Stamatics
IIT Kanpur

Mathemania 2022

26 December, 2022 **ROUND 1**

Panem. The far, far future. The games have been re-established, much to the horror of the people. The year is 2022 which, coincidentally also marks the 4^{th} Quarter Quell, the 100^{th} Hunger Games. This year, the twist is that the Game makers are mathematicians and so the Games will not only test your survival skills but also your mathematical skills....

Problem 1

Since it is the Quarter Quell the number of tributes this year is not the traditional 24. You watch the customary announcement by the President on the television. However, the President decides to have some fun. He gives the following equation: $f(x^3 + y^3) = xf(x^2) + yf(y^2)$. Given that $f'(0) = 5$, and $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Find $(f'(5))^2 + (f'(0))^2$, the number of tributes.

Solution format: An integer which represents the number of tributes.

Problem 2

It's Sorting Day, you are waiting for the names to be called out. Every citizen between the ages of $12 - 18$ has been assigned an identity number. The slips in the giant glass bowls contain the numbers. Effie rummages around the bowl and pulls out a slip. You watch with bated breath, waiting for her to announce the number. However, she says the following:

The identity number of the tribute is the number of 7×7 grids whose each square is either black or white. Two grids are considered to be different if it is not possible to rotate one of them so that they look the same.

Solution format: The **sum of digits** of the identity number of the tribute.

Problem 3

You realize that the answer matches with your identity number. You walk to the stage and from their you are taken to the Capitol. In the training arena, you are drawn to the puzzle's stall. The trainer comes to you with three boxes A, B, C with 0, 0 and 27 balls respectively. He wants you to sort the balls such that all boxes have the same number of balls. But, he says, at every n^{th} move, you should move exactly n balls from one box to another. What is the minimum number of moves needed to sort the balls?

Solution format: Integer representing minimum number of moves.

Problem 4

You have your individual session with the Game makers today and you are determined to show your best. You confidently ask them to pose any puzzle. They ask you the following question: Let A and B are two non singular matrices such that $3ABA^{-1} + A = 2A^{-1}BA$, then find the value of $\det(A + B)$.

Solution format: Integer representing the value of $\det(A + B)$.

Problem 5

It is the day of the interviews!! You are anxiously waiting for your turn, and go nervously to the stage when Caesar, the interviewer, calls out your name. "We heard about your amazing feat in the training arena", he says. "A solid 12! I am sure the audience would like to see you work that brain." He then defines the following legal operations on a quadratic polynomial $P(x) = ax^2 + bx + c$,

- O_1: Switch a and c
- O_2: Replace x by $x + t$, $t \in \mathbb{R}$

He then asks you to transform $x^2 - x - 2$ into $x^2 - x - 1$ using these operations.
Your reputation is at stake, you must answer quickly before your time is up.

Solution Format: Write 0 if the transformation is not possible and 1 if it is possible.

Problem 6

The time has come and all of today's tributes are being dropped into the arena. You rush to the cornucopia to grab some supplies. At the cornucopia, there are a large number of boxes labelled with numbers 105 to 210. You see the other tributes approaching and you have to pick fast without looking at the labels. A set of boxes is called **good** if it contains **atleast** two boxes whose numbers are not coprime. You can carry only a small number of boxes so find out the least number of boxes you need to pick such that the set is good.

Solution Format: The least number of boxes needed to guarantee that the set is good

Problem 7

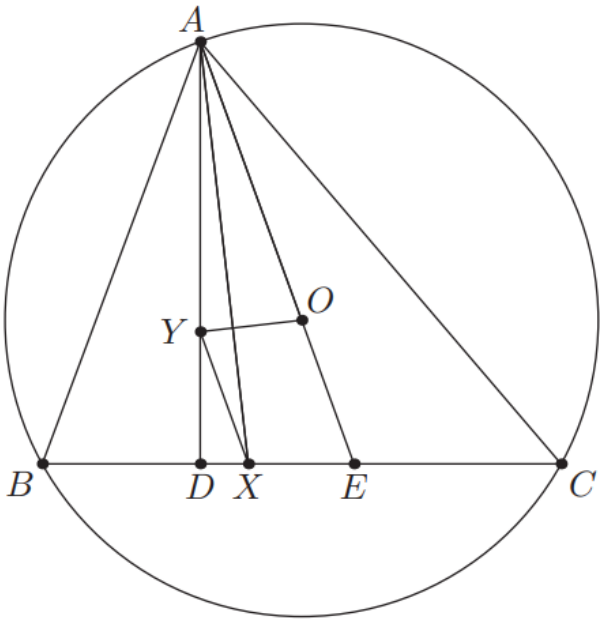
A day of harsh battle later, it is finally night time. But how many exactly died, you wonder? This time, the Game-makers decided to keep the names of those deceased a secret. Instead of the traditional cannons they posed the following puzzle:
Suppose that a, b, c, d are positive real numbers satisfying $(a + c)(b + d) = ac + bd$. Find the smallest possible value of $S = \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$
Where S is the number of contestants that died the first day?

Solution Format: Integer representing the value of S.

Problem 8

You are tired after a long day of hunting for food and are looking for some shelter when you come across a cave. Surprisingly, you find a geometry problem on the wall of the cave. Sleepless and ready for challenges, you decide to solve it...

Let ABC be an acute triangle with circumcenter O such that $AB = 7$, $AC = 8$, $BC = 10$. Let D be the foot of the altitude from A to BC and E be the intersection of lines AO and BC. Suppose that X is on BC between D and E such that there is a point Y on AD satisfying $XY \parallel AO$ and $YO \perp AX$. Determine the length of BX.



Solution Format: Value of $p + q$ where $p, q \in \mathbb{N}$ with $\gcd(p, q) = 1$ and $BX = \frac{p}{q}$.

Problem 9

The next day, you wake up to find the tribute from District 12, Posy staring at you. Alarmed, you jump up with your knife. You're just about to kill her when you realise that she has been keeping a watch on you. She wants to strike a deal with you. She has received food from her district but her parachute also has a lock. She is willing to share her food with you if you help her with the lock. Realising you haven't eaten for two days, you readily agree. This is what is written:
"Congratulations! That sure was not a cakewalk
$1! + 2! + 3! \ldots .2022!$ Is the hint for this lock!
Enter the last two digits and the lock will open,
But wait too long and death you have chosen"
Find the answer to the riddle to open the box!

Solution Format: Correct Integer for last two digits.

Problem 10

After a hasty meal, you and Posy go in search of some water. After a while, you come across a

strange hole that emits blue light. Posy bends over to look into the hole, causing both of you to tumble and fall into it. You find yourself in a mirror world, except the sky is red, much to your surprise. A deep voice echoes "Turn me blue with 2022 or the redness will take over you" You are supposed to somehow turn the sky to blue for this task. You see a pattern marked by sticks on the ground `0, 1, 2, 10, 11, 12, 20, 21, 22, 100, 101, 102`. You look around for help and suddenly some words start to appear beneath the marks - `NO, NO, NO, NO, NO, YES, NO, YES, NO, NO, NO, YES`. A stone was floating above the sticks, and it was embossed with "Sum the first 2022 numbers marked with a `YES` and the red sky shall suppress, but don't you dare add the numbers here!". You guys connect the dots. Turn the sky blue again or get trapped in utter dark redness!

Solution Format: Correct Integer
Hint: The given numbers represent the first 12 natural numbers in base n where n is a particular natural number. Find n first and then convert the ones marked with a `YES` to base 10 before adding!

Problem 11

Whew, The sky is blue again. Once you can see, you realize you are stuck in a room with a digital lock and a note which says:
Find the sum of xy over all positive integral solutions (x, y) of $x^2 + 7y^2 = 991$ or stay stuck forever. The answer may be the password for the lock.

Solution Format: Integer representing the password of the lock.

Problem 12

Finally free you and Posy return to your cave for the night. At around dawn, a nearby cry jolts you awake and Posy who had been keeping watch whispers to you, "Hey, I guess that's someone out there". You both creep out of your cave and notice that someone had been caught in a trap that you had not noticed before. The poor victim had tripped and fallen into a rapidly filling ditch with a digital gate. The gate is square shaped with 100 square divisions. A inscription inside the ditch reads...
Here are 100 squares for you to play with. Let each square be denoted by (i, j), $1 \leq i, j \leq 100$. To escape use the digital palette available, colour the grid such that squares (i, j), (j, k), (j, l), $1 \leq i < j \leq 100$ and $1 \leq k < l \leq 100$, do not have the same colours."

Since the victim is unconscious you both decide to help him out. How many minimum number of colours would you require for this task?

Solution Format: Correct integer for the minimum number of colours.

Right as you help the poor tribute out of the death trap, you hear a booming voice echo throughout the arena...

"All the remaining tributes are kindly requested to make their way, without killing each other, to the main arena in the arena centre!"