Mathemania 2022

Stamatics
IIT Kanpur

26 December, 2022 **ROUND 3**

Problem 1 (300 Points)

For some fixed $n \in \mathbb{N}$, find all the *Pythagorean Triplets* (x, y, z) of natural numbers satisfying:

$$\left(\frac{1}{x} + \frac{1}{y}\right)^2 = \frac{n}{z^2}$$

Problem 2 (300 Points)

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(2023) = 2022$ and the following holds,

$$f(x)f(f(x)) = 1 \qquad \forall x \in \mathbb{R}$$

Show that there exists $a, b \in \mathbb{R}$ such that f is differentiable on (a, b).

Problem 3 (400 Points)

Let $A = \{k \in \mathbb{N} : 1 \leq k \leq 2022, gcd(k, 2022) = 1\}$. Evaluate the following :

$$S = \prod_{k \in A} \cos\left(\frac{k\pi}{2022}\right)$$

Problem 4 (500 Points)

Let $\{a_n\}$ be a sequence of positive integers such that $a_k = a + k^2$, where $a \in \mathbb{N}$. Define a function $f : \mathbb{N} \to \mathbb{N}$, such that $f(t) = gcd(a_t, a_{t+1})$. Maximum of $f(t)$ occurs at $t = g(a)$ for some function $g : \mathbb{N} \to \mathbb{N}$. Define: $P(z) = \frac{1}{2^{\frac{z}{2}}(1+\sqrt[2^{\frac{z}{2}}]{2})}$ Evaluate:

$$\sum_{a=1}^{\infty} P(g(a))$$

Problem 5 (500 Points)

Find the number of ways in which the binary operators $+, -, \times$ or $\div$ can be used to fill the following n blanks such that the overall expression evaluates to 0 or 1.

$$\underbrace{1\_1\_1\_\,\ldots\ldots\ldots\,\_1\_1\_1}_{n+1 \text{ ones}}$$

The expression is evaluated according to the usual *BODMAS* rule.

Problem 6 (500 Points)

Find a closed form expression for the following series:

$$\sum_{n=1}^{\infty} \frac{1+2022^n}{2023^n} \frac{1}{n^2}$$

Problem 7 (500 Points)

The only allowed operations in straightedge and pencil constructions are the following -

- You can mark a point $Q \in \mathbb{R}^2$ on the plane using pencil.
- Given two points $R \in \mathbb{R}^2$ and $S \in \mathbb{R}^2$, you can construct the line segment RS.

Show that it is impossible to construct a perpendicular from a point $P \in \mathbb{R}^2$ to a straight line segment $AB \subset \mathbb{R}^2$, $P \notin AB$, using a straightedge and a pencil only.

END OF PAPER