_action14a.png")
```



And here is the generic map.

```
1 sage: q.save("sageoutput/plot_action15.png")
2 sage: p.save("sageoutput/plot_action15a.png")
```



SVD

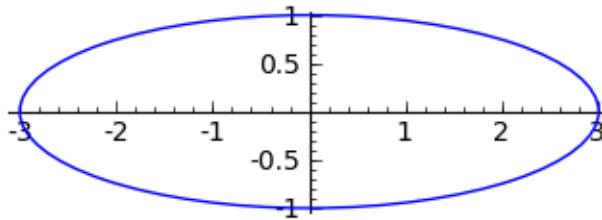
The above pictures show the unit circle mapping to ellipses. Recall that in \mathbb{R}^2 an ellipse has a *major axis*, the longer one, and a *minor axis*.¹ Write σ_1 for the length of the semi-major axis, the distance from the center to the furthest-away point on the ellipse, and write σ_2 for the length of the semi-minor axis.

¹If the two axes have the same length then the ellipse is a circle. If one axis has length zero then the ellipse is a line segment and if both have length zero then it is a point.

```

1 sage: plot.options['aspect_ratio'] = 1
2 sage: sigma_1=3
3 sage: sigma_2=1
4 sage: E = ellipse((0,0), sigma_1, sigma_2)
5 sage: E.save("sageoutput/svd01.png", figsize=3.5)

```



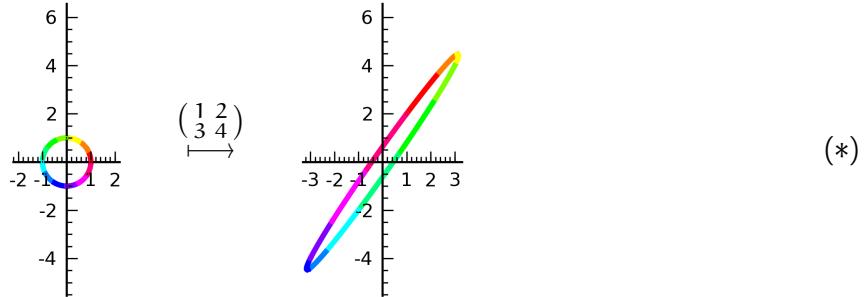
The two axes are orthogonal. In the above graph the major axis lies along the x -axis while the minor axis lies along the y -axis.

Under any linear map $t: \mathbb{R}^n \rightarrow \mathbb{R}^m$ the unit sphere maps to a hyperellipse. This is a version of the *Singular Value Decomposition* of matrices: for any linear map $t: \mathbb{R}^m \rightarrow \mathbb{R}^n$ there are bases $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_m \rangle$ for the domain and $D = \langle \vec{\delta}_1, \dots, \vec{\delta}_n \rangle$ for the codomain such that $t(\vec{\beta}_i) = \sigma_i \vec{\delta}_i$, where the *singular values* σ_i are scalars. The next section states that result more fully and sketches the proof but first we illustrate the central idea by using the generic matrix. Here is its action again, this time on a full circle.

```

1 sage: q.save("sageoutput/plot_action16.png")
2 sage: p.save("sageoutput/plot_action16a.png")

```



Sage will find the SVD of this example matrix.

```

1 sage: M = matrix(RDF, [[1, 2], [3, 4]])
2 sage: U,Sigma,V = M.SVD()
3 sage: U
4 [ -0.404553584834 -0.914514295677]
5 [ -0.914514295677 0.404553584834]
6 sage: Sigma
7 [ 5.46498570422 0.0]
8 [ 0.0 0.365966190626]
9 sage: V
10 [ -0.576048436766 0.81741556047]
11 [ -0.81741556047 -0.576048436766]
12 sage: U*Sigma*(V.transpose())

```

```

13 [ 1.0  2.0 ]
14 [ 3.0  4.0 ]

```

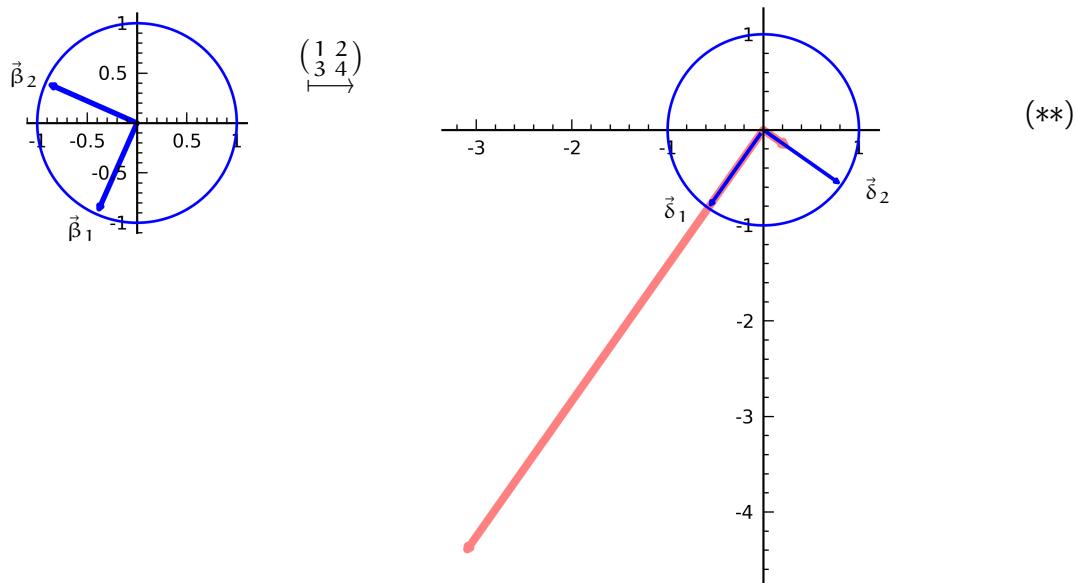
The Singular Value Decomposition describes M as the product of three matrices, $U\Sigma V^T$. The basis vectors $\vec{\beta}_1$, $\vec{\beta}_2$, $\vec{\delta}_1$, and $\vec{\delta}_2$ are the columns of U and V . The singular values are the diagonal entries of Σ . Sage will plots the effect of the transformation on the basis vectors for the domain so we can compare those with the basis vectors for the codomain.

```

1 sage: U, Sigma, V = M.SVD()
2 sage: beta_1 = vector(RR, [U[0][0], U[1][0]])
3 sage: beta_2 = vector(RR, [U[0][1], U[1][1]])
4 sage: delta_1 = vector(RR, [V[0][0], V[1][0]])
5 sage: delta_2 = vector(RR, [V[0][1], V[1][1]])
6 sage: plot.options['axes_pad'] = 0.05
7 sage: plot.options['fontsize'] = 7
8 sage: plot.options['dpi'] = 600
9 sage: plot.options['aspect_ratio'] = 1
10 sage: C = circle((0,0), 1)
11 sage: P = C + plot(beta_1) + plot(beta_2)
12 sage: P.save("sageoutput/svd02a.png", figsize=2, axes_pad=0.05)
13 sage: image_color=Color(1,0.5,0.5) # color for t(beta_1), t(beta_2)
14 sage: Q = C + plot(beta_1*M, width=3, color=image_color)
15 sage: Q = Q + plot(delta_1, width=1.4, color='blue')
16 sage: Q = Q + plot(beta_2*M, width=3, color=image_color)
17 sage: Q = Q + plot(delta_2, width=1.4, color='blue')
18 sage: Q.save("sageoutput/svd02b.png", figsize=4.467, axes_pad=0.05)

```

In the picture below, the domain's blue $\vec{\beta}$'s on the left map to the light red vectors on the right. Also on the right, in blue, are the $\vec{\delta}$'s. The red $t(\vec{\beta}_1)$ does look to be about 5.5 times $\vec{\delta}_1$, and $t(\vec{\beta}_2)$ does look something like 0.4 times $\vec{\delta}_2$.



Note also that the two bases are *orthonormal*—the unit circles help us see that the bases are comprised of unit vectors, and further, the two members of each basis are orthogonal.

Compare this diagram to the one before it labelled (*), which shows the effect of the matrix on the unit circle. We used the whole circle in (*) to spotlight the ellipse, to make it easier to see that in (***) the longest red vector is a semi-major axis of that ellipse and the shortest red vector is a semi-minor axis.

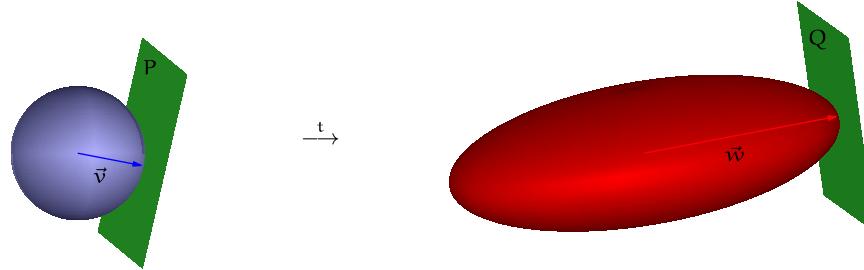
Proof sketch

This argument, adapted from [Blank et al. \[1989\]](#), is a sketch because it uses results that a typical reader has only seen in a less general version and because it relies on material from the book that is in an optional section. In addition, we'll consider only the case of a nonsingular matrix and map; it shows the main idea, which is the point of a sketch.

Consider an $n \times n$ matrix T that is nonsingular, and the nonsingular transformation $t: \mathbb{R}^n \rightarrow \mathbb{R}^n$ that is represented by T with respect to the standard bases.

Recall the Extreme Value Theorem from Calculus I: for a continuous function f , if $D \subset \mathbb{R}$ is a closed and bounded set then the image $f(D)$ is also a closed and bounded set (see [Wikipedia \[2012a\]](#)). A generalization of that result gives that because the unit sphere in \mathbb{R}^n is closed and bounded then its image under t is closed and bounded. The image is in fact an ellipsoid, although we won't prove it, so we will call it that.

Because the ellipsoid is closed and bounded it has a point furthest from the origin. Let \vec{w} be a vector extending from the origin to that furthest point. Let \vec{v} be the member of the unit sphere that maps to \vec{w} . Let P be the plane that touches the sphere only at the endpoint of \vec{v} . Let Q be the image of P under t . Since t is one-to-one, Q touches the ellipsoid only at \vec{w} .



The tangent plane P has the property that the set of vectors whose bodies lie in P are the set of vectors perpendicular to \vec{v} . That is, if in the picture above we slide P along the vector \vec{v} to the origin then we have the subspace of \mathbb{R}^n of vectors perpendicular to \vec{v} . This subspace has dimension $n - 1$. We will argue below that if we similarly slide Q to the origin then we have the set of vectors perpendicular to \vec{w} . With that we will have an argument by induction: we start constructing the bases B and D by taking $\vec{\beta}_1$ to be \vec{v} , taking σ_1 to be the length $\|\vec{w}\|$, and taking $\vec{\delta}_1$ to be $\vec{w}/\|\vec{w}\|$.

So consider Q . First observe that there is a unique plane that touches the ellipsoid only at \vec{w} since if there were another \hat{Q} then $t^{-1}(\hat{Q})$ would be a second plane that touches the sphere only at \vec{v} , which is impossible. To see that Q is perpendicular to \vec{w} consider a sphere centered at the origin whose radius is $\|\vec{w}\|$. This sphere has a plane tangent at the endpoint of \vec{w} , which is perpendicular to \vec{w} . Because \vec{w} ends at a point on the ellipsoid furthest from the origin, the ellipsoid is entirely contained in the sphere, so this plane touches the ellipsoid only at \vec{w} . Therefore, from the second sentence of this paragraph, this plane is Q . That ends the proof.

Matrix factorization

We can express those geometric ideas in an algebraic form (for a proof see [Trefethen and Bau \[1997\]](#)).

The *singular value decomposition* of an $m \times n$ matrix A is a factorization $A = U\Sigma V^T$. The $m \times n$ matrix Σ is all zeroes except for diagonal entries, the singular values, $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ where r is the rank of A . The $m \times m$ matrix U and the $n \times n$ matrix V are unitary, meaning that their columns form an orthogonal basis of unit vectors, the left and right *singular vectors* for A , respectively.

```

1 sage: M = matrix(RDF, [[0, 1, 2], [3, 4, 5]])
2 sage: U, Sigma, V = M.SVD()
3 sage: U
4 [-0.274721127897 -0.961523947641]
5 [-0.961523947641 0.274721127897]
6 sage: Sigma
7 [7.34846922835 0.0 0.0]
8 [0.0 1.0 0.0]
9 sage: V
10 [-0.392540507864 0.824163383692 0.408248290464]
11 [-0.560772154092 0.137360563949 -0.816496580928]
12 [-0.72900380032 -0.549442255795 0.408248290464]

```

The product $U\Sigma V^T$ simplifies. To see how, consider the case where all three matrices are 2×2 . Write \vec{u}_1, \vec{u}_2 for the columns of U and \vec{v}_1, \vec{v}_2 for the columns of V , so that the rows of V^T are \vec{v}_1^T and \vec{v}_2^T .

$$\begin{aligned}
U\Sigma V^T &= (\vec{u}_1 \quad \vec{u}_2) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{pmatrix} \\
&= (\vec{u}_1 \quad \vec{u}_2) \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{pmatrix} + (\vec{u}_1 \quad \vec{u}_2) \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{pmatrix} \\
&= \sigma_1 \cdot (\vec{u}_1 \quad \vec{u}_2) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{pmatrix} + \sigma_2 \cdot (\vec{u}_1 \quad \vec{u}_2) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{v}_1^T \\ \vec{v}_2^T \end{pmatrix} \tag{***}
\end{aligned}$$

In the first term right multiplication by the $1,1$ unit matrix picks out the first column of U and left multiplication by the $1,1$ unit matrix picks out first row of V so those are the only parts that remain after the product. In short, we get this.

$$\begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{1,1} & v_{2,1} \\ v_{1,2} & v_{2,2} \end{pmatrix} = \begin{pmatrix} u_{1,1}v_{1,1} & u_{1,1}v_{2,1} \\ u_{2,1}v_{1,1} & u_{2,1}v_{2,1} \end{pmatrix} = \vec{u}_1 \vec{v}_1^T$$

Thus, equation (***) simplifies to $U\Sigma V^T = \sigma_1 \cdot \vec{u}_1 \vec{v}_1^T + \sigma_2 \cdot \vec{u}_2 \vec{v}_2^T$. Cases other than 2×2 simplify in the same way.

Application: data compression

We can write any matrix as a sum $M = \sigma_1 \cdot \vec{u}_1 \vec{v}_1^T + \sigma_2 \cdot \vec{u}_2 \vec{v}_2^T + \dots$ where the vectors have unit size and the σ_i 's decrease in size.

Suppose that the matrix is $n \times n$. If we need to store it or transmit it then we need to work with n^2 real numbers. For instance, if $n = 500$ then we have $50^2 = 250\,000$ reals. If we instead express the matrix as a sum then each term requires n real numbers for \vec{u}_i , another n reals for \vec{v}_i , and one more real for σ_i . So keeping all the terms would require $500 * (2 * 500 + 1) = 500\,500$ reals, which is twice the data of the original matrix. But if we store only some of the terms, say the first 50 of them, then it would be a savings: $50 \cdot (2n + 1) = 50\,050$, which is about 20% of the 500^2 size of the full matrix. Thus, if we have data as a matrix then we could hope to compress it with that formula by dropping terms with small σ 's. The question is whether we lose too much of the information by only retaining the information associated with 10% of the singular values.

To illustrate that we can succeed while retaining at least some aspects of data we will do image compression. Meet Lenna. This top third of a pinup is a standard test image [Wikipedia \[2012b\]](#).



The code we will use is in the `img_squeeze` function that is listed at the end of this chapter. In that code we have a peek at the eight largest singular values (in the red matrix): 93233.7882139,

11193.1186308, 8660.12549856, 7054.38662024, 5916.89456484, 5742.14618029, 4125.16604075, and 3879.15303685. The code also reports the singular value where we make the cutoff 606.389279688, and the eight smallest: 0.547288023466, 0.470266870056, 0.1988515109, with five more that are so small they are essentially zero.¹ The large singular values are much larger than the small singular values, even setting aside the ones at the end that are zero but for numerical issues.

We want to see how badly the image degrades for various cutoffs. The code below sets it to 10%. This image is 512×512 so that will sum the terms associated with the first 51 singular values.

```
1 sage: load "img_squeeze.sage"
2 sage: img_squeeze("Lenna.png", "Lena_squeezed.png", 0.10)
```

Below is the squeezed image.



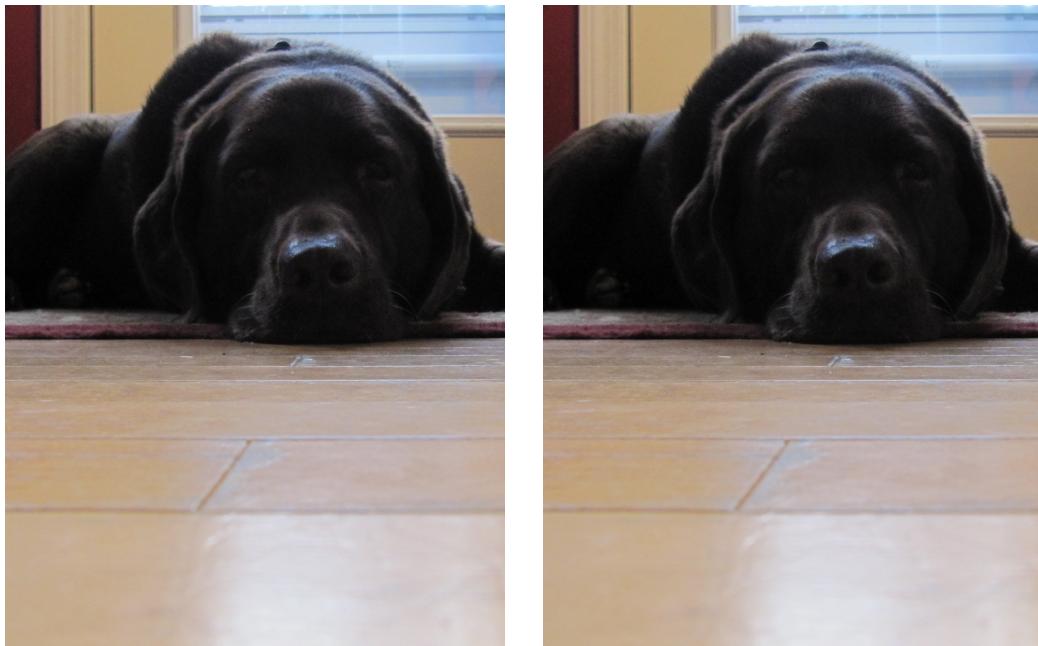
It definitely shows loss; its image quality is less than the original. The colors are not as good,

¹These are four values of $1.024\,723\,452\,83 \times 10^{-11}$ and one of $2.325\,339\,864\,91 \times 10^{-12}$.

the edges are not sharp, and there are a few artifacts, including a horizontal line across the top of Lenna's forehead and another halfway up her hat. But certainly the image is entirely recognizable.

Where the image is $n \times n$, every additional term in the summation adds to the storage and transmission requirements by about $2n$ reals. The original image requires n^2 reals so we want to choose percentage values less than 0.50. On the other hand, higher values give better fidelity. So selecting an optimal percentage parameter is an engineering decision of making the tradeoff. The above image has the advantage is that it needs only 20% of the storage and transmission requirements of the original image.

Experimenting with percentage value shows that setting the parameter to 20% is enough to make the output image hard to tell from the original. For a final contrast, below is the picture of Suzy from this manual's cover, on the left as the camera took it and on the right squeezed with a cutoff of 0.20.



Source of `plot_action.sage`

The `plot_circle_action` routine that we call above takes the four entries of the 2×2 matrix and returns a list of graphics. Other parameters are the number of colors, and a flag giving whether to plot a full circle or whether to plot just the top half circle (this is the default).

This routine determines the list of colors and calls a helper `color_circle_list`, given below, which returns a list of graphics. Finally, this routine plots those graphics.

```

1 NUM_CIRCLE PTS = 12
2 def plot_circle_action(a, b, c, d, n = NUM_CIRCLE PTS, full_circle = False):
3     """Show the action of the matrix with entries a, b, c, d on half
4     of the unit circle, as the circle and the output curve, broken into
5     a number of colors.
6     a, b, c, d  reals Entries are upper left, ur, ll, lr.
7     n = 12 positive integer Number of colors.
8     full_circle=False boolean Show whole circle, or top half

```

```

9 """
10 colors = rainbow(n)
11 G = Graphics() # holds graph parts until they are to be shown
12 for g_part in color_circle_list(a,b,c,d,colors,full_circle):
13     G += g_part

```

The helper does the heavy lifting. It produces a parametrized curve $(x(t), y(t))$, and uses Sage's `parametric_plot` function to get the resulting graphic.

```

1 DOT_SIZE = .02
2 CIRCLE_THICKNESS = 2
3 def color_circle_list(a, b, c, d, colors, full_circle=False):
4     """Return list of graph instances for the action of a 2x2 matrix on
5     half of the unit circle. That circle is broken into chunks each
6     colored a different color.
7     a, b, c, d reals entries of the matrix ul, ur, ll, lr
8     colors list of rgb tuples; len of this list is how many chunks
9     full_circle=False Show a full circle instead of a half circle.
10 """
11 r = []
12 if full_circle:
13     p = 2*pi
14 else:
15     p = pi
16 t = var('t')
17 n = len(colors)
18 for i in range(n):
19     color = colors[i]
20     x(t) = a*cos(t)+c*sin(t)
21     y(t) = b*cos(t)+d*sin(t)
22     g = parametric_plot((x(t), y(t)),
23                           (t, p*i/n, p*(i+1)/n),
24                           color = color, thickness=CIRCLE_THICKNESS)
25     r.append(g)
26     r.append(circle((x(p*i/n), y(p*i/n)), DOT_SIZE, color=color))
27 if not(full_circle): # show (x,y)=(-1,0) is omitted
28     r.append(circle((x(pi), y(pi)), 2*DOT_SIZE, color='black',
29                     fill = 'true'))
30     r.append(circle((x(pi), y(pi)), DOT_SIZE, color='white',
31                     fill = 'true'))
32 return r

```

If this routine is plotting the upper half circle then it adds a small empty circle at the end to show that the image of $(-1, 0)$ is not part of the graph.

The global variable `CIRCLE_THICKNESS` sets the thickness of the plotted curve, in points (a printer's unit, here 1/72 inch). Similarly `DOT_SIZE` sets the size of the small empty circle.

Source of img_squeeze.sage

We will use the Python Image Library for help in reading the figure, getting the color value at each pixel, etc. The function `img_squeeze` takes three arguments, the names of the input and output functions, and a real number between 0 and 1 which determines the percentage of the singular values to include in the sum before the cutoff.

```

1 # Compress the image
2 import Image
3
4 def img_squeeze(fn_in, fn_out, percent):
5     """Squeeze an image using Singular Value Decomposition.
6         fn_in, fn_out string name of file
7         percent real in 0..1 Fraction of singular values to use
8     """

```

This function first brings the input data to a format where each pixel is a triple (red, green, blue) of integers that range from 0 to 255. It uses those numbers to gradually build three Python arrays `rd`, `gr`, and `bl`, which then initialize the three Sage matrices `RD`, `GR`, and `BL`.

```

1 img = Image.open(fn_in)
2 img = img.convert("RGB")
3 rows, cols = img.size
4 cutoff = int(round(percent*min(rows,cols),0))
5 # Gather data into three arrays, then give to Sage's matrix()
6 rd, gr, bl = [], [], []
7 for row in range(rows):
8     for a in [rd, gr, bl]:
9         a.append([])
10    for col in range(cols):
11        r, g, b = img.getpixel((int(row), int(col)))
12        rd[row].append(r)
13        gr[row].append(g)
14        bl[row].append(b)
15 RD, GR, BL = matrix(RDF, rd), matrix(RDF, gr), matrix(RDF, bl)

```

The next step finds the Singular Value Decomposition of those three. Out of curiosity, we have a peek at the eight largest singular values in the red matrix, the singular value where we make the cutoff, and the eight smallest.

```

1 # Get the SVDs
2 print "about to get the svd"
3 U_RD, Sigma_RD, V_RD = RD.SVD()
4 U_GR, Sigma_GR, V_GR = GR.SVD()
5 U_BL, Sigma_BL, V_BL = BL.SVD()
6 # Have a look
7 for i in range(8):
8     print "sigma_RD",i,"=",Sigma_RD[i][i]
9 print " :"
10 print "sigma_RD",cutoff,"=",Sigma_RD[cutoff][cutoff]
11 print " :"
12 for i in range(Sigma_RD.nrows()-8,Sigma_RD.nrows()):
13     print "sigma_RD",i,"=",Sigma_RD[i][i]

```

Finally, for each matrix we compute the sum $\sigma_1 \cdot \vec{u}_1 \vec{v}_1^T + \sigma_2 \cdot \vec{u}_2 \vec{v}_2^T + \dots$ up through the cutoff index.

```

1 # Compute sigma_1 u_1 v_1^trans +
2 a = []
3 for i in range(rows):
4     a.append([])
5     for j in range(cols):
6         a[i].append(0)
7 A_RD, A_GR, A_BL = matrix(RDF, a), matrix(RDF, a), matrix(RDF, a)
8 for i in range(cutoff):
9     sigma_i = Sigma_RD[i][i]
10    u_i = matrix(RDF, U_RD.column(i).transpose())
11    v_i = matrix(RDF, V_RD.column(i))
12    A_RD = copy(A_RD)+sigma_i*u_i*v_i
13    sigma_i = Sigma_GR[i][i]
14    u_i = matrix(RDF, U_GR.column(i).transpose())
15    v_i = matrix(RDF, V_GR.column(i))
16    A_GR = copy(A_GR)+sigma_i*u_i*v_i
17    sigma_i = Sigma_BL[i][i]
18    u_i = matrix(RDF, U_BL.column(i).transpose())
19    v_i = matrix(RDF, V_BL.column(i))
20    A_BL = copy(A_BL)+sigma_i*u_i*v_i

```

The code asks for the transpose of the `U_RD.column(i)`, etc., which may seem to be a mistake. Remember that Sage favors rows for vectors, so this is how we make the \vec{u} vector a column, while \vec{v} is already a row. Because of this operation on the vector, the first time you run this code in a Sage session you may get a depreciation warning.

To finish, we put the data in the .PNG format and save it to disk.

```

1 # Make a new image
2 img_squeeze = Image.new("RGB", img.size)
3 for row in range(rows):
4     print "transferring over row=", row
5     for col in range(cols):
6         p = (int(A_RD[row][col]),
7               int(A_GR[row][col]),
8               int(A_BL[row][col]))
9         img_squeeze.putpixel((int(row), int(col)), p)
10 img_squeeze.save(fn_out)

```

This part of the routine function takes a very long time to run, in part because the code is intended to be easy to read rather than fast. Consequently, in the function source there are some status lines that help convince a person executing the code that it is still working; those busy-work lines are left out of some of the earlier output listings.

Geometry of Linear Maps

Sage can illustrate the geometric effect of linear transformations. The graphs here picture transformations of the plane \mathbb{R}^2 because those fit on the paper but the principles extend to higher-dimensional spaces.

Lines map to lines

The pictures in this chapter are based on the observation that under a linear map, the image of a line in the domain is a line in the range.

To verify that, consider a domain space \mathbb{R}^d and codomain space \mathbb{R}^c , along with the linear map h . We get a line in the domain by fixing a vector of slopes $\vec{m} \in \mathbb{R}^d$ and a vector of offsets from the origin $\vec{b} \in \mathbb{R}^d$, and then considering the set $\ell = \{\vec{v} = \vec{m} \cdot s + \vec{b} \mid s \in \mathbb{R}\}$. The image of ℓ is this set.

$$h(\ell) = \{h(\vec{m} \cdot s + \vec{b}) \mid s \in \mathbb{R}\} = \{h(\vec{m}) \cdot s + h(\vec{b}) \mid s \in \mathbb{R}\}$$

This is a line in the codomain \mathbb{R}^c with the vector of slopes $h(\vec{m})$ and the vector of offsets $h(\vec{b})$.

For example, consider the transformation $t: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that rotates vectors counterclockwise by $\pi/6$ radians.

$$\text{Rep}_{E_2, E_2}(t) = \begin{pmatrix} \cos(\pi/6) & \sin(\pi/6) \\ -\sin(\pi/6) & \cos(\pi/6) \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$$

Also consider the line $y = 3x + 2$, described here as a set of vectors.

$$\ell = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \cdot s + \begin{pmatrix} 2 \\ 0 \end{pmatrix} \mid s \in \mathbb{R} \right\}$$

That line when rotated by t is this set.

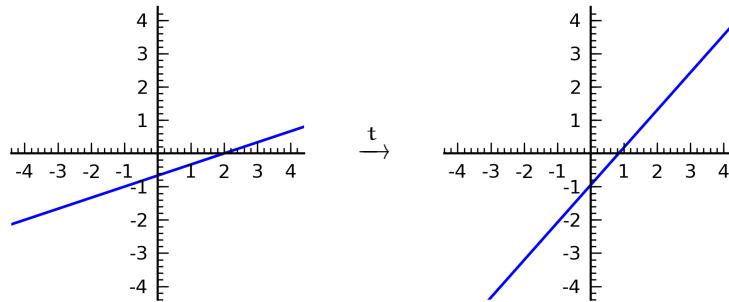
$$t(\ell) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3\sqrt{3}-1 \\ 3+\sqrt{3} \end{pmatrix} \cdot s + \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \mid s \in \mathbb{R} \right\}$$

```
1 sage: s = var('s')
2 sage: plot.options['figsize'] = 2.5
3 sage: plot.options['axes_pad'] = 0.05
4 sage: plot.options['fontsize'] = 7
5 sage: plot.options['dpi'] = 500
6 sage: plot.options['aspect_ratio'] = 1
```

```

7 sage: ell = parametric_plot((3*s+2, 1*s), (s, -10, 10))
8 sage: ell.set_axes_range(-4, 4, -4, 4)
9 sage: ell.save("sageoutput/plot_action0.png", fontsize=7)
10 sage: t_x(s) = ((3*sqrt(3)-1)/2)*s+sqrt(3)
11 sage: t_y(s) = ((3+sqrt(3))/2)*s+1
12 sage: t_ell = parametric_plot((t_x(s), t_y(s)), (s, -10, 10))
13 sage: t_ell.set_axes_range(-4, 4, -4, 4)
14 sage: t_ell.save("sageoutput/plot_action0a.png", fontsize=7)

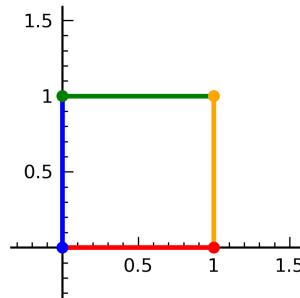
```



(The limits on the parameter s of 10 and -10 are arbitrary, just chosen to be large enough that the line segment covers the entire domain and codomain intervals shown, from -4 to 4 .) For example, the vector that ends at $(2, 0)$ is rotated to the vector that ends at $(\sqrt{3}, 1)$.

The unit square

We first illustrate the effect of transformations $t: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by applying them to this unit square.



That was generated by this Sage session.

```

1 sage: load "plot_square.sage"
2 sage: p = plot_square_action(1, 0, 0, 1) # identity matrix
3 sage: p.set_axes_range(-0.25, 1.5, -0.25, 1.5)
4 sage: p.save("sageoutput/plot_action1.png")

```

The `plot_square_action` routine plots the action on the square of a matrix representing the transformation.

$$(x \ y) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax + cy \ bx + dy)$$

The routine's source is at the end of this chapter but the observation above that linear maps send lines to lines makes it easy: the matrix has this action on the four corners of the square

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \xrightarrow{\text{t}} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{t}} \begin{pmatrix} a \\ b \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\text{t}} \begin{pmatrix} a+c \\ b+d \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{\text{t}} \begin{pmatrix} c \\ d \end{pmatrix}$$

and the routine plots the four line segments. In the Sage session above that routine is given the identity matrix, so it plots the unit square unchanged.

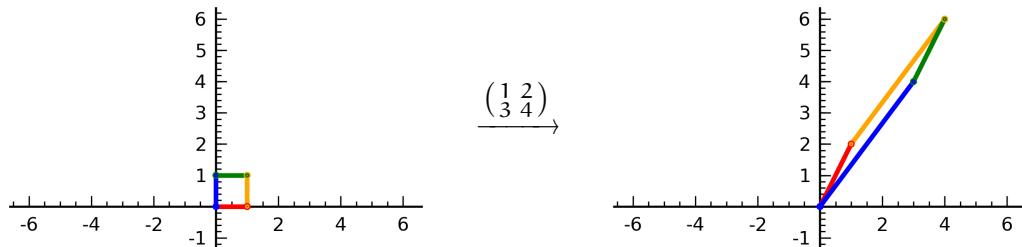
This Sage session

```

1 sage: load "plot_action.sage"
2 sage: q = plot_square_action(1, 0, 0, 1)
3 sage: q.set_axes_range(-6, 6, -1, 6)
4 sage: q.save("sageoutput/plot_action2.png")
5 sage: p = plot_square_action(1, 2, 3, 4)
6 sage: p.set_axes_range(-6, 6, -1, 6)
7 sage: p.save("sageoutput/plot_action2a.png")

```

generates these before and after pictures of the effect of the generic matrix.¹



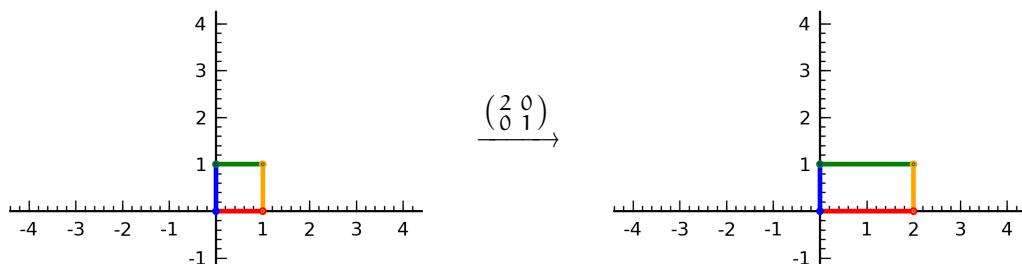
The colors are there to show that transformations can change orientations. Take the colors in their natural order of red, orange, green, and blue. Then the domain square is a counterclockwise shape, while the transformed square is clockwise.

We can develop an understanding of complex actions by building from simple ones. This transformation doubles the x components of all vectors.

```

1 sage: q.save("sageoutput/plot_action3.png")
2 sage: p.save("sageoutput/plot_action3a.png")

```

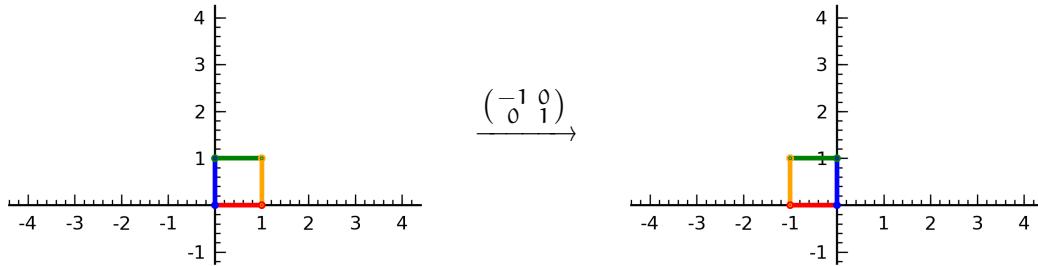


Linear maps send the zero vector to the zero vector so the new shape is also anchored at the origin. But it has been stretched horizontally: it has the same orientation as the starting square, but twice the area.

¹The remaining examples in this chapter omit the fiddly lines that load, save, set the axis ranges, etc.

If we triple the x coordinate then we get a similar shape but with three times the area of the starting one. What about if we take -1 times the x -coordinate?

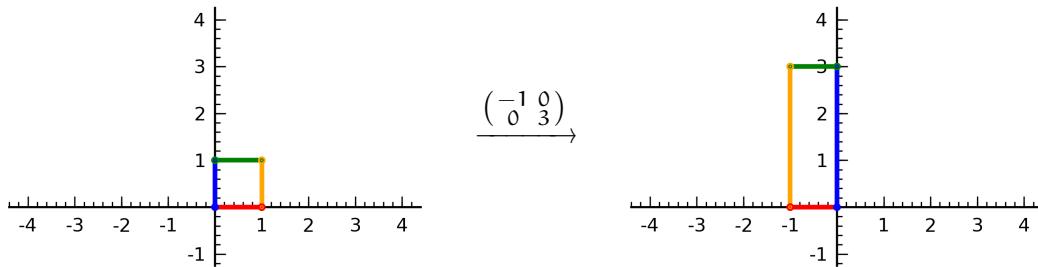
```
1 sage: q.save("sageoutput/plot_action4.png")
2 sage: p.save("sageoutput/plot_action4a.png")
```



It changes the orientation. We say that the above shape has an *oriented area* of -1 . The motivation for the sign is: we start with the right-hand figure from the example before this one and slide the orange side in from the right, from 2 to 1 , to 0 and then to -1 . The area falls from 2 to 1 , to 0 , and it is in some way natural to assign the figure above an area measure of -1 . The prefix ‘oriented’ is just there to distinguish this idea from the usual idea of area.¹ The usual area is the absolute value of the oriented area.

The next transformation combines action in two axes, tripling the y components and multiplying x components by -1 .

```
1 sage: q.save("sageoutput/plot_action5.png")
2 sage: p.save("sageoutput/plot_action5a.png")
```

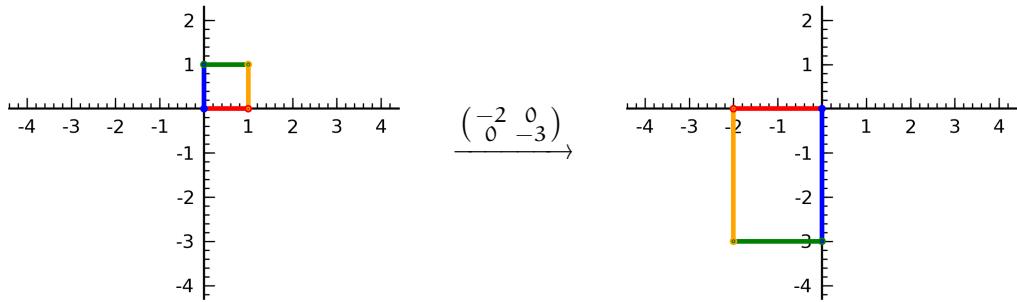


The colors show that this transformation also changes the orientation, so the new shape has an oriented area of -3 .

What if we change the orientation twice?

```
1 sage: q.save("sageoutput/plot_action6.png")
2 sage: p.save("sageoutput/plot_action6a.png")
```

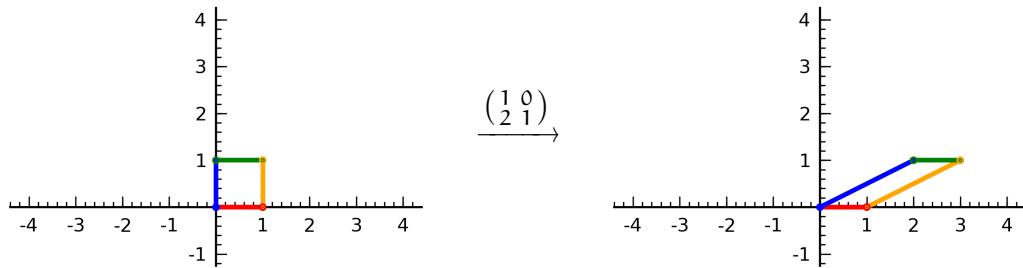
¹The oriented area is the same as the *determinant* of the matrix, which the book covers in the Chapter Four.



The colors are the same as the original shape's counterclockwise red, orange, green, and blue. Thus the new shape has an oriented area of 6.

We next show the effect of putting in off-diagonal entries.

```
1 sage: q.save("sageoutput/plot_action7.png")
2 sage: p.save("sageoutput/plot_action7a.png")
```



This transformation is a *skew*¹ (or *shear*). The line segment sides of the original square map to line segments, but these sides are not at right angles. The action

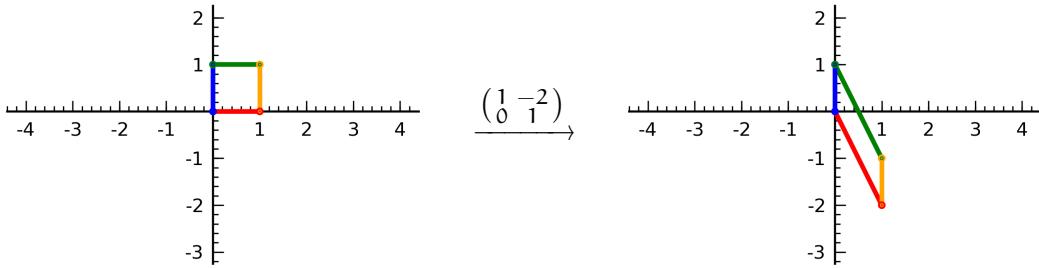
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + 2y \\ y \end{pmatrix}$$

means that a vector with a y component of 1 is shifted right by 2 while a vector with a y component of 2 is shifted right by 4, so vectors are shifted depending on how far they are above or below the x-axis. This transformation preserves orientation and the shape has a base of 1 with a height of 1 so its oriented area is 1.

Putting a nonzero value in the other off-diagonal entry, the upper right, has much the same effect except that it skews parallel to the y-axis.

```
1 sage: q.save("sageoutput/plot_action8.png")
2 sage: p.save("sageoutput/plot_action8a.png")
```

¹The word skew means “to distort from a symmetrical form.”



The action

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ -2x + y \end{pmatrix}$$

means that vectors are shifted depending on how far they are from the x axis. For instance, a vector with an x component of 1 is shifted by -2 . The oriented area of this shape is 1.

Turing's factorization $PA=LDU$ We will now see how the action of any matrix can be decomposed into the actions shown in the prior section. This will give us a complete geometric description of any linear map.

Recall that the row operations of Gauss's Method can be done with matrix multiplication. For instance, multiplication from the left by this matrix has the effect of the row operation $2\rho_1 + \rho_2$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 4 \\ -6 & 1 & -8 \\ 0 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 3 & 0 \\ 0 & -3 & 2 \end{pmatrix}$$

In general, as described in the book, the *elementary reduction matrices* come in three types $M_i(k)$, $P_{i,j}$, and $C_{i,j}(k)$, and arise from applying a row operation to an identity matrix.

$$I \xrightarrow{k\rho_i} M_i(k) \text{ for } k \neq 0 \quad I \xrightarrow{\rho_i \leftrightarrow \rho_j} P_{i,j} \text{ for } i \neq j \quad I \xrightarrow{k\rho_i + \rho_j} C_{i,j}(k) \text{ for } i \neq j$$

With these we can use matrix multiplication to do Gauss's Method: for a matrix H we can do row scaling $k\rho_i$ with $M_i(k)H$, we can swap rows $\rho_i \leftrightarrow \rho_j$ with $P_{i,j}H$, and we can add a multiple of one row to another $k\rho_i + \rho_j$ with $C_{i,j}(k)H$. The prior paragraph used the 3×3 matrix $C_{1,2}(2)$.

We can continue the process from the equation above and use $C_{2,3}(-1)$ to perform $-\rho_2 + \rho_3$, producing echelon form.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 4 \\ -6 & 1 & -8 \\ 0 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad (*)$$

As in this example, using matrix multiplication alone we can do Gauss's Method and produce echelon form. One way to express this is that equation $(*)$ is a factorization of the starting matrix into a product of some elementary matrices and an echelon form matrix.

Observe further that if the starting matrix is such that we don't need to do any row swapping then we can stick elementary operations that involve only using a row to work on a row below it, that is, $k\rho_i + \rho_j$ where $j > i$. The elementary matrices that perform those operations are *lower*

triangular since all of their nonzero entries are in the lower left. (Matrices with all of their nonzero entries in the upper right are *upper triangular*.) Thus our factorization so far is into a product of lower triangular elementary matrices and an echelon form matrix.

We can factor further. We can use a diagonal matrix to ensure that the leading entries of the nonzero rows of the echelon form matrix are 1's. Here we've done that additional step to equation (*).

$$\begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 4 \\ -6 & 1 & -8 \\ 0 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1/3 & 4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (**)$$

We can use matrix multiplication to factor all the way to a block partial identity matrix. The idea is to use column operations. Here we use right-multiplication on the right-hand side of (**) to add $-1/3$ times the first column to the second column.

$$\begin{pmatrix} 1 & 1/3 & 4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Adding $-4/3$ times the first column to the third column leave us with an identity matrix.

$$\begin{pmatrix} 3 & 1 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (***)$$

Thus, if we start with a matrix A that does not require any row swaps, we get this matrix equation.

$$L_1 L_2 \dots L_k \cdot A \cdot U_1 U_2 \dots U_r = D$$

Here D is a partial identity matrix, the L_i are lower-triangular row combination matrices, and the U_j are upper-triangular column combination matrices.

We are focused on transformations so we assume all of these matrices are square. All of the row operations can be undone (for instance, $2\rho_1 + \rho_2$ is undone with $-2\rho_1 + \rho_2$), so each of those lower triangular matrices has an inverse. Likewise, each of the upper-triangular matrices has an inverse. Therefore, if we don't need any swaps in a Gauss-Jordan reduction of A then we have this factorization.

$$A = L_k^{-1} \dots L_1^{-1} \cdot D \cdot U_r^{-1} \dots U_1^{-1}$$

To fix the swap issue we can pre-swap. That is, before we factor the starting matrix we can first swap its rows with a permutation matrix P .

$$P \cdot A = L_k^{-1} \dots L_1^{-1} \cdot D \cdot U_r^{-1} \dots U_1^{-1} \quad (**)$$

A product of lower triangular matrices is lower-triangular, and a product of upper triangular matrices is upper-triangular so we could combine all the L 's and all the U 's, and this result is often known as $PA = LDU$.

We illustrate with the generic 2×2 transformation of \mathbb{R}^2 represented with respect to the standard basis in this way.

$$T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Gauss's Method is straightforward.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow{-3\rho_1+\rho_2} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} \xrightarrow{-(1/2)\rho_2} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \xrightarrow{-2x_1+x_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(We use χ_i for the columns.) This is the associated factorization.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

The product is as expected.

```

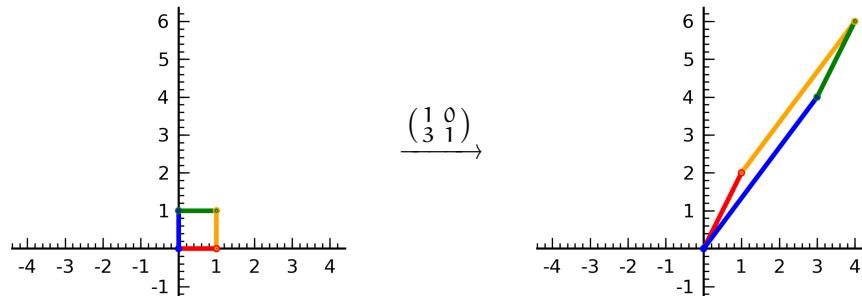
1 sage: L = matrix(QQ, [[1, 0], [3, 1]])
2 sage: D = matrix(QQ, [[1, 0], [0, -2]])
3 sage: U = matrix(QQ, [[1, 2], [0, 1]])
4 sage: L*D*U
5 [1 2]
6 [3 4]
```

We got into this to understand the geometric effect of the transformation.

$$(x \ y) \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

```

1 sage: q.save("sageoutput/plot_action20.png")
2 sage: p.save("sageoutput/plot_action20a.png")
```



We can expand it using the above factorization

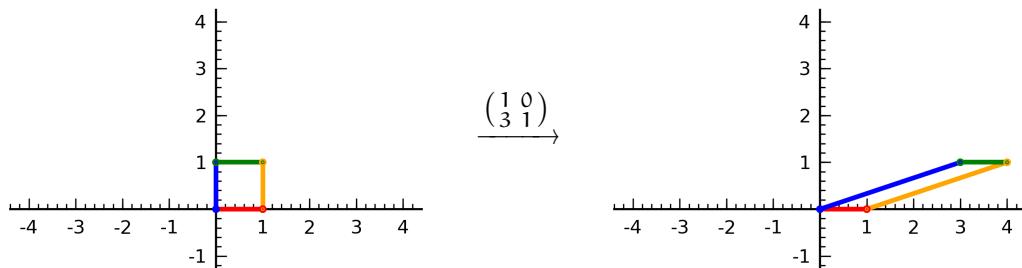
$$(x \ y) \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and look at the effect of the three component matrices. We will do this first by looking at the effect of the three transformations L , D , and U , separately. After that we will look at the cumulative effect: that of L , then LD , then LDU .

The first matrix is a skew parallel to the x -axis.

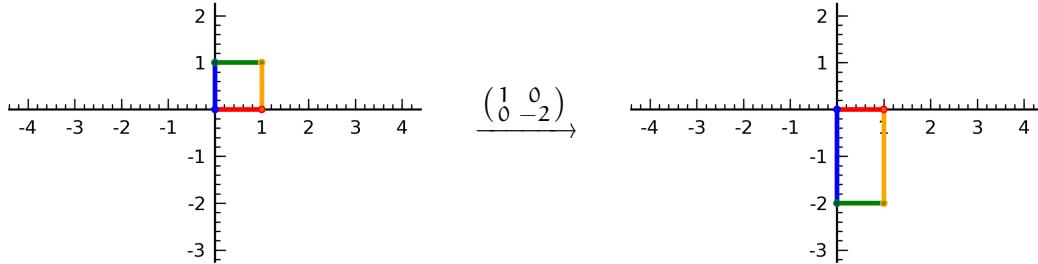
```

1 sage: q.save("sageoutput/plot_action21.png")
2 sage: p.save("sageoutput/plot_action21a.png")
```



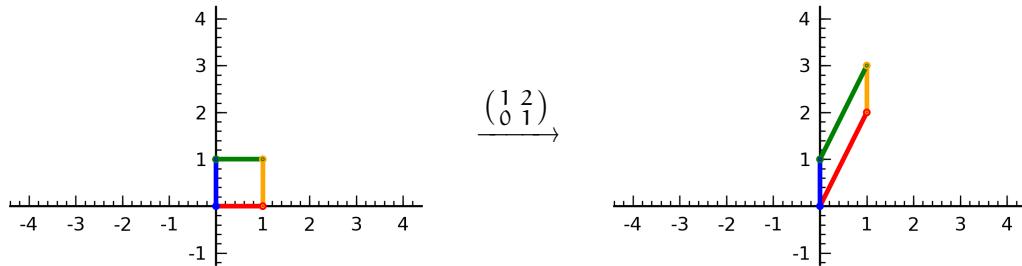
The second matrix rescales, and changes the orientation.

```
1 sage: q.save("sageoutput/plot_action22.png")
2 sage: p.save("sageoutput/plot_action22a.png")
```



Finally, the third matrix is a skew parallel to the y-axis.

```
1 sage: q.save("sageoutput/plot_action23.png")
2 sage: p.save("sageoutput/plot_action23a.png")
```



Note that the lower-triangular and upper-triangular matrices do not change the orientation. So, absent any row swaps, any orientation changing happens when one of the diagonal entries is negative. (*Comment.* This matrix does not require any row swaps. However, if a matrix does require row swaps then the situation is more subtle. Each row swap toggles the orientation, from counterclockwise to clockwise or from clockwise to counterclockwise. Thus if the matrix requires an odd number of swaps then the orientation changes, but with an even number of swaps the orientation stays the same.)

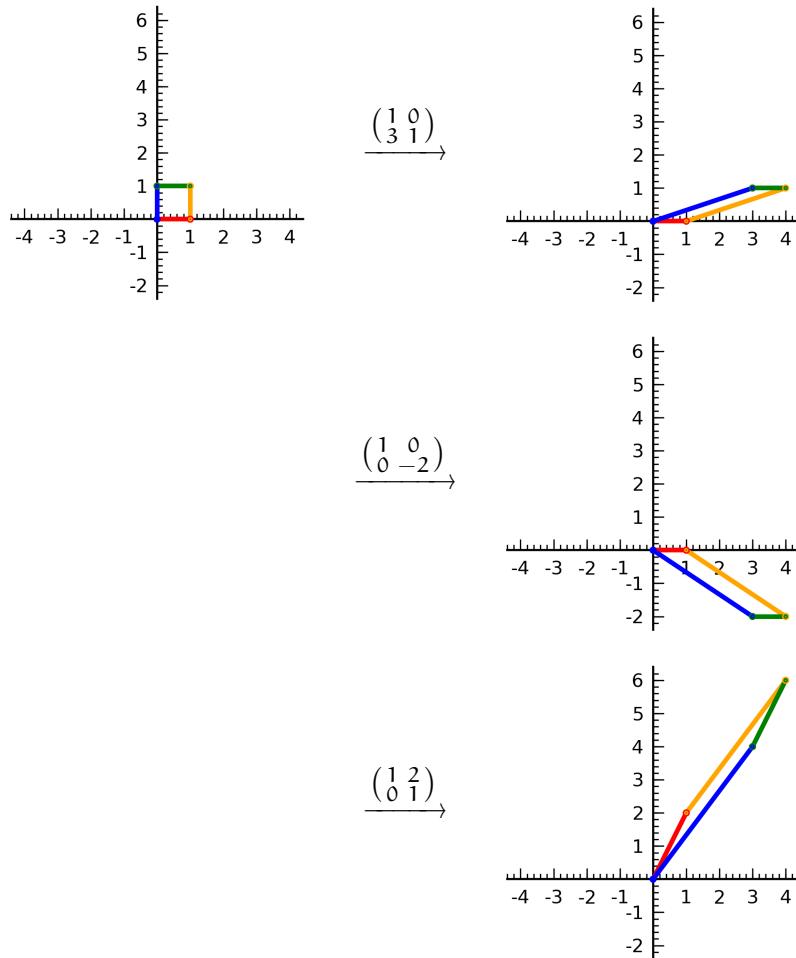
Now for the cumulative effect. For example, the cuulative effect of the maps on the unit square's upper right corner is here.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{L} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \xrightarrow{D} \begin{pmatrix} 4 \\ -2 \end{pmatrix} \xrightarrow{U} \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

```

1 sage:
2 sage: L = matrix(QQ, [[1, 0], [3, 1]])
3 sage: D = matrix(QQ, [[1, 0], [0, -2]])
4 sage: U = matrix(QQ, [[1, 2], [0, 1]])
5 sage: LD = L*D
6 sage: load "plot_action.sage"
7 sage: p.save("sageoutput/plot_action24.png")
8 sage: p.save("sageoutput/plot_action24a.png")
9 sage: p.save("sageoutput/plot_action24b.png")
10 sage: p.save("sageoutput/plot_action24c.png")

```



Source of `plot_action.sage`

```
1 # plot_action.sage
2 # 2012-Dec-04 JH
3
4 DOT_SIZE = .02
5 CIRCLE_THICKNESS = 2
6 def color_circle_list(a, b, c, d, colors, full_circle=False):
7     """Return list of graph instances for the action of a 2x2 matrix on
8     half of the unit circle. That circle is broken into chunks each
9     colored a different color.
10    a, b, c, d  reals  entries of the matrix ul, ur, ll, lr
11    colors  list of rgb tuples; len of this list is how many chunks
12    full_circle=False  Show a full circle instead of a half circle.
13 """
14 r = []
15 if full_circle:
16     p = 2*pi
17 else:
18     p = pi
19 t = var('t')
20 n = len(colors)
21 for i in range(n):
22     color = colors[i]
23     x(t) = a*cos(t)+c*sin(t)
24     y(t) = b*cos(t)+d*sin(t)
25     g = parametric_plot((x(t), y(t)),
26                           (t, p*i/n, p*(i+1)/n),
27                           color = color, thickness=CIRCLE_THICKNESS)
28     r.append(g)
29     r.append(circle((x(p*i/n), y(p*i/n)), DOT_SIZE, color=color))
30 if not(full_circle):    # show (x,y)=(-1,0) is omitted
31     r.append(circle((x(pi), y(pi)), 2*DOT_SIZE, color='black',
32                      fill = 'true'))
33     r.append(circle((x(pi), y(pi)), DOT_SIZE, color='white',
34                      fill = 'true'))
35 return r
36
37 NUM_CIRCLE PTS = 12
38 def plot_circle_action(a, b, c, d, n = NUM_CIRCLE PTS, full_circle = False):
39     """Show the action of the matrix with entries a, b, c, d on half
40     of the unit circle, as the circle and the output curve, broken into
41     a number of colors.
42     a, b, c, d  reals  Entries are upper left, ur, ll, lr.
43     n = 12  positive integer  Number of colors.
44     full_circle=False  boolean  Show whole circle, or top half
45 """
46 colors = rainbow(n)
47 G = Graphics()  # holds graph parts until they are to be shown
48 for g_part in color_circle_list(a,b,c,d,colors,full_circle):
49     G += g_part
50 return plot(G)
```

```

52 SQUARE_THICKNESS = 1.75 # How thick to draw the curves
53 ZORDER = 5 # Draw the graph over the axes
54 def color_square_list(a, b, c, d, colors):
55     """Return list of graph instances for the action of a 2x2 matrix
56     on a unit square. That square is broken into sides, each colored a
57     different color.
58     a, b, c, d  reals entries of the matrix
59     colors  list of rgb tuples; len of this list is at least four
60 """
61 r = []
62 t = var('t')
63 # Four sides, ccw around square from origin
64 r.append(parametric_plot((a*t, b*t), (t, 0, 1),
65                         color = colors[0], zorder=ZORDER,
66                         thickness = SQUARE_THICKNESS))
67 r.append(parametric_plot((a+c*t, b+d*t), (t, 0, 1),
68                         color = colors[1], zorder=ZORDER,
69                         thickness = SQUARE_THICKNESS))
70 r.append(parametric_plot((a*(1-t)+c, b*(1-t)+d), (t, 0, 1),
71                         color = colors[2], zorder=ZORDER,
72                         thickness = SQUARE_THICKNESS))
73 r.append(parametric_plot((c*(1-t), d*(1-t)), (t, 0, 1),
74                         color = colors[3], zorder=ZORDER,
75                         thickness = SQUARE_THICKNESS))
76 # Dots make a cleaner join between edges
77 r.append(circle((a, b), DOT_SIZE,
78                 color = colors[0], zorder = 2*ZORDER,
79                 thickness = SQUARE_THICKNESS*1.25, fill = True))
80 r.append(circle((a+c, b+d), DOT_SIZE,
81                 color = colors[1], zorder = 2*ZORDER+1,
82                 thickness = SQUARE_THICKNESS*1.25, fill = True))
83 r.append(circle((c, d), DOT_SIZE,
84                 color = colors[2], zorder = ZORDER+1,
85                 thickness = SQUARE_THICKNESS*1.25, fill = True))
86 r.append(circle((0, 0), DOT_SIZE,
87                 color = colors[3], zorder = ZORDER+1,
88                 thickness = SQUARE_THICKNESS*1.25, fill = True))
89 return r
90
91 def plot_square_action(a, b, c, d, show_unit_square = False):
92     """Show the action of the matrix with entries a, b, c, d on half
93     of the unit circle, as the circle and the output curve, broken into
94     colors.
95     a, b, c, d  reals Entries are upper left, ur, ll, lr.
96 """
97 colors = ['red', 'orange', 'green', 'blue']
98 G = Graphics() # hold graph parts until they are to be shown
99 if show_unit_square:
100     for f_part in color_square_list(1,0,0,1,colors):
101         G += f_part
102 for g_part in color_square_list(a,b,c,d,colors):

```

```
103     G += g_part
104
105     p = plot(G)
106
107     return p
108
109 EPSILON = 0.25
110 ARROW_THICKNESS = .25
111
112 def point_list(a, b, c, d, pts, colors=None):
113     """Show the action of the matrix with entries a, b, c, d on the
114     points.
115     a, b, c, d  reals  Upper left, ur, ll, lr or matrix.
116     pts  list of pairs of reals
117     colors = None  list of colors
118     """
119
120     r = []
121     for dex, pt in enumerate(pts):
122         x, y = pt
123         f_x = a*x + c*y
124         f_y = b*x + d*y
125         if colors:
126             color = colors[dex]
127         else:
128             color = 'lightgray'
129         if ((abs(x-f_x) < EPSILON) and (abs(y-f_y) < EPSILON)):
130             r.append(circle(pt, DOT_SIZE, color=color,
131                             thickness=ARROW_THICKNESS))
132         else:
133             r.append(arrow(pt, (f_x,f_y), color=color,
134                           width=ARROW_THICKNESS))
135
136     return r
137
138
139 def point_grid(max_x, max_y):
140     """Return a grid of points (x,y).
141     max_x, max_y  positive integers
142     """
143
144     r = []
145     for x in range(-1*max_x, max_x+1):
146         for y in range(-1*max_y, max_y+1):
147             r.append((x,y))
148
149     return r
150
151
152 def plot_point_action(a, b, c, d, pts, colors=None):
153     """Show the action of the matrix with entries a, b, c, d on the
154     points.
155     a, b, c, d  reals  Upper left, ur, ll, lr or matrix.
156     pts  list of pairs of reals
157     """
158
159     if colors is None:
160         colors = ["gray"]*len(pts)
161
162     G = Graphics()
163     for action_arrow in point_list(a,b,c,d,pts,colors=colors):
164         G += action_arrow
```

```

154     p = plot(G)
155     return p
156
157 BA_THICKNESS = 1.5
158 def before_after_list(a, b, c, d, pts, colors=None):
159     """Show the action of the matrix with entries a, b, c, d on the
160     points by showing the vector before and the vector after in the
161     same color.
162     a, b, c, d  reals Upper left, ur, ll, lr or matrix.
163     pts  list of pairs of reals
164     colors = None  list of colors
165     """
166
167     r = []
168     for dex, pt in enumerate(pts):
169         x, y = pt
170         v = vector(RR, pt)
171         M = matrix(RR, [[a, b], [c, d]])
172         f_x, f_y = v*M
173         if colors:
174             color = colors[dex]
175         else:
176             color = 'lightgray'
177         if ((abs(x-f_x) < EPSILON) and (abs(y-f_y) < EPSILON)):
178             r.append(circle(pt, DOT_SIZE, color=color,
179                         thickness=BA_THICKNESS))
180         else:
181             r.append(arrow((0,0), (x,y), color=color,
182                           width=BA_THICKNESS))
183             r.append(arrow((0,0), (f_x,f_y), color=color,
184                           width=BA_THICKNESS))
185
186     return r
187
188 def plot_before_after_action(a, b, c, d, pts, colors=None):
189     """Show the action of the matrix with entries a, b, c, d on the
190     points.
191     a, b, c, d  reals Upper left, ur, ll, lr or matrix.
192     pt_list  list of pairs of reals; the before pts to show
193     """
194
195     if colors is None:
196         colors = ["gray"]*len(pts)
197     G = Graphics()
198     for ba in before_after_list(a,b,c,d,pts,colors=colors):
199         G += ba
200     p = plot(G)
201     return p
202
203 def find_angles(a,b,c,d,num_pts):
204     """Apply the matrix to points around the upper half circle, and
205     return the angles between the input and output vectors.
206     a, b, c, d  reals Upper left, ur, ll, lr or matrix.
207     num_pts  positive integer number of points

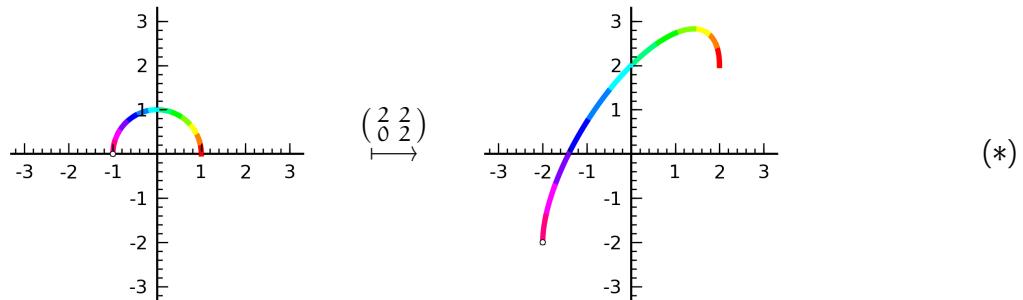
```

```
205 """
206 r = []
207 M = Matrix(RR, [[a, b], [c, d]])
208 for i in range(num_pts):
209     t = i*pi/num_pts
210     pt = (cos(t), sin(t))
211     v = vector(RR, pt)
212     w = v*M
213     try:
214         angle = arccos(w*v/(w.norm()*v.norm()))
215         # angle_v = atan2(v[1], v[0])
216         # angle_w = atan2(w[1], w[0])
217         # angle = angle_w - angle_v
218     except:
219         angle = None
220     r.append((t, angle))
221 return r
222
223 plot.options['figsize'] = 2.5
224 plot.options['axes_pad'] = 0.05
225 plot.options['fontsize'] = 7
226 plot.options['dpi'] = 500
227 plot.options['aspect_ratio'] = 1
```


Eigenvalues

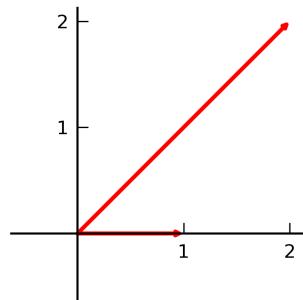
Consider again the action of a skew map on the unit circle.

```
1 sage: p = plot_circle_action(2, 2, 0, 2)
```



There are twelve colors; on the left they begin with red at $(x, y) = (1, 0)$ while on the right they begin with red at $(1, 2)$. Here are the before and after vectors for that red point.

```
1 sage: p = plot_before_after_action(2, 2, 0, 2, [(1,0)], ['red'])
```



The matrix's action on the red vector is to both resize and rotate.

```
1 sage: v = vector(RR, [1,0])
2 sage: M = matrix(RR, [[2, 2], [0, 2]])
3 sage: w = v*M
4 sage: w.norm(), v.norm()
5 (2.82842712474619, 1.00000000000000)
6 sage: angle = arccos(w*v/(w.norm()*v.norm()))
7 sage: angle
8 0.785398163397448
```

The map resizes and rotates the vector at the other end of the half circle, at $(-1, 0)$, by the same amount as it does the vector ending at $(1, 0)$. This is because a linear map has the same action on all vectors that lie on the same line through the origin. But intermediate vectors are not resized and rotated by the same amount. Here is the next of the twelve points on the half-circle.

```

1 sage: v = vector(RR, [cos(pi/12), sin(pi/12)])
2 sage: M = matrix(RR, [[2, 2], [0, 2]])
3 sage: w = v*M
4 sage: w.norm(), v.norm()
5 (3.11962350413778, 1.00000000000000)
6 sage: angle = arccos(w*v/(w.norm()*v.norm()))
7 sage: angle
8 0.641199650524028

```

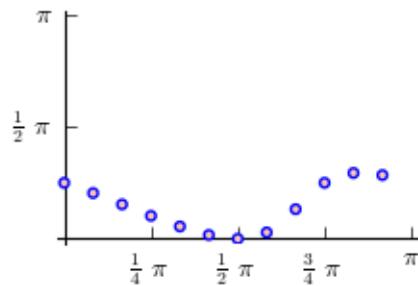
In Chapter 6 on Singular Value Decomposition we studied how transformations resize vectors; the maximum resizing factors are the singular values. In this chapter we consider the turning.

Sage will show us the rotation of each of the twelve points under the transformation. At the end of this chapter is the source for a routine `find_angles` that takes in the matrix entries and a list of input vectors, then for each vector finds the angle between itself and its image and returns a list of those angles. Also there is the source of `circle_points` that takes in a number and returns that many vectors evenly spaced around the half-circle. We can use those two to make a graph.

```

1 sage: NUM_CIRCLE PTS
2 12
3 sage: angles = find_angles(2, 2, 0, 2, NUM_CIRCLE PTS)
4 sage: angles[0:2]
5 [(0, 0.785398163397448), (1/12*pi, 0.641199650524028)]
6 sage: ticks = ([0, pi/4, pi/2, 3*pi/4, pi], [0, pi/2, pi])
7 sage: p = scatter_plot(angles, marker='.', edgecolor='blue', ticks=ticks)
8 sage: p.set_axes_range(0, pi, 0, pi)
9 sage: p.save("sageoutput/plot_action103.png", figsize=2.5, tick_formatter=pi)

```



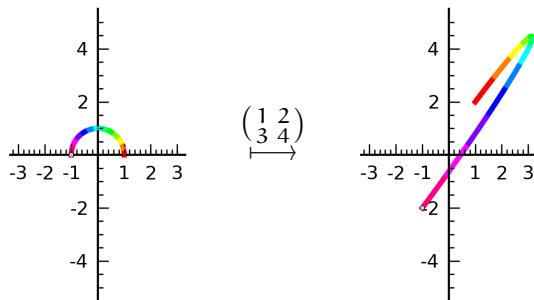
The most interesting point on this graph is where the angle is 0—for this transformation there is a input vector that does not turn at all. It is the vector halfway around the circle, with endpoint $(0, 1)$. We can see it on the graphic above labelled (*), where the green input vector lies on the y-axis and so does its green image. It is resized but it isn't rotated.

Generic matrix We can do the same analysis for our usual generic 2×2 matrix. Here is its action on the unit circle.

```

1 sage: p = plot_circle_action(1, 2, 3, 4)

```

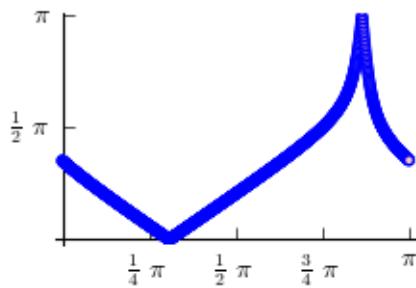


This graphs the angle between the each before arrow and its associated after arrows.

```

1 sage: #vecs = circle_points(1000)
2 sage: angles = find_angles(1,2,3,4,1000)
3 sage: ticks = ([0,pi/4,pi/2,3*pi/4,pi], [0,pi/2,pi])
4 sage: p = scatter_plot(angles, marker='.', edgecolor='blue', ticks=ticks)
5 sage: p.save("sageoutput/plot_action113.png", figsize=2.5, tick_formatter=pi)

```



This graph has two interesting points, where $y = 0$ and where $y = \pi$. In the second case the angle of π radians means that the vector remains on the same line through the origin but is multiplied by a negative scalar. In both places the vector is not turned at all, only resized.

Vectors that are not turned by a transformation t but are only resized are *eigenvectors* for t , and the amount by which it is resized is the *eigenvalue* associated with that vector. For a fixed eigenvalue λ , the set of vectors associated with it is an *eigenspace*.

```

1 sage: M = matrix(RDF, [[1, 2], [3, 4]])
2 sage: evs = M.eigenvectors_left()
3 sage: evs
4 [(-0.372281323269, [(-0.909376709132, 0.415973557919)], 1),
5  (5.37228132327, [(-0.565767464969, -0.824564840132)], 1)]
6 sage: evs[0]
7 (-0.372281323269, [(-0.909376709132, 0.415973557919)], 1)
8 sage: evs[1]
9 (5.37228132327, [(-0.565767464969, -0.824564840132)], 1)

```

Explain the RDF and the left.

The first information that Sage gives, `evs[0]`, is that the set of vectors with a basis consisting of the single unit vector with endpoint $(-0.909376709132, 0.415973557919)$ is an eigenspace associated with the eigenvalue -0.372281323269 . Sage also finds with `evs[1]` that a second eigenspace has a

basis consisting of the vector with endpoint $(-0.565767464969, -0.824564840132)$, associated with the eigenvalue 5.37228132327 .

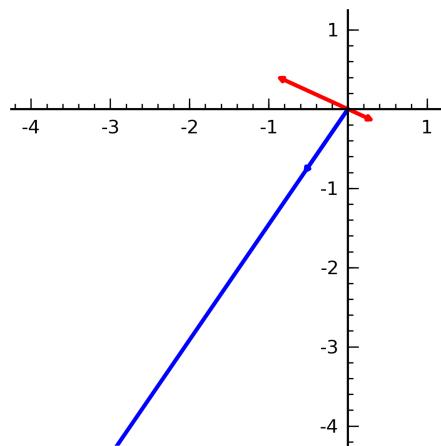
We can draw those two as before and after pictures.

```

1 sage: p1 = plot_before_after_action(1,2,3,4, [evs[0][1][0]], ['red'])
2 sage: p2 = plot_before_after_action(1,2,3,4, [evs[1][1][0]], ['blue'])
3 sage: p = p1+p2
4 sage: p.set_axes_range(-4, 1, -4, 1)
5 sage: p.save("sageoutput/plot_action114.png", figsize=3.5)

```

Each before vector is a unit vector, so it is easy to pick it from among the two. The after vector is scaled from the before vector.



Characteristic polynomial

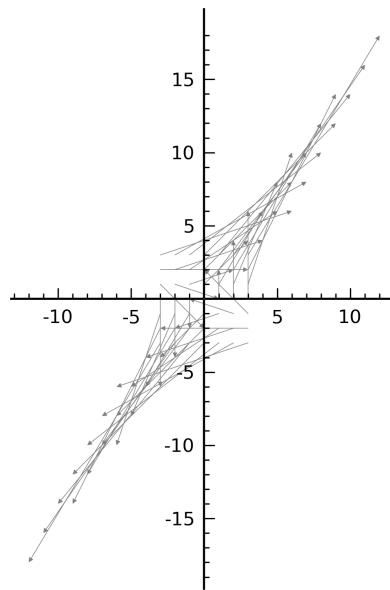
Attraction Another natural picture to draw is the action of linear maps on some points.

We start with the action of the linear transformation $t: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ represented with respect to the standard bases by the generic matrix with entries 1, 2, 3, and 4.

```

1 sage: load "plot_action.sage"
2 sage: plot.options['figsize']=4.5
3 sage: g = point_grid(3, 3)
4 sage: p = plot_point_action(1, 2, 3, 4, g)
5 sage: p.save("sageoutput/plot_action110.png")

```



Source of find_angles

This routine uses a formula for the angle between two vectors that always gives a positive value, that is, it is the unoriented angle. That's fine for purpose here, which is to use the graph to roughly locate places where the action of the matrix does not turn the vector.

```

1 def find_angles(a,b,c,d,num_pts):
2     """Apply the matrix to points around the upper half circle, and
3     return the angles between the input and output vectors.
4     a, b, c, d  reals  Upper left, ur, ll, lr or matrix.
5     num_pts    positive integer  number of points
6 """
7 r = []
8 M = Matrix(RR, [[a, b], [c, d]])
9 for i in range(num_pts):
10     t = i*pi/num_pts
11     pt = (cos(t), sin(t))
12     v = vector(RR, pt)
13     w = v*M
14     try:
15         angle = arccos(w*v/(w.norm()*v.norm()))
16         # angle_v = atan2(v[1], v[0])
17         # angle_w = atan2(w[1], w[0])
18         # angle = angle_w - angle_v
19     except:
20         angle = None
21     r.append((t,angle))
22 return r

```

Source of plot_before_after_action

The only perhaps unexpected point in this routine, and the helper routine before it, is that if the vector is not mapped very far then the helper routine does not show an arrow but instead shows a circle.

```

1 BA_THICKNESS = 1.5
2 def before_after_list(a, b, c, d, pts, colors=None):
3     """Show the action of the matrix with entries a, b, c, d on the
4     points by showing the vector before and the vector after in the
5     same color.
6     a, b, c, d  reals  Upper left, ur, ll, lr or matrix.
7     pts  list of pairs of reals
8     colors = None  list of colors
9 """
10
11    r = []
12    for dex, pt in enumerate(pts):
13        x, y = pt
14        v = vector(RR, pt)
15        M = matrix(RR, [[a, b], [c, d]])
16        f_x, f_y = v*M
17        if colors:
18            color = colors[dex]
19        else:
20            color = 'lightgray'
21        if ((abs(x-f_x) < EPSILON) and (abs(y-f_y) < EPSILON)):
22            r.append(circle(pt, DOT_SIZE, color=color,
23                           thickness=BA_THICKNESS))
24        else:
25            r.append(arrow((0,0), (x,y), color=color,
26                           width=BA_THICKNESS))
27            r.append(arrow((0,0), (f_x,f_y), color=color,
28                           width=BA_THICKNESS))
29
30    return r
31
32 def plot_before_after_action(a, b, c, d, pts, colors=None):
33     """Show the action of the matrix with entries a, b, c, d on the
34     points.
35     a, b, c, d  reals  Upper left, ur, ll, lr or matrix.
36     pts  list of pairs of reals; the before pts to show
37 """
38
39     if colors is None:
40         colors = ["gray"]*len(pts)
41     G = Graphics()
42     for ba in before_after_list(a,b,c,d,pts,colors=colors):
43         G += ba
44     p = plot(G)
45     return p

```

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