# Chapter 3

⬀⬀⬀Geometric model of Scheimpflug Imaging

### Introduction

TO DO: State the motivation and assumptions.

TO DO: Preview of what is coming in the following sections.

TO DO: State the novelty of this approach, and why needed to develop this model

TO DO: Notation

* A left super-script indicates the frame of reference. For example, indicates that the variable is w.r.t. the world coordinate frame . If no reference is explicitly stated it implies that the variable is w.r.t. the world coordinate frame (or the camera coordinate frame if the camera coordinate frame and the world coordinate frame are the same.
* A subscript is used to associate a variable to a particular xxx like entrance-pupil position (), image plane (), for example is used to represent the 3D rotation matrix applied to the entrance pupil plane in the camera frame . The same notation is also used to indicate a transformed variable, for example is used to represent under the rotational transformation by in the camera coordinate frame . As also mentioned earlier, if the camera coordinate frame is the same as the world coordinate frame, then the notation shall be used.
* represents the pose of frame w.r.t. frame TO DO: mention how a point in one frame is represented in another frame.

### Relation between the direction cosines of the chief rays in the object and image side

Let be the direction cosine of the chief ray from a world point to the entrance pupil (ENP) in the object side, and let be the direction cosine of the chief ray emerging from the exit pupil (EXP) to the corresponding image point in the image side. If the angles between the chief ray and the optical axis at the object side and image side be and respectively, then the relation between the angles is given by the airy tangent condition [ref]:

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Where, is the pupil magnification defined as the ratio between the paraxial exit-pupil to the paraxial entrance-pupil diameter.

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| ***Figure 3.1*** *Schematic showing the chief ray from the object point to the image point when the optical axis is aligned with the z-axis of the reference frame with its origin at the entrance-pupil (ENP) position.* |

If and are the zenith and azimuthal angles of the chief ray in the object side, and and the corresponding angles in the image side, the direction cosines are represented in the spherical coordinates as:

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When the optical axis is aligned with the z-axis, then and . Substituting the expressions for from equation (3.2) into equation (3.1) we get:

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By definition the chief ray (a meridional ray) is …. Therefore and the above relations reduces to:

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Also, from (3.1)

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Simplifying further,

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Therefore, the expressions for output direction cosines in terms of the input direction cosines and the pupil magnification is obtained as:

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Since the chief ray emerging from the exit pupil spans the plane formed by the input chief ray and the optical axis, the chief ray in the image side may be represented as a linear combination of the input chief ray and the optical axis (represented in vector form) as follows:

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Where, and are the weights and is direction cosine of the line representing the optical axis. when the optical axis is aligned with the z-axis (present case). The weight is readily obtained by comparing equations (3.7) and (3.8):

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Substituting the expression for into and comparing with (3.6) we get the expression for :

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We would like to derive the relationship between the input and output direction cosines of the chief ray for an arbitrary rotation of the optical axis centered about the ENP as shown in figure 3.2.

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| ***Figure 3.2*** *Schematic showing the chief ray from the object point to the image point when the optical axis is rotated about an arbitrary axis about the ENP.* |

Write appropriate text here.

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Where

Write appropriate text here.

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Letting and

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By comparing the above equation with (3.8) the expression for the weights and are obtained as:

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Equation (3.13) can then be written as:

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Furthermore, using and , the general expression for the direction cosines of the chief ray in the image side in terms of the pupil magnification and direction cosines in the object side is:

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Where

### Relation between object coordinate and image coordinate for arbitrary orientation of the lens and image plane

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| ***Figure 3.3*** *Schematic showing the chief ray from the object point to the image point when the optical axis is rotated about an arbitrary axis about the ENP.* |

Explain the different coordinate frames.

Let the position vector of the exit pupil (EXP) in the reference frame be . Since the EXP is located at a distance from the entrance pupil (ENP) along the optical axis, . The optical axis (along with the cardinal planes of the lens) is tilted in the camera frame using a rotation matrix . Therefore the location of the EXP in the camera frame is . The chief ray emerging from the exit pupil (EXP) with direction cosine vector is then represented by the parametric equation:

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Where represents any point along the chief ray in the image side.

The equation of the image plane with unit normal   in Hessian normal form is written as:

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Where is the perpendicular distance between the plane and the origin of the reference frame , and is a point on the plane.

The expression for is obtained by multiplying equation (3.17) by   and rearranging the terms:

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Substituting (3.19) in to (3.17):

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Let the point of intersection of the z-axis of the camera frame with the image plane be the origin of the image plane’s reference frame . The orientation of the image plane can be described by applying a rotation to the image plane, with its plane normal nominally equal to , about the origin of as shown in figure 3.3. If is the rotation matrix then

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The expression for is obtained as follows (refer to figure 3.4):

The equation of the image plane is

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Since is a point on the plane, therefore

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| ***Figure 3.4*** *xxx.* |

Using the above result the expression for the point of intersection of the chief-ray with the image plane in terms of the input direction cosines is

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The direction cosines and the world point are related as

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Which can be compactly written as

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Substituting equation (3.25) into equation (3.23) a general relation between the world point and its corresponding image point is obtained:

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