Multi-Level Graph Spanners (CSC 620)

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Overview

Introduction

- Given a graph $G(V, E)$, a subgraph $G'(V, E')$ is a t-spanner of G, if for every $u, v \in V$, the distance from u to v in G' is at most t times longer than the distance in G. We refer to t as the *stretch factor* of G'.

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- Peleg et al. [1] have shown that this is an NP-complete problem.
- A reduction from the Edge Dominating Set problem for bipartite graph.

MLST Problem

Definition 1 (Multi-Level Spanner Problem)

Given a sequence of positive integers $t_1, t_2, \ldots, t_l$ (Note that we have not mentioned the relationship among the stretch factors. We will consider either $t_1 < t_2 < \ldots < t_l$ or $t_1 > t_2 > \ldots > t_l$.), a multi-level $(t_1, t_2, \ldots, t_l)$-spanner of G is a set of spanners $\{t_i\text{-spanner} : i \in \{1, 2, \ldots, l\}\}$ such that t_i-spanner $\subseteq t_{i+1}$-spanner and $\sum_{i \in \{1,2,\ldots,l\}} c(t_i\text{-spanner})$ is minimum where $c(G')$ is the sum of the weights of all edges of G' for weighted graphs and for unweighted graphs it is the number of edges.

Multi-Level Spanners

Definition 2 (Multi-Level Spanner with Nested Terminal Sets)

Given an input graph G, a sequence of positive integers $t_1, t_2, \ldots, t_l$ ($t_1 < t_2 < \ldots < t_l$ or $t_1 > t_2 > \ldots > t_l$) and l nested terminal sets $T_1 \subset T_2 \subset \ldots \subset T_l \subset V$, a $(t_1, t_2, \ldots, t_l)$-spanner with nested terminal sets is a series of spanners $G_1, G_2, \ldots, G_l$ such that

1. The stretch factor of G_i is equal to t_i
2. $G_{i-1} \subseteq G_i$
3. The vertex set of G_i contains T_i
4. The sum $\sum_{i \in \{1,2,\ldots,l\}} c(t_i\text{-spanner})$ is minimum.

Multi-Level Spanners

Definition 3 (Additive Spanner)

Let $G(V, E)$ be an unweighted graph. We denote the shortest path distance between two vertices $u, v \in V$ in G by $d_G(u, v)$. A subgraph $G'(V, E')$ is an additive t-spanner of G, if for every $u, v \in V$, $d_{G'}(u, v) \leq d_G(u, v) + t$. We refer to t as the *additive stretch factor* of G'.

Multi-Level Spanners

Definition 4 (Multi-Level Additive Spanner)

Given an input graph G, an integer $t_1 < t_2 < \cdots < t_l$ and l nested terminal sets $T_1 \subset T_2 \subset \ldots \subset T_l \subset V$, a multi-level additive $(t_1, t_2, \cdots, t_l)$-spanner is a series of spanners $G_1, G_2, \ldots, G_l$ such that

1. The additive stretch factor of G_i is equal to t_i
2. $G_{i-1} \subseteq G_i$
3. The vertex set of G_i contains T_i
4. The sum $\sum_{i \in \{1,2,\ldots,l\}} c(t_i\text{-spanner})$ is minimum.

Multi-Level Spanners

Definition 5 (A Simple Multi-level Additive Spanner)

1. Every level have same additive stretch: $t_1 = t_2 = \cdots = t_l = 2$.
2. Also there is a pattern on the size of the terminal sets. The size of the terminal sets increase exponentially as we go from top to bottom: $|T_1| = 2, |T_2| = 4, \ldots, |T_l| = 2^l$.

Known Results

- Cygan et al. [2] have shown how to construct $O(n\sqrt{|S|})$ size subset-wise additive spanner for stretch 2 where $|S|$ is the size of the subset.
- Kavitha [3] has studied pairwise spanners with additive stretch factors 4 and 6.

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Compute Independent Spanners

- First we assume that the vertex set of every level is same.
- We run the greedy algorithm l times independently with stretch factors $t_1, t_2, \ldots, t_l$ where every time the input graph of the algorithm is G.
- If we assume $t_i = i$, then the total number of edges is $n^{1+\frac{1}{1}} + n^{1+\frac{1}{2}} + \cdots + n^{1+\frac{1}{l}} \leq n^{1+\frac{1}{2}}(1 + \frac{1}{n^{\frac{1}{6}}} + \frac{1}{n^{\frac{1}{4}}} + \frac{1}{n^{\frac{3}{10}}} + \cdots) \leq (1+\epsilon)n^{1+\frac{1}{2}}$, where $\epsilon \leq 2$.

Compute Independent Spanners

Lemma 6

There exists a graph having two independent spanners t_i-spanner and t_{i+1}-spanner such that t_{i+1}-spanner $\not\subseteq$ t_i-spanner.

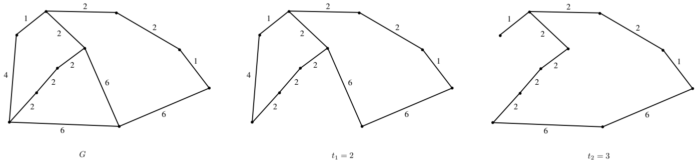

Figure: Independent inconsistent spanners

An Iterative Approach

- Generate spanner from G having stretch t_1.

An Iterative Approach

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- To generate a t_2-spanner, instead of giving G as input we input the t_1-spanner we have computed in the previous step.

An Iterative Approach

Filtser et al. [4] have provided the following theorem that provides the relation between iterative spanners.

Lemma 7

Let $G = (V, E, w)$ be any weighted graph, let $t > 1$ be any stretch parameter, and let H be the greedy t-spanner of G. If H' is a t-spanner for H, then $H' = H$.

An Iterative Approach

- What happens if the stretch factor of two levels are different ($t_1 \neq t_2$)?

An Iterative Approach

- What happens if the stretch factor of two levels are different ($t_1 \neq t_2$)?
- We want to find the relationship between the spanner of second level and the input graph G (Is it a t_2-spanner of G?).

An Iterative Approach

Lemma 8

Let $G = (V, E, w)$ be any weighted graph, let $1 < t_1 < t_2$ be two stretch parameters, and let G_1 be the greedy t_1-spanner of G. If we compute a greedy spanner G_2 from G_1 with stretch parameter t_2, then G_2 may not be a t_2-spanner of G.

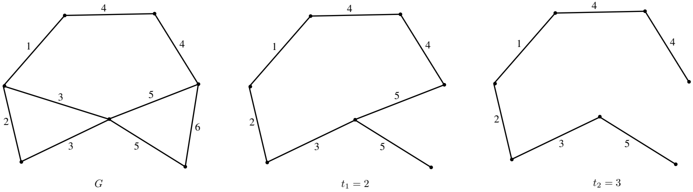

Figure: Spanners generated by iterative approach

An Iterative Approach

Lemma 9

Let $G = (V, E, w)$ be any weighted graph, let $1 < t_1 < t_2$ be two stretch parameters, and let G_1 be the greedy t_1-spanner of G. If we compute a greedy spanner G_2 from G_1 with stretch parameter t_2, then G_2 is a $(t_1 t_2)$-spanner of G.

An Iterative Approach

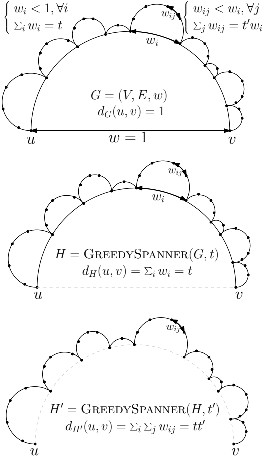

Figure: Spanners generated by iterative approach

Top-down and bottom-up approaches

Given: a nested set of vertices $V_1 \subset V_2 \subset \cdots \subset V_l$ where V_i is the terminal set of level l. We propose two algorithms to solve this general problem:

A Bottom-up Approach

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- Decreasing stretch factors: $t_1 > t_2 > \cdots > t_l$ where t_i is the stretch factor of level i.
- Idea: find the spanner G_l of the bottom-most level.
- G_l is a t_l-spanner of G and also a valid t_{l-1}-spanner, as $t_{l-1} > t_l$.

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- Problem: possibly extra edges in G_l. We may be able to remove some and still have a t_{l-1}-spanner.
- Try to remove edges from G_l to get a spanner for level $l-1$. We repeat for the remaining spanners $G_{l-2}, G_{l-3}, \cdots, G_1$.

Bottom-up: Method

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- For a t_{l-1} spanner, spanning V_{l-1}: divide edges in G_l into three types: neither vertex belongs to V_{l-1}, exactly one belongs, or both do.

Bottom-up: Method

- Consider every edge by descending order of weight.
- Remove it from the t_l-spanner and find all pair shortest paths. If there are two vertices in V_{l-1} such that the shortest path distance between them is more than t_{l-1} times the shortest path in G, then we put back the removed edge.
- Repeat for the second and third types of edges to get a t_{l-1}-spanner. Iteratively, we find spanners for each level on top.

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- Idea: ignore stretch factor by always taking the shortest paths.
- Find a t_1-spanner that spans the vertices in V_1.
- Use APSP to retain edges that appear in the shortest path between any pair of vertices in V_1.
- Repeat the procedure for V_2 by using the retained edges in the t_2-spanner, and so on.

Note on Steiner Tree Approximation

- The Steiner tree 2-approximation algorithm is similar. But it does not always give us a valid 1-spanner, since it finds the MST from the APSP.
- The APSP is always a valid 1-spanner. But after computing the MST, this is not true.

Note on Steiner Tree Approximation

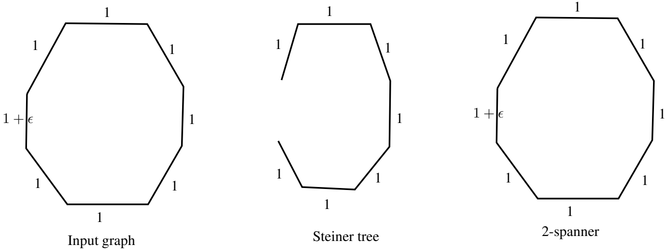

Figure: Here the Steiner tree is a 8-spanner but not a 2-spanner. And if we increase the number of vertices in the cycle, then we can show that the Steiner tree is not a valid t-spanner for any value of t. Note that here the edges represent a path, and the vertices represent terminals.

ILP Formulation

Definition 10 (Spanner Using ILP)

Given a connected, undirected graph $G = (V, E)$ with edge weights c_e, $e \in E$, a set of node pairs $K = \{(u_i, v_i)\}, i = 1, \cdots, k$, and a stretch factor $t > 1$:

Find $G' = (V, E')$ such that:

1. the size of $E' \in E$ is minimum
2. for every pair $(u_i, v_i) \in K$: the shortest path between (u_i, v_i) in G' is at most t times the shortest path in G.

If $K = V \times V$, then it is the spanner problem.

ILP Terminology

- P_{uv}: set of paths between u and v with length less than or equal to t times the shortest path in G, for all $(u, v) \in K$
- $P = \cup_{(u,v)\in K} P_{uv}$.
- δ_p^e: indicator variable stating whether the edge e is present in path p or not.
- x_e and y_p: decision variables, indicating whether the edge e and the path p is present in the spanner.

IP Formulation

$$\text{minimize} \sum_{e \in E} c_e x_e \tag{1}$$

Subject to:

$$\sum_{p \in P_{uv}} y_p \delta_p^e \leq x_e, \forall e \in E, \forall (u,v) \in K \tag{2}$$

$$\sum_{p \in P_{uv}} y_p \geq 1, \forall (u,v) \in K \tag{3}$$

$$x_e = \{0,1\}, \forall e \in E \tag{4}$$

$$y_p = \{0,1\}, \forall p \in P \tag{5}$$

Restricted Master Problem

- The number of variables may be exponential; thus, we solve the restricted master problem (RMP) instead:

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$$\text{minimize} \sum_{e \in E} c_e x_e \tag{6}$$

Subject to:

$$\sum_{p \in P_{uv}} y_p \delta_p^e \leq x_e, \forall e \in E, \forall (u,v) \in K \tag{7}$$

$$\sum_{p \in P_{uv}} y_p \geq 1, \forall (u,v) \in K \tag{8}$$

$$x_e \geq 0, \forall e \in E \tag{9}$$

$$y_p \geq 0, \forall p \in P' \tag{10}$$

where $P'_{uv} \subseteq P_{uv}, P'_{uv} \neq \emptyset, \forall (u,v) \in K$ and $P' = \cup_{(u,v) \in K} P'_{uv}$.

Solving the LP

- Idea: delayed column generation
- Can start with only one path in P'_{uv}.
- Iteratively, add paths to P'.
- Now consider the dual LP with variables π_e^{uv} and $\sigma_{uv}, \forall e \in E, \forall (u, v) \in K$:

Dual LP Formulation

$$\text{maximize} \sum_{(u,v)\in K} \sigma_{uv} \tag{11}$$

Subject to:

$$\sum_{(u,v)\in K} \pi_e^{uv} \leq c_e, \forall e \in E \tag{12}$$

$$-\sum_{e\in E} \delta_p^e \pi_e^{uv} + \sigma_{uv} \leq 0, \forall p \in P_{uv}, \forall (u,v) \in K \tag{13}$$

$$\pi_e^{uv} \geq 0, \forall e \in E, \forall (u,v) \in K \tag{14}$$

$$\sigma_{uv} \geq 0, \forall (u,v) \in K \tag{15}$$

Solving the LP

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- Else we add a path in P' by solving the resource-constrained shortest path problem.
- For multi-level spanners, an edge selected at the ith level should also be selected in all lower levels.
- Denote the selection of an edge e at level l by x_e^l where $x_e^l \in \{0, 1\}$. Add the constraint:

$$x_e^l \geq x_e^{l-1}, l \in \{2, 3, \ldots, k\} \tag{16}$$

Solving the LP

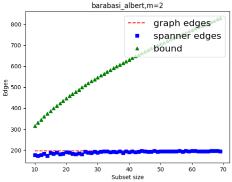
The primal LP remains the same for this formulation. We introduce another dual variable m_l^e, and the corresponding constraints become:

$$\sum_{(u,v)\in K} \pi_e^{uv} + m_l^e \leq c_e, \forall e \in E \tag{17}$$

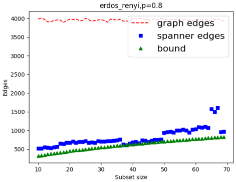
Experimental Results

- Test the algorithm for additive stretch 2 provided by Cygan et al. [2]
- Four graph generation models used
  - Erdös-Renyi, random geometric, Watts-Strogatz, Barabási-Albert
- Number of vertices $|V| = 100$
- Subset size $k = 10, \cdots, 100$

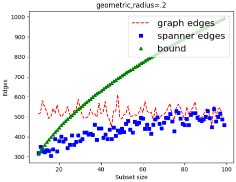
Plots



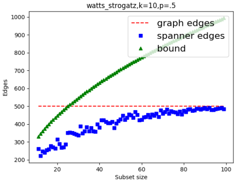
(a) BarabsiAlbert



(b) Erdős–Renyi



(c) Random Geometric



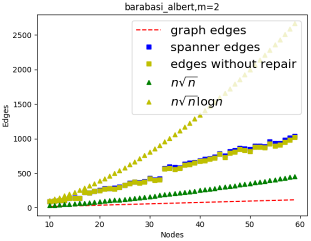
(d) Watts-Strogatz

Figure: Results for the single level spanner with additive stretch 2

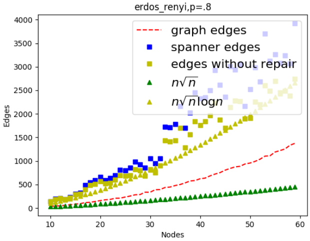
Experimental Results

- Consider the simple multi-level additive spanner
  - The size of the terminal sets increases exponentially
  - Every level has stretch 2
- Four graph generation models used
  - Erdös-Renyi, random geometric, Watts-Strogatz, Barabási-Albert
- Number of vertices $|V|$ up to 60

Plots



(a) BarabsiAlbert



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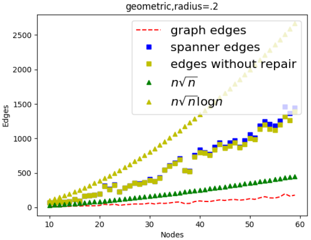

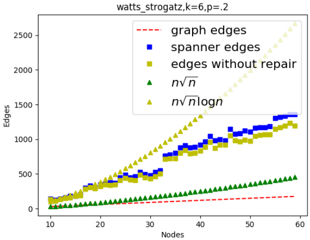

Plots

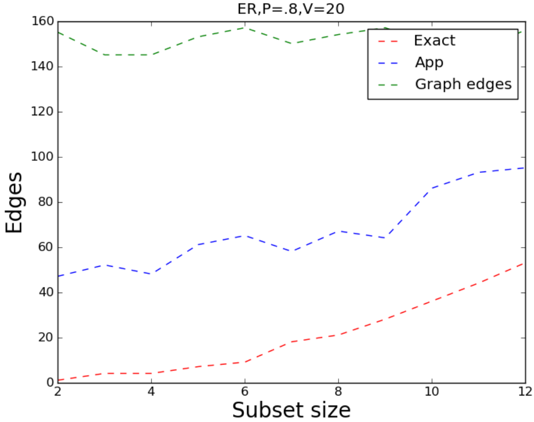

Figure: A comparison between the exact solution and the algorithm provided by Cygan et al. [2]

Future Work

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- Implement the exact algorithm for multi-level.
- Run experiment for large graphs.
- Consider the multiplicative version.

P. David and S. A. A., "Graph spanners," *Journal of Graph Theory*, vol. 13, no. 1, pp. 99–116.

M. Cygan, F. Grandoni, and T. Kavitha, "On Pairwise Spanners," in *30th International Symposium on Theoretical Aspects of Computer Science (STACS 2013)* (N. Portier and T. Wilke, eds.), vol. 20 of *Leibniz International Proceedings in Informatics (LIPIcs)*, (Dagstuhl, Germany), pp. 209–220, Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2013.

T. Kavitha, "New pairwise spanners," *Theory of Computing Systems*, vol. 61, pp. 1011–1036, Nov 2017.

A. Filtser and S. Solomon, "The greedy spanner is existentially optimal," in *Proceedings of the 2016 ACM Symposium on Principles of Distributed Computing*, PODC '16, (New York, NY, USA), pp. 9–17, ACM, 2016.