UNIT-1

Probability

Introduction:

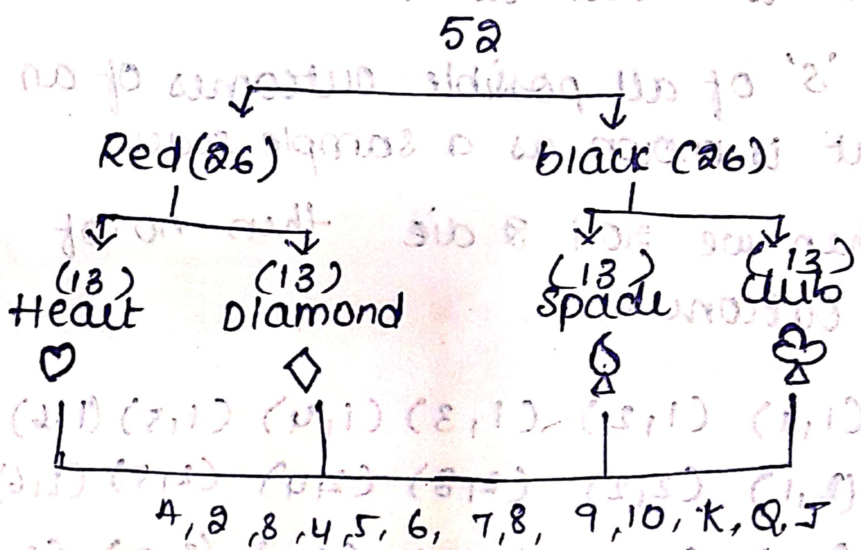
A measure of uncertainity is provided by the branch of maltematics theory of probability

⇒ uncertainity plays an impoetant role in business and probability is a concept which measures the degree of uncertainity

some basic teems :-

1. Die :- A die is a small homogenous queue with 6 faces marked with numbers or dots from 1 to 6

2. cards :- A standard pack or a well shuffled packe has 52 cards divide into 4 suits



Random experiment: An experiment can said to be Random experiment if it satisfies the following condition

1. → It has more than possible outcome
2) It is not possible to predict the outcome in advance.

Eg:- rolling a die is a random experiment because if a die is rolled then are 6 possible outcomes 1, 2, 3, 4, 5, 6

Trial:- when we perform an experiment then it is known as trial

→ picking a card is trial
→ tossing a ~~card~~ coin is trial

Sample space:- A possible result of a random experiment is known as an outcome.

The set 'S' of all possible outcomes of an experiment is known as a sample space

eg:- when we roll 2 die then no. of possible outcomes

$$
\begin{aligned}
S = \{ & (1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6) \\
& (2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6) \\
& (3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6) \\
& (4,1)\ (4,2)\ (4,3)\ (4,4)\ (4,5)\ (4,6) \\
& (5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5)\ (5,6) \\
& (6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6) \}
\end{aligned}
$$

Event :- The possible outcome of a ~~trai~~ trail is known as event

⇒ Events are classified into various types.

1. Impossible and sure event

The empty set ϕ and a sample space S itself are events here ϕ is known as an impossible event and S is known as sure event or certain event.

if $S = \{1, 2, 3, 4, 5, 6\}$

the event of getting a number >6 is imposible event. and the event of appearance of one of the numbers from 1 to 6 is sure event.

Mutually Exclusive Events :-

If 2 events are said to be mutually exclusive if the occurance of one ~~depends~~ prevent on the occurance of other ie 2 events A & B are said to be mutually exclusive if they are disjoint $A \cap B = \phi$

Equally likely events:

Events are said to be Equally likely if there is no reason to expect any of the events in reference to the others.

Eg:- In rolling a die the 6 possible outcomes are equally likely

Exhaustive events:

Events are said to be E.E when all they include all possible outcomes in an experiment

Eg:- In tossing a coin the events of getting head and tail are exhaustive.

Mutually Exclusive & Exhaustive Events:

The events $E_1, E_2 \dots E_n$ are said to be mutually exclusive and exhaustive if $E_1 \cup E_2 \cup \dots \cup E_n = S$

and $E_1 \cap E_2 \cap \dots \cap E_n = \phi$

Eg:- In an experiment of rolling a die

Let A be the event of getting an even number and B be the event of getting an odd number

$A = \{2, 4, 6\}$

$B = \{1, 3, 5\}$

$A \cap B = \{\} = \phi$

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

$= S$

∴ They are mutually Exclusive and exhaustive events

Favourable cases:-

The cases which ensure the occurance of an event is said to be favourable to that event

Defination of probability:- The probability of an event E is defined as a ratio of no. of favourable events with respect to E to the total no. of events of a random experiment

$$P(E) = \frac{\text{No. of favourable events}}{\text{Total no. of events.}}$$

⇒ If $\bar{E}$ denotes the event of non occurance of E then $P(\bar{E}) = 1 - P(E)$

$$\boxed{P(E) + P(\bar{E}) = 1}$$

Note : The probability of an event E is a numerical value always lies b/w 0 and 1 ie

$$0 \leq P(E) \leq 1$$

- If $P(E) = 1$ then the event is a sure event or certain event.
- If $P(E) = 0$ then the event is an impossible event

Axioms of probability :-

If S is sample space and E is any event in a random experiment

(i) Axiom of positivity : $P(E) \geq 0$ for every subset E of S.

(ii) Axiom of certainity : $P(S) = 1$

(iii) Axiom of union : If E_1 and E_2 are mutually exclusive events in S then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

Note :- A combination is a selection of 'r' objects from 'n' objects where order does not count ie

$$\boxed{{}^nC_r = \frac{n!}{(n-r)!\,r!}}$$

• An arrangment of a set of n objects in a given order is a permutation.

$$\boxed{{}^nP_r = \frac{n!}{(n-r)!}}$$

Problems.

1. A card is drawn at random from a well shuffled pack of 52 cards what is the probability that the card is a King.

Sol:- In a well shuffled pack there are 52 cards

Let A be the event of getting a king

∴ No. of favourable events = 4

Total No. of events = 52.

Probability of getting a king

$$P(A) = \frac{\text{No. of favourable events}}{\text{Total No. of events}}$$

$$= \frac{4}{52} = \frac{1}{13}$$

2. If a die is thrown what is the probability of getting an even number and an odd number.

If a die is thrown there are 6 possibilities

Let A be the event of getting even number

Let B be the event of getting odd number.

No. of favorable events of getting even = 3

NO of favourable events of getting odd = 3

$$P(A) = \frac{\text{No. of favourable events}}{\text{Total no. of event}}$$

$$= \frac{3}{6} = \frac{1}{2}$$

$$P(B) = \frac{\text{No. of favourable events}}{\text{Total no. of events}}$$

$$= \frac{3}{6} = \frac{1}{2}$$

3. A card is drawn from a well shuffled pack of 52 cards find the probability that the card ~~first~~ (i) diamond (ii) not diamond.

In a well shuffled pack there are 52 cards

Let A be the event of getting a diamond

Let B be the event of getting not diamond.

NO of favourable events to get diamond = 13

NO. of favourable event to get not diamond = 39

$$P(A) = \frac{\text{No. of favourable events}}{\text{Total no. of events}} = \frac{13}{52}$$

$$P(B) = \frac{\text{No. of favourable events}}{\text{Total no. of event}} = \frac{39}{52}$$

4. Find the probability of getting a number of sum ten if we throw two die's

When two dice are thrown there are 36 outcomes

= (1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

⇒ No. of favourable outcomes = 3
Total no. of outcomes = 36

$$P(E) = \frac{\text{No. of favourable outcomes}}{\text{Total no. of outcomes}} = \frac{3}{36} = \frac{1}{12}$$

5. 3 coins are tossed at once find the probability of getting (i) 3 heads (ii) 2 heads (iii) no heads.

When 3 coins are tossed 8 possibilities

= { H H H
H H T
H T H
T H H
T T H
T H T
H T T
T T T }

(i) No. of favourable events to get 3 heads = 1

Ans.

Let A be the event to get 3 heads

$$P(A) = \frac{\text{No. of favourable outcomes}}{\text{Total no. of outcomes}}$$

$$= \frac{1}{8}$$

Let B be the event to get 2 heads

No. of favourable outcomes to get 2 heads = 3

$$P(B) = \frac{\text{No. of favourable Outcomes}}{\text{Total no. of outcomes}}$$

$$= \frac{3}{8}$$

Let C be the event to get no heads.

No. of favourable outcomes to get no heads = ϕ

$$P(C) = \frac{\text{No. of favourable outcomes}}{\text{Total no. of outcome}}$$

$$= \frac{\phi}{8}$$

6. A bag contains 7 white, 6 red and 5 black balls 2 balls are drawn at random find the probability they both will be white.

Total no. of balls $7+6+5 = 18$

There are 7 white, 6 red and 5 black balls in a bag

we have to draw balls from 18 balls

$$\text{Total no. of events} = {}^{18}C_2 = \frac{\cancel{18}^{9} \times 17 \times \cancel{16}}{\cancel{16} \times \cancel{2} \times 1}$$

$$= 153$$

we have to draw 2 balls of white colour from 7 white balls.

No. of favourable events $= {}^7C_2$

$$= \frac{7\times\cancel{6}\times\cancel{5}\times 3\times\cancel{2}}{\cancel{5!}\times\cancel{2}} = 21$$

The probability of drawing 2 white balls.

$$= \frac{{}^7C_2}{{}^{18}C_2} = \frac{21}{153} = \frac{7}{51}$$

7. In a class there are 10 boys and 5 girls a committee of 4 students is to be selected from the class. Find the probability for the comitte to contain atleast 3 girls.

Sol:- Boys -10 girls -5

Total $= 15$

There are 10 boys and 5 girls in a committee

Total no. of student $= 10+5$

$= 15$

we have to select 4 students from 15 students to form a comitte

T.E $= {}^{15}C_4 = 1365$

There are 5 girls in a class

we have to select atleast 3 girls

$\therefore$ No. of favourable events $= ({}^5C_3 \times {}^{10}C_1) + {}^5C_4$

$= 105$

$\therefore$ Probability for the comitte containing atleast 3 girls

$\because$ atleast 3

so 3 & greater than 3

atmost 3

3 & less than 3

exactly 3 → equal to 3

$= \frac{\text{No. of favourable}}{\text{Total no. of events}}$

$= \frac{105}{1365} = \frac{1}{13}$

8. A class consists of 6 girls and 10 boys. If a comitte of 3 students is choosen at random from the class find the probability that 3 boys are selected, exactly 2 girls are selected.

Girls = 6
Boys = 10

we have to select 3 students

Total no. of events $= 16C_3 = 560$

(i) we have to select 3 boys

no. of favourable events $= 10C_3$
$= 120$

Probability of selecting 3 boys $= \frac{\text{no. of favourable}}{\text{Total no. of events}}$

$= \frac{120}{560} = \frac{3}{14}$

(ii) we have to select exactly 2 girls

no. of favourable event $= {}^6C_2 \times {}^{10}C_1$

$= 150$

The probability of selecting exactly 2 girls $= \frac{150}{560} = \frac{15}{56}$

9. Out of 15 items 4 are not in good condition 4 are selected at random. Find the probability (i) All are not good (ii) two are not good.

Total no. of outcomes $= 15C_4$
$= 1365$

(i) All are not good

No. of favourable event $= {}^4C_4 = 1$

Total no of events $= 1365$

probability of all are not good $= \frac{1}{1365}$

(ii) two are not good

No. of favourable events $= {}^4C_2 \times {}^{11}C_2$

$= 330$

= No. of favourable event

probability two are not good $= \frac{22}{91}$

Addition theorem:-

If S is a samplespace and E_1, E_2 are any events in S

$P(E_1 \text{ or } E_2) = P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Proof:-

case(i): $E_1 \cap E_2 = \phi$

Now $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

$= P(E_1) + P(E_2) - 0$

$= P(E_1) + P(E_2) - P(\phi)$

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Case 2 :- $E_1 \cap E_2 \neq \phi$

Let E_1 & E_2 contains some sample points

$a_1, a_2 \dots a_k, a_{k+1} \dots a_{k+l}$ &

$a_{k+1}, a_{k+2} \dots a_{k+l}, a_{k+l+1} \dots a_{k+l+m}$ respectively

$E_1 = \{a_1, a_2, \dots a_k, a_{k+1}, \dots a_{k+l}\}$

$E_2 = \{a_{k+1}, a_{k+2}, \dots, a_{k+l}, a_{k+l+1}, \dots a_{k+l+m}\}$

Now $E_1 \cap E_2 = \{a_{k+1}, a_{k+2}, \dots, a_{k+l}\}$

$E_1 \cup E_2 = \{a_1, a_2, \dots a_k, a_{k+1}, \dots a_{k+l}, a_{k+l+1} \dots a_{k+l+m}\}$

$\therefore P(E_1) + P(E_2) - P(E_1 \cap E_2) = P(a_1) + P(a_2) + \dots + P(a_k) + P(a_{k+1}) + \dots + P(a_{k+l})$

$+ P(a_{k+1}) + P(a_{k+2}) + \dots + P(a_{k+l}) + P(a_{k+l+1})$

$+ P(a_{k+l+m}) - [P(a_{k+1}) + P(a_{k+2}) + \dots + P(a_{k+l})]$

$= P(a_1) + P(a_2) + \dots + P(a_k) + P(a_{k+1}) + \dots + P(a_{k+l}) + P(a_{k+l+1}) + \dots + P(a_{k+l+m})$

$= P(E_1 \cup E_2)$

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$P(A \cup (B \cup C)) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Note:-

Let A and B be the events of an experiment whose samplespace is S.

$\Rightarrow$ $A \cup B$ is an event that occurs if A occurs or B occurs or both

$\Rightarrow$ $A \cap B$ is an event that occurs if A occurs and B occurs or both

$\Rightarrow$ A^c or A' or $\bar{A}$ the compliment of an event A is the event that occurs if A does not occur

$$\boxed{A' = S - A}$$

$\Rightarrow$ $A - B$ denotes the event A but not B

$$\boxed{A - B = A \cap B'}$$

$\Rightarrow$ $P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B)$

$\Rightarrow$ $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

$\Rightarrow$ $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

Problem

1. $\Rightarrow$ Let A and B be the events with $P(A) = \frac{1}{5}$, $P(B) = \frac{2}{3}$ and $P(A \cap B) = \frac{1}{15}$ then find

(i) $P(A \cup B)$ (ii) $P(A^c \cap B^c)$ (iii) $P(\bar{A} \cup \bar{B})$ (iv) $P(A^c \cap B)$ (v) $P(A \cap B')$

Given $P(A) = \frac{1}{5}$ $P(B) = \frac{2}{3}$ $P(A \cap B) = \frac{1}{15}$

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{5} + \frac{2}{3} - \frac{1}{15}$

$= 4/5$

(ii) $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

$= 1 - \frac{4}{5}$

$= \frac{1}{5}$

(iii) $P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B)$

$= 1 - \frac{1}{15}$

$= \frac{14}{15}$

(iv) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

$= \frac{2}{3} - \frac{1}{15}$

$= \frac{3}{5}$

(v) $P(A \cap B') = P(A) - P(A \cap B)$

$= \frac{1}{5} - \frac{1}{15}$

$= \frac{2}{15}$

2 A card is drawn from a well shuffled pack of cards what is the probability that it is either a spade or an ace

There are 52 cards in a well shuffled pack

Let A be the event of drawing a spade

Let B be the event of drawing an ace

Let $A \cap B$ be the event of drawing a spade and ace

$P(A) = \frac{13}{52} = \frac{1}{4}$

$P(B) = \frac{4}{52} = \frac{1}{13}$

$P(A \cap B) = \frac{1}{52}$ (since in space only one ace)

∴ The probability of drawing either a space or an ace

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{4} + \frac{1}{13} - \frac{1}{52}$

$= \frac{4}{13}$

3. A bag contains 12 balls numbered from 1 to 12 if a ball is taken randomly what is the probability that having a ball with a number which is a multiple of either 2 or 3

Sol:- let A be event of drawing multiple of 2

$A = \{2, 4, 6, 8, 10, 12\}$

$P(A) = \frac{6}{12} = \frac{1}{2}$

Let B be the event of drawing multiple of 3

$B = \{3, 6, 9, 12\}$

$P(B) = \frac{4}{12} = \frac{1}{3}$

$P(A \cap B)$ be the common multiple of 2 and 3

$A \cap B = \{6, 12\}$

$P(A \cap B) = \frac{2}{12} = \frac{1}{6}$

NOW, the probability of a number is multiple of either 2 or 3

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{6}$$

$$= \frac{2}{3}$$

4 Among 150 students ~~80~~ 80 students studying maths 40 studying physics and 30 are studying physics and maths. If a student is choosen at random find the probability (i) studing maths or physics (ii) studing neither maths nor physics.

Total = 150

A - maths = 80

B - physics = 40.

$$P(A) = \frac{80}{150} = \frac{8}{15}$$

$$P(B) = \frac{40}{150} = \frac{4}{15}$$

$$P(A\cap B) = \frac{30}{150} = \frac{1}{15}$$

(i) studing maths or physics

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$= \frac{8}{15} + \frac{4}{15} - \frac{1}{15}$$

$$= \frac{3}{5}$$

(iii) $P(\overline{A \cup B}) = 1 - P(A \cup B)$

$$= 1 - \frac{3}{5}$$

$$= 2/5$$

5. Three students A, B, C are in running race. A and B have same probability of wining and each is twice as likely to win as C. find the probability that B or C wins.

Sol:

$$P(A) = P(B) = 2P(C)$$

we know that

$$P(A) + P(B) + P(C) = 1$$

$$2P(C) + 2P(C) + P(C) = 1$$

$$5 \cdot P(C) = 1$$

$$\boxed{P(C) = \frac{1}{5}}$$

$$P(A) = \frac{2}{5} \quad P(B) = \frac{2}{5}$$

$$P(B \cap C) = 0$$

$$P(B \cup C) = P(B) + P(C) - P(B \cap C)$$

$$= \frac{2}{5} + \frac{1}{5} - 0$$

$$= 3/5$$

Conditional event :- If E_1 and E_2 are events of a sample space S and if E_2 occurs after the occurance of E_1 then the event of occurance of E_2 after the event E_1 is known as condition event of E_2 given E_1

It is denoted by E_2/E_1

Eg:- 2 coins are tossed the event of getting two tails given that there is atleast one tail

Conditional probability: If E_1 and E_2 are events of sample space S and probability of E_1 is > 0 then $p(E_2)$ after the event E_1 is known as conditional probability.

It is denoted by $P(E_2/E_1)$ and is defined as

$$\frac{P(E_1 \cap E_2)}{P(E_1)}$$

$$\therefore \quad P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}$$

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

Multiplication theorem :-

In a random experiment if E_1 and E_2 are two events such that $P(E_1) \neq 0$ and $P(E_2) \neq 0$ then $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2/E_1)$

$P(E_2 \cap E_1) = P(E_2) P(E_1/E_2)$

Proof :-

Let S be a sample space

Let E_1 and E_2 be any 2 events in the sample spaces such that $P(E_1) \neq 0$ $P(E_2) \neq 0$

since $P(E_1) \neq 0$

$$P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}$$

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2/E_1)$$

$$\boxed{P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}}$$

Since $P(E_2) \neq 0$

By defn of conditional probability

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

$$P(E_1 \cap E_2) = P(E_2) \cdot P(E_1/E_2)$$

Dependent Event :-

If the occurance of the event E_2 is effected by the occurance of E_1 then the events E_1 and E_2 are said to be dependent ie $P(E_2/E_1) \neq P(E_2)$

Independent event :

If the occurance of the event E_2 is not effected by the occurance of E_1 or non occurance of the event E_1 then the events E_2 is said to be independent of E_1 ie $P(E_2/E_1) = P(E_2)$

Note :-

1. Two events E_1 and E_2 are said to be independent if $P(E_1 \cap E_2) = P(E_1)P(E_2)$
2. Two events E_1 and E_2 are said to be dependent if $P(E_1 \cap E_2) \neq P(E_1)P(E_2)$

Problems.

1. Determine (i) probability of $P(B/A)$ (ii) $P(A/\bar{B})$ if A and B are events where $P(A) = \frac{1}{3}$ $P(B) = \frac{1}{4}$ $P(A \cup B) = \frac{1}{2}$

Sol: (i) $P(A) = \frac{1}{3}$ $\quad P(B) = \frac{1}{4}$ $\quad P(A \cup B) = \frac{1}{2}$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{3}$$

$$= \frac{1}{12}$$

$$P(B/A) = \frac{\frac{1}{12}}{\frac{1}{3}} = \frac{1}{4}$$

(ii) $P(A/B^c) = \dfrac{P(A\cap B^c)}{P(B^c)}$

$= \dfrac{P(A) - P(A\cap B)}{1-P(B)}$

$= \dfrac{\frac{1}{3} - \frac{1}{12}}{1-\frac{1}{4}}$

$= 1/3$

(2) If $P(A^c) = 3/8$, $P(B^c) = \frac{1}{2}$, $P(A\cap B) = 1/4$

then find $P(A/B)$ (ii) $P(B/A)$ (iii) $P(A^c/B^c)$

(iv) $P(B^c/A^c)$

sol:- $P(A^c) = 3/8$ $\quad P(B^c) = \frac{1}{2}$ $\quad P(A\cap B) = 1/4$

$P(A^c) = 1-P(A) \Rightarrow \frac{3}{8} = 1-P(A)$ $\qquad P(B) = \frac{1}{2}$

$\Rightarrow P(A) = 1-\frac{3}{8}$

$= \frac{5}{8}$

(i) $P(A/B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{1/4}{\frac{1}{2}} = \frac{1}{2}$

(ii) $P(B/A) = \dfrac{P(A\cap B)}{P(A)} = \dfrac{1/4}{5/8} = \frac{2}{5}$

(iii) $P(A^c/B^c) = \dfrac{P(A^c\cap B^c)}{P(B^c)} =$

(iv) $P(B^c/A^c) = \frac{P(A^c \cap B^c)}{P(A^c)} =$

(3) 2 Dies are thrown let $\overset{\text{Abedthe}}{\text{the}}$ $\overset{\text{whee}}{}$ sum on the faces be 9. Let B be the event that atleast one number is 6 find the (i) $P(A \cap B)$ (ii) $P(A \cup B)$ (iii) $P(A^c \cup B^c)$

Sol:- Let A be the event the sum on the faces is 9 $A = \{(3,6)\ (4,5)\ (5,4)\ (6,3)\}$

Let B be the event atleast one number is 6

$B = \{(1,6)\ (2,6)\ (3,6)\ (4,6)\ (5,6)$
$(6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6)\}$

$P(A) = \frac{4}{36} = \frac{1}{9}$

$P(B) = \frac{11}{36}$

(i) $A \cap B = \{(3,6)\ (6,3)\}$

$P(A \cap B) = \frac{2}{36} = \frac{1}{18}$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{9} + \frac{11}{36} - \frac{1}{18}$

$P(A \cup B) = \frac{13}{36}$

(iii) $P(A^c \cup B^c) = P(A^c) + P(B^c) - P(A^c \cap B^c)$

(ii

$P(A^c) = 1 - \frac{1}{9}$ $\quad P(B^c) = 1 - \frac{11}{36}$

$= 8/9$ $\quad = \frac{25}{36}$

$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

$= 1 - \frac{13}{36}$

$= \frac{23}{36}$

$= P(\overline{A \cap B})$

$= 1 - P(A \cap B)$

$= 1 - \frac{1}{18}$

$= \frac{17}{18}$

④ If E_1 and E_2 are independent events of a sample space S then prove that (i) $\bar{E_1}$ & $\bar{E_2}$ are independent.
(ii) $\bar{E_1}$ & E_2 are independent (iii) E_1 & $\bar{E_2}$ "

sol:- $\boxed{P(E_1 \cap E_2) = P(E_1)P(E_2)}$

(i) $P(\bar{E_1} \cap \bar{E_2}) = P(\overline{E_1 \cup E_2})$

$= 1 - P(E_1 \cup E_2)$

$= 1 - [P(E_1) + P(E_2) - P(E_1 \cap E_2)]$

$= 1 - P(E_1) - P(E_2) + P(E_1) P(E_2)$

$= - P(E_1)[1 - P(E_2)] + 1[1 - P(E_2)]$

$$= [1 - P(E_2)]\,[1 - P(E_1)]$$

$$= P(\bar{E}_2)\,P(\bar{E}_1)$$

$$= P(\bar{E}_1) \cdot P(\bar{E}_2)$$

(ii) $P(\bar{E}_1 \cap E_2) = P(E_2) - P(E_1 \cap E_2)$

$$= P(E_2) - P(E_1)\,P(E_2)$$

$$= P(E_2)\,[1 - P(E_1)]$$

$$= P(\bar{E}_1)\,P(E_2)$$

⑤ What is the probability that a card drawn at random from a pack of playing cards may be either a queen or a king.

Sol:- There are 52 cards.

Let A be the event of drawing a queen.

Let B be the event of drawing a king.

$$P(A) = \frac{4}{52} = \frac{1}{13}$$

$$P(B) = \frac{4}{52} = \frac{1}{13}$$

$P(A \cap B) = 0$ (there is no common)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{13} + \frac{1}{13} - 0 \Rightarrow \frac{2}{13}$$

⑥ 6 cards are drawn from a pack of 52 cards find probability (i) atleast 3 diamonds (ii) ~~atleast~~ 4 are diamonds.

Sol:- Total cards $- {}^{52}C_6 - 20358520$

(i) atleast 3 diamond cards, diamond - 13, remaining - 39

$$= ({}^{13}C_3 \times {}^{39}C_3) + ({}^{13}C_4 \times {}^{39}C_2) + ({}^{13}C_5 \times {}^{39}C_1) + {}^{13}C_6$$

$$= 3193762$$

No. of favourable events $= 3193762$

$$P(A) = \frac{3193762}{20358520} = 0.156$$

(ii) atleast 4 card are diamond.

diamond - 13
remaining - 39

$= ({}^{13}C_4 \times {}^{39}C_2) \cancel{+ ({}^{13}C_5 \times {}^{39}C_1) + {}^{13}C_6}$

$= \cancel{581724}$ 529815

No. of ~~favourable~~ Total events = 20358520

$P(B) = \frac{\cancel{581724}\ 529815}{20358520} = 0.026$

(7) In a single or pair of die what is probability of obtaining sum of 2 faces of the dice is atleast 10

sol:- Total events = 36

favourable events = {(4,6) (6,4) (5,6) (6,5) (6,6) (5,5)}

$P(A) = \frac{6}{36} = \frac{1}{6}$

(8) When two dies are thrown simultaneously if x and y denote the numbers on the first and second respectively. Find the probability for x+y to be greater than or equal to 8.

sol:- Total events = 36

favourable events = {(4,4) (5,3) (3,5) (2,6) (6,2), (3,6)
(4,5) (4,6) (5,4) (5,5) (5,6)
(6,3) (6,4) (6,5) (6,6)}

$P(A) = \frac{15}{36}$

$= 5/12$

⑨ A fair die is thrown twice find the probability of getting 4, 5, 6 on first toss and 1, 2, 3 or 4 on second toss.

Sol:- Let A be the event of getting 4, 5, 6

$A = \{4, 5, 6\}$

$P(A) = \frac{3}{6} = \frac{1}{2}$

Let B be the event of getting 1, 2, 3, 4

$B = \{1, 2, 3, 4\}$

$P(B) = \frac{4}{6} = 2/3$

$P(A \cap B) = P(A) \cdot P(B)$

$= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{3}$

⑩ In a group consisting of equal no. men and women 10% of the men and 45% of women are unemployed. If a person is selected randomly from the group then find the probability that the person is an employee

Sol:- There are equal no. of men and women in a group.

Let M be the event of selecting men and W be the event of selecting women

$P(M) = 1/2$

$P(W) = 1/2$ $\because$ equal no. of men & women.

	M	W
unem.	10%	45%
em	90%	55%

Probability of unemployee in Men

(A) → unem - $P(A/M) = \frac{10}{100} = 0.1$

P(employ)

(B) → em - $P(B/M) = \frac{90}{100} = 0.9$.

Probability of unemploye in women $P(A/w) = 0.45$

Probability of employe. in women $P(B/w) = 0.55$

$\therefore$ The probability of selecting an employee

$$= P(M) \cdot P(B/M) + P(W) P(B/W)$$

$$= 1/2 (0.9) + 1/2 (0.55)$$

~~t 0.1/2 ...~~ to

$$= 0.725$$

(11) Let A, B, C are aiming to shoot a ballon. A will succeed 4 times out of 5 attemps. The chance of being B to shoot the ballon is 3 out of 4 and that of C is 2 out of 3. If 3 aim the balloon simultaneously then find the probability that atleast 2 of them hit the balloon.

Sol:- Given

$P(A) = \frac{4}{5}$

$P(B) = \frac{3}{4}$

$P(C) = 2/3$

The probability of A to hit a ballon is:

$$P(A) = \frac{4}{5}$$

The probability of B to hit a balloon is

$$P(B) = \frac{3}{4}$$

The probability of C to hit a balloon is

$$P(C) = \frac{2}{3}$$

The probability of not hitting ballon $= 1 - P(A)$

$$P(\bar{A}) = \frac{1}{5}$$

The probability of not hitting balloon $= 1 - P(B)$

$$P(\bar{B}) = \frac{1}{4}$$

The probability of not hitting balloon $= 1 - P(C)$

$$P(\bar{C}) = \frac{1}{3}$$

$\therefore$ The probability that atleast 2 members hit the balloon $= P(A \cap B \cap \bar{C}) + P(\bar{A} \cap B \cap C) + P(A \cap \bar{B} \cap C) + P(A \cap B \cap C)$

$$= P(A) \cdot P(B) \cdot P(\bar{C}) + P(\bar{A}) P(B) P(C) + P(A) \cdot P(\bar{B}) P(C) + P(A) P(B) P(C)$$

$$= \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} + \frac{1}{5} \frac{3}{4} \frac{2}{3} + \frac{4}{5} \cdot \frac{1}{4} \frac{2}{3} + \frac{4}{5} \frac{3}{4} \frac{2}{3}$$

$$= \frac{1}{5} + \frac{1}{10} + \frac{2}{15} + \frac{2}{5} = \frac{5}{6} = 0.83$$

(10) A box A contains 5 red and 3 white marbles and box B contains 2 red and 6 white marbles. If a marble is drawn from each box what is the probability that they are both of same colour

Sol:- Given

A → 5 red
3 white

B → 2 red
6 white

Let E_1 be the event of drawing a marble which is red color from box A

Let E_2 be the event of drawing a marble red from box B

($\frac{1}{2}$ box probability)

$$P(E_1) = \frac{1}{2} \cdot \frac{5}{8} = \frac{5}{16}$$

$$P(E_2) = \frac{1}{2} \cdot \frac{2}{8} = \frac{1}{8}$$

Probability of drawing red marbles from A and B.

$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$

$$= \frac{5}{16} \cdot \frac{1}{8} = \frac{5}{128}$$

Let E_3 be the event of drawing a white marble from box A.

" E_4 be the event of drawing a white marble from box B

$$E_3 = \frac{1}{2} \cdot \frac{^3C_1}{^8C_1} = \frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16}$$

$$E_4 = \frac{1}{2} \cdot \frac{^6C_1}{^8C_1} = \frac{1}{2} \cdot \frac{6}{8} = 3/8$$

The probability that they are both white

$$P(E_3 \cap E_4) = P(E_3)\,P(E_4)$$

$$= \frac{3}{16} \cdot \frac{3}{8} = \frac{9}{128}$$

∴ The probability of drawing balls of same colour from both boxes $= P(E_1 \cap E_2) + P(E_3 \cap E_4)$

$$= \frac{5}{128} + \frac{9}{128} = \frac{14}{128} = \frac{7}{64} = 0.109.$$

Baye's theorem :-

Let $E_1, E_2 \dots E_n$ are 'n' mutually exclusive and Exhaustive event such that the probability of $P(E_i) > 0$ $(i = 1, 2, 3 \dots n)$. In a sample space 'S' and A is any other event in 'S' intersecting with E_i occuring combination with any one of event $E_1 E_2 \dots E_n$ such that $P(A) > 0$. If E_i is any of the events $E_1 E_2 \dots E_n$ where $P(E_1), P(E_2) \dots P(E_n)$ and $P(A/E_1), P(A/E_2) \dots P(A/E_n)$ are known then

$$P(E_K/A) = \frac{P(E_K)\,P(A/E_K)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + \dots + P(E_n)P(A/E_n)}$$

Proof :- Let $E_1, E_2 \dots E_n$ are n events in a samplespace S

Since $E_1, E_2 \dots E_n$ are mutually exclusive and exhaustive events.

ie $E_i \cap E_j \neq 0$ where $i, j = 1, 2, 3 \dots n$

(mutually exclusive)

$E_i \cup E_j = S$ ie

$E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n = S$

Let A be any other event in sample space S intersecting with every E_i

$A \cap (B \cup C) = (A \cap B) \cup A(\cap C)$

consider

$$A = A \cap S$$
$$= A \cap [E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n]$$
$$= (A \cap E_1) \cup (A \cap E_2) \cup \dots \cup (A \cap E_n)$$

Here $(A \cap E_1)(A \cap E_2) \dots (A \cap E_n)$ are mutually exclusive events.

consider $$P\left(\frac{E_k}{A}\right) = \frac{P(E_k \cap A)}{P(A)}$$

$$= \frac{P(E_k \cap A)}{P((A \cap E_1) \cup (A \cap E_2) \dots \cup (A \cap E_n))}$$

$$= \frac{P(E_k \cap A)}{P(A \cap E_1) + P(A \cap E_2) + \dots + P(A \cap E_n)}$$

$$= \frac{P(E_k) P(A/E_k)}{P(E_1) P(A/E_1) + P(E_2) P(A/E_2) + \dots + P(E_n) P(A/E_n)}$$

$E_n)$

1. In a certain college 25% of boys and 10% of girls are studying mathematics. The girls constitute 60% of the students.

(a) What is the probability that maths is begin studied

(b) If a student is selected at random and is formed to be studying mathematics. find the probability that a student is i girl ii boy.

Sol:- Given

$B \to 25\%$

$G \to 10\%$

girls – 60%.

Girls percentage is.

$$P(G) = 60\% = \frac{60}{100} = \frac{6}{10} = 0.6$$

$$P(B) = 40\% = 0.4$$

Let M be the event of studying maths

Prob of maths studying boys

$$P(M/B) = 25\% = \frac{25}{100} = 0.25$$

Girls $P(M/G) = 10\% = \frac{10}{100} = 0.1$

(a) The proba of maths studying student.

$$P(M) = P(B)\,P(M/B) + P(G)\,P(M/G)$$

$$= (0.4)(0.25) + (0.6)(0.1)$$

$$= 0.1 + 0.06$$

$$P(M) = 0.16$$

(b)(i) The probability of girls in maths students

$$P(G/M) = \frac{P(G)\,P(M/G)}{P(G)\,P(M/G) + P(B)\,P(M/B)}$$

$$= \frac{0.06}{0.16}$$

$$= 0.375$$

(ii) $$P(B/M) = \frac{P(B)\,P(M/B)}{P(G)\,P(M/G) + P(B)\,P(M/B)}$$

$$= \frac{0.1}{0.16} = 0.625$$

2. In a bolt factory machines A, B, C manufacture 20%, 30% and 50% of their total output and 6%, 3% and 2% are defective. A bolt is drawn at random and found to be defective. find the probability that it is manufactured from

(i) A machine

(ii) B machine

(iii) C machine.

given

$A \to 20\% = \frac{20}{100} = \frac{2}{10} = 0.2 \Rightarrow P(A)$

$B \to 30\% = \frac{3}{100} = \frac{3}{10} = 0.3 \Rightarrow P(B)$

$C \to 50\% = \frac{50}{100} = \frac{5}{10} = 0.5 \Rightarrow P(C)$

3. First box contains 2 black, 3 red and 1 white
second box contains 1 black, 1 red and 2 white
third box contains 5 black, 3 red and 4 white

a box is selected at random from it a red ball is drawn randomly. If the ball is red. Find probability that it is from second box

sol:- Given

box A → black − 2
Red − 3
white − 1

BOX A → Black − 1
Red − 1
white − 2

BOX C → black − 5
Red − 3
white − 4.

$P(A) = \frac{1}{3}$ $P(B) = \frac{1}{3}$ $P(C) = \frac{1}{3}$

4. The chances that a politician a business man or an academician ~~academision~~ will be appointed as as Vice chancellor of university are 0.5, 0.3, 0.2 respectively Prob. that research is promoted by these persons if they are appointed as vice chancellor are 0.3, 0.7 & 0.8 respectively

(i) determine the Prob that the research is promoted

(ii) If the research is promoted what is the prob. that vice chancellor is an academision.

sol:-

$P(A) = 0.5$

$P(B) = 0.3$

$P(C) = 0.2$

$P(R/A) = 0.3$

$P(R/B) = 0.7$

$P(R/C) = 0.8$

(i) The prob that the research is promoted is

$$P(R) = P(R/A)\cdot P(A) + P(R/B)\cdot P(B) + P(R/C)P(C)$$

$$= (0.5)(0.3) + (0.7)(0.3) + (0.8)(0.2)$$

$$= 0.52.$$

(ii) If the research is promoted.

The prob that Vice Chancellor.

$$P(C/R) = \frac{P(C)P(R/C)}{P(R/A)P(A) + P(R/B)P(B) + P(R/C)P(C)}$$

$$= \frac{(0.2)(0.8)}{0.52}$$

$$= \frac{0.16}{0.52} = 0.307$$

5- Suppose 5 men out of 100 and 25 women out of 10,000 are colour blind a color blind person is choosen at random what is the probability of the person being a male. Assuming that male and female to be in equal numbers.

Sol:- Probability of seleleting Men $P(M) = \frac{1}{2}$

" " " women $P(W) = \frac{1}{2}$

The selecting color blind be $P(C)$

Probability of colour blind in men $P(C/M) = \frac{5}{100}$

" " " in women $P(C/W) = \frac{25}{10.000}$

Now

probability of men being color blind

$$P(M/C) = \frac{P(M)P(C/M)}{P(M)P(C/M) + P(W)P(C/W)}$$

$$= \frac{(\frac{1}{2})(\frac{5}{100})}{(\frac{1}{2})(\frac{5}{100}) + (1/2)(\frac{25}{10000})}$$

$$= \frac{0.025}{0.02625} = 0.66$$

6. Companies B_1 B_2 B_3 produce 30%, 45%, 25% of the cars respectively. It is known that 2%, 3% and 2% of the cars produced from B_1 B_2 B_3 are defective (i) what is the prob a car produced is defective (ii) If a is car is ~~produced~~ purchased found to be defective.

sol:- Given that

B_1 B_2 and B_3 companies produce car

$$P(B_1) = \frac{30}{100} = \frac{3}{10} = 0.03$$

$$P(B_2) = \frac{45}{100} = \frac{9}{20}$$

$$P(B_3) = \frac{25}{100} = \frac{1}{4}$$

* **Random Variable** :- A random variable 'X' in a sample space 'S' is a function from the set of sample space to the set of real numbers ie, a function $X : S \rightarrow R$ is known as random variable.

→ It is also defined as "a real variable 'X' whose value is determined by the outcome of a random experiment".

Ex When we toss 2 coins at a time, the sample space is $S = \{HH, HT, TH, TT\}$.
Let 'X' be the no. of heads then the random variable 'X' is given by

$$X = \{2, 1, 0\} = \{0, 1, 2\}.$$

* **Types of Random Variable** :-

There are two types of random variables.

(1) Discrete Random variable

(2) Continuous random variable

1. **Discrete R.V** :- A random variable 'X' which can take only a finite no. of discrete values in an interval is known as discrete random variable.

Ex A random variable denoting no. of students in a class is a discrete random variable

2. **Continuous R.V :-** A random variable which can take values continuously in the interval is known as continuous random variable

Exn The random variable denoting the heights of the students in a particular class may be b/w 4 feet and 6 feet.

$$X = \{ x : 4 \leq x \leq 6 \}$$

* **Probability distribution function :-**

Let 'x' be a random variable then the probability distribution function associated with 'x' is defined as the probability that the outcome of an experiment will be one of the outcomes for which $X(s) \leq x \ \forall \ x \in R$

ie., the function $F(x)$ defined by $F(x) = P(X \leq x) = P\{s : X(s) \leq x\}$ is known as distribution function of 'x'.

* ~~Note~~ **Discrete probability distribution :-**

Suppose 'x' is a discrete random variable with possible outcomes $x_1, x_2, \ldots, x_3$. The probability of each possible outcome x_i is $P_i = P(X = x_i) = P(x_i)$ for $i = 1, 2, 3, \ldots$

If the numbers $P(x_i), i = 1, 2, 3, \ldots$ satisfy the two conditions

(i) $P(x_i) > 0 \quad \forall$ 'i' ie, $0 < P(x_i) \leq 1$

(ii) $\Sigma P(x_i) = 1$; $i = 1, 2, 3, ---$

Then the function $P(x)$ is known as probability mass function of R.V. 'x' & the set $\{P(x_i): i = 1, 2, --\}$ is known as discrete probability distribution.

~~continuous probability distribution~~

* Expectation, mean, variance of Discrete R.V

Suppose a random variable 'X' assumes the values $x_1, x_2, ---, x_n$ w.r.t probabilities $P_1, P_2, ---, P_n$ then the mathematical expectation (or) mean (or) Expected value of 'x' denoted by $E(X)$ is defined as the sum of products of different values of 'x' & the corresponding probabilities.

$$\therefore E(X) = x_1 P_1 + x_2 P_2 + --- + x_n P_n$$

$$\text{ie, } E(X) = \sum_{i=1}^{n} P_i x_i$$

* Properties of Expectation

1. If 'x' is a R.V. & 'k' is a constant then $E(X+k) = E(X) + k$.

Proof: By def'n,

$$E(X) = \sum_{i=1}^{n} x_i P_i$$

$$\therefore \quad E(x+k) = \Sigma (x_i + k) P_i$$
$$= \Sigma x_i P_i + k \Sigma P_i$$
$$= E(x) + k \qquad | \because \Sigma P_i = 1$$

2. If 'x' is a R.V. & 'a' & 'b' are constants then
$$E(ax \pm b) = aE(x) \pm b$$

Proof:- By def'n,
$$E(x) = \sum_{i=1}^{n} x_i P_i$$

$$\therefore \quad E(ax \pm b) = \Sigma (a x_i \pm b) P_i$$
$$= a \Sigma x_i P_i \pm b \Sigma P_i$$
$$= aE(x) \pm b$$

3. If 'x' & 'y' are any two random variables then $E(x+y) = E(x) + E(y)$.

Proof:- By def'n,
$$E(x) = \sum_{i=1}^{n} x_i P_i \quad ; \quad E(y) = \sum_{j=1}^{m} P_j y_j$$

Consider $E(x+y) = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i + y_j) P_{ij}$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \left[x_i P_{ij} + y_j P_{ij} \right]$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} x_i P_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{m} y_j P_{ij}$$
$$= \sum_{i=1}^{n} x_i P_i \left[\sum_{j=1}^{m} P_j \right] + \sum_{j=1}^{m} y_j P_j \left[\sum_{i=1}^{n} P_i \right]$$
$$= \sum_{i=1}^{n} x_i P_i + \sum_{j=1}^{m} y_j P_j = E(x) + E(y)$$

4. If 'X' & 'Y' are two independent R.V. then

$$E(XY) = E(X).E(Y).$$

<u>Proof</u>: Let $E(X) = \sum_{i=1}^{n} x_i p_i$; $E(Y) = \sum_{j=1}^{m} y_j p_j$

$$E(XY) = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i y_j) p_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j p_i p_j$$ | ∵ 'X' & 'Y' indep $p_{ij} = p_i \cdot p_j$

$$= \left[\sum_{i=1}^{n} x_i p_i\right] \left[\sum_{j=1}^{m} y_j p_j\right]$$

$$= E(X).E(Y)$$

* Mean:

The mean value 'μ' of discrete distribution function is given by

$$\mu = \frac{\sum p_i x_i}{\sum p_i} = \sum p_i x_i = E(X).$$

$$\therefore \boxed{\mu = E(X) = \sum p_i x_i}$$

* Variance:

Variance of probability distribution of a R.V. 'X' is the mathematical expectation of $[X - E(X)]^2$.

$$\text{ie, } var(X) = E[X - E(X)]^2.$$

$$= E(X-\mu)^2 = E(X^2) - [E(X)]^2$$

* Properties of variance :-

1. Prove that $V(X) = E(X^2) - [E(X)]^2$

Proof:- By def^n, $V(X) = E(X-\mu)^2$

$$= E[X^2+\mu^2-2X\mu]$$

$$= E(X^2) + E(\mu^2) - 2\mu(E(X))$$

$$= E(X^2) + \mu^2 - 2\mu \cdot E(X)$$

$$= E(X^2) + \mu^2 - 2\mu^2 \quad | \because E(X) = \mu$$

$$= E(X^2) - \mu^2$$

$$= E(X^2) - [E(X)]^2$$

2. If 'X' is a R.V. then prove that $V(aX+b) = a^2 \cdot V(X)$.

Sol: By def^n, $V(X) = E(X^2) - [E(X)]^2$

$$\therefore V(aX+b) = E((aX+b)^2) - [E(aX+b)]^2$$

$$= E[a^2X^2+b^2+2abX] - [aE(X)+b]^2$$

$$= a^2E(X^2) + b^2 + 2ab\,E(X) - [a^2(E(X))^2 + b^2 + 2ab\,E(X)]$$

$$= a^2E(X^2) + \cancel{b^2} + \cancel{2abE(X)} - a^2[E(X)]^2 - \cancel{b^2} - \cancel{2abE(X)}$$

$$= a^2[E(X^2) - (E(X))^2] = a^2 \cdot V(X)$$

3. $V(X+a) = V(X)$
4. $V(aX) = a^2 V(X)$
5. $V(X+Y) = V(X) + V(Y)$ where 'X' & 'Y' are independent.

* Standard Deviation (σ)

It is a +ve square root of variance & is denoted by 'σ'

ie, $S.D\ (\sigma) = \sqrt{\text{Variance}}$

* Continuous probability Distribution

Consider the small interval $\left[x-\frac{dx}{2}, x+\frac{dx}{2}\right]$ of length dx around the point 'x'. Let $f(x)$ be any continuous function of x so that $f(x)dx$ represents the probability that the variable 'x' falls in the small interval $\left[x-\frac{dx}{2}, x+\frac{dx}{2}\right]$.

Symbolically, $f(x)dx = P\left[x-\frac{dx}{2} \leq x \leq x+\frac{dx}{2}\right]$

then $f(x)$ is known as probability density function when

(i) $f(x) \geq 0 \quad \forall\ x \in R$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad \forall\ x \in R$ [Total prob. = 1]

Mean, Variance of Continuous R.V

1. **Mean** Mean of the distribution is given by

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx.$$

* Mode is a value where $f(x)$ is maximum ie, $f'(x)=0$, $f''(x)<0$

*Median $\Rightarrow \int_{a}^{M} f(x)dx = \int_{M}^{b} f(x)dx = 1/2$

2. Variance :- Variance of the distribution is

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \quad (\text{or}) \quad \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

* Problems :-

① Let 'x' denotes the no. of heads in a single toss of 4 fair coins. Determine (i) $P(x<2)$
(ii) $P(1<x\leq 3)$

Sol: Let 'x' denotes the no. of heads when we toss 4 coins, at a time.

$\therefore\ x = \{4, 3, 2, 1, 0\}.$

The probability distribution is given by

X	0	1	2	3	4
P(X)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

(i) $P(x<2) = P(0) + P(1)$

$= \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$

$P(1<x\leq 3) = P(2) + P(3)$

$= \frac{6}{16} + \frac{4}{16} = \frac{10}{16} = \frac{5}{8}$

Sample space (S) = { HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, TTHH, THTH, TTTH, TTHT, THTT, HTTT, THHT, HTTH, TTTT }

2. A random variable 'X' has following probability function

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$.

Determine (i) k (ii) $P(X<6)$ (iii) $P(X\geq 6)$ (iv) $P(0<X<5)$

(v) $P(0\leq X\leq 4)$ (vi) Mean (vii) Variance.

Soln We know that

(i) Sum of probabilities $=1$

ie, $\Sigma P(x) = 1$

$\Rightarrow k+2k+3k+k^2+2k^2+7k^2+k=1$

$\Rightarrow 10k^2+9k-1=0 \Rightarrow (10k-1)(k+1)=0$

$\Rightarrow k = 1/10 = 0.1$ | $k \neq -1$ since $P(x)\geq 0$

$\therefore \boxed{k=0.1}$

The probability distribution is given by

x	0	1	2	3	4	5	6	7
$P(x)$	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

(ii) $P(X<6) = P(0)+P(1)+P(2)+P(3)+P(4)+P(5)$

$= 0+0.1+0.2+0.2+0.3+0.01 = \underline{0.81}$

(iii) $P(X\geq 6) = P(6)+P(7) = 0.02+0.17 = 0.19$

(iv) $P(0<X<5) = P(1)+P(2)+P(3)+P(4) = 0.1+0.2+0.2+0.3$

$= 0.8$

(v) $P(0 \le x \le 4) = P(0) + P(1) + P(2) + P(3) + P(4)$

$= 0 + 0.1 + 0.2 + 0.2 + 0.3 = 0.8$

(vi) Mean $(\mu) = \sum_{i=0}^{7} x_i p_i$

$= 0(0) + 1(0.1) + 2(0.2) + 3(0.2) + 4(0.3)$

$+ 5(0.01) + 6(0.02) + 7(0.17)$

$= 3.66$

(vii) Variance $(\sigma^2) = \sum p_i x_i^2 - \mu^2$

$= 0(0)^2 + 0.1(1)^2 + 0.2(2)^2 + 0.2(3)^2 + 0.3(4)^2$

$+ 0.01(5)^2 + 0.02(6)^2 + 0.17(7)^2 - (3.66)^2$

$= 3.4044$

3. A sample of 4 items is selected at random from a box containing 12 items of which 5 are defective. Find the expected no. of defective items.

Soln Let 'x' denote the no. of defective items

$\therefore$ Random variable $(x) = \{0, 1, 2, 3, 4\}$

No. of good items $= 7$

No. of defective items $= 5$

$\therefore P(x=0) = P(\text{no defective}) = \dfrac{{}^7C_4}{{}^{12}C_4} = \dfrac{7}{99}$

$P(x=1) = P(\text{one def \& 3 good}) = \dfrac{{}^7C_3 \times {}^5C_1}{{}^{12}C_4} = \dfrac{35}{99}$

$P(X=2) = P(2 \text{ def } \& \text{ 2 good}) = \frac{{}^7C_2 \times {}^5C_2}{{}^{12}C_4} = \frac{42}{99}$

$P(X=3) = P(3 \text{ def } \& \text{ 1 good}) = \frac{{}^7C_1 \times {}^5C_3}{{}^{12}C_4} = \frac{14}{99}$

$P(X=4) = P(\text{all are def}) = \frac{{}^5C_4}{{}^{12}C_4} = \frac{1}{99}$

The probability distribution is

X	0	1	2	3	4
$P(x)$	$7/99$	$35/99$	$42/99$	$14/99$	$1/99$

$\therefore$ Expected no. of defective items $E(X) = \Sigma x_i P_i$

$= 0\left(\frac{7}{99}\right) + 1\left(\frac{35}{99}\right) + 2\left(\frac{42}{99}\right) + 3\left(\frac{14}{99}\right) + 4\left(\frac{1}{99}\right)$

$= \frac{165}{99}$

4. The probability distribution of variate 'X' is

X	0	1	2	3	4	5	6
$P(X)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find (i) k (ii) $P(X<4)$ (iii) $P(X \geq 5)$ (iv) $P(3 < X \leq 6)$

(v) Mean (vi) Variance.

5. From a lot of 10 items containing 3 defectives, a sample of 4 items is drawn at random. Let the random variable 'x' denote the no. of defective items in the sample. Find prob. distribution of 'x' when the sample is drawn without replacement.

6. Let 'x' denotes the minimum of two numbers that appear when a pair of fair dice is thrown once. Determine (i) Discrete probability distribution (ii) Expectation (iii) Variance.

7. Identify the value of 'k' if the function $f(x) = kx$ in $0 \le x < 1$ is a valid probability density function.

Sol: Given that,

$$f(x) = kx \quad ; \quad 0 < x < 1$$

we know that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \qquad | \because \text{sum of prob.} = 1$$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{1} kx\,dx = 1 \quad \Rightarrow k\int_{0}^{1} x\,dx = 1$$

$$\Rightarrow k\left[\frac{x^2}{2}\right]_0^1 = 1$$

$$\Rightarrow \frac{k}{2} = 1 \quad \Rightarrow \boxed{k = 2}$$

8. If $f(x) = k e^{-x/5}$; $x \ge 0$ is probability density function then find 'k'.

9. If p.d.f $f(x) = kx^3$; $0 \le x \le 3$ find the value of 'k'.

10. The probability density $f(x)$ of a continuous random variable is given by $f(x) = c.e^{-|x|}; -\infty < x < \infty$ show that $c = \frac{1}{2}$ & find (i) mean & (ii) Variance.

Sol: Given that

$$f(x) = c.e^{-|x|} \quad ; -\infty < x < \infty$$

we know that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} c\,e^{-|x|}\,dx = 1 \Rightarrow c\int_{-\infty}^{\infty} e^{-|x|}\,dx = 1$$

$$\Rightarrow 2c\int_{0}^{\infty} e^{-|x|}\,dx = 1 \quad \Big|\because e^{-|x|} \text{ is even func}$$

$$\Rightarrow 2c\int_{0}^{\infty} e^{-x}\,dx = 1$$

$$\Rightarrow 2c\left[-e^{-x}\right]_0^{\infty} = 1 \Rightarrow -2c\left[e^{-\infty} - e^{-0}\right] = 1$$

$$\Rightarrow -2c\left[0-1\right] = 1 \Rightarrow +2c = 1$$

$$\Rightarrow \boxed{c = \frac{1}{2}}$$

$$\therefore f(x) = c.e^{-|x|} = \frac{1}{2}e^{-|x|}$$

(i) Mean $(\mu) = \int_{-\infty}^{\infty} x\,f(x)\,dx$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x\,e^{-|x|}\,dx = 0 \quad \Big|\because x e^{-|x|} \text{ is odd fun}$$

(ii) Variance $(\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

$$= \int_{-\infty}^{\infty} (x-0)^2 \frac{1}{2} e^{-|x|}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|}\,dx$$

$$= 2\left(\frac{1}{2}\right)\int_0^\infty x^2 e^{-|x|}\,dx \quad \Big|\because x^2 e^{-|x|} \text{ is even}$$

$$= \int_0^\infty x^2 e^{-x}\,dx$$

$$= \left\{x^2\left[\frac{e^{-x}}{-1}\right] - 2x\left[\frac{e^{-x}}{1}\right] + 2\left[\frac{e^{-x}}{-1}\right]\right\}_0^\infty$$

$$= 0-(-2) = 2$$

11. Probability density function of random variable 'x'

is $f(x) = \begin{cases} \frac{1}{2}\sin x \ ; & 0 \le x \le \pi \\ 0 \ ; & \text{elsewhere} \end{cases}$

Find mean, mode, median of the distribution & also find Probability b/w 0 & $\pi/2$.

Sol: Given that

$$f(x) = \begin{cases} \frac{1}{2}\sin x \ ; & 0 \le x \le \pi \\ 0 \ ; & \text{elsewhere} \end{cases}$$

(i) Mean $= \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_{-\infty}^{0} x(0)\,dx + \int_0^\pi x\left(\frac{1}{2}\right)\sin x\,dx + \int_\pi^\infty x(0)\,dx$$

$$= \frac{1}{2}\int_0^\pi x\sin x\,dx = \frac{1}{2}\Big[-x\cos x + \sin x\Big]_0^\pi$$

$$= \pi/2$$

(ii) Mode is the value of x where $f(x)$ is max.

Now, $f'(x) = \frac{1}{2}\cos x$

If $f'(x)=0 \Rightarrow \cos x = 0$

$\therefore x = \pi/2$

$$f''(x) = -\frac{1}{2}\sin x$$

At $x = \pi/2 \Rightarrow f''(x) = -\frac{1}{2} < 0$

Hence, $f(x)$ is max. at $x = \pi/2$

$\therefore$ Mode $= \pi/2$

(iii) Median is given by,

$$\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_0^M \frac{1}{2}\sin x\,dx = \int_M^\pi \frac{1}{2}\sin x\,dx = \frac{1}{2}$$

Consider, $\int_0^M \frac{1}{2}\sin x\,dx = \frac{1}{2}$

$$\Rightarrow \frac{1}{2}\left(-\cos x\right)_0^M = \frac{1}{2}$$

$$\Rightarrow -\cancel{\tfrac{1}{2}}\left[\cos M - 1\right] = \cancel{\tfrac{1}{2}}$$

$$\Rightarrow 1 - \cos M = 1 \Rightarrow \cos M = 0$$

$$\Rightarrow M = \pi/2$$

$\therefore$ Median $= \pi/2$

Hence, Mean $=$ Median $=$ Mode $= \pi/2$.

(iv) $P(0 < x < \pi/2) = \int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} \frac{1}{2}\sin x\,dx$

$$= \frac{1}{2}\left[-\cos x\right]_0^{\pi/2} = \frac{1}{2}[0-1] = \frac{1}{2}$$

12 If a random variable has probability density

$$f(x) = \begin{cases} 2e^{-2x} ; & x > 0 \\ 0 ; & x \le 0 \end{cases}$$ find probabilities

(i) between 1 & 3 (ii) greater than 0.5

* Chebyshev's inequality :-

If 'μ' and 'σ' are the mean and standard deviation of a random variable 'x' and $\sigma \neq 0$ then for any positive constant 'k'

$$P\left[|X-\mu| < k\sigma\right] \geq 1-\frac{1}{k^2}$$

ie, the probability that 'x' will take on a value with in 'k' standard deviation of the mean is atleast $1-\frac{1}{k^2}$.

Note: chebyshev's theorem places bound on the probability that the values of a distribution will be within a certain interval around the mean.

Ex: Atleast 75% of all values of a distribution fall within 2 standard deviations of the mean.

ie, $k=2$.

1. According to chebyshev's theorem atleast what % of data values lies b/w $\bar{X}-4\sigma$ & $\bar{X}+4\sigma$.

Sol: By taking $k=4$.

using chebyshev's theorem,

$$P\left[|X-\mu| < k\sigma\right] \geq 1-\frac{1}{k^2}$$

$$1-\frac{1}{k^2} = 1-\frac{1}{4^2}$$

$$= 1-\frac{1}{16}$$

$$= \frac{15}{16} = 0.937$$

$$= 93.7\%$$