# Chapter 3

⬀⬀⬀**GEOMETRIC MODEL OF SCHEIMPFLUG IMAGING**

Scheimpflug cameras provide the greatest flexibility for image composition; however, that flexibility is traded for complexity. Truthful modeling of Scheimpflug imaging is quite involved, and its art of operation is often left to experts who frequently employ approximate methods. These cameras find use in few scientific imaging applications, but the vast majority of them contemporarily are used for landscape and studio photography.

Existing models of Scheimpflug camera employ simple....by imposing/restricting…. While these models work quite well for documentary photography, they are often restrictive and inaccurate for scientific purpose. A rich description of such cameras requires the development of a more general model. We aim to develop a model for Scheimpflug imaging using the axioms of *geometric optics* (*ray optics*). Like any model, ray optics in itself does not provide a complete picture of imaging; yet the definitions and postulates therein provide useful tools for the analysis and synthesis of optical systems.

Assumptions are crucial and necessary ingredients of modeling that enable its expediency and impose limits on its applicability. We have assumed paraxial imaging, isotropy (directional uniformity) and homogeneity (positional uniformity) of the lens elements and the interstitial media, rotational symmetry, and aberration free optics. Additionally, the lens is surrounded by air of refractive index one. Consequently, the front and back focal lengths of the lens in our model are equivalent and the two nodal points coincide with the corresponding principal points.

TODO: Describe the Scheimpflug camera in two sentences.

TODO: Review what is out there. Type of models that are there, their limitations. Also, comment on the existing process of “focus-transfer” why that is erroneous. Point out, without explicitly stating, that this method has several advantages (and explicitly point out the advantages), the new insights that it provides and not a re-engineering of existing knowledge just for the sake of being different.

TO DO: State the novelty of this approach, and why needed to develop this model. Is there any relation to eikonal equations?

TODO: Preview of what is coming in the following sections.

### 3.1 Introduction

*Pupils* are the sine qua non for describing imaging systems in both domains of *ray* and *wave optics*. Most imaging systems consist of several groups of elements; those with optical power bends rays of light. The tiniest orifice in the system is the *system aperture* or *stop*. Its size and position affect image resolution, brightness, and geometry.

The stop viewed through the elements preceding it is the *entrance pupil* (). The exit pupil () is the stop viewed through the elements following it. That is, the pupils are the images of the stop produced by the elements on either side of the stop. We call the region anterior to the entrance pupil, which includes the object and light sources, as the object space, and the region posterior to the exit pupil, which includes the image plane, as the image space.

Rotationally symmetric lenses have an axis of symmetry (the optical axis). Planes passing through the axis of such lenses are the meridional planes. The patterns formed by rays confined to the meridional planes—the *meridional rays*—on either side of the optical axis are mirror-reversed, exhibiting bilateral symmetry. Figure 3.1 shows two types of meridional rays, traced in Zemax, that are fundamental to geometric analysis. The *marginal ray* (MR) originates from the axial object position and skirts the edges of the aperture and pupils (virtually); the *chief ray* (CR) starts at an off-axis object point and pierces the centers of the aperture and pupils[[1]](#footnote-1) (virtually). This pair of rays determines the location and size of the pupils, the position of the image, and the magnification. Furthermore, the bundle of chief rays from the object space converge at the center of the entrance pupil ()—thus *homocentric*—forming the vertex of the object-space perspective cone; in the image space, the bundle of chief rays diverge from the center of the exit pupil () producing the vertex of the image-space perspective cone.

Imagine a film projector working backwards. Imagine the stream of light rays flowing from the illuminated portion of the scene towards a small circular hole in the projector. This *pencil of rays* creates a conical volume of light—the perspective cone—with its vertex at the hole and its base towards the scene. The “illuminated portion” is the angular extent of the scene that is visible in the image, confined by the circumferential chief rays. These extreme chief rays determine the opening angle of the cone. The “small hole” represents the entrance pupil of a camera or the pupil at the center of the iris in an eye. In the image space (behind the hole), the rays form another cone with the vertex at the center of the exit pupil. This image-space perspective cone projects the light from the scene onto the film surface or the retina in the eye. This process of image formation, known as the *central projection*, is fundamental to all imaging systems including the camera and the eye. While the opening angle of the object-space perspective cone determines the field-of-view, its counterpart in the image space determines the angular dimension of the image. The ratio of the pupil sizes—the *pupil magnification*—determines the relationship between the image and object-space opening angles of the two perspective cones.

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| **Figure 3.1** Fundamental rays and pupils in a typical Double Gauss Lens. The chief rays—close to the optical axis (0°, ±5° in the object space measured at)—appear to converge at the entrance pupil center and diverge from the center of the exit pupil. The marginal ray appear to skirt the edges of the two pupils. The red circles specify the vertices of the perspective cones. The rays were traced in Zemax. |

### 3.2 Notation (this section is currently a placeholder for the notation, skip reading this for now. I will write it appropriately after completion of the major math)

* Right-handed coordinate system with the +z along the direction of travel of light
* In general, non-primed quantities are used to indicate input or object space (e.g.) and primed quantities are used to indicate output or image space (e.g.).
* Unit vectors are represented using a hat (), with the exception of the direction cosine vectors (e.g. although the norm of the direction cosine vectors are unity.
* A left superscript indicates the frame of reference. For example, indicates that the variable is w.r.t. the world coordinate frame. If no reference is explicitly stated it implies that the variable is w.r.t. the world coordinate frame (or the camera coordinate frame if the camera coordinate frame and the world coordinate frame are the same.
* A subscript is used to associate a variable with a particular xxx like entrance pupil position (), image plane (), for example is used to represent the 3D rotation matrix applied to the entrance pupil plane in the camera frame. The same notation is also used to indicate a transformed variable, for example is used to represent under the rotational transformation by in the camera coordinate frame. As also mentioned earlier, if the camera coordinate frame is the same as the world coordinate frame, then the notation shall be used.
* represents the pose of frame w.r.t. frame TO DO: mention how a point in one frame is represented in another frame.
* The one-based indexing of matrices and vectors. Also a matrix is also represented as where are the columns of .
* Describe what I mean by lens plane.
* Overloading of the term “direction cosine(s)” and “direction cosine vector”. It should be clear from the context

### 3.3 Relation between pupil magnification and chief ray angle

The *pupil magnification*, is defined as the ratio of the paraxial exit pupil () size to the entrance pupil () size [refref]. Figure 3.2 illustrates the meridional and sagittal planes associated with an arbitrarily located object of height units above the optical axis (OA) and its image of height in a typical optical system. The figure also shows the chief ray (CR) from the object’s edge further from the OA, the marginal ray (MR) from the axial point in the object, and the two pupils contained in the meridional plane. The schematic, although simple, is quite general as a (meridional) plane always exist for a given object point irrespective of its position in the three-dimensional space, if the lens is rotationally symmetric.

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| **Figure 3.2** Schematic of chief and marginal rays. The ratio of the tangents of the chief ray angles in the object space to the image space yields the pupil magnification. |

Let the angles produced by the CR with the OA (called the *ray angle*) in the object- and image-space be and respectively. Also, let the angles produced by the MR with the OA in the object- and image-space be and respectively. Then, we can obtain the relation between the CR ray angles and the pupil magnification as follows:

From the Figure 3.2,

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Eliminating and after dividing by , we have

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As per the *Lagrange invariant* [ref] property of the two rays,. Therefore

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The equation (3.2), has been previously derived in [ref] using a different formulation. For a given optics the pupil magnification is constant. This constancy of the ratio of the tangents of the CR angles for varying object (and image) heights is a necessary and sufficient condition for distortion-free imaging known as the *Airy’s Tangent-Condition* [ref]. Equation (3.2) also suggests that when the perspective cones in the object- and image-space are symmetric. In the following section, we will use equation (3.2) to derive the relationship between the direction cosines of the object-space (input) chief rays and direction cosines of the image-space (output) chief rays.

### 3.4 Transfer of chief ray’s direction cosines between the pupils

The direction cosines, a 3-tuple of unit norm, specify the direction of a ray. Its elements are the cosines of the angles the ray makes with the three coordinate axes. Given the direction cosine of a chief ray in the object space what is the direction cosine of the corresponding chief ray in the image space? What is the relation between the input and output chief ray’s direction cosines for an imaging system in which the lens is tilted (rotated about the x-axis) or swung (rotated about the y-axis) about a pivot point?

We begin by solving a specific problem of the *transfer* of the direction cosines between the pupils in which the optical axis (OA) coincides with the z-axis of the camera frame, as show in Figure 3.3. Later, we will apply the method of induction to yield the solution of the general *transfer* problem—in which the OA is free to swivel about the origin of. Let be the direction cosine of the chief ray (CR) from a world point to the center of the entrance pupil (), and let be the corresponding direction cosine of the CR from the exit pupil (). The parameters , ,and are specified with respect to frame .

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| **Figure 3.3** Specific problem—optical axis coincides with reference frame’s z-axis. If and are the angles of the CR with the OA in the object- and image-space respectively, then and. |

If and are the zenith and azimuthal angles of the CR in the object space, and and the corresponding angles in the image space, then the direction cosines, in the camera frame, are represented as:

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Since the OA is aligned with the z-axis, and. Substituting the expressions for from equation (3.3) into equation (3.2) we get:

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As the input and out chief rays are confined to the same meridional plane [ref], , yielding and in terms of and , the ratios of to , and :

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From (3.2) we have

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which after simplification yields in terms of the pupil magnification and input

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Combining equations (3.5) and (3.7), we obtain the expression for output direction cosine of the chief ray in terms of the input direction cosines and the pupil magnification as:

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Our objective is to derive the expressions for the transfer of direction cosines of the CR from to for arbitrary orientation of the OA. Equation (3.8) accurately represents the *transfer* for the specific problem; however, we will cast the expression in a slightly different form whose raison d'être is to enable generalization—through direct application of the result. Specifically, we can express the output CR as a linear combination of the input CR and the OA since the two rays and the OA span the same (meridional) plane. Let , the standard basis vector along z-axis of , represent the OA since the OA is coincident with the z-axis. Then,

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where and are the weights, and.

Rewriting the above equation as

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the weight is readily obtained by comparing equations (3.8) and (3.10):

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Substituting the expression for into and comparing with (3.7) yields:

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We are now ready to apply the result of the specific problem to the general problem. Figure 3.4 shows the schematic of the general problem—the OA pivots about the origin of . Let us describe the general orientation of the OA by the action of the rotation matrix on. The matrix may be a composition of two or more matrices that denotes a sequence of rotations about the x-axis and/or y-axis. Then, , the unit vector representing the new orientation of the OA is obtained as: or .

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| **Figure 3.4** Configuration of the general problem—optical axis (OA) pivots freely about the origin of. |

As the output direction cosine , the input direction cosine , and the optical axis lie on the same plane,

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Note that the input direction cosine in equation (3.13) is different from the corresponding in equation (3.9) even for the same object-point . This difference is due to the displacement of following the rotation of the OA; in fact, the designation of a ray as the CR (from to ) keeps altering as we keep displacing . Multiplying equation (3.13) by:

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Letting and,

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Comparing equations (3.9) and (3.15) the expressions for the weights and are obtained as:

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If we write the matrix where are the columns of . Then,

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Therefore,

Rewriting equation (3.15) as:

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which can be compactly written as:

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The positive or negative sign of the direction cosine determines the forward or backward direction of light-travel along a rectilinear path. Under the assumptions of isotropy and homogeneity, the only condition under which a ray of light emerges in an antipodal path from an interface is if it encounters a mirror surface *normally*. This condition does not arise within the context of our problem. Therefore, without any loss of generality, we can drop the negative sign in equation (3.17); accordingly, the output direction cosines assume the sign of the corresponding input direction cosines. Furthermore, using and yields the general expression for the direction cosines of the CR in the image space in terms of the pupil magnification and direction cosines in the object space as:

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where.

Note that equation (3.18) only describes the output CR’s direction cosines—a free vector. The ouput CR is obtained from the knowledge of the direction cosine and the location of the exit pupil in the appropriate reference frame.

Although it is not obvious from the expression (3.18), we expect to have unit magnitude. We have provided a proof in [Appendix] that shows the (magnitude) of equal one, and is the normalizing term.

We can draw the following inferences about from the equation (3.18):

1. If the pupil magnification, , then , which implies that the opening angles of the image- and object-space perspective cones are equal, irrespective of the orientation of the optical axis. Therefore, we can find a plane perpendicular to the optical axis about which the lens is symmetric. Such lenses are can be reversed without affecting system properties, and are called symmetric lenses [ref].
2. If we let , such that , then we can write , where is the scalar normalization term. Furthermore, as is a diagonal matrix, and is orthonormal, can immediately recognize the form as the Eigen value decomposition of the symmetric matrix, with —the columns of —as the eigenvectors and the corresponding eigenvalues. As is a diagonal matrix,

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In equation (3.20) the terms are the projections of the input direction cosine along the eigenvectors . Also, , which is the 3rd column of the rotation matrix , is the direction of the optical axis. The effect of the transformation is a *shearing* of the input direction cosine along the optical axis.

### 3.5 Image formation for arbitrary orientation of the lens and image plane

Geometric imaging is a mapping (bijective in projective space) between points in the world to points on a mathematical surface that we call the *image*. Here we aim to study the nature of this mapping on a planar surface—the image plane—for arbitrary orientations of either the lens and image planes. To that effect, we will use the knowledge of the transfer of direction cosines of the chief ray (CR) derived previously.

An extended object emanates a multitude of chief rays that reach the image space through the pupils and the stop. The locus of points formed by the intersection of these rays with the image plane constitutes the *projection* of the object in the image plane [ref]. Furthermore, we identify the projection of the world-point as an “image” when the pencil of rays from the world-point, filling the pupils and stop geometrically, converge at a single point in the image space.

We assume that the lens is unencumbered by radial distortions and optical aberrations. Figure 3.5 represents a schematic of the problem, in which we have introduced an image plane whose orientation is described by the unit surface normal. Two local frames are introduced: the frame is fixed to the optical axis with its origin at, and the frame fixed to the image plane with its origin at the intersection of the image plane with the z-axis of the camera frame. The origin of also represents the pivot point about which the image plane is free to swivel (tilt or swing about its local x-axis or y-axis respectively).

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| **Figure 3.5** Schematic of geometric image formation. is the *central* *projection* of the world point on image plane. The optical axis and image plane are free to swivel about the origins of coordinate frames and respectively. |

Let the exit pupil () be located units from the pivot point along the OA. Following the rotation of the optical axis, by applying the matrix, the position of in is given as .

The parametric equation of the CR emerging from with direction cosine is then represented as:

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where represents any point along the output CR in . The first term on the R.H.S. of (3.22) the the initial position of the ray (at the center of ) and is a real number that determines the length of the ray.

The equation of the image plane in Hessian normal form is given as:

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where is the unit normal to the image plane, is the perpendicular distance from the origin (of frame ) to the plane, and is an arbitary point on the plane.

We obtain the expression for for which the ray intersects the image plane by equating to, multiplying equation (3.22) by, and rearranging the terms:

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Substituting (3.24) into (3.22):

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As previously stated, the origin of , the image plane’s local reference frame, is located at the intersection of the z-axis of with the image plane. The orientation of the image plane can be described by applying a rotation matrix (or a composition of successive rotation matrices) to the image plane, with its unit plane-normal nominally equal to , about the origin of as shown in Figure 3.5. If is the rotation matrix then

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Referring to Figure 3.6, the expression for is obtained as follows:

The equation of the image plane is

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Since is a point on the plane,

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| **Figure 3.6** Schematic of the image plane. The image plane having surface normal is located at units from the origin of camera frame along the z-axis that intersects the plane at . is the perpendicular distance from the origin to the plane. The local image coordinate frame with its origin at the intersection of the image plane and z-axis of the camera frame is represented by . |

Using the above result, the expression for the point of intersection of the chief ray with the image plane in terms of the input direction cosines is

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Similar to the exit pupil (), let the entrance pupil () be located at a distance from the pivot point along the OA in the camera frame. Then, the location of the in is. The direction cosines and the world point are related as

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which can be written compactly as:

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Substituting equation (3.30) into equation (3.28) a general relation between the world point and its corresponding image point is obtained:

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Equation (3.31) represents the image point in the camera frame. Once an image is formed, we specify positions and dimensions within the image independent of the position and orientation of the sensor and lenses. We can transform the image coordinates in the camera frame to the image frame by observing that the origin of is displaced from by , and the standard basis vectors of are rotated by . Therefore a point in relative to may be expressed as:

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Let and be the homogeneous representation [ref] of the and respectively. The equation (3.32) can be expressed as a linear matrix operation as:

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where is the pose of with respect to frame . Then, it follows that

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where.

Finally, we obtain the image coordinates in the image frame as:

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TO DO:

1. Table showing different known equations that fall out of it. For example, a) thin lens, fronto-parallel imaging, b) thin-lens image plane tilt model, c) thin-lens image and lens tilt model, d) thick lens model, etc.
2. Zemax based verification
3. Analysis of the equation, especially with regards to collinear transformation
4. Tilt about entrance pupil vs. tilt about the principal point.

# Appendix⬀

**Claim** **3.1** The direction cosine in the image space, obtained by the linear transformation of the direction cosinein the object space, has unit , and is the normalization term.

*Proof*.

The expression for the direction cosine in the image space is

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where , is the column of the rotation matrix applied to the optical axis, , and is the pupil magnification.

Our objective is to prove .

For the convenience of notation within the proof, let

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where, and , the diagonal matrix with non-negative real values.

Also, let us represent the columns of as

Then, , and

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Now, since is a rotation matrix, it is orthonormal (the column of, having unit length, are orthogonal to each other). Therefore, .

Then,

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where

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where and .

As is a diagonal matrix, we can rewrite as

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Now,

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Substituting in equation (4.5),

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Substituting into equation (4.4) we have

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It follows that the scalar quantity is the normalization term. Q.E.D.

1. In the presence of spherical aberrations, the chief ray goes through the center of the aperture but may not exactly go through the center of the pupils [Ref Mirrors, Prisms … Southall, Lens design by Kingslake]. [↑](#footnote-ref-1)