# Chapter 3

⬀⬀⬀**GEOMETRIC MODEL OF SCHEIMPFLUG IMAGING**

We would like to investigate the capabilities and limitations of Scheimpflug imaging for extending the imaging volume of iris-image acquisition for biometrics.

Imaging systems primarily consists of a lens, a light sensitive surface called the sensor, and a surface in sharp focus called the object. The object and its image formed by the lens are called *conjugates*. In this work, we restrict the object and image surfaces to planes. Further, we define the plane containing the sensor as the *image plane*, the object surface (in focus) as the *object plane*, and a plane representing the lens as the *lens plane*. Amongst the several possible planes that pass through a lens, we cherry-pick a plane that provides some advantage in our model to be the lens plane.

|  |
| --- |
|  |
| **Figure 3.1** Scheimpflug camera movements. The image and lens planes are free to swivel about their pivots. The figure depicts a superposition of several photographs of a Sinar P3 camera in a multitude of lens and image plane positions. The lens standard (left) and the sensor standard (right) are indicated by the two attached steel rulers. The red and blue cross hairs indicate the physical point of rotation (pivot point) for each standard. The arrows indicate the various moments of the camera. |

In the *fronto-parallel* configuration used in common camera design, the lens and the sensor planes are parallel to each other and perpendicular to the optical axis. In such designs, the physics of optical imaging ("Gaussian Imaging equation") dictates that the object plane is also parallel to the lens and image planes. In contrast, the lens- and image- planes in a Scheimpflug camera are free to swivel about their pivots (as shown in Figure 3.1) resulting in a corresponding swivel of the object plane. We would like to exploit this feature—the freedom to arbitrarily orient the object plane—of Scheimpflug configuration to improve the depth of field of the iris acquisition devices.

Scheimpflug cameras provide the greatest flexibility for image composition; however, that flexibility is traded for complexity. Accurate modeling of Scheimpflug imaging is quite involved, and its art of operation is often left to experts who frequently employ approximate methods. While these cameras have been used in few scientific imaging applications, but the vast majority of them contemporarily are used for landscape and studio photography.

Existing models of Scheimpflug camera commonly employ overtly simple models that work quite well for documentary photography but are often restrictive and inaccurate for scientific purpose. A rich description of such cameras requires the development of a more general model. We aim to develop, in this chapter, a model for Scheimpflug imaging using the axioms of *geometric optics* (*ray optics*).

Ray optics does not provide a complete picture of imaging; yet, the definitions and postulates therein provide useful tools for the analysis and synthesis of optical systems.

Assumptions are crucial and necessary for modeling that enable its expediency and limits its applicability. For the model described herein, we have assumed paraxial imaging, rotational symmetry and aberration-free optics in order to make the problem tractable. Additionally, justified by environment in which …., we have assumed the refractive index of the lens elements and the interstitial medium to be isotropic (uniform along all directions) and homogeneous (uniform at all positions); this assumption imposes rectilinear propagation of light. Further, we have assumed the lens is surrounded by air of refractive index one. Consequently, the front and back focal lengths are equivalent, and the two nodal points coincide with the corresponding principal points.

TODO: Review what is out there. Type of models that are there, their limitations. Also, comment on the existing process of “focus-transfer” why that is erroneous. Point out, without explicitly stating, that this method has several advantages (and explicitly point out the advantages), the new insights that it provides and not a re-engineering of existing knowledge just for the sake of being different.

TO DO: State the novelty of this approach, and why needed to develop this model. Is there any relation to eikonal equations?

TODO: Preview of what is coming in the following sections.

### 3.1 Introduction

Optical imaging systems consist of several groups of elements; those elements endowed with optical power bends rays of light. The tiniest orifice in the system is called the *system aperture* or *stop*. Its interaction with the elements in the system gives rise to the pupils.

*Pupils* are the sine qua non of optical systems. They are indispensable in the design and specification of all optical systems, in both domains of *ray* and *wave optics*. The *entrance pupil* () is the “image” of the stop seen through the elements preceding it is. The *exit pupil* () is the “image” of the stop seen through the elements following it is. That is, the pupils are the images of the stop produced by the elements on either side of it. The region preceding the entrance pupil, which includes the objects and light sources, is called the object space; and the region following the exit pupil, which includes the image plane, is called the image space. The size and position of the stop (and hence the pupils) affect image resolution, aberration, brightness, and geometry.

Rotationally symmetric lenses have an axis of symmetry—the optical axis. A ray coincident with the optical axis traverses undeviated through the lens. Planes passing through the axis of such lenses are the meridional planes. Rays restricted to the meridional planes are *meridional rays*. Patterns formed by the meridional rays on either side of the optical axis are mirror-reversed, exhibiting bilateral symmetry. Figure 3.2 shows two types of meridional rays, traced in Zemax, that are fundamental to geometric analysis. The *marginal ray* (MR) originates from the axial object position and skirts the edges of the aperture and pupils (virtually); the *chief ray* (CR) starts at an off-axis object point and pierces the centers of the aperture and pupils[[1]](#footnote-1) (virtually). This pair of rays determines the location and size of the pupils, the position of the image, and the magnification. Furthermore, the bundle of chief rays from the object space converge at the center of the entrance pupil—thus *homocentric*—forming the vertex of the object-space perspective cone; in the image space, the bundle of chief rays diverge from the center of the exit pupil producing the vertex of the image-space perspective cone.

Imagine a film projector working backwards. Imagine the stream of light rays flowing from the illuminated portion of the scene towards a small circular hole in the projector. This pencil of rays forms a conical volume of light—the perspective cone—with its vertex at the hole and its base towards the scene. The “illuminated portion” is the angular extent of the scene visible in the image, confined by the circumferential chief rays. These extreme chief rays determine the opening angle of the cone. The “small hole” represents the entrance pupil of a camera or the pupil at the center of the iris in an eye. In the image space (behind the hole), the ray-pencil form another cone with the vertex at the center of the exit pupil. This image-space perspective cone projects the light from the scene onto the film surface or the retina in the eye. This process of image formation, known as the *central projection*, is fundamental to all imaging systems—inanimate and animate—including the camera and the eye. While the opening angle of the object-space perspective cone determines the field-of-view, its counterpart in the image space determines the angular dimension of the image. The ratio of the pupil sizes (pupil magnification) determines the relationship between the image and object-space opening angles of the two perspective cones.

|  |
| --- |
|  |
| **Figure 3.2** Fundamental rays (contained within the meridional plane) and pupils in a Double Gauss lens for an object at infinity. The chief rays—close to the optical axis (0°, ±5° in the object space at entrance pupil)—appear to converge at the center of entrance pupil and diverge from the center of exit pupil. The marginal ray, which is parallel to the optical axis since the object is at infinity, appear to skirt the edges of the two pupils. The red circles specify the vertices of the perspective cones (centers of the pupils). The rays were traced in Zemax. |

### 3.2 Notation (this section is currently a placeholder for the notation, skip reading this for now. I will write it appropriately after completion of the major math)

* Right-handed coordinate system with the +z along the direction of travel of light, (left to right in the plane of drawing)
* In general, non-primed quantities are used to indicate input or object space (e.g.) and primed quantities are used to indicate output or image space (e.g.).
* Unit vectors are represented using a hat (), with the exception of the direction cosine vectors (e.g. although the norm of the direction cosine vectors are unity.
* A left superscript indicates the frame of reference. For example, indicates that the variable is w.r.t. the world coordinate frame. If no reference is explicitly stated it implies that the variable is w.r.t. the world coordinate frame (or the camera coordinate frame if the camera coordinate frame and the world coordinate frame are the same.
* A subscript is used to associate a variable with a particular xxx like entrance pupil position (), image plane (), for example is used to represent the 3D rotation matrix applied to the entrance pupil plane in the camera frame. The same notation is also used to indicate a transformed variable, for example is used to represent under the rotational transformation by in the camera coordinate frame. As also mentioned earlier, if the camera coordinate frame is the same as the world coordinate frame, then the notation shall be used.
* represents the pose of frame w.r.t. frame TO DO: mention how a point in one frame is represented in another frame.
* The one-based indexing of matrices and vectors. Also a matrix is also represented as where are the columns of .
* Describe what I mean by lens plane.
* Overloading of the term “direction cosine(s)” and “direction cosine vector”. It should be clear from the context

### 3.3 Relation between pupil magnification and chief ray angle

The *pupil magnification* is defined as the ratio of the paraxial exit-pupil size to the entrance-pupil size [refref].

Figure 3.3 illustrates the meridional and sagittal planes associated with an arbitrarily located object of height above the optical axis and its image of height in a typical optical system. The figure also shows the chief ray from the object’s edge further from the optical axis, the marginal ray from the axial point in the object, and the two pupils contained in the meridional plane. The schematic, although simple, is quite general as a (meridional) plane always exist for a given object point irrespective of its position in the three-dimensional space, if the lens is rotationally symmetric.

|  |
| --- |
|  |
| **Figure 3.3** Schematic of chief and marginal rays. The ratio of the tangents of the chief ray angles in the object space to the image space yields the pupil magnification. |

Let the angles between the chief ray and the optical axis (called the *ray-angle*) in the object- and image-space be and respectively. Also, let the angles produced by the marginal ray with the optical axis in the object- and image-space be and respectively. Then, we can obtain the relation between the chief ray ray-angles— and —and the pupil magnification as follows:

From the Figure 3.3,

|  |  |
| --- | --- |
|  |  |

Eliminating and after dividing by, we have

|  |  |
| --- | --- |
|  |  |

A common observation in imaging is that we can increase the transverse magnification () by increasing the lens-to-image-plane distance while correspondingly decreasing the lens-to-object-plane distance in order to maintain focus on the object. However, increasing (decreasing) the image plane distance proportionally decreases (increases) the marginal ray angle (see Figure 3.3). Consequently, the angular magnification () decreases with increase in lens-to-image-plane distance. Therefore, a large transverse magnification is associated with a correspondingly small angular magnification. This result follows from a more general theory called the *Lagrange invariant* [ref] property of the two rays (the chief ray and the marginal ray) when applied between conjugate locations. As per the invariant property, the product of the transverse magnification and the angular magnification equals to one, i.e., or. Cancelling the corresponding terms in Eq. (3.2) yields the relationship between the pupil magnification and the object- and image-space chief ray angles:

|  |  |
| --- | --- |
|  |  |

The above relation (Eq. (3.3)) has been previously derived in [ref] using a different formulation.

For a given optical system, the pupil magnification is constant. This constancy of the ratio of the tangents of the chief ray angles for varying object (and image) heights is a necessary and sufficient condition for distortion-free imaging known as the *Airy’s Tangent-Condition* [ref]. Eq. (3.3) also suggests that when the perspective cones in the object- and image-space are symmetric. In the following section, we will use Eq. (3.3) to derive the relationship between the direction cosines of the object-space (input) chief rays and direction cosines of the image-space (output) chief rays.

### 3.4 Transfer of chief ray’s direction cosines between the pupils

The direction cosines, a unit vector of cardinality three, specify the direction of a ray. Its elements are the cosines of the angles the ray makes with the three coordinate axes. In other words, the elements of the direction cosine vector are the projections of the unit vector in the direction of the ray on the x-, y-, and z-axes. Given the direction cosine of the chief ray in the object space, we would like to know the direction cosine of the corresponding ray in the image space. Furthermore, what is the relation between the input and output chief ray’s direction cosines if the lens is swiveled about a pivot point along the optical axis?

We begin by solving a specific problem of the *transfer* of the direction cosines between the pupils in which the optical axis coincides with the z-axis of the camera frame, as show in Figure 3.4. Subsequently, we will deduce the general *transfer* expression in which the optical axis is free to swivel about the origin of. Let be the direction cosine of the chief ray from a world point to the center of the entrance pupil, and let be the corresponding direction cosine of the chief ray from the exit pupil. The parameters , ,and are specified with respect to frame .

|  |
| --- |
|  |
| **Figure 3.4** Specific problem—optical axis coincides with reference frame’s z-axis. If and are the angles of the CR with the OA in the object- and image-space respectively, then and. |

If and are the zenith and azimuthal angles of the chief ray in the object space, and and the corresponding angles in the image space, then the direction cosines, in , are:

|  |  |
| --- | --- |
|  |  |

Since the optical axis is aligned with the z-axis, and. Substituting the expressions for from Eq. (3.4) into Eq. (3.3) we obtain:

|  |  |
| --- | --- |
|  |  |

Further, since the input and out chief rays are confined to the same meridional plane [ref], , yielding and in terms of and , the ratios of to , and :

|  |  |
| --- | --- |
|  |  |

From (3.3) we have

|  |  |
| --- | --- |
|  |  |

which after simplification yields in terms of the pupil magnification and input

|  |  |
| --- | --- |
|  |  |

Combining Eqs. (3.6) and (3.8), we obtain the expression for output direction cosine of the chief ray in terms of the input direction cosines and the pupil magnification as:

|  |  |
| --- | --- |
|  |  |

which we can write compactly as:

|  |  |
| --- | --- |
|  |  |

Our objective is to derive the expression for the transfer of the chief ray’s direction cosines from entrance pupil to exit pupil for arbitrary orientation of the optical axis as shown in Figure 3.5. Although a formal derivation is provided in [Appendix], we can readily infer the general *transfer* expression from Eq. (3.10) as follows:

Suppose we swivel the optical axis about the origin of the camera frame. This rotation can be described by the matrix. As before, we designate the ray from the object-point to the (new position of the) center of the entrance pupil as the chief ray. Let us also suppose that we have another coordinate frame,, sharing its origin with and its z-axis coincident with the optical axis. If be the direction cosine of the chief ray from the object-point in the frame , then the direction cosine in the frame is and the third element of the direction cosine is , where is the third column of . Representing, the direction cosine of the chief ray emerging from the exit pupil is obtained by substituting for and for in Eq. (3.10):

|  |  |
| --- | --- |
|  |  |

The above expression represents the output direction cosine in the coordinate frame In order to transform the output direction cosine from the coordinate frame to the camera frame we need to multiply the direction cosine vector by to obtain:

|  |  |
| --- | --- |
|  |  |

|  |
| --- |
|  |
| **Figure 3.5** Configuration of the general problem—optical axis (OA) pivots freely about the origin of. |

The positive or negative sign of the direction cosine determines the forward or backward direction of light-travel along a rectilinear path. Under the assumptions of isotropy and homogeneity, the only condition under which a ray of light emerges in an antipodal path from an interface is if it encounters a mirror surface *normally*. This condition does not arise within the context of our problem. Therefore, without any loss of generality, we can drop the negative sign in Eq. (3.11); accordingly, the output direction cosines assume the sign of the corresponding input direction cosines. Therefore, the general expression for the direction cosines of the chief ray in the image space is obtained as:

|  |  |
| --- | --- |
|  |  |

where.

Note that Eq. (3.12) only describes the output chief ray’s direction cosines—a free vector. The output chief ray is obtained from the knowledge of the direction cosine and the location of the exit pupil in the appropriate reference frame.

Although it is not obvious from the expression (3.12), we expect to have unit magnitude. We have provided a proof in [Appendix] that shows the (magnitude) of is indeed equal to one, and is the normalizing term.

We can draw the following inferences about from the Eq. (3.12):

1. If the pupil magnification, , then , which implies that the opening angles of the image- and object-space perspective cones are equal, irrespective of the orientation of the optical axis. Then, the lens is symmetric about a plane perpendicular to the optical axis (in addition to the symmetry about the optical axis). Such symmetric lenses are can be reversed without affecting system properties [ref].
2. If we let, such that, then we can write , where is the scalar normalization term. Furthermore, as is a diagonal matrix, and is orthonormal, can immediately recognize the form as the Eigen value decomposition of the symmetric matrix, with —the columns of —as the eigenvectors and the corresponding eigenvalues. As is a diagonal matrix,

|  |  |
| --- | --- |
|  |  |

and

|  |  |
| --- | --- |
|  |  |

In Eq. (3.13) the terms are the projections of the input direction cosine along the eigenvectors. Also,, which is the third column of the rotation matrix , is the direction of the optical axis. The effect of the transformation is a *shearing* of the input direction cosine along the optical axis.

### 3.5 Image formation for arbitrary orientation of the lens and image plane

Geometric imaging is a mapping (*bijective* in projective space) between points in the three-dimensional world space to corresponding points on a mathematical surface that we call the *image*. Here we aim to study the nature of this mapping on a planar surface—the image plane—for arbitrary orientations of either the lens and image planes. To that effect, we will use the knowledge of the transfer of direction cosines of the chief ray derived previously.

An extended object emanates a multitude of chief-rays that reach the image space through the pupils and the stop. The locus of points formed by the intersection of these rays with the image plane constitutes the *projection* of the object in the image plane [ref]. Further, we identify the projection of the world-point as an “image” if the pencil of rays (including the chief-ray) from the world-point, filling the pupils and stop, geometrically converge at a single point in the image space.

For simplicity, we assume that the lens is unencumbered by radial distortions and optical aberrations. Figure 3.6 represents a schematic of the problem in which we have introduced an image plane whose orientation is described by the unit surface normal. Two local frames are also introduced: the frame is attached to the optical axis with its origin at entrance pupil (), and the frame attached to the image plane with its origin at the image plane pivot. The image plane is free to swivel (tilt or swing about its local x-axis or y-axis respectively) about the image plane pivot.

|  |
| --- |
|  |
| **Figure 3.6** Schematic of geometric image formation. is the *central* *projection* of the world point on image plane. The optical axis and image plane are free to swivel about the origins of coordinate frames and respectively. |

Let the exit pupil () be located units from the pivot point along the optical axis. Following the rotation of the optical axis, by applying the matrix, the position of the exit pupil in is given as.

We can represent the chief ray emerging from the exit pupil with direction cosine by the parametric equation:

|  |  |
| --- | --- |
|  |  |

where represents any point along the output chief ray in. The first term on the R.H.S. of (3.14) is the initial position of the ray (at the center of) and is a real number that determines the length of the ray.

The equation of the image plane in Hessian normal form is

|  |  |
| --- | --- |
|  |  |

where is the unit normal to the image plane, is the perpendicular distance from the origin of frame to the plane, and is an arbitrary point on the image plane.

We obtain the expression for for which the ray intersects the image plane by equating to, multiplying Eq. (3.14) by, and rearranging the terms:

|  |  |
| --- | --- |
|  |  |
|  |  |

Substituting (3.16) into (3.14)

|  |  |
| --- | --- |
|  |  |

As previously stated, the origin of, the image plane’s local reference frame, is located at the intersection of the z-axis of with the image plane. The orientation of the image plane can be described by applying a rotation matrix to the image plane about the point in, with its unit plane-normal nominally equal to as shown in Figure 3.6. The normal is

|  |  |
| --- | --- |
|  |  |

Referring to Figure 3.7, the expression for is obtained as follows:

The equation of the image plane is

|  |  |
| --- | --- |
|  |  |

Since is a point on the plane,

|  |  |
| --- | --- |
|  |  |

|  |
| --- |
|  |
| **Figure 3.7** Schematic of the image plane. The image plane having surface normal is located at units from the origin of camera frame along the z-axis that intersects the plane at . is the perpendicular distance from the origin to the plane. The local image coordinate frame with its origin at the intersection of the image plane and z-axis of the camera frame is represented by . |

Using the above result, the expression for the point of intersection of the chief ray with the image plane in terms of the input direction cosines is

|  |  |
| --- | --- |
|  |  |

Similar to the exit pupil, let the entrance pupil be located at a distance from the pivot point along the optical axis in the camera frame. Then, the location of the entrance pupil in is. The direction cosines and the world point are related as

|  |  |
| --- | --- |
|  |  |

which can be written compactly as:

|  |  |
| --- | --- |
|  |  |

Substituting Eq. (3.22) into Eq. (3.20) a general relation between the world point and its corresponding image point is obtained:

|  |  |
| --- | --- |
|  |  |

Eq. (3.23) represents the image point in the camera frame. Once an image is formed, we specify positions and dimensions within the image independent of the position and orientation of the sensor and lenses. We can transform the image coordinates in the camera frame to the image frame by observing that the origin of is displaced from by , and the standard basis vectors of are rotated by . Therefore a point in relative to may be expressed as:

|  |  |
| --- | --- |
|  |  |

Let and be the homogeneous representation [ref] of the and respectively. The Eq. (3.24) can be expressed as a linear matrix operation as:

|  |  |
| --- | --- |
|  |  |

where is the pose of with respect to frame . Then, it follows that

|  |  |
| --- | --- |
|  |  |

where.

Finally, we obtain the image coordinates in the image frame as:

|  |  |
| --- | --- |
|  |  |

### 3.6 Object, lens and image plane relationships in Scheimpflug imaging

In this section, we will derive a general expression that relates the object, lens and image planes. We will further show that the general expression yields more specific relationships between the object, lens and image planes for specific cases. In order to keep the problem tractable, we will impose the constraint that three pivots (for the three planes) lie along the z-axis of the camera frame. The origin of the frame is co-located with optical axis’ pivot. We begin by deriving an expression for an arbitrary chief ray joining a point in the object plane to a point in the image plane. In order for to be the geometric image of , the chief rays conjoining the conjugate points must satisfy the Gaussian imaging equation. This constraint allows us to uniquely determine the position (of the image plane along the z-axis of) and orientation of the three planes in Scheimpflug configuration. The setup is shown in Figure 3.8.

|  |
| --- |
|  |
| **Figure 3.8** Schematic of Scheimpflug imaging. The figure shows the object plane, optical axis and image plane pivoted about points along the z-axis of the camera frame. The local object plane and image plane coordinates ( and) are centered at the object- and image- plane pivots. |

The object plane is located at a distance of from the origin of camera frame, along the z-axis. Pivoted about the point in the camera frame, it is completely described by the object plane pivot and the plane normal,. We describe the normal itself as the product of the rotation matrix and ().

That is, the equation of the object plane normal is

|  |  |
| --- | --- |
|  |  |

We can obtain the perpendicular distance to the object plane from the origin as

|  |  |
| --- | --- |
|  |  |

Then, the equation of the object plane in Hessian Normal form is

|  |  |
| --- | --- |
|  |  |

where is any point on the object plane.

Similar to the object plane normal, we describe the image plane normal as

|  |  |
| --- | --- |
|  |  |

where is the rotation matrix applied to the image plane at the image plane pivot point .

Repeating the steps used to derive the object plane equation, the equation for the image plane is

|  |  |
| --- | --- |
|  |  |

where is any point on the image plane.

Suppose the entrance () and exit () pupils are located at distances and from the lens plane pivot point (origin of) respectively, along the optical axis. Also, let us describe the rotation of the optical axis by applying the matrix. Then, following the rotation, the positions of the pupils in are

|  |  |
| --- | --- |
|  |  |

|  |  |
| --- | --- |
|  |  |

Consider the chief ray from the object-point , to the corresponding image point , passing through the center of the entrance and exit pupils. Let the direction cosines of ray in the object- and image- space be and respectively. We can describe a line along the ray originating at the entrance pupil and extending backwards towards the object plane in parametric form as

|  |  |
| --- | --- |
|  |  |

where is any point on the line (chief ray), is the position of the entrance pupil, and the parameter determines the length of the ray.

Let the length of the ray between the object-point and the entrance pupil be. Then, in Eq. (3.35), when, which implies that

|  |  |
| --- | --- |
|  |  |

Since is a point on the object plane, it satisfies the object plane Eq. (3.30). Substituting into Eq. (3.30) we get

|  |  |
| --- | --- |
|  |  |

The chief ray emerges from the exit pupil, having a direction cosine vector, and intersecting the image plane at to the right (positive z direction) of the exit pupil. Therefore, the parametric equation of this chief ray is

|  |  |
| --- | --- |
|  |  |

where is any point on the ray, is the position of the exit pupil, and is the length of the ray.

If we let the length of the chief ray in the image space be, then when in Eq. (3.38), implying

|  |  |
| --- | --- |
|  |  |

Substituting in Eq. (3.32) we get

|  |  |
| --- | --- |
|  |  |

Eqs. (3.37) and (3.40) are expressions for the lengths of the chief ray between points and in the object and image space respectively. In order for to be the geometric image of , the expressions must satisfy the Gaussian imaging equation.

The well-known Gaussian imaging equation () relates the focal length and the conjugate plane distances measured from the principal planes. Instead, if the distances are measured from the pupils (entrance pupil to object plane and exit pupil to image plane) in lieu of the principal planes, then a variant of the Gaussian imaging equation is used. The updated imaging equation, which incorporates the pupil magnification into the formula, is

|  |  |
| --- | --- |
|  |  |

where and are distances along the optical axis measured from the entrance pupil to the object plane and from the exit pupil to the image plane, is the pupil magnification, and is the focal length. We have provided a derivation and a brief exposition of the above formula in [Appendix].

The most common application of Eq. (3.41) is for fronto-parallel imaging in which the conjugate planes are parallel to each other and perpendicular to the optical axis; moreover, all pairs of object-image conjugate points satisfy this relation even if the ensemble of object points and image points belong to planes on object and image spaces respectively that are *not parallel* to each other.

The ray vector of length and direction in the object space is. The projection of this ray vector on the optical axis () is. Similarly, the ray projection of the image space ray vector on the optical axis is. Substituting and in to Eq. (3.41) we have:

|  |  |
| --- | --- |
|  |  |

Further, substituting the expressions for and (Eqs. (3.37) and (3.40)) into the above equation, we have

|  |  |
| --- | --- |
|  |  |

The direction cosine of the chief-ray in the image space,, is related to the direction cosine of the chief-ray in the object space as (Eq. (3.12))

|  |  |
| --- | --- |
|  |  |

Substituting, into Eq. (3.43) we have

|  |  |
| --- | --- |
|  |  |

To simplify the above expression, let us consider as

|  |  |
| --- | --- |
|  |  |

Therefore, we can write in Eq. (3.44) as. Similarly, we can also write as. Then, Eq. (3.44) can be written as

|  |  |
| --- | --- |
|  |  |

We can further simplify the above equation by noting that:

1. ,
2. ,
3. ,
4. .

Therefore, Eq. (3.46) reduces to

|  |  |
| --- | --- |
|  |  |

Further, following a trivial algebraic manipulation we get

|  |  |
| --- | --- |
|  |  |

Note that in the above expressions, the third column of the rotation matrix, is the unit vector along the optical axis.

Since is a direction cosine vector with -Norm equal to one, the above equation is satisfied if the second vector,, is either perpendicular to or identically equal to zero. Let us consider the possibility of the former case. *All* chief rays—an infinitude of vectors within the object and image space perspective cones—must satisfy Eq. (3.48). Are all possible vectors perpendicular to the second vector? Since the second vector is a linear combination of (the object plane normal), (the transformed image plane normal), and (unit vector along the optical axis) whose weights are constant for a given system, we can conclude that is not perpendicular to the second vector *in general*. Therefore,

|  |  |
| --- | --- |
|  |  |

Further, if we let and, then we can write Eq. (3.49) as

|  |  |
| --- | --- |
|  |  |

Eq. (3.50) is most general in the sense that it readily yields the various expressions for specific cases of object, lens and image plane orientations as we now demonstrate in the following set of examples.

**Example 3.1:** If the object plane is perpendicular to the optical axis and units from the origin of camera frame, what is the orientation and position of the image plane?

Assuming, the distance of the entrance pupil from the origin of, is non-zero, the object plane is located at a distance of () from the entrance pupil, where is numerically negative.

1. Since the lens is not tilted, ,
2. Since the object plane is also not tilted,
3. Let, and. Then,

Substituting the above parameters in Eq. (3.50) we have

|  |  |
| --- | --- |
|  |  |

which yields the following relations from each row of Eq. (3.51):

|  |  |
| --- | --- |
|  |  |

|  |  |
| --- | --- |
|  |  |

|  |  |
| --- | --- |
|  |  |

From Eqs. (3.52) and (3.53) we see that the image plane normal is equal to. That is, as expected, the image plane is parallel to the lens and object plane and perpendicular to the optical axis if the object plane is parallel to the lens plane. Eq. (3.54) is the image plane distance from the origin of camera frame along the z-axis (which, in this case, is also along the optical axis). Note that the distance from the exit pupil to the image plane is ().

For thin lens model (, ), Eq. (3.54) reduces to the Gaussian imaging equation for thin lenses:

|  |  |
| --- | --- |
|  |  |

For think lens model, in which the distances to the object and image plane are specified from the object and image space principal points (), equation (3.54) reduces to

|  |  |
| --- | --- |
|  |  |

in which, is the distance of the object plane from the object-space principal point, and and represent the locations of the object- and image-space principal points in the camera frame . While is numerically negative, the signs of and depend on the position of the principal points with respect to the origin of.

**Example 3.2:** If the object plane is tilted about a pivot that is located at in the camera frame in the direction from +z to +y by an angle (see figure) while the lens plane remains static, what is the orientation and position of the image plane?

1. Since the lens is not tilted, ,
2. Let, and. Then,
3. Let, and. Then,

Substituting the above parameters in Eq. (3.50) we have

|  |  |
| --- | --- |
|  |  |

As in the Example 3.1, the distance of the image plane pivot in the camera frame, obtained from the third of Eq. (3.57), is

|  |  |
| --- | --- |
|  |  |

From the first and second rows of Eq. (3.57), we have

|  |  |
| --- | --- |
|  |  |

and

|  |  |
| --- | --- |
|  |  |

Since the object plane is rotated by an angle (from +z-axis towards +y-axis) the object plane pivot, we can represent the rotation matrix as

|  |  |
| --- | --- |
|  |  |

Then, the object plane normal is obtained as (Eq. (3.28))

|  |  |
| --- | --- |
|  |  |

from which, we can get by dividing by

|  |  |
| --- | --- |
|  |  |

Substituting into Eqs. (3.59) and (3.60) we get

|  |  |
| --- | --- |
|  |  |

**Example 3.3:** If the object plane is swiveled about a pivot that is located at in the camera frame while the lens plane remains static, what is the orientation and position of the image plane?

# Appendix⬀

### Derivation 3.1

Later, we will apply the method of induction to yield the solution of the general *transfer* problem—in which the optical axis is free to swivel about the origin of.

Eq. (3.9) accurately represents the *transfer* for the specific problem; however, we will cast the expression in a slightly different form whose raison d'être is to enable generalization—through direct application of the result. Specifically, we can express the output chief ray as a linear combination of the input chief ray and the optical axis since the two rays and the optical axis span the same (meridional) plane. Let, the standard basis vector along z-axis of, represent the optical axis since the optical axis is coincident with the z-axis. Then,

|  |  |
| --- | --- |
|  |  |

where and are the weights, and.

Rewriting the above equation as

|  |  |
| --- | --- |
|  |  |

the weight is readily obtained by comparing Eqs. (3.9) and (4.2):

|  |  |
| --- | --- |
|  |  |

Substituting the expression for into and comparing with (3.8) yields:

|  |  |
| --- | --- |
|  |  |

We are now ready to apply the result of the specific problem to the general problem. Figure 3.4 shows the schematic of the general problem—the optical axis pivots about the origin of. Let us describe the general orientation of the optical axis by the action of the rotation matrix on. The matrix may be a composition of two or more matrices that denotes a sequence of rotations about the x-axis and/or y-axis. Then,, the unit vector representing the new orientation of the optical axis, is obtained as: or .

As the output direction cosine, the input direction cosine, and the optical axis lie on the same plane,

|  |  |
| --- | --- |
|  |  |

Note that the input direction cosine in Eq. (4.5) is different from the corresponding in Eq. (4.1) even for the same object-point . This difference is due to the displacement of entrance pupil () following the rotation of the optical axis; in fact, the designation of a ray as the chief ray (from to) keeps altering as we keep displacing the entrance pupil. Multiplying Eq. (4.5) by:

|  |  |
| --- | --- |
|  |  |

Letting and,

|  |  |
| --- | --- |
|  |  |

Comparing Eqs. (4.1) and (4.7) the expressions for the weights and are obtained as:

|  |  |
| --- | --- |
|  |  |

Where represents the projection of the direction cosine vector, **,** on the rotated optical axis. If we write the matrix where are the columns of . Then,

|  |  |
| --- | --- |
|  |  |

and

|  |  |
| --- | --- |
|  |  |

Therefore, since is the third element of.

Rewriting Eq. (4.7) as:

|  |  |
| --- | --- |
|  |  |

which can be compactly written as:

|  |  |
| --- | --- |
|  |  |

Finally, substituting and yields the general expression for the direction cosines of the chief ray in the image space in terms of the pupil magnification and direction cosines in the object space as:

|  |  |
| --- | --- |
|  |  |

where.

### Claim 3.1 The output direction cosine, originating from exit pupil, has unit -Norm

The direction cosine in the image space, obtained by the linear transformation of the direction cosinein the object space, has unit , and is the normalization term.

*Proof*.

The expression for the direction cosine in the image space is

|  |  |
| --- | --- |
|  |  |

where , is the column of the rotation matrix applied to the optical axis, , and is the pupil magnification.

Our objective is to prove.

For the convenience of notation within the proof, let

|  |  |
| --- | --- |
|  |  |

where, and , the diagonal matrix with non-negative real values.

Also, let us represent the columns of as

Then, , and

|  |  |
| --- | --- |
|  |  |

Now, since is a rotation matrix, it is orthonormal (the column of, having unit length, are orthogonal to each other). Therefore, .

Then,

|  |  |
| --- | --- |
|  |  |

where

|  |  |
| --- | --- |
|  |  |

where and .

As is a diagonal matrix, we can rewrite as

|  |  |
| --- | --- |
|  |  |

Now,

|  |  |
| --- | --- |
|  |  |

Also

|  |  |
| --- | --- |
|  |  |

Substituting in Eq. (4.15),

|  |  |
| --- | --- |
|  |  |

Substituting into Eq. (4.14) we have

|  |  |
| --- | --- |
|  |  |

It follows that the scalar quantity is the normalization term. ☐

### Derivation 3.2 Gaussian lens formula incorporating pupil magnification

The familiar Gaussian lens formula, , shows the relationship between the conjugate planes in which is the distance between the object plane (perpendicular to the optical axis) and the principal plane () in the object space, is the distance between the in-focus image plane and the principal plane () in the image space, and is the focal length of the lens. The distances being measured along the optical axis. If the distances of the object and image planes are specified from the entrance () and exit pupil () instead of the principal planes, then the Gaussian lens formula needs to be slightly modified to incorporate the pupil magnification (). Here we derive the modified formula starting from the Gaussian lens formula. The same result is derived in [Ref] using a slightly different approach.

Figure x.x shows a schematic of the entrance and exit pupils, the object and image space principal planes, and the object and image points.

|  |
| --- |
|  |
| **Figure x.x** Schematic of imaging through a lens. The figure shows the object () and its image (), the object space principal plane () and the image side principal plane (), the entrance () and exit () pupils, and the associated distances along the optical axis. |

In the figure, and are the distances from the principal planes to the object and image planes, and are distances from the principal planes to the entrance and exit pupils and and are the distances from the entrance and exit pupils to the object and image planes. Since the entrance and exit pupil planes are conjugates, like the object and image planes, the Gaussian lens formula holds as follows:

|  |  |
| --- | --- |
|  |  |

and

|  |  |
| --- | --- |
|  |  |

The transverse magnification, , between the object and image planes is:

|  |  |
| --- | --- |
|  |  |

The pupil magnification, , defined as the ratio of the exit pupil diameter to the entrance pupil diameter, is also the ratio between the exit pupil and entrance pupil distances (measured from the principal planes) just like the transverse magnification between any conjugate planes:

|  |  |
| --- | --- |
|  |  |

Equating Eqs. (4.20) and (4.21), we obtain

|  |  |
| --- | --- |
|  |  |

Substituting and in the above equation, and using Eqs. (4.22) and (4.23):

|  |  |
| --- | --- |
|  |  |

Further, we can also substitute and in Eq. (4.20) and equate with Eq. (4.21):

|  |  |
| --- | --- |
|  |  |

which after cross-multiplication and cancellations of common terms produces

|  |  |
| --- | --- |
|  |  |

Dividing throughout by, and substituting by the pupil magnification, and by we obtain:

|  |  |
| --- | --- |
|  |  |

Eq. (4.25) is valid even if the and denote distances from the principal planes provided we let. This outcome is indeed consistent with geometric optics theory, according to which the magnification between the principal planes is unity. In fact, Eq. (3.41) is more general than the Gaussian Lens formula in that it relates a pair of conjugate planes with any other pair of conjugate planes for which the transverse magnification (between the planes) is known. When one of the pairs happen to be the principal planes ( and), between which the magnification is one, we obtain the Gaussian Lens formula.

TO DO

1. Table showing different known equations that fall out of it. For example, a) thin lens, fronto-parallel imaging, b) thin-lens image plane tilt model, c) thin-lens image and lens tilt model, d) thick lens model, etc.
2. Zemax based verification
3. Analysis of the equation, especially with regards to collinear transformation
4. Tilt about entrance pupil vs. tilt about the principal point.

1. In the presence of spherical aberrations, the chief ray goes through the center of the aperture but may not exactly go through the center of the pupils [Ref Mirrors, Prisms … Southall, Lens design by Kingslake]. [↑](#footnote-ref-1)