# Chapter 3

⬀⬀⬀**GEOMETRIC MODEL OF SCHEIMPFLUG IMAGING**

We aim to develop a model for Scheimpflug imaging from the elements of *geometric optics* (*ray optics*). Like any model, the ray optics in itself does not provide a complete picture of imaging; yet the definitions and postulates therein provide useful tools for the analysis and synthesis of optical systems. Assumptions are both crucial and necessary ingredients of modeling that enable its expediency and impose limits on its applicability. We have assumed paraxial imaging, isotropy (uniformity along all directions) and homogeneity (uniformity along all positions) of the medium, rotational symmetry, and aberration free optics.

TODO: Preview of what is coming in the following sections.

TODO: Review what is out there. Type of models that are there, their limitations. Also, comment on the existing process of “focus-transfer” why that is erroneous. Point out, without explicitly stating, that this method has several advantages (and explicitly point out the advantages), the new insights that it provides and not a re-engineering of existing knowledge just for the sake of being different.

TO DO: State the novelty of this approach, and why needed to develop this model. Is there any relation to eikonal equations?

### 3.1 Introduction

*Pupils* are the sine qua non for describing imaging systems in both domains of *ray* and *wave optics*. Most imaging systems consist of several groups of elements; those with optical power bends rays of light. The tiniest orifice in the system is the *system aperture* or *stop*. Its size and position affect the image resolution, brightness, and geometry.

The stop viewed through the elements preceding it is called the *entrance pupil* (). The exit pupil () is the stop viewed through the elements following it. That is, the pupils are the images of the stop produced by the elements on either side of the stop.

Rotationally symmetric lenses have an axis of symmetry (the optical axis). Planes passing through the axis of such lenses are called meridional planes. The patterns formed by rays confined to the meridional planes—the *meridional rays*—on either side of the optical axis are mirror-reversed, exhibiting bilateral symmetry. Figure 3.1 shows two types of meridional rays, traced in Zemax, that are fundamental to geometric analysis. The *marginal ray* (MR) originates from the axial object position and skirts the edges of the aperture and pupils (virtually); the *chief ray* (CR) starts at an off-axis object point and pierces the centers of the aperture and pupils[[1]](#footnote-1) (virtually). This pair of rays determines the location and size of the pupils, the position of the image, and the magnification. Furthermore, the bundle of chief rays from the object side converge at the center of the entrance pupil ()—thus *homocentric*—forming the vertex of the object side perspective cone; on the image side, the bundle of chief rays diverge from the center of the exit pupil () producing the vertex of the image side perspective cone.

Imagine a film projector working in reverse. The infinite set of rays flowing from the illuminated portion of the scene towards a central hole in the projector creates a perspective cone on the object side. The “illuminated portion” is the angular extent of the scene, bounded by the extreme chief rays, visible in the image. The extreme chief rays determine the opening angle of the cone. The “central hole” is the entrance pupil of a camera or the pupil at the center of the iris in an eye. On the image side, the perspective cone with its vertex at the center of the exit pupil projects the scene onto the film surface or the retina in the eye. This process of image formation, known as the *central projection*, is fundamental to all imaging systems including the camera and the eye. While the opening angle of the object-side perspective cone determines the field-of-view, its counterpart on the image side determines the angular dimension of the image. The ratio of the pupil sizes—the *pupil magnification*—determines the relationship between the image and object side opening angles of the two perspective cones.

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| **Figure 3.1** Fundamental rays and pupils in a typical Double Gauss Lens. The chief rays—close to the optical axis (0°, ±5° on the object side measured at)—appear to converge at the entrance pupil center and diverge from the center of the exit pupil. The marginal ray appear to skirt the edges of the two pupils. The vertices of the perspective cones are shown as red circles. The rays were traced in Zemax. |

### 3.2 Notation (this section is currently a placeholder for the notation, skip reading this for now. I will write it appropriately after completion of the major math)

* Right-handed coordinate system with the +z along the direction of travel of light
* In general, non-primed quantities are used to indicate input or object space (e.g.) and primed quantities are used to indicate output or image space (e.g.).
* Unit vectors are represented using a hat (), with the exception of the direction cosine vectors (e.g. although the norm of the direction cosine vectors are unity.
* A left superscript indicates the frame of reference. For example, indicates that the variable is w.r.t. the world coordinate frame. If no reference is explicitly stated it implies that the variable is w.r.t. the world coordinate frame (or the camera coordinate frame if the camera coordinate frame and the world coordinate frame are the same.
* A subscript is used to associate a variable with a particular xxx like entrance pupil position (), image plane (), for example is used to represent the 3D rotation matrix applied to the entrance pupil plane in the camera frame. The same notation is also used to indicate a transformed variable, for example is used to represent under the rotational transformation by in the camera coordinate frame. As also mentioned earlier, if the camera coordinate frame is the same as the world coordinate frame, then the notation shall be used.
* represents the pose of frame w.r.t. frame TO DO: mention how a point in one frame is represented in another frame.
* The zero-based indexing of matrices and vectors

### 3.3 Relation between pupil magnification and chief ray angle

The *pupil magnification*, is defined as the ratio of the paraxial exit pupil () size to the entrance pupil () size [refref]. Figure 3.2 illustrates the chief ray (CR) from an object of height above the optical axis (OA), the marginal ray (MR) from the axial point on the object, the two pupils, and the image of height in a typical optical system. Note that for a rotationally symmetric lens, a (meridional) plane always exists that contains the two fundamental rays from the object and the optical axis.

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| **Figure 3.2** Schematic of chief and marginal rays. The ratio of the tangents of the chief ray angles in the object side to the image side yields the pupil magnification. |

Let the angles produced by the CR with the OA (called the *ray angle*) in the object and image side be and respectively. Also, let the angles produced by the MR with the OA in the object and image side be and respectively. Then, the relation between the CR ray angles and the pupil magnification is obtained as follows:

From the Figure 3.2,

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Eliminating and after dividing by , we have

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As per the *Lagrange invariant* [ref] property of the two rays,. Therefore

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The equation (3.2), has been previously derived in [ref] using a different formulation. For a given optics the pupil magnification is constant. This constancy of the ratio of the tangents of the CR angles for varying object (and image) heights is a necessary and sufficient condition for distortion-free imaging known as the *Airy’s Tangent-Condition* [ref]. Equation (3.2) also suggests that when the perspective cones on the object and image side are symmetric. In the following section, we will use equation (3.2) to derive the relationship between the direction cosines of the object-side (input) chief rays and direction cosines of the image-side (output) chief rays.

### 3.4 Transfer of chief ray’s direction cosines between the pupils

The direction cosines, a 3-tuple set of unit norm, specify the direction of a ray. The elements are the cosines of the angles the ray makes with the three coordinate axes. Given the direction cosine of a chief ray on the object side what is the direction cosine of the corresponding chief ray on the image side? What is the relation between the input and output chief ray’s direction cosines for an imaging system in which the lens is tilted (rotated about the x-axis) or swung (rotated about the y-axis) about a pivot point?

We begin by solving a specific problem of the *transfer* of the direction cosines between the pupils in which the optical axis (OA) coincides with the z-axis of the camera frame, as show in Figure 3.3. Later, the method of induction will be applied to yield the solution of the general *transfer* problem—in which the OA is free to swivel about the origin of. Let be the direction cosines of the chief ray (CR) from a world point to the center of the entrance pupil (), and let be the corresponding direction cosines of the CR from the exit pupil (). The parameters , ,and are specified with respect to frame .

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| **Figure 3.3** Specific problem—optical axis coincides with reference frame’s z-axis. If and are the angles of the CR with the OA in the object and image side respectively, then and. |

If and are the zenith and azimuthal angles of the CR in the object side, and and the corresponding angles in the image side, then the direction cosines, in the camera frame, are represented as:

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Since the OA is aligned with the z-axis, and. Substituting the expressions for from equation (3.3) into equation (3.2) we get:

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As the input and out chief rays are confined to the same meridional plane [ref], , yielding and in terms of and , the ratios of to , and :

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From (3.2) we have

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which after simplification yields in terms of the pupil magnification and input

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Combining equations (3.5) and (3.7), we obtain the expression for output direction cosines of the chief ray in terms of the input direction cosines and the pupil magnification as:

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Our objective is to derive the expressions for the transfer of direction cosines of the CR from to for arbitrary orientation of the OA. Equation (3.8) accurately represents the *transfer* for the specific problem; however, we will cast the expression in a slightly different form whose raison d'être is to enable generalization—through direct application of the result. Specifically, the output CR can be expressed as a linear combination of the input CR and the OA since the two rays and the OA span the same (meridional) plane. Let , the standard basis vector along z-axis of , represent the OA since the OA is coincident with the z-axis. Then,

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where and are the weights, and.

Rewriting the above equation as

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the weight is readily obtained by comparing equations (3.8) and (3.10):

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Substituting the expression for into and comparing with (3.7) yields:

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We are now ready to apply the result of the specific problem to the general problem. Figure 3.4 shows the schematic of the general problem—the OA pivots about the origin of . The new orientation of the OA is described by applying the rotation matrix to. The matrix may be a composition of two or more rotation matrices that incorporates a sequence of rotations about the x-axis and/or y-axis. Then, , the unit vector denoting the new orientation of the OA is given as: or .

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| **Figure 3.4** Configuration of the general problem—optical axis (OA) pivots freely about the origin of. |

As the output direction cosine , the input direction cosine , and the optical axis lie on the same plane,

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Note that the input direction cosine in equation (3.13) is different from the input direction cosine in equation (3.9), even for the same world point due to the displacement of after the rotation of the OA. Multiplying equation (3.13) by:

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Letting and,

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Comparing equations (3.9) and (3.15) the expressions for the weights and are obtained as:

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where or equivalently, .

Rewriting equation (3.15) as:

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which can be compactly written as:

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The positive or negative sign of the direction cosine vector determines the forward or backward direction of light travel along a rectilinear path. Under the assumptions of isotropy and homogeneity, a ray of light does not emerge in an antipodal path upon meeting an interface unless the posterior surface happens to be a mirror, and the angle of incidence is zero. In the context of our problem setting such a situation does not arise. Therefore, without any loss of generality, we can drop the negative sign in equation (3.17). The elements of the output direction cosine vector will assume the sign of the corresponding elements of the input direction cosine vector. Furthermore, using and yields the general expression for the direction cosines of the chief ray in the image side in terms of the pupil magnification and direction cosines in the object side as:

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where.

Also, note that the equation (3.18) only describes the output CR’s direction cosines—a free vector. A complete description of the chief ray is obtained from the knowledge of the direction cosine and the location of the exit pupil in the camera frame.

### 3.5 Image formation for arbitrary orientation of the lens and image plane

Geometric imaging is a mapping (bijective in projective space) between points in the world to points on a mathematical surface that we call the *image*. Here we aim to study the nature of this mapping on a planar surface—the image plane—for arbitrary orientation of both the lens and image planes. To that effect, we will use the knowledge of the transfer of direction cosines of the chief ray (CR) derived previously.

There are a multitude of chief rays for an extended object. The locus of points formed by the intersection of the chief rays, in the image side, with the image plane constitutes the *projection* of the object on the image plane [ref]. We prefer to differentiate the term “image” from “projection”. The term “image” will be used only when all the points in the projection are in focus optically.

It is assumed that the lens is unencumbered by radial distortions and other optical aberrations. Figure 3.5 represents a schematic of the problem, in which we have introduced an image plane whose orientation is described by the unit surface normal. Two local frames are introduced: the frame is fixed to the optical axis with its origin at, and the frame fixed to the image plane with its origin at the intersection of the image plane with the z-axis of the camera frame. The origin of also represents the pivot point about which the image plane is free to roll (tilt about local x-axis) and/or pitch (swing about local y-axis).

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| **Figure 3.5** Schematic of geometric image formation. is the *central* *projection* of the world point on image plane. The the optical axis and image plane are free to swivel about the origins of coordinate frames and respectively. |

Let the exit pupil () be located units from the pivot point along the OA. The orientation is represented by the rotation matrix. Consequently, the position of in is . The parametric equation of the CR emerging from with direction cosine is then represented as:

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where represents a point in the output CR in .

The equation of the image plane with unit normal   in Hessian normal form is written as:

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where is the perpendicular distance between the plane and the origin of the reference frame , and is a point on the plane.

The expression for for which the ray intersects the image plane is obtained by equating to, multiplying equation (3.19) by, and rearranging the terms as follows:

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Substituting (3.21) into (3.19):

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The point of intersection of the z-axis of with the image plane is origin of the image plane’s local reference frame. The orientation of the image plane can be described by applying a rotation matrix (or a composition of successive rotation matrices) to the image plane, with its unit plane-normal nominally equal to , about the origin of as shown in Figure 3.5. If is the rotation matrix then

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Referring to Figure 3.6, the expression for is obtained as follows:

The equation of the image plane is

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Since is a point on the plane, therefore

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| **Figure 3.6** Schematic of the image plane. The image plane having surface normal is located at units from the origin of camera frame along the z-axis that intersects the plane at . is the perpendicular distance from the origin to the plane. The local image coordinate frame with its origin at the intersection of the image plane and z-axis of the camera frame is represented by . |

Using the above result, the expression for the point of intersection of the chief ray with the image plane in terms of the input direction cosines is

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Similar to the exit pupil (), let the entrance pupil () be located at a distance from the pivot point along the OA in the camera frame. Then, the location of the in is. The direction cosines and the world point are related as

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which can be written compactly as:

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Substituting equation (3.27) into equation (3.25) a general relation between the world point and its corresponding image point is obtained:

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Equation (3.28) represents the image point in the camera frame. Once an image is formed, we specify positions and dimensions within the image independent of the position and orientation of the sensor and lenses. We can transform the image coordinates in the camera frame to the image frame by observing that the origin of is displaced from by , and the standard basis vectors of are rotated by . Therefore a point in relative to may be expressed as:

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Let and be the homogeneous representation [ref] of the and respectively. The equation (3.29) can be expressed as a linear matrix operation as:

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where is the pose of with respect to frame . Then, it follows that

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where.

Finally, we obtain the image coordinates in the image frame as:

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TO DO:

1. Table showing different known equations that fall out of it. For example, a) thin lens, fronto-parallel imaging, b) thin-lens image plane tilt model, c) thin-lens image and lens tilt model, d) thick lens model, etc.
2. Zemax based verification
3. Analysis of the equation, especially with regards to collinear transformation
4. Tilt about entrance pupil vs. tilt about the principal point.

1. In the presence of spherical aberrations, the chief ray goes through the center of the aperture but may not exactly go through the center of the pupils [Ref Mirrors, Prisms … Southall, Lens design by Kingslake]. [↑](#footnote-ref-1)