# Chapter 3

⬀⬀⬀Geometric model of Scheimpflug Imaging

### Introduction

TO DO: State the motivation and assumptions.

TO DO: Preview of what is coming in the following sections.

TO DO: State the novelty of this approach, and why needed to develop this model

TO DO: Notation

* A left super-script indicates the frame of reference. For example, indicates that the variable is w.r.t. the world coordinate frame . If no reference is explicitly stated it implies that the variable is w.r.t. the world coordinate frame (or the camera coordinate frame if the camera coordinate frame and the world coordinate frame are the same.
* A subscript is used to associate a variable to a particular xxx like entrance-pupil position (), image plane (), for example is used to represent the 3D rotation matrix applied to the entrance pupil plane in the camera frame . The same notation is also used to indicate a transformed variable, for example is used to represent under the rotational transformation by in the camera coordinate frame . As also mentioned earlier, if the camera coordinate frame is the same as the world coordinate frame, then the notation shall be used.
* represents the pose of frame w.r.t. frame TO DO: mention how a point in one frame is represented in another frame.
* The zero-based indexing of matrices and vectors

### Relation between the direction cosines of the chief rays in the object and image side

Let be the direction cosine of the chief ray from a world point to the entrance pupil (ENP) in the object side, and let be the direction cosine of the chief ray emerging from the exit pupil (EXP) to the corresponding image point in the image side. The , , , and are specified in the camera frame as shown in figure 3.1. If the angles between the chief ray and the optical axis at the object and image side are and respectively, then the relation between the angles is given by the airy tangent condition [ref]:

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Where, is the pupil magnification, defined as the ratio between the paraxial exit-pupil to the paraxial entrance-pupil diameter.

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| ***Figure 3.1*** *Schematic showing the chief ray from the object point to the image point when the optical axis is aligned with the z-axis of the reference frame with its origin at the entrance-pupil (ENP) position.* |

If and are the zenith and azimuthal angles of the chief ray in the object side, and and the corresponding angles in the image side, then the direction cosines, in the camera frame , are represented as:

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When the optical axis is aligned with the z-axis (as shown in figure 3.1), then and . Substituting the expressions for from equation (3.2) into equation (3.1) we get:

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By definition the chief ray (a meridional ray) is …. Therefore and the above relations reduces to:

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Also, from (3.1)

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Simplifying further,

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Therefore, the expressions for output direction cosines in terms of the input direction cosines and the pupil magnification is obtained as:

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Since the chief ray emerging from the exit pupil spans the plane formed by the input chief ray and the optical axis, the chief ray in the image side may be represented as a linear combination of the input chief ray and the optical axis. This linear relationship in terms of the direction cosines of the chief-ray and the unit vector representing the optical axis is shown below.

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Where, and are the weights and , is unit-length vector representing the optical axis that is coincident with the z-axis of the camera frame. The subscript, , is used to indicate this special, coincident, case. The weight is readily obtained by comparing equations (3.7) and (3.8):

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Substituting the expression for into and comparing with (3.6) we get the expression for :

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We would like to derive the relationship between the input and output direction cosines of the chief ray for an arbitrary rotation of the optical axis centered about the ENP as shown in figure 3.2.

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| ***Figure 3.2*** *Schematic showing the chief ray from the object point to the image point when the optical axis is rotated about an arbitrary axis about the ENP.* |

Write appropriate text here… explain .

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Where or

Note that the input direction cosine in equation (3.11) is different from the input direction cosine in equation (3.8), even for the same world point as a result of the displacement of the entrance pupil. Multiplying equation (3.11) by :

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Letting and

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By comparing equations (3.8) and (3.13) the expressions for the weights and are obtained as:

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Where .

Re-writing equation (3.13) as:

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Furthermore, using and , the general expression for the direction cosines of the chief ray in the image side in terms of the pupil magnification and direction cosines in the object side is:

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Where .

Comment on considering just the positive sign.

### Relation between object coordinate and image coordinate for arbitrary orientation of the lens and image plane

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| ***Figure 3.3*** *Schematic showing the chief ray from the object point to the image point when the optical axis is rotated about an arbitrary axis about the ENP.* |

Explain the different coordinate frames.

Let the exit pupil (EXP) be located at a distance from the pivot point along the optical axis that is tilted in the camera frame , represented by the 3D rotation matrix . Then, the location of the EXP in the camera frame is . The chief ray emerging from the EXP with direction cosine vector is then represented by the parametric equation:

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Where represents any point along the chief ray in the image side in the camera frame.

The equation of the image plane with unit normal   in Hessian normal form is written as:

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Where is the perpendicular distance between the plane and the origin of the reference frame , and is a point on the plane.

The expression for for which the ray intersects the image plane is obtained by equating to , multiplying equation (3.17) by  , and rearranging the terms as follows:

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Substituting (3.19) in to (3.17):

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Let the point of intersection of the z-axis of the camera frame with the image plane be the origin of the image plane’s reference frame . The orientation of the image plane can be described by applying a 3D rotation matrix (or a composition of successive rotation matrices) to the image plane, with its unit plane-normal nominally equal to , about the origin of as shown in figure 3.3. If is the rotation matrix then

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The expression for is obtained as follows (refer to figure 3.4):

The equation of the image plane is

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Since is a point on the plane, therefore

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| ***Figure 3.4*** *xxx.* |

Using the above result, the expression for the point of intersection of the chief-ray with the image plane in terms of the input direction cosines is

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Similar to the EXP, let the entrance pupil (ENP) be located at a distance from the pivot point along the optical axis in the camera frame . Then, the location of the ENP in is . The direction cosines and the world point are related as

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Which can be compactly written as

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Substituting equation (3.25) into equation (3.23) a general relation between the world point and its corresponding image point is obtained:

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