Limits & colimits

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CONTENTS

1 DEFINITION

Albeit this may cause uneasy nights, we present ex abrupto the general notion of (co)limits.

Definition 1.1 (Limits & colimits)**.** Let $\mathcal{I}$ and $\mathcal{C}$ be two categories. For every $v \in |\mathcal{C}|$ we have the *constant functor*

$$k_v : \mathcal{I} \to \mathcal{C}$$

where $k_v(i) := v$ for every $i \in |\mathcal{I}|$ and $k_v(f) := 1_v$ for every morphism f of $\mathcal{I}$. A *limit* of a functor $F : \mathcal{I} \to \mathcal{C}$ is some $v \in |\mathcal{C}|$ with a natural transformation $\lambda : k_v \Rightarrow F$ such that:

> for any $a \in |\mathcal{C}|$ and $\mu : k_v \Rightarrow F$ there exists one and only one $f \in \mathcal{C}(a, v)$ such that
>
> $$\begin{tikzcd} a \arrow[dr, "\mu_i"] \arrow[dd, "f"'] & \\ & F(i) \\ v \arrow[ur, "\lambda_i"'] & \end{tikzcd}$$
>
> commutes for every $i \in |\mathcal{I}|$.

A *colimit*, instead, is an $u \in |\mathcal{C}|$ together with a natural transformation $\chi : F \Rightarrow k_u$ that has the property:

> for every $b \in |\mathcal{C}|$ and $\xi : F \Rightarrow k_b$ there exists one and only one $g \in \mathcal{C}(u, b)$ that makes
>
> $$\begin{tikzcd} u \arrow[dd, "g"'] & \\ & F(i) \arrow[ul, "\chi_i"'] \arrow[dl, "\xi_i"] \\ b & \end{tikzcd}$$

commute for every $i \in |\mathcal{I}|$.

It may be of aid to expand a little bit some parts of the definition. For example, what is a natural transformation $\eta : k_v \Rightarrow F$? By definition of natural transformation, it is a collection $\{\eta_i : v \to F(i) \mid i \in |\mathcal{I}|\}$ of morphisms of $\mathcal{C}$ that has the property:

$$\begin{tikzcd} & F(i) \arrow[dd, "F(f)"] \\ v \arrow[ur, "\eta_i"] \arrow[dr, "\eta_j"'] & \\ & F(j) \end{tikzcd}$$

commutes for every $i, j \in |\mathcal{I}|$ and $f \in \mathcal{I}(i, j)$.

Exercise 1.2. And what is a natural transformation $\theta : F \Rightarrow k_u$?

Yes, for a functor $\mathcal{I} \to \mathcal{C}$, the category $\mathcal{C}$ has its share, but it is $\mathcal{I}$ who has the last say in the research of (co)limits. The role of $\mathcal{I}$ is to give a 'shape' of the limits we are looking for, indeed.

Example 1.3. Let $\mathcal{C}$ be a category and $\mathbf{1}$ a category that has one object and one morphism, and take a functor $f : \mathbf{1} \to \mathcal{C}$, some $v \in \mathcal{C}$ and the corresponding constant functor $k_v : \mathbf{1} \to \mathcal{C}$. A natural transformation $\zeta : k_v \Rightarrow f$ amounts of a single morphism $v \to \widetilde{f}$ of $\mathcal{C}$, where $\widetilde{f}$ indicates the image of the unique object of $\mathbf{1}$ via f. Thus, a limit of f is some $v \in |\mathcal{C}|$ and a morphism $\lambda : v \to \widetilde{f}$ of $\mathcal{C}$ such that: for every object u and morphism $\mu : u \to \widetilde{f}$ in $\mathcal{C}$, there is a unique morphism $u \to v$ of $\mathcal{C}$ that makes

$$\begin{tikzcd} u \arrow[dr, "\mu"] \arrow[dd] & \\ & \widetilde{f} \\ v \arrow[ur, "\lambda"'] & \end{tikzcd}$$

commute.

Exercise 1.4. What are colimts of functors $\mathbf{1} \to \mathcal{C}$?

Sandbox 1.5. Consider a monoid (viz a single object category) $\mathcal{G}$: for the scope of this example we write G for the set of the morphisms of $\mathcal{G}$. Let $F : \mathcal{G} \to \mathbf{Set}$ be a functor, and let $\widehat{F}$ indicate the F-image of the unique object of $\mathcal{G}$ whilst, for $f \in G$, $\widehat{f}$ the function $F(f) : \widehat{F} \to \widehat{F}$. Now, being $k_X : \mathcal{G} \to \mathbf{Set}$ the functor constant at X, with X a set, a natural transformation $\lambda : F \Rightarrow k_X$ is a morphism $\lambda : \widehat{F} \to X$ such that $\lambda = \lambda\widehat{f}$ for every $f \in G$. These two things, the set X and the function λ, together are a colimit of F whenever

> for every set Y and function $\mu : \widehat{F} \to Y$ such that $\mu = \mu\widehat{f}$ for every $f \in G$ there exists one and only one function $h : X \to Y$ such that $\mu = h\lambda$.

Is that thing even interesting? [...]

[Write about functors $\mathcal{G} \to \mathbf{Set}$...]

2 Noteworthy limits and colimits

Fortunately, there are few shapes that are both ubiquitous and simple. This section is dedicated to them, while in the successive one we will prove (Proposition 3.2) that if some simple functors have limits, then all the functors do have limits.

2.1 Terminal and initial objects

Definition 2.1. For $\mathcal{C}$ category, the limits of the empty functor $\varnothing \to \mathcal{C}$ are called *terminal objects* of $\mathcal{C}$, whereas the colimits *initial objects*.

Let us expand the definition above so that we can can look inside things. A cone over the empty functor $\varnothing \to \mathcal{C}$ with vertex $a \in |\mathcal{C}|$ is a natural transformation

$$\begin{tikzcd} \varnothing \arrow[r, bend left, "k_a"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathcal{C} \end{tikzcd} .$$

Here, the empty functor is k_a because there is at most one functor $\varnothing \to \mathcal{C}$. Again, because there must be a unique one, our natural transformation is the empty transformation, viz the one devoid of morphisms. A similar reasoning leads us to the following explicit definition of terminal and initial object.

Definition 2.2. Let $\mathcal{C}$ be a category. Then

- a terminal object of $\mathcal{C}$ is an $a \in |\mathcal{C}|$ such that for every $x \in |\mathcal{C}|$ there exists one and only one $f \in \mathcal{C}(x, a)$;
- an initial object of $\mathcal{C}$ is an $a \in |\mathcal{C}|$ such that for every $x \in |\mathcal{C}|$ there exists one and only one $f \in \mathcal{C}(a, x)$.

Examples time.

Example 2.3 (Recursion)**.** In Set Theory, there is a nice theorem, the *Recursion Theorem*:

> Let $(\mathbb{N}, 0, s)$ be a Peano Model, where $0 \in \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{N}$ is its successor function. For every pointed set X, $a \in X$ and $f : X \to X$ there exists one and only one function $x : \mathbb{N} \to X$ such that $x_0 = a$ and $x_{s(n)} = f(x_n)$ for every $n \in \mathbb{N}$.

Here, by Peano Model we mean a set $\mathbb{N}$ that has one element, we write 0, stood out and a function $s : \mathbb{N} \to \mathbb{N}$ such that, all this complying some rules:

1. s is injective;
2. $s(x) \neq 0$ for every $x \in \mathbb{N}$;
3. for if $A \subseteq \mathbb{N}$ has 0 and $s(n) \in A$ for every $n \in A$, then $A = \mathbb{N}$.

We show now how we can involve Category Theory in this case. First of all, we need a category where to work.
The statement is about things made as follows:

> a set X, one distinguished $x \in X$ and one function $f : X \to X$.

[Is there a name for these things?] We may refer to such new things by barely a triple (X, a, f), but we prefer something like this:

$$1 \xrightarrow{x} X \xrightarrow{f} X,$$

where 1 is any singleton, as usual. Peano Models are such things, with some additional properties. It is told about the existence and the uniqueness of a certain function. We do not want mere functions, of course: given

$$1 \xrightarrow{x} X \xrightarrow{f} X \text{ and } 1 \xrightarrow{y} Y \xrightarrow{g} Y,$$

we take the functions $r : X \to Y$ such that

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "r"] & X \arrow[d, "r"] \\ 1 \arrow[ur, "x"] \arrow[dr, "y"'] & & \\ & Y \arrow[r, "g"'] & Y \end{tikzcd}$$

commutes and nothing else. [Is there a name for such functions?] These ones are the things we want to be morphisms. Suppose given

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "p"] & X \arrow[d, "p"] \\ 1 \arrow[ur, "x"] \arrow[r, "y"] \arrow[dr, "z"'] & Y \arrow[r, "g"] \arrow[d, "q"] & Y \arrow[d, "q"] \\ & Z \arrow[r, "h"'] & Z \end{tikzcd}$$

where all the squares and triangles commute: thus we obtain the commuting

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "qp"] & X \arrow[d, "qp"] \\ 1 \arrow[ur, "x"] \arrow[dr, "z"'] & & \\ & Z \arrow[r, "h"'] & Z \end{tikzcd}$$

This means that composing two morphisms as functions in **Set** produces a morphism. This is how we want composition to defined in this context. This choice makes the categorial axioms automatically respected. We call this category **Peano**. [Unless there is a better naming, of course.]
Being the environment set now, the Recursion Theorem becomes more concise:

> Peano Models are initial objects of **Peano**.

Exercise 2.4 (Induction $\Leftrightarrow$ Recursion)**.** In **Set**, suppose you have $1 \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$, where s is injective and $s(n) \neq 0$ for every $n \in \mathbb{N}$. Demonstrate that the following statements are equivalent:

1. for if $A \subseteq \mathbb{N}$ has 0 and $s(n) \in A$ for every $n \in A$, then $A = \mathbb{N}$;
2. $1 \xrightarrow{0} \mathbb{N} \xrightarrow{s} \mathbb{N}$ is an initial object of $(\mathbb{N}, 0, s)$.

(1) $\Rightarrow$ (2) proves the Recursion Theorem, whereas (2) $\Rightarrow$ (1) requires you to codify a proof by induction into a recursion. Try it, it could be nice.

Remark 2.5. In general, suppose you have a category $\mathcal{C}$ that has a terminal object, that we denote by 1. [Explain why a terminal object is required.] We have then the category $\mathbf{Peano}_{\mathcal{C}}$, that is **Peano** with objects and morphisms picked from $\mathcal{C}$ and compositions performed in $\mathcal{C}$ — really, there is nothing special in **Set** that hinders us to do so. A *natural number object* of $\mathcal{C}$ is any of the initial objects of $\mathbf{Peano}_{\mathcal{C}}$, that is an assignment of one object $\mathbb{N}$, a morphism $0 : 1 \to \mathbb{N}$ and a morphism $f : X \to X$ of $\mathcal{C}$ such that they satisfy the *recursion property*, the Recursion Theorem but for $\mathcal{C}$. **Set** has natural number objects, as we have seen in the previous example, but another categories may have none. [For instance?] [This is quite interesting… An 'arithmetic' for categories, perhaps…]

Construction 2.6. Let us introduce a nice category that allows us to express some nice and simple facts in Mathematics. Let $\mathcal{C}$ and $\mathcal{J}$ two categories, a one of its objects and take a functor $F : \mathcal{J} \to \mathcal{C}$. We have the category $(a \downarrow F)$, this made:

- the objects are the morphisms $a \to F(x)$ of $\mathcal{C}$, for $x \in |\mathcal{J}|$;
- the morphisms from $f : a \to F(x)$ to $g : a \to F(y)$ are the morphisms $h : x \to y$ of $\mathcal{J}$ such that

$$\begin{tikzcd} & F(x) \arrow[d, "F(h)"] \\ a \arrow[ur, "f"] \arrow[r, "g"'] & F(y) \end{tikzcd}$$

  commutes;
- the composition is that of $\mathcal{J}$.

Example 2.7. In Linear Algebra (or when dealing with free modules) we have a nice theorem:

> Let V be a vector space over a field k and $S \subseteq V$ a base. For every vector space W over k and function $\phi : S \to W$ there exists a unique linear function $f : V \to W$ such that
>
> $$\begin{tikzcd} S \arrow[r, hook, "i"] \arrow[dr, "\phi"'] & V \arrow[d, "f"] \\ & W \end{tikzcd}$$
>
> commutes.

In other words, this statement says that a linear function is completely determined by what it does with the vectors of S. We will consider now two functors

$$\mathbf{Set} \xrightarrow{\langle \cdot \rangle} \mathbf{Vect}_k \xrightarrow{U} \mathbf{Set}.$$

The first one takes a set S and produces the vector space on k

$$\langle S \rangle := \left\{ \sum_{x \in S} \lambda_x x \,\middle|\, \lambda : S \to k,\ \lambda_x \neq 0 \text{ for finitely many times} \right\}$$

(considered with two obvious operations). Furthermore, a function of sets $f : S \to T$ induces a linear function $\langle f \rangle : \langle S \rangle \to \langle T \rangle$ defined by

$$\langle f \rangle \left(\sum_{x \in S} \lambda_x x \right) := \sum_{x \in S} \lambda_x f(x)$$

where $\lambda : S \to k$ is almost always null. The functoriality of $\langle \cdot \rangle$ is just a matter of quick controls. [Do we really need all that machinery?] The functor U instead takes vector spaces and returns the correspondent set of vectors; we write $U(V) := V$, but observe that in **Set** we don't care anymore of the vector structure of V. Similarly, it takes linear functions and the return them: but, since U lands onto **Set**, who cares about linearity there? (We may say that U is the 'inclusion' of $\mathbf{Vect}_k$ into **Set**.) [Talk about forgetful functors elsewhere...] All this words allow us restate the aforementioned theorem as:

For if S is a set, the inclusion $S \hookrightarrow \langle S \rangle$ is an initial object of $(S \downarrow U)$.

Exercise 2.8. In the previous example some details are omitted: you can be more talkative, though. However, it is really worth to think about it — not only because we will meet such pattern later under the vest of adjunctions. You may also look for another examples of similar kind, I'm sure you will find some.

Construction 2.9 (Category of cones)**.** For $\mathcal{C}$ category, let $F : \mathcal{I} \to \mathcal{C}$ be a functor. Then we define the *category of cones* over F as follows.

- The objects are the cones over F.
- For $\alpha := \left\{a \xrightarrow{\alpha_i} F(i)\right\}_{i \in |\mathcal{I}|}$ and $\beta := \left\{b \xrightarrow{\beta_i} F(i)\right\}_{i \in |\mathcal{I}|}$ two cones, the morphisms from α to β are the morphisms $f : a \to b$ of $\mathcal{C}$ such that

$$\begin{tikzcd} a \arrow[d, "f"'] \arrow[dr, "\alpha_i"] & \\ b \arrow[r, "\beta_i"'] & F(i) \end{tikzcd}$$

  commutes for every $i \in |\mathcal{I}|$.
- The composition of morphisms here is the same as that of $\mathcal{C}$.

We write such category as $\mathbf{Cn}_F$. We define also the *category of cocones* over F, written as $\mathbf{CoCn}_F$.

- The objects are the cocones over F.
- For $\alpha := \left\{F(i) \xrightarrow{\alpha_i} a\right\}_{i \in |\mathcal{I}|}$ and $\beta := \left\{F(i) \xrightarrow{\beta_i} b\right\}_{i \in |\mathcal{I}|}$ cocones, the morphisms from α to β are the morphisms $f : a \to b$ of $\mathcal{C}$ such that

$$\begin{tikzcd} & a \arrow[dl, "\alpha_i"'] \arrow[d, "f"] \\ F(i) & b \arrow[l, "\beta_i"] \end{tikzcd}$$

  commutes for every $i \in |\mathcal{I}|$.
- The composition of morphisms here is the same as that of $\mathcal{C}$.

It is quite immediate in either of the cases to show that categorial axioms are verified.

Proposition 2.10. For $\mathcal{C}$ category and $F : \mathcal{I} \to \mathcal{C}$ functor,

- limits of F are terminal objects of $\mathbf{Cn}_F$ and viceversa.
- colimits of F are initial objects of $\mathbf{CoCn}_F$ and viceversa.

Proof. Simple. □

2.2 PRODUCTS AND COPRODUCTS

Definition 2.11. Let $\mathcal{C}$ be a category and $\mathcal{S}$ a discrete category. Then (co)limits of the functors $X : \mathcal{S} \to \mathcal{C}$ are called *(co)products*.

Again, let us put this definition into more explicit terms. First of all, what are cones over $F : \mathcal{S} \to \mathcal{C}$? For $p \in |\mathcal{C}|$ and $k_p : \mathcal{S} \to \mathcal{C}$ the functor constant at p, a natural transformation

$$\begin{tikzcd} \mathcal{S} \arrow[r, bend left, "k_p"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\pi"] & \mathcal{C} \end{tikzcd}$$

is exactly a collection

$$\left\{p \xrightarrow{\pi_x} F(x)\right\}_{x\in|\mathcal{S}|}.$$

In this fortunate case, in fact, the naturality condition automatically holds because $\mathcal{S}$ has no morphism other than the identities. Similarly, one can easily deduce what cocones are. Thus the explicit definition is:

Definition 2.12 (Explicit)**.** [...] [Introduce some new locutions...]

Example 2.13 (Cartesian product)**.** Given a family of sets $\{X_i \mid i \in I\}$, we have the *Cartesian product*

$$\prod_{i\in I} X_i := \left\{ f : I \to \bigcup_{i\in I} X_i \,\middle|\, f(i) \in X_i \text{ for every } i \in I \right\}.$$

with the *component functions*, one for each $j \in I$,

$$p_j : \prod_{i\in I} X_i \to X_j\,,\ p_j(f) := f(j).$$

(In other words, p_j takes an f and evaluates it at j.) Taken now any set A and functions $f_i : A \to X_i$ for $i \in I$, we have

$$\prod_{i\in I} f_i : A \to \prod_{i\in I} X_i$$

with $\left(\prod_{i\in I} f_i\right)(a)$ mapping $k \in I$ to $f_k(a)$. It is simple to show that

$$\begin{tikzcd} A \arrow[d, "\prod_{i\in I} f_i"'] \arrow[dr, "f_\alpha"] & \\ \prod_{i\in I} X_i \arrow[r, "p_\alpha"'] & X_\alpha \end{tikzcd}$$

commutes for every $\alpha \in I$. Moreover, $\prod_{i\in I} f_i$ is the only one that does this. Consider any function $g : A \to \prod_{i\in I} X_i$ with $f_\alpha = p_\alpha g$ for every $\alpha \in I$: then for every $x \in A$ we have

$$\big(g(x)\big)(\alpha) = p_\alpha\big(g(x)\big) = f_\alpha(x) = p_\alpha\left(\left(\prod_{i\in I} f_i\right)(x)\right) = \left(\left(\prod_{i\in I} f_i\right)(x)\right)(\alpha).$$

Hence, we can conclude $\prod_{i\in I} X_i$ with the collection of functions $\{p_i \mid i \in I\}$ is a product in **Set**.

2.3 EQUALIZERS AND COEQUALIZERS

2.4 PULLBACKS AND PUSHOUTS

3 (CO)COMPLETENESS

[Rewrite]

Definition 3.1. A category $\mathcal{C}$ is said (*co*)*complete* whenever any functor $\mathcal{I} \to \mathcal{C}$ has a (co)limit.

Proposition 3.2 (Completeness Theorem)**.** Categories that have products and equalizers are complete.

Proof. Yet to write... □