Functors

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Last revision: 24th February 2022.

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1 DEFINITION

Definition 1.1 (Functors)**.** A functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is having the following functions, all indicated with F:

- a 'function on objects'

$$F : |\mathcal{C}| \to |\mathcal{D}| \ , \ x \to F(x)$$

- for every objects a and b, one 'function on morphisms'

$$F : \mathcal{C}(a,b) \to \mathcal{D}(F(a),F(b)) \ , \ f \to F(f)$$

such that

1. for every object x of $\mathcal{C}$ we have $F(1_x) = 1_{F(x)}$;
2. for every objects x, y, z and morphisms $f : x \to y$ and $g : y \to z$ of $\mathcal{C}$ we have $F(g)F(f) = F(gf)$.

To say that F is a functor from $\mathcal{C}$ to $\mathcal{D}$ we use $F : \mathcal{C} \to \mathcal{D}$, a symbolism that recalls that one of morphism in categories.

Example 1.2 (Set functions)**.** Take a category whose class of objects is an actual set and devoid of morphisms. In that case, compositions are functions between empty sets and the categorial axioms are vacuous truths. So, for X and Y two sets, regarded as categories in the sense just outlined, functors $F : X \to Y$ are practically reduced to a function on objects, that is elements; in that case a functor between sets 'is' a function of sets. Observe that, in this — legal! — case not having any morphism yields that the functorial axioms are bare vacuous truths: so, also the converse holds, that is set funtions can be seen as functors.

Example 1.3 (Monotonic functions)**.** We have met before, how a preordered set is a category; recall also the pure set-theoretic definition of this notion. For $(A, \leq_A)$ and $(B, \leq_B)$ preordered sets, a function $f : A \to B$ is said *monotonic* whenever for every $x, y \in A$ we have $f(x) \leq_B f(y)$ provided that $x \leq_A y$. In bare set-theoretic terms, this can be rewritten as follows: for every $x, y \in A$ such that $(x, y) \in \leq_A$, then $(f(x), f(y)) \in \leq_B$, where we make explicit the pairs, that are morphisms of the preordered sets seen as categories. [...]

Example 1.4 (Monoid homomorphsisms)**.** We have previously shown that a monoid 'is' a single-object category. Consider now two such categories $\mathcal{G}$ and $\mathcal{H}$, so that we can see what a functor $f : \mathcal{G} \to \mathcal{H}$ is. By writing $\bullet_{\mathcal{G}}$ and $\bullet_{\mathcal{H}}$ the object of $\mathcal{G}$ and $\mathcal{H}$ respectively, there is a unique possibility:

$$f(\bullet_{\mathcal{G}}) = \bullet_{\mathcal{H}}. \tag{1.1}$$

For morphisms happens this:

$$f(xy) = f(x)f(y) \tag{1.2}$$

for every morphisms x and y of $\mathcal{G}$ and

$$f(1_{\mathcal{G}}) = 1_{\mathcal{H}}, \tag{1.3}$$

where $1_{\mathcal{G}}$ and $1_{\mathcal{H}}$ are the identities of $\mathcal{G}$ and $\mathcal{H}$ respectively. [...]

Example 1.5 (Monoid actions)**.** [...]

Example 1.6 (A functor $\mathbf{Mat}_k \to \mathbf{FDVect}_k$)**.** For k field, consider the functor

$$M_k : \mathbf{Mat}_k \to \mathbf{FDVect}_k$$

that maps $n \in |\mathbf{Mat}_k| = \mathbb{N}$ to $M_k(n) := k^n$ and $A \in \mathbf{Mat}_k(p, q)$ to the homomorphism of vector spaces $M_k(A) : k^p \to k^q$ defined by $(M_k(A))(x) = Ax$. (Here the elements of k^n 'are' matrices of type $n \times 1$.)

2 CONTRAVARIANT FUNCTORS

Traditionally, functors of Definition 1.1 above are called 'covariant', because there are 'contra'-variant functors too. There is no sensible reason to maintain these two adjectives; at least, almost all agree to not use the first adjective, whilst the second one still survives.

For $\mathcal{C}$ and $\mathcal{D}$ categories, a *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. It is best that we say what functors $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ do. They map objects to objects, nothing special; they map morphisms $f : a \to b$ of $\mathcal{C}^{\mathrm{op}}$ to morphisms $F(f) : F(a) \to F(b)$ of $\mathcal{D}$. But, remembering how dual categories are defined, what F actually does is this: it maps objects of $\mathcal{C}$ to objects of $\mathcal{D}$, and morphisms $f : b \to a$ of $\mathcal{C}$ to morphisms $F(f) : F(a) \to F(b)$ of $\mathcal{D}$. Notice how a and b have their roles flipped. Now, what about funtoriality axioms? Neither with identities F does something different. The composite gf of $\mathcal{C}^{\mathrm{op}}$ is mapped to the composite $F(g)F(f)$ of $\mathcal{D}$; by definition of dual categories, what happens is: the composite fg of $\mathcal{C}$ is mapped to $F(g)F(f)$. Notice here how f and g have their places switched.

Example 2.1. The set of natural numbers $\mathbb{N}$ has the order relation of divisibility that we denote $|$: regard this poset as a category. By Group Theory we know that for every $m, n \in \mathbb{N}$ such that $m \mid n$ there is a homomorphism

$$f_{m,n} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}\,, \; f_{m,n}(a + n\mathbb{Z}) := a + m\mathbb{Z}.$$

In fact, $\mathbb{Z}/m\mathbb{Z}$ is the kernel of the homomorphism $\pi_m : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$, $\pi_m(x) := x + m\mathbb{Z}$ and, because $m \mid n$, we have $n\mathbb{Z} \subseteq m\mathbb{Z}$. In that case, some Isomorphism Theorem[1] justifies the existence of $f_{m,n}$. This offers us a nice functor:

$$F : (\mathbb{N}, \mid)^{\mathrm{op}} \to \mathbf{Grp}$$

that maps naturals n to groups $\mathbb{Z}/n\mathbb{Z}$ and $m \mid n$ to the homomorphism $f_{m,n}$ defined above.

3 EQUIVALENT CATEGORIES

[1] How theorems are named sometimes varies, so for sake of clarity let us explicit the statement we are referring to: Let G and H be two groups, $f : G \to H$ an homomorphism and N some normal subgroup of G. Consider also the homomorphism $p_N : G \to G/N$, $p_N(x) := xN$. If $N \subseteq \ker f$ then there exists one and only one homomorphism $\overline{f} : G/N \to H$ such that $f = \overline{f} p_N$. (Moreover, $\overline{f}$ is surjective if and only if so is f.)