

Functors

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1 Definition

Definition 1.1 (Functors). A functor F from a category \mathcal{C} to a category \mathcal{D} is having the following functions, all indicated with F :

- a ‘function on objects’

$$F : |\mathcal{C}| \rightarrow |\mathcal{D}|, x \rightarrow F(x)$$

- for every objects a and b , one ‘function on morphisms’

$$F : \mathcal{C}(a, b) \rightarrow \mathcal{D}(F(a), F(b)), f \rightarrow F(f)$$

such that

1. for every object x of \mathcal{C} we have $F(1_x) = 1_{F(x)}$;
2. for every objects x, y, z and morphisms $f : x \rightarrow y$ and $g : y \rightarrow z$ of \mathcal{C} we have $F(g)F(f) = F(gf)$.

To say that F is a functor from \mathcal{C} to \mathcal{D} we use $F : \mathcal{C} \rightarrow \mathcal{D}$, a symbolism that recalls that one of morphism in categories.

Example 1.2 (Set functions). Take a category whose class of objects is an actual set and devoid of morphisms. In that case, compositions are functions between empty sets and the categorial axioms are vacuous truths. So, for X and Y two sets, regarded as categories in the sense just outlined, functors $F : X \rightarrow Y$ are practically reduced to a function on objects, that is elements; in that case a functor between sets ‘is’ a function of sets. Observe that, in this — legal! — case not having any morphism yields that the functorial axioms are bare vacuous truths: so, also the converse holds, that is set funtions can be seen as functors.

Example 1.3 (Monotonic functions). We have met before, how a preordered set is a category; recall also the pure set-theoretic definition of this notion. For (A, \leq_A) and (B, \leq_B) preordered sets, a function $f : A \rightarrow B$ is said *monotonic*

whenever for every $x, y \in A$ we have $f(x) \leq_B f(y)$ provided that $x \leq_A y$. In bare set-theoretic terms, this can be rewritten as follows: for every $x, y \in A$ such that $(x, y) \in \leq_A$, then $(f(x), f(y)) \in \leq_B$, where we make explicit the pairs, that are morphisms of the preordered sets seen as categories.

Example 1.4 (Monoid homomorphisms). We have previously seen that a monoid ‘is’ a single-object category. Consider now two such categories, say \mathcal{G} and \mathcal{H} , and a functor $f : \mathcal{G} \rightarrow \mathcal{H}$ is. Denoting by $\bullet_{\mathcal{G}}$ and $\bullet_{\mathcal{H}}$ the object of \mathcal{G} and \mathcal{H} respectively, there is a unique possibility: mapping $\bullet_{\mathcal{G}}$ to $\bullet_{\mathcal{H}}$. The functorial axioms in that case are:

$$f(xy) = f(x)f(y)$$

for every morphisms x and y of \mathcal{G} and

$$f(1_{\mathcal{G}}) = 1_{\mathcal{H}},$$

with $1_{\mathcal{G}}$ and $1_{\mathcal{H}}$ being the identities of \mathcal{G} and \mathcal{H} respectively. These two properties say that f is a monoid homomorphism; in this case there is also an equation that about objects but these two are a mere subtlety that adds nothing. It is easy to do the converse: a monoid homomorphism is a functor.

Example 1.5 (The category **Eqv**). A *setoids* [nlab uses this term...], that is sets together with an equivalence relation defined on it; if X is a set and \sim an equivalence relation over X , the corresponding setoid is written simply as (X, \sim) . Any set X has naturally its equality relation $=_X$ defined on it¹, and for X and Y sets, a function $f : X \rightarrow Y$ respects this rule by definition:

$$\text{for every } a, b \in X, \text{ if } a =_X b \text{ then } f(a) =_Y b.$$

We would like to replace the equalities above with equivalence relations: for if (X, \sim_X) and (Y, \sim_Y) are setoids, a *functoid* [ok, let me find/craft a nicer name...] from (X, \sim_X) to (Y, \sim_Y) is exactly a set function $f : X \rightarrow Y$ such that

$$\text{for every } a, b \in X, \text{ if } a \sim_X b \text{ then } f(a) \sim_Y f(b).$$

Functoids are certain type of functions, so composing two of them as such returns a functoid. Categorical axioms hold for free, so we really have a *category of setoids and functoids*, **Eqv**.

There is a nice theorem:

Let X and Y be two sets with \sim_X and \sim_Y equivalence relations on X and Y respectively. Then for every $f : X \rightarrow Y$ such that $f(a) \sim_Y f(b)$ for every $a, b \in X$ such that $a \sim_X b$, there exists one and only one $\phi : X/\sim_X \rightarrow Y/\sim_Y$ that makes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \lambda a.[a]_X \downarrow & & \downarrow \lambda b.[b]_Y \\ X/\sim_X & \xrightarrow{\phi} & Y/\sim_Y \end{array}$$

commute. (The vertical functions are the canonical projections.)

This underpins the functor

$$\pi : \mathbf{Eqv} \rightarrow \mathbf{Set}$$

¹ In Set Theory, $=_X$ is the set $\{(a, a) \mid a \in X\}$.

that maps setoids (X, \sim) to the sets X/\sim and functoids $f : (X, \sim_X) \rightarrow (Y, \sim_Y)$ to functions

$$\begin{aligned}\pi_f : X/\sim_X &\rightarrow Y/\sim_Y \\ \pi_f([a]_X) &:= [f(a)]_Y,\end{aligned}$$

whose existence and uniqueness is claimed by the just mentioned Proposition.

Example 1.6 (Free group functor). Suppose given a *group alphabet* S , that is a set of ‘letters’; then *group words* with system S are strings obtained by juxtaposition of a finite amount of ‘ x ’ and ‘ x^{-1} ’, where $x \in S$. The *empty word* is obtained by writing no letter, and we shall denote it by something, say e ; instead, the other words appear as

$$x_1^{\phi_1} \cdots x_n^{\phi_n},$$

with $x_1, \dots, x_n \in S$ and $\phi_1, \dots, \phi_n \in \{-1, 1\}$.^{2 3}

The length of a word is the number of letters it is made of. We define equality only on words having the same length: we say $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is equal to $y_1^{\beta_1} \cdots y_n^{\beta_n}$ whenever $x_i = y_i$ and $\alpha_i = \beta_i$ for every $i \in \{1, \dots, n\}$.

A group word $x_1^{\phi_1} \cdots x_n^{\phi_n}$ is called *irreducible* whenever $x_i^{\phi_i} \neq x_{i+1}^{-\phi_{i+1}}$ for every $i \in \{1, \dots, n-1\}$; the empty word is irreducible by convention. Let us write $\langle S \rangle$ the set of all irreducible words written using the alphabet S . It is natural to join two words by bare juxtaposition, but the resulting word may not be irreducible; this issue has to be fixed:

$$\begin{aligned}\cdot : \langle S \rangle \times \langle S \rangle &\rightarrow \langle S \rangle \\ e \cdot w &:= w, \quad w \cdot e := w \\ (x_1^{\lambda_1} \cdots x_m^{\lambda_m}) \cdot (y_1^{\mu_1} \cdots y_n^{\mu_n}) &:= \begin{cases} (x_1^{\lambda_1} \cdots x_{m-1}^{\lambda_{m-1}}) \cdot (y_2^{\mu_2} \cdots y_n^{\mu_n}) & \text{if } x_m^{\lambda_m} = y_1^{-\mu_1} \\ x_1^{\lambda_1} \cdots x_m^{\lambda_m} y_1^{\mu_1} \cdots y_n^{\mu_n} & \text{otherwise.} \end{cases}\end{aligned}$$

Let us define a function that either reverses the order of the letters and changes each exponent to the other one:

$$i : \langle S \rangle \rightarrow \langle S \rangle, \quad i(x_1^{\xi_1} \cdots x_i^{\xi_i} x_{i+1}^{\xi_{i+1}} \cdots x_n^{\xi_n}) := x_n^{-\xi_n} \cdots x_{i+1}^{-\xi_{i+1}} x_i^{-\xi_i} \cdots x_1^{-\xi_1}.$$

It is immediate to show that $w \cdot i(w) = i(w) \cdot w = e$ for every $w \in \langle S \rangle$. Only the associativity of \cdot is a bit tricky to prove. At this point we have endowed $\langle S \rangle$ with a group structure.

Thus from a set S we are able to build a group $\langle S \rangle$, that is called *free group* with base S , or group generated by S . Now, if take two sets S and T and a function $f : S \rightarrow T$, we have the group homomorphism

$$\langle f \rangle : \langle S \rangle \rightarrow \langle T \rangle, \quad \langle f \rangle(x_1^{\delta_1} \cdots x_n^{\delta_n}) := (f(x_1))^{\delta_1} \cdots (f(x_n))^{\delta_n}.$$

It is immediate to demonstrate that we ended up with having a functor

$$\langle \rangle : \mathbf{Set} \rightarrow \mathbf{Grp}.$$

² Something that may irk you is that our words can be redundant, being consecutive repetitions of the same letter allowed. If you want, you can let exponents range over all the integers, but this needs you to modify what comes after.

³ Here, we can choose any pair of symbols instead of -1 and 1 . If we do so, we need a function that maps each of them into the other one. In this presentation we employ the function that takes one integer and returns its opposite.

Exercise 1.7. There is a plenty of ‘free stuff’ around that can give arise to functors like the one above. Find and illustrate some of them.

Traditionally, functors of Definition 1.1 above are called ‘covariant’, because there are *contravariant* functors too. However, there is no sensible reason to maintain these two adjectives; at least, almost everyone agrees to not use the first adjective, whilst the second one still survives.

For if \mathcal{C} and \mathcal{D} are categories, a *contravariant functor* from \mathcal{C} to \mathcal{D} is just a functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$. It is best that we say what functors $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ do. They map objects to objects and morphisms $f : a \rightarrow b$ of \mathcal{C}^{op} to morphisms $F(f) : F(a) \rightarrow F(b)$ of \mathcal{D} . But, remembering how dual categories are defined, what F actually does is this:

it maps objects of \mathcal{C} to objects of \mathcal{D} , and morphisms $f : b \rightarrow a$ of \mathcal{C} to morphisms $F(f) : F(a) \rightarrow F(b)$ of \mathcal{D} (mind that a and b have their roles flipped).

Now, what about functoriality axioms? Neither with identities F does something different and the composite gf of \mathcal{C}^{op} is mapped to the composite $F(g)F(f)$ of \mathcal{D} . Again by definition of dual categories, this can be translated as follows:

the composite fg of \mathcal{C} is mapped to $F(g)F(f)$ (notice here how f and g have their places switched).

You can think of contravariant functors as a trick to do what we want.

Example 1.8. The set of natural numbers \mathbb{N} has the order relation of divisibility, that we denote $|$: regard this poset as a category. From Group Theory, we know that for every $m, n \in \mathbb{N}$ such that $m | n$ there is a homomorphism

$$f_{m,n} : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}, \quad f_{m,n}(a + n\mathbb{Z}) := a + m\mathbb{Z}.$$

In fact, $\mathbb{Z}/m\mathbb{Z}$ is the kernel of the homomorphism

$$\pi_m : \mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}, \quad \pi_m(x) := x + m\mathbb{Z}$$

and, because $m | n$, we have $n\mathbb{Z} \subseteq m\mathbb{Z}$. In that case, some Isomorphism Theorem⁴ justifies the existence of $f_{m,n}$. This offers us a nice functor:

$$F : (\mathbb{N}, |)^{\text{op}} \rightarrow \mathbf{Grp}$$

that maps naturals n to groups $\mathbb{Z}/n\mathbb{Z}$ and $m | n$ to the homomorphism $f_{m,n}$ defined above.

2 Category of categories?

Functors can be composed, and I think at this point it is not a secret. Take \mathcal{C} , \mathcal{D} and \mathcal{E} categories and functors

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}.$$

⁴ How theorems are named sometimes varies, so for sake of clarity let us explicit the statement we are referring to: Let G and H be two groups, $f : G \rightarrow H$ an homomorphism and N some normal subgroup of G . Consider also the homomorphism $p_N : G \rightarrow G/N$, $p_N(x) := xN$. If $N \subseteq \ker f$ then there exists one and only one homomorphism $\bar{f} : G/N \rightarrow H$ such that $f = \bar{f}p_N$. (Moreover, \bar{f} is surjective if and only if so is f .)

The sensible way to define the composite functor $GF : \mathcal{C} \rightarrow \mathcal{E}$ is mapping the objects x of \mathcal{C} to the objects $GF(x)$ of \mathcal{E} , and the morphisms $f : x \rightarrow y$ of \mathcal{C} to the morphisms $GF(f) : GF(x) \rightarrow GF(y)$ of \mathcal{E} . That being set, the composition is associative and there is an identity functor too.

So what prevents us to consider a category — we can call \mathbf{Cat} — that has categories as objects and functors as morphisms? If we work upon NBG, we can think of any proper class as a category, for this statement have a closer look at Example 1.2. What happens now is that the class of objects of \mathbf{Cat} has an element that is a proper class, which isn't clearly legal in NBG.

Is a category of *locally small* categories and functors problematic? Take \mathcal{C} such that $\mathcal{C}(a, b)$ is a proper class for some a and b objects: consider \mathcal{C}/b . In this case $|\mathcal{C}/b|$ is a proper class too.

So, now what? If we stick to NBG, this is a limit we have to take into account. From now on, \mathbf{Cat} is the category of *small* categories and functors between small categories.

3 Equivalent categories

Let us give a definition that will motivate our discourse.

Definition 3.1 (Full- and faithfulness). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said *full*, respectively *faithful*, whenever for every $a, b \in |\mathcal{C}|$ the functions

$$F : \mathcal{C}(a, b) \rightarrow \mathcal{D}(F(a), F(b))$$

are surjective, respectively injective; we say that F is *fully faithful* [how lame, lol...] whenever it is both full and faithful.

What do we want ‘two categories are the same’ to mean? [Craft a nicer exposition... Let us try with categories being isomorphic first, and then with *essentially surjective* functors. Talk about *skeletons* of categories, and how can help to say whether two categories are equivalent.]

Example 3.2 (A functor $\mathbf{Mat}_k \rightarrow \mathbf{FDVect}_k$). For k field, consider the functor

$$M : \mathbf{Mat}_k \rightarrow \mathbf{FDVect}_k$$

that maps $n \in |\mathbf{Mat}_k| = \mathbb{N}$ to $M(n) := k^n$ and $A \in \mathbf{Mat}_k(r, s)$ to the linear function

$$\begin{aligned} M_A : k^r &\rightarrow k^s \\ M_A(x) &= Ax. \end{aligned}$$

(Here the elements of k^n are matrices of type $n \times 1$.) [...]