# Chapter 3

⬀⬀⬀**GEOMETRIC MODEL OF SCHEIMPFLUG IMAGING**

A model of Scheimpflug imaging is developed from the basic principles of *geometric optics* (*ray optics*). Like any model, the ray optics model in itself doesn’t provide a complete picture of imaging; yet the definitions and postulates therein provide useful tools for the analysis and synthesis of optical systems. Assumptions are both critical and necessary ingredients of modeling that enables its expediency, while simultaneously limiting its applicability. The imaging model developed in this chapter uses axioms of paraxial ray optics theory, assumes a homogeneous medium for propagation of light, rotationally symmetric and aberration-free optics.

TO DO: Preview of what is coming in the following sections.

### 3.1 Introduction

The concept of *pupils* are the sine qua non of imaging system design in both domains of *ray* and *wave optics*. In general, an imaging system consists of several groups of elements, some of which has power that bend light rays. The most severely limiting orifice in the system is called the *system aperture* or *stop*. Its size and position determine important system parameters such as resolution, brightness, image quality, and image perspective. The image of the stop viewed through the elements preceding it in the object space is called the *entrance pupil* (). The exit pupil () is the image of the stop viewed through the elements following it in the image space. The pupils and the stop are geometric images of one another. Rotationally symmetric lenses have an axis of symmetry. Planes passing through the axis of such lenses are called *meridional planes* about which rays exhibit *bilateral symmetry*, and the rays **confined** to the meridional planes are called *meridional rays*. Two meridional rays—the *marginal ray* that originates at the axial object position and skirts the edges of the aperture and pupils (virtually), and the chief ray that starts at an off-axis object point and goes through the center of the aperture and pupils[[1]](#footnote-1) (virtually)—are fundamental to geometric analysis of optical systems. They define the location and size of the pupils, site of the image, and magnification. Furthermore, as shown in Figure 3.1, the homocentric bundle of incoming chief rays forms the vertex of the object space perspective cone at the center of the entrance pupil (), while the bundle of chief-rays diverging from the center of the exit pupil () produces the vertex of the image space perspective cone.

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| **Figure 3.1** Fundamental rays and pupils in a typical Double Gauss Lens. The chief rays—close to the optical axis (0°, ±5° on the object side measured at )—appear to converge at the entrance pupil center and diverge from the center of the exit pupil. The marginal ray appear to skirt the edges of the two pupils. The vertices of the perspective cones are shown as red circles. The rays were traced in Zemax. |

TO DO: State the novelty of this approach, and why needed to develop this model

### 3.2 Notation (this section is currently a place holder for the notation, skip reading this for now. It will be appropriately written after completion of the major math)

* Right handed coordinate system with the +z along the direction of travel of light
* In general, non-primed quantities are used to indicate input or object space (e.g. ) and primed quantities are used to indicate output or image space (e.g. ).
* A left super-script indicates the frame of reference. For example, indicates that the variable is w.r.t. the world coordinate frame . If no reference is explicitly stated it implies that the variable is w.r.t. the world coordinate frame (or the camera coordinate frame if the camera coordinate frame and the world coordinate frame are the same.
* A subscript is used to associate a variable to a particular xxx like entrance-pupil position (), image plane (), for example is used to represent the 3D rotation matrix applied to the entrance pupil plane in the camera frame . The same notation is also used to indicate a transformed variable, for example is used to represent under the rotational transformation by in the camera coordinate frame . As also mentioned earlier, if the camera coordinate frame is the same as the world coordinate frame, then the notation shall be used.
* represents the pose of frame w.r.t. frame TO DO: mention how a point in one frame is represented in another frame.
* The zero-based indexing of matrices and vectors

### 3.3 Relation between pupil magnification and chief ray angle

The *pupil magnification*, is defined as the ratio of the exit pupil () size to the entrance pupil () size. Figure 3.2 shows a notional illustration of an optical system with just the chief ray (CR), the marginal ray (MR) and the two pupils. An arbitrary object of height units above the optical axis (OA) is located at units from , and the resulting image of height units is produced at a distance from .

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| **Figure 3.2** Schematic of chief and marginal rays. The pupil magnification is obtained as the ratio of the tangents of the chief ray angles in the object side to the image side. |

Let the angles produced the CR with the OA in the object and image space be and respectively. Also, let the angles produced by the MR with the OA in the object and image space be and respectively. Then the relation between the chief ray angles and the pupil magnification may be derived as follows:

From the Figure 3.2,

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Eliminating and from the above equations, we have

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As per the *Lagrange invariant* [Ref] property of the two rays, . Therefore

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The equation (3.2), has been previously derived in [Ref] using a different formulation. For a given optics the pupil magnification is constant. This constancy of the ratio of the tangents of the CR angles for varying object (and image) heights is a necessary and sufficient condition for distortion-free imaging known as the *Airy’s Tangent-Condition* [Ref]. In the following section, the relationship (3.2) is used to derive the expression for the direction cosines of the image space CR emerging from in terms of the object space direction cosines and the pupil magnification.

### 3.4 Transfer of chief ray’s direction cosines between the pupils

We begin by solving a specific problem of the *transfer* of the direction cosines between the pupils in which the optical axis (OA) is coincident with the z axis of the camera frame , as show in Figure 3.3. Subsequently, the result of the specific problem will be directly use to yield the general *transfer* problem—in which the OA is arbitrarily rotated about the origin of . Let be the direction cosine of the chief ray (CR) from a world point to the center of the entrance pupil (), and let be the direction cosine of the “transferred” CR from the exit pupil () to the image point . The parameters—, , , and —are specified in .

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| **Figure 3.3** Specific problem—optical axis coincident with reference frame’s z axis. Therefore and . |

If and are the zenith and azimuthal angles of the CR in the object space, and and the corresponding angles in the image space, then the direction cosines, in the camera frame , are represented as:

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Since the OA is aligned with the z axis, and . Substituting the expressions for from equation (3.3) into equation (3.2) we get:

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The input and out chief rays are **confined** to the same meridional plane [Ref]. Therefore, , yielding and in terms of inputs and , the ratios of output to input , and pupil magnification :

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Also, from (3.2) we have

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Which after simplification yields in terms of the pupil magnification and input

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Combining the above results, the expressions for output direction cosines in terms of the input direction cosines and the pupil magnification is obtained as:

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Our objective is to derive the expressions for the transfer of direction cosines of the CR from to for arbitrary orientation of the OA. Although, the above equations accurately represent the *transfer* for the specific problem, we will cast the results in a slightly different form whose raison d'être is to enable generalization by the direct application of the obtained results. Specifically, since the output CR originating at spans the (meridional) plane formed by the input CR and the OA, the output CR can be represented as a linear combination of the input CR and the OA. This linear relationship in terms of the direction cosines of the CR and the unit vector representing the optical axis () is shown below:

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where and are the weights, and , is unit vector representing the OA, which is coincident with the z axis of . The subscript signifies this special, coincident, case.

Rewriting the above equation

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The weight is readily obtained by comparing equations (3.8) and (3.10):

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Substituting the expression for into and comparing with (3.7) yields :

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We are now ready to apply the results of the specific problem to the general problem. Figure 3.4 shows the schematic of the general problem—the OA, pivoted at the origin of is free to have any orientation (generally restricted to roll and/or pitch). The orientation is described by the application of the rotation matrix to , which is the unit vector coincident with the z axis of . If we impart both roll (tilt about x axis) and pitch (swing about y axis), then represents a composition of two or more rotation matrices. Let denote the unit vector in the arbitrary orientation of the OA. Then or .

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| **Figure 3.4** Configuration of the general problem—optical axis (OA) has arbitrary orientation about the origin of . |

The output direction cosine vector of the CR is a linear combination of the input direction cosine vector and , the unit vector representing the OA:

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Note that the input direction cosine in equation (3.13) is different from the input direction cosine in equation (3.9), even for the same world point due to the displacement of after the rotation of the OA. Multiplying equation (3.13) by :

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Letting and ,

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Comparing equations (3.9) and (3.15) the expressions for the weights and are obtained as:

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where .

Re-writing equation (3.15) as:

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The above equation can be written compactly as:

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Furthermore, using and yields the general expression for the direction cosines of the chief ray in the image space in terms of the pupil magnification and direction cosines in the object space as:

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where .

Comment on considering just the positive sign. Or is it enough to consider only the positive sign? Also note that the equation (3.18) only describes the output CR’s direction cosines—a free vector. The output chief ray is determined from the knowledge of the direction cosine and the location of exit pupil in the camera frame.

### 3.5 Image formation for arbitrary orientation of the lens and image plane

Geometric imaging is a mapping (bijective in projective space) between points in the world to points on a mathematical surface that we call the *image*. Here we aim to determine the nature of this mapping on a planar surface—the image plane—using the knowledge of the transfer of direction cosines of the chief ray (CR), derived in the preceding section. The locus of points formed by the intersection of the chief rays with the image plane constitutes the image. It is assumed that the lens is unencumbered by radial distortions and other aberrations. Figure 3.5 represents a schematic of the problem, in which we have introduced an image plane whose orientation is described by its unit surface normal  . Two local frames are introduced: the frame is fixed to the optical axis at , and the frame fixed to the image plane at the intersection of the image plane with the z axis of the camera frame . The origin of also represents the pivot point about which the image plane is free to roll (tilt about local x axis) and/or pitch (swing about local y axis).

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| **Figure 3.5** Schematic of geometric image formation. is the image of the object point formed by an imaging system whose lens and image planes are oriented arbitrarily about the origin of the camera frame and image frame respectively. |

Let the exit pupil () be located units from the pivot point along the OA. The orientation is represented by the rotation matrix . Consequently, the position of in is . The parametric equation of the CR emerging from with direction cosine is then represented as:

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where represents a point in the output CR in .

The equation of the image plane with unit normal   in Hessian normal form is written as:

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where is the perpendicular distance between the plane and the origin of the reference frame , and is a point on the plane.

The expression for for which the ray intersects the image plane is obtained by equating to , multiplying equation (3.19) by  , and rearranging the terms as follows:

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Substituting (3.21) in to (3.19):

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The point of intersection of the z axis of the with the image plane is origin of the image plane’s local reference frame . The orientation of the image plane can be described by applying a rotation matrix (or a composition of successive rotation matrices) to the image plane, with its unit plane-normal nominally equal to , about the origin of as shown in Figure 3.5. If is the rotation matrix then

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Referring to Figure 3.6, the expression for is obtained as follows:

The equation of the image plane is

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Since is a point on the plane, therefore

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| **Figure 3.6** Schematic of image plane. The image plane having surface normal is located at units from the origin of camera frame along the z axis that intersects the plane at . is the perpendicular distance from the origin to the plane. |

Using the above result, the expression for the point of intersection of the chief-ray with the image plane in terms of the input direction cosines is

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Similar to the exit pupil (), let the entrance pupil () be located at a distance from the pivot point along the OA in the camera frame . Then, the location of the in is . The direction cosines and the world point are related as

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Which can be compactly written as

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Substituting equation (3.27) into equation (3.25) a general relation between the world point and its corresponding image point is obtained:

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1. In the presence of spherical aberrations the chief ray goes through the center of the aperture but may not exactly go through the center of the pupils [Ref Mirrors, Prisms … Southall, Lens design by Kingslake]. [↑](#footnote-ref-1)