Stamatics
IIT Kanpur

Mathemania 2022

26 December, 2022 **ROUND 2**

"WELCOME, WELCOME!" You hear the same booming voice as you, Posy and the victim enter the empty field. You notice the remaining tributes all staring at each other with baffled expressions.

"For the following two trials, we would like to call out five people who have impressed us the most so far, and since the games aren't progressing as fast as we would like it to, two of the three will be eliminated. "

"Impressing us with their mathematical and logical prowess we call ahead Posy, Sparkle, and ....". As you squeeze your eyes tight praying for your name not to be called, you hear your name booming throughout the arena."

Problem 1

You are dropped onto an arena which has the form of a 2023 by 2023 square grid.
"Pick a square to start on. As soon as I fire this gun, you must start running to save your life. From what? You ask? From the Mechanical Snake Drillerton 3000 of course, which will spawn on a square adjacent to yours. This beast will chase you all over the arena, and the only way to beat it, will be to come back to the square you started on. My snake monster is wicked fast with an amazing memory however. If you ever try to take a step forward, it will strike and kill you, and if you ever come back to a square that it remembers you stepping on, it will strike and kill you. Its memory has a fixed size however, so provided you run a path long enough, it should forget where you started, and then you should be able to make it back home to live another day. Just to be safe, run the **longest** possible path following the given rules and end on the tile you started on. Here are the rules:

- You can only step on tiles adjacent to your current tile and not those diagonal to it.
- You cannot step on tiles that you have already visited earlier, except to satisfy the win condition in rule 4.
- You aren't allowed to take two steps in the same direction continuously, ie. the directions of two consecutive moves must be perpendicular.
- The way to win, is to go back to the tile you first started on.

Solution format: Submitting the number of squares in the longest path carries 100 points and uploading a brief explanation/ proof of the same carries 200 points.

Problem 2

"For this round you three will be given a calculator."
The MK-97 calculator can perform the following three operations on numbers in its memory:

- Determine whether two chosen numbers are equal.
- Add two chosen numbers together.
- For chosen numbers a and b, find the real roots of $x^2 + ax + b$, or announce that no real roots exist.

The results of each operation are accumulated in memory. Initially the memory contains a single number x. How can one determine, using the MK-97, whether x is equal to 1?

Solution format: Explain the steps used to reach the conclusion.

Problem 3

"This next trial is a duel".
You sit opposite to Posy on a table.
"Player 1 will pick n numbers from the first $2n$ natural numbers. Player 2 must find a pair of numbers from these n numbers which are either co-prime or their *gcd* is 2. If player 2 can do so, they win, and player 1 dies, else player 2 dies. Let's flip a coin to decide who gets to be player 1".
You call heads instinctively, as the coin flies in the air. Which player should you be to have a better chance at winning? Prove it.

Solution format: Mention the player number and prove why it is favourable.

Problem 4

The coin does not land in your favour in the previous round, and you lose the duel, but your sponsor decides to give you one last chance and decides to give you an easy question as your next trial.
"Can you find all such numbers n such that $2n^2+1, 3n^2+1$ and $6n^2+1$ are all perfect squares?", he asks you.

Solution format: For this question we need a proof. Either prove that the list of numbers you come up with are the only numbers which satisfy this condition, or prove that no such number satisfies this condition.

Problem 5

"In Hilbert's Grand Hotel are an uncountable infinite number of rooms and guests. Everything in this hotel has a real number ID. Every guest has one, every room has one and anything else you can imagine has one. And it all connects to each other using a bunch of mappings, and a rickety old PC from the 80s. Yikes! The programmer must have been a legend. But the rickety old PC does not support parallel processing, and can compute only one thing at a time, which makes things slow. But hey, in an infinite hotel, surely an infinite wait time isn't a problem.

Except, in comes a pandemic to mess things up. The guests anxiously want to hoard toilet paper rolls and simply asking the computer to compute the toilet paper function for every guest simply will not do. You decide to look up the wiki.

The toilet-paper-get function is a function $f : \mathbb{R} \to \mathbb{R}$, that takes in one real number input x, the guest ID, and returns the number of toilet paper rolls allotted to them, which may be any real number. After getting this number, use the toilet-paper-fetch function to get the IDs of the toilet paper you need to send to guest x. Interestingly enough, the toilet-paper-get function always obeys this relation -

$$f(x)f(y) = 2f(x + yf(x))$$

"But what does this have to do with the trial? And why would people hoard toilet paper rolls in a pandemic?" You ask the voice.
"Well, a team of brilliant minds has calculated that this is exactly what would happen in a pandemic and the Capitol would like to be prepared for such a situation. And you, dear tribute, are the latest intern at Hilbert's Grand Hotel, and you need to tell me all the possible toilet-paper-get functions."

Solution format: We expect a proof for this answer. Find all possible functions that satisfy this relation, and prove that these are the only such functions possible.

Problem 6

You are dropped onto an arena which has the form of a grid. Every point in this grid has a colour - pink, yellow, blue, black or red.
"The Graph Colouring Theorem states that one only needs four colours to assign every point in a loopless planar graph a colour such that no two neighbouring points have the same colour. I, however, in my infinite generosity will colour every point in my infinite grid with one of the 5 colours. In a short while, all these points will spawn spikes, impaling anyone standing anywhere near them. The only areas safe will be those rectangular areas whose vertices are all of the same colour. Go on now, find your safe zones!" The voice commands, and the three of you scramble off to find such rectangles. Can you find such a rectangle all of whose vertices are the same colour? Prove it.

Solution format: Explain if such a rectangle is possible to find with proof.

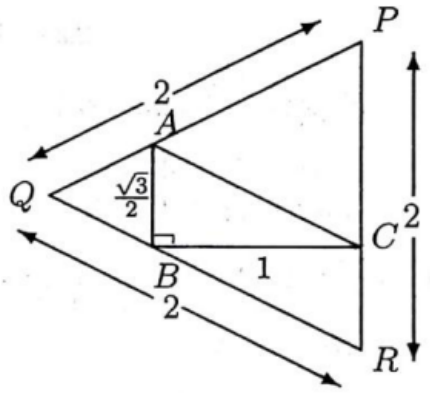
Problem 7

"Onwards we go! You're running out of time you see. Remember that challenge from the previous round? The one with too many ones? So you see I am not quite over that, a sequence of 1's is just so engaging. Now, what lengths of such a sequence are possible, such that the number they form are divisible by the length? I mean 1 is divisible by 1, 11 is not divisible by 2, but 111 is divisible by 3. So for what numbers n is the number formed by n 1's divisible by n? "
You make a face naturally, to which the voice responds, "What? Scared of infinity? Oh come on you're no fun. Alright just find which prime numbers satisfy this question, and tell me the sum of all such primes." The set of prime numbers is still infinite, but your gut tells you that this much information may be enough. Give the sum of such prime numbers as your answer. If the list is infinitely large, then write -1.

Solution format: Report the integer answer.

Problem 8

"Every mathematician loves good old equilateral triangles. They form such beautifully symmetric systems. Here, look what happens when I inscribe a right triangle inside one."



"Do you see the beauty of segment BR?" You nod along, this does not seem like the place to express disregard. "Hmm, I don't think you do. Find its length, then I'll believe you. Oh and if you can't, you'll die for your dishonesty." Find the length of segment BR.

Solution format: Report the length as a decimal rounded to three decimal places. In case there are multiple lengths possible, find their product, but report your answer as a decimal rounded to three decimal places.

Problem 9

Your parents stand in front of you. In a mix of shock and awe and surprise, you hug them, before the voice announces, "The touching reunion must be interrupted as you haven't won yet and this is a trial. All of you need to cross the river in this small boat that can carry only two people at once." "That's easy!", you begin to shout before you realise the catch, as three demons approach you. "Yes, except these 3 devils also need to cross the river, and if at any point, on either side, if the demons outnumber the humans, they will kill the humans. I'm very curious to see how these demons hunt their prey." Come up with a plan to ensure all 6 of you can cross the river safely. Assume that the demons will follow your instructions and can row the boat.

Solution format: Explain the plan in detail

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