Categories

Indrjo Dedej

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Contents

1 Definition

Let us start with some examples you should be familiar with.

Example 1.1 (Set Theory)**.** Here we have *sets* and *functions*. Whereas these of set and membership are assumed as primitive, the concept of function has a precise definition:

> For A and B sets, a function from A to B is any $f \subseteq A \times B$ such that for every $x \in A$ there exists one and only one $y \in B$ such that $(x, y) \in f$.

We write '$f : A \to B$' to mean 'f is a function from A to B'; for $x \in A$, we denote by $f(x)$ the element of B bound to x by f. Observe the following ways to introduce a function are equivalent:

- telling the pairs that make f;
- for every $x \in A$ saying which is the $y \in B$ such that $(x, y) \in f$; I'm sure you are pretty used to introduce functions by writing something like

$$f(x) := \text{formula that may contain } x.$$

That being said, let us deal with the operation of composing consecutive functions: for A, B and C sets and functions

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

the *composite* of g and f is the function posed in this way:

$$\begin{gathered} g \circ f : A \to C \\ g \circ f(x) := g\big(f(x)\big). \end{gathered}$$

Such operation has some remarkable properties.

1. For every set A the function $1_A : A \to A$ defined by $1_A(x) = x$ is such that for every set B and function $g : B \to A$ we have
$$1_A \circ g = g$$
and for every set C and function $h : A \to C$ we have
$$h \circ 1_A = h.$$
Here, 1_A is the *identity* for A.
2. $\circ$ is associative, that is for A, B, C and D sets and
$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$
functions, we have the identity
$$(h \circ g) \circ f = h \circ (g \circ f).$$

Example 1.2 (Topology)**.** A *topological space* is a set where some of its subsets have the status of 'open sets'; *continuous functions* are set functions that care about such label: that is for X and Y topological spaces a function $f : X \to Y$ is said *continuous* iff for every open set U of Y the set $f^{-1}U$ is an open set of X. In general, for X, Y and Z sets and $f : X \to Y$ and $g : Y \to Z$ functions, we have
$$(g \circ f)^{-1}U = f^{-1}\left(g^{-1}U\right) \text{ for every } U \subseteq Z$$
Now, if X, Y and Z are topological spaces and f and g continuous, for if U is open, then so is $(g \circ f)^{-1}U$: that is $g \circ f$ is continuous as well. Being continuous functions functions, the associativity comes for free; moreover, the identity functions are continuous. Take the properties listed in the previous example and replace 'set' with 'topological space' and 'function' with 'continuous function' and notice how things work fine.

Now it's time to give a definition of what we have been highlighting so far.

Definition 1.3 (Categories)**.** A *category* amounts at assigning some things called *objects* and, for each couple of objects a and b, of some other things named *morphisms* from a to b. We write $f : a \to b$ to say that f is a morphism from a to b, where a is the *domain* of f and b the codomain. Besides, for a, b and c objects and $f : a \to b$ and $g : b \to c$ morphisms, there is associated the *composite morphism*
$$gf : a \to c.$$
All those things are regulated by the following axioms:

1. for every object x there is a morphism, 1_x, from x to x such that for every object y and morphism $g : y \to x$ we have
$$1_x g = g$$
and for every object z and morphism $h : x \to z$ we have
$$h 1_x = h;$$
2. for a, b, c and d objects and morphisms
$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$
we have the identity
$$(hg)f = h(gf).$$

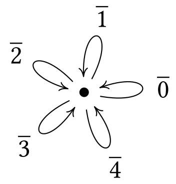

Figure 1. The group $\mathbb{Z}_5$ pictured as a category. Actually, you do not need to represent a monoid in this way; the picture is just to give a visual representation of the shift required by Category Theory.

Sometimes, instead of 'morphism' you may find written 'map' or 'arrow'. The former is quite used outside Category Theory, whereas the latter refers to the fact that '$\rightarrow$' is employed.

Exercise 1.4 (Set Theory, again)**.** Take sets and relations. For A and B sets, by relation from A to B we mean a subset of $A \times B$. Define the composition of two relations — the first example may suggest to you the 'right' way to proceed. Can you stand some properties out akin to the case of sets and functions?

Exercise 1.5 (Topology, again)**.** This is a little variation of the previous example. Take topological spaces and *open functions*: now, can you stand out some pattern, like that of the previous two examples? Here, for X and Y topological spaces, a function $f : X \rightarrow Y$ is said open whenever for every open set $U \subseteq X$ the set $fU \subseteq Y$ is open too.

Exercise 1.6. At this early stage, one can provide a generous amount of examples of contexts presenting that leitmotif when things are built upon Set Theory. After all, groups are sets with some additional ingredients, homomorphisms are functions that cares about the group structure, composing two such functions yields a homomorphism; and composition complies the same laws highlighted in the previous two examples. The same applies to vector spaces, measure spaces, probability spaces, ... Make some examples by yourself.

The above definition has the aim to generalize the 'objects-morphisms-compositionality' pattern to a broader class of situations.

Example 1.7 (Monoids are categories)**.** Consider a category $\mathcal{G}$ with a single object, that we indicate with a bare $\bullet$. All of its morphisms have $\bullet$ as domain and codomain: this fact implies the composite of two morphisms $\bullet \rightarrow \bullet$ is a morphism $\bullet \rightarrow \bullet$ too. This motivates us to proceed as follows: let G be the collection of the morphisms of $\mathcal{G}$ and consider the function

$$G \times G \rightarrow G, \ (x, y) \rightarrow xy,$$

that is the operation of composing morphisms. Being $\mathcal{G}$ a category implies this function is associative and $\mathcal{G}$ has the identity of $\bullet$, that is G has one element we call 1 and such that $f1 = 1f = f$ for every $f \in G$. In other words, we are saying G is a monoid.
Conversely, take a monoid G and any thing you want: make such thing acquire the status of object and the elements of G that of morphisms; in that case, the operation of G has the right to be called composition because the axioms of monoid say so.

The conclusion is: a monoid ‘is’ a category with a single object. $\bullet$ is something we cared of only because by definition morphisms need objects and it has no role other than this.

In Mathematics, a lot of things are monoids, so this is nice.

Example 1.8 (Preordered sets are categories)**.** A *preordered set* (sometimes contracted as *proset*) consists of a set A and a relation $\leq$ on A such that:

1. $x \leq x$ for every $x \in A$;
2. for every $x, y, z \in A$ we have that if $x \leq y$ and $y \leq z$ then $x \leq z$.

Now we do this: for $x, y \in A$, whenever $x \leq y$ take $(a, b) \in A \times A$. We operate with these couples as follows:

$$(y, z)(x, y) := (x, z), \tag{1.1}$$

where $x, y, z \in A$. This definition is perfectly motivated by (2): in fact, if $x \leq y$ and $y \leq z$ then $x \leq z$, and so there is (x, z). By (1), for every $x \in A$ we have the couple (x, x), which has the following property: for every $y \in A$

$$\begin{aligned} (x, y)(x, x) &= (x, y) \quad \text{for every } y \in A \\ (x, x)(z, x) &= (z, x) \quad \text{for every } z \in A. \end{aligned} \tag{1.2}$$

Another remarkable feature is that for every $x_1, x_2, x_3, x_4 \in A$

$$\big((x_3, x_4)(x_2, x_3)\big)(x_1, x_2) = (x_3, x_4)\big((x_2, x_3)(x_1, x_2)\big) \tag{1.3}$$

We have a category indeed: its objects are the elements of A, the morphisms are the couples (x, y) such that $x \leq y$ and (1.1) gives the notion of composition; (1.2) says what are identities while (1.3) tells the compositions are associative.

A lot of things are prosets, so this is nice.

Example 1.9 (Matrices)**.** We need to clarify some terms and notations before. Fixed some field k, for m and n positive integers, a *matrix* of type $m \times n$ is a table of elements of k arranged in m rows and n columns:

$$\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,n} \end{pmatrix}$$

If A is the name of a matrix, then $A_{i,j}$ is the element on the intersection of the ith row and the jth column. Matrices can be multiplied: if A and B are matrices of type $m \times n$ and $n \times r$ respectively, then AB is the matrix of type $m \times r$ where

$$(AB)_{i,j} := \sum_{p=1}^{n} A_{i,p} B_{p,j}.$$

Our experiment is this: consider the positive integers in the role of objects and, for m and n integers, the matrices of type $m \times n$ as morphisms from n to m; now, take AB as the composition of A and B. Let us investigate whether categorial axioms hold.

- For n positive integer, we have the *identity matrix* I_n, the one of type $n \times n$ defined by
$$(I_n)_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$
One, in fact, can verify that such matrix is an 'identity' in categorial sense: for every positive integer m, an object, and for every matrix A of type $m \times n$, a morphism from n to m, we have
$$AI_n = A,$$
that is composing A with I_n returns A; similarly, for every positive integer r and for every matrix B of type $r \times n$ we have
$$I_n B = B.$$
- For A, B and C matrices of type $m \times n$, $n \times r$ and $r \times s$ respectively, we have
$$(AB)C = A(BC).$$
Again, this identity can be regarded under a categorial light.

The category of matrices on a field k just depicted is written $\mathbf{Mat}_k$.

Remark 1.10. Though the previous example may seem quite useless, it really does matter. Just wait until we talk about equivalence of categories.

Example 1.11 (Chain complexes)**.** [...]

2 The language of diagrams

A diagram is a drawing made of 'nodes', that is empty slots, and 'arrows', that part from some nodes and head to other ones. Here is an example:

$$\begin{tikzcd} & {} \arrow[dr] & \\ {} \arrow[ur, bend left] \arrow[ur, bend right] & & {} \\ & {} \arrow[uu] & \end{tikzcd} \tag{2.4}$$

Nodes are the places where to put objects' names and arrows are to be labelled with morphisms' names. The next step is putting labels indeed, something like this:

$$\begin{tikzcd} & b \arrow[dr, "q"] & \\ a \arrow[ur, bend left, "f"] \arrow[ur, bend right, "g"'] & & d \\ & c \arrow[uu, "h"'] & \end{tikzcd} \tag{2.5}$$

The idea we want to capture is: having a scheme of nodes and arrows, as in (2.4), and then assigning labels, as in (2.5). Since diagrams serve to graphically show some categorial structure, there should exist the possibility to 'compose' arrows: two consecutive arrows

$$\begin{tikzcd} {} \arrow[r] & {} \arrow[r] & {} \end{tikzcd} \tag{2.6}$$

naturally yields that one that goes from the first node and heads to the last one; if in (2.6) we label the arrows with f and g, respectively, then the composite arrow is to be labelled with the composite morphism gf. That operation shall be associative and there should exist identity arrows too, that is arrows that represent and behave exactly as identity morphisms. In other words, our drawings shall care of the categorial structure.

If we want to formalize the idea just outlined, the definition of diagram sounds something like this:

Definition 2.1 (Diagrams)**.** A *diagram* in a category $\mathcal{C}$ is having:

- a scheme of nodes and arrows, that is a category $\mathcal{I}$;
- labels for nodes, that is for every object i of $\mathcal{I}$ one object x_i of $\mathcal{C}$;
- labels for arrows, that is for every pair of objects i and j of $\mathcal{I}$ and morphism $\alpha : i \to j$ of $\mathcal{I}$, one morphism $f_\alpha : x_i \to x_j$ of $\mathcal{C}$

with all this complying the following rules:

1. $f_{1_i} = 1_{x_i}$ for every i object of $\mathcal{I}$;
2. $f_\beta f_\alpha = f_{\beta\alpha}$, for α and β two consecutive morphisms of $\mathcal{I}$.

[A finer formalisation of commutativity?] Consecutive arrows form a 'path'; in that case, we refer to the domain of its first arrow as the domain of the path and to the codomain of the last one as the codomain of the path. Two paths are said *parallel* when they share both domain and codomain. A diagram is said to be *commutative* whenever any pair of parallel paths yields the same composite morphism.

Example 2.2 (Identities via commutative diagrams)**.** Let $\mathcal{C}$ be a category and x an object of $\mathcal{C}$. The fact that 1_x the identity of x can be translated as follows: the diagrams

$$\begin{tikzcd} & x \arrow[dd, "1_x"] \\ a \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & x \end{tikzcd} \qquad \begin{tikzcd} x \arrow[dd, "1_x"'] \arrow[dr, "g"] & \\ & b \\ x \arrow[ur, "g"'] & \end{tikzcd} \tag{2.7}$$

commute for every a and b objects and f and g morphisms of $\mathcal{C}$.

Example 2.3 (Associativity via commutative diagrams)**.** Associativity can be rephrased by saying:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "gf"'] & b \arrow[dr, "hg"] & \\ & c \arrow[r, "h"'] & d \end{tikzcd}$$

commutes for every a, b, c and d objects and f, g and h morphisms in $\mathcal{C}$.

3 Isomorphisms

[This section has to be rewritten.]

Let us step back to the origins. The categorial axioms state identities that deals with morphisms, since equality between morphisms is involved. For that reason, we shall regard these axioms as ones about morphisms, since objects barely appear as start/end point of morphisms.

Thus categories have a notion of sameness between morphisms, the equality, but nothing is said about objects. Of course, if a category has also the equality for objects, it is fine, but we can craft a better notion of sameness of objects. Not because equality is bad, but we shall look for something that can be stated solely in categorial terms.

As usual, simple examples help us to isolate this notion.

Example 3.1 (Set Theory, equinumerousity)**.** Cantor, the father of Set Theory, conducted its enquiry on cardinalities and not on equality of sets. For A and B sets, the following statements are equivalent:

1. there exists a bijective function $A \to B$;
2. there exist two functions $\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \end{tikzcd}$ such that $gf = 1_A$ and $fg = 1_B$.

Though they are logically equivalent, they differ in some sense. In Set Theory, the adjective 'bijective' is defined by referring of the fact that sets are things that have elements:

for every $y \in B$ there is one and only one $x \in A$ such that $f(x) = y$.

In contrast, (2) is a statement written in terms of functions and compositions of functions: so (2) is written in a categorial language.

Exercise 3.2. Demonstrate the equivalence above.

This is enough to formulate a definition.

Definition 3.3 (Isomorphic objects)**.** Let $\mathcal{C}$ be a category and a and b two of its objects. A morphism $f : a \to b$ of $\mathcal{C}$ is an *isomorphism* whenever there is in the same category a morphism $g : b \to a$ such that $gf = 1_a$ and $fg = 1_b$. In that case, a is said *isomorphic* to b when there is an isomorphism $a \to b$ in $\mathcal{C}$.

Definition 3.4 (Skeletal categories)**.** [...]

Example 3.5 (Inverse matrices)**.** [Yet to be TEX-ed...]

Definition 3.6 (Skeleton)**.** [...]

4 Duality

For $\mathcal{C}$ a category, its *dual* (or *opposite*) category is denoted $\mathcal{C}^{\mathrm{op}}$ and is described as follows. Here, the objects are the same of $\mathcal{C}$ and 'being a morphism $a \to b$' exactly means 'being a morphism $b \to a$ in $\mathcal{C}$'. In other words, passing from a category to its dual leaves the objects unchanged, whereas the morphisms have their verses reversed. To dispel any ambiguity, by 'reversing' the morphisms we mean that morphisms $f : a \to b$ of $\mathcal{C}$ can be found among the morphisms $b \to a$ of $\mathcal{C}^{\mathrm{op}}$ and, vice versa, morphisms $a \to b$ of $\mathcal{C}^{\mathrm{op}}$ among the morphisms $b \to a$ of $\mathcal{C}$. Nothing is actually constructed out of the blue. Some authors suggest to write f^{op} to indicate that one f once it has domain and codomain interchanged, but we do not do that here, because they really are the same thing but in different places. So, if f is the name of a morphism of $\mathcal{C}$, the name f is kept to indicate that morphism as a morphism of $\mathcal{C}^{\mathrm{op}}$; obviously, the same convention applies in the opposite direction. It may seem we are going to nowhere, but it makes sense when it comes to define the compositions in $\mathcal{C}^{\mathrm{op}}$: for $f : a \to b$ and $g : b \to c$ morphsisms of $\mathcal{C}^{\mathrm{op}}$ the composite arrow is so defined

$$gf := fg.$$

This is not a commutative property, though. Such definition is to be read as follows. At the left side, f and g are to be intended as morphisms of $\mathcal{C}^{\mathrm{op}}$ that are to be composed therein. Then the composite gf is calculated as follows:

1. look at f and g as morphisms of $\mathcal{C}$ and compose them as such: so $f : b \to a$ and $g : c \to b$ and $fg : c \to a$ according to $\mathcal{C}$;
2. now regard fg as a morphism of $\mathcal{C}^{\mathrm{op}}$: this is the value gf is bound to.

Let us see now whether the categorial axioms are respected. For x object of $\mathcal{C}^{\mathrm{op}}$ there is 1_x, which is a morphism $x \to x$ in either of $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$. For every object y and morphism $f : y \to x$ of $\mathcal{C}^{\mathrm{op}}$ we have

$$1_x f = f 1_x = f.$$

Similarly, we have that

$$g 1_x = g$$

for every object z and morphism $g : x \to z$ of $\mathcal{C}^{\mathrm{op}}$. Hence, 1_x is an identity morphism in $\mathcal{C}^{\mathrm{op}}$ too. Consider now four objects and morphisms of $\mathcal{C}^{\mathrm{op}}$

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

and let us parse the composition

$$h(gf).$$

In $h(gf)$ regard both h and gf as morphisms of $\mathcal{C}$. In that case, $h(gf)$ is exactly $(gf)h$, where gf is fg once f and g are taken as morphisms of $\mathcal{C}$ and composed there. So $h(gf) = (fg)h$, where at left hand side compositions are performed in $\mathcal{C}$: being the composition is associative, $h(gf) = (fg)h = f(gh)$. We go back to $\mathcal{C}^{\mathrm{op}}$, namely $f(gh)$ becomes $(gh)f$ and gh becomes hg, so that we eventually get the associativity

$$h(gf) = (hg)f.$$

It may seem hard to believe, but duality is one of the biggest conquest of Category Theory. [...]

5 Foundations

Let us return at the beginning, namely the definition of category. Why not formulate it in terms of sets? That is, why don't muster the objects into a set, for any pair of objects, the morphisms into a set and writing compositions as functions?

Let us analyse what happens if we do that. A basic and quite popular fact that fatally crushes our hopes is:

there is no set of all sets.[1]

The first aftermath is that the existence of **Set** would not be legal, because otherwise a set would gather all sets.

Another example comes from both Algebra and Set Theory. In general, it's not a so profound result, but it is interesting for our discourse:

[1] If we want a set X to be the set of all sets, then it has all its subsets as elements, which is an absurd. In fact, Cantor's Theorem states that for every set X there is no surjective function $f : X \to 2^X$.

every pointed set $(X, 1)$ has an operation that makes it a group.[2]

Viz there exists no set of all groups, and then neither **Grp** would be supported.

As if the previous examples were not enough, Topology provides another irreducible case. Any set has the corresponding powerset, thus any set gives rise to at least one topological space. Our efforts are doomed, again: there is no set of all topological spaces, and so also **Top** would not be allowed!

It seems that using Set Theory requires the sacrifice of nice categories; and we do not want that, of course. From the few examples above one could surmise it is a matter of *size*: sets sometimes are not appropriate for collecting all the stuff that makes a category. Luckily, there is not a unique Set Theory and, above all, there is one that could help us.

The *von Neumann-Bernays-Gödel approach*, usually shortened as NBG, was born to solve size problems, and may be a good ground for our purposes. In NBG we have *classes*, the most general concept of 'collection'. But not all classes are at the same level: some, the *proper classes*, cannot be element of any class, whilst the others are the *sets*. Here is how the definition of category would look like.

Definition 5.1 (Categories)**.** A category $\mathcal{C}$ consists of:

- a class of objects, denoted $|\mathcal{C}|$;
- for every $a, b \in |\mathcal{C}|$, a class of morphisms from a to b, written as $\mathcal{C}(a, b)$;
- for every $a, b, c \in |\mathcal{C}|$, a composition, viz a function

$$\mathcal{C}(b, c) \times \mathcal{C}(a, b) \to \mathcal{C}(a, c)\,, \ (g, f) \to gf$$

with the following axioms:

1. for every $x \in |\mathcal{C}|$ there exists a $1_x \in \mathcal{C}(x, x)$ such that for every $y \in |\mathcal{C}|$ and $g \in \mathcal{C}(y, x)$ we have
$$1_x g = g$$
and for every $z \in |\mathcal{C}|$ and $h \in \mathcal{C}(x, z)$ we have
$$h1_x = h;$$
2. for $a, b, c, d \in |\mathcal{C}|$ and $f \in \mathcal{C}(a, b)$, $g \in \mathcal{C}(b, c)$ and $h \in \mathcal{C}(c, d)$ we have the identity
$$(hg)f = h(gf).$$

How does this double ontology of NBG actually apply at our discourse? For example, in NBG the class of all sets is a legit object: it is a proper class, because it cannot be an actual set. Thus, **Set** exists on NBG, and so exists **Grp**, **Top** and other big categories. Which is nice.

Hence, it is sensible to introduce some terms that distinguish categories by the size of their class of objects. [...]

[What can go wrong if $\mathcal{C}(a, b)$ are proper classes?]

6 Monomorphisms and Epimorphisms

[This section has to be rewritten.]

[2] Actually, this fact is equivalent to the Axiom of Choice.

Definition 6.1 (Monomorphisms and epimorphisms)**.** A morphism $f : a \to b$ of a category $\mathcal{C}$ is said to be:

- a *monomorphism* whenever if

$$\begin{tikzcd} c \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & a \arrow[r, "f"] & b \end{tikzcd}$$

  commutes for every object c and morphisms $g_1, g_2 : c \to a$ of $\mathcal{C}$, then $g_1 = g_2$;
- an *epimorphism* whenever if

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, bend left, "h_1"] \arrow[r, bend right, "h_2"'] & c \end{tikzcd}$$

  commutes for every object d and morphisms $h_1, h_2 : c \to a$ of $\mathcal{C}$, then $h_1 = h_2$;

[...]

Another way to express the things of the previous definition is this: $f : a \to b$ is a monomorphism whenever for every $c \in |\mathcal{C}|$ the function

$$\mathcal{C}(c, a) \to \mathcal{C}(c, b)\,,\ g \to fg \tag{6.8}$$

is injective. Similarly, $f : a \to b$ is an epimorphism when for every $d \in |\mathcal{C}|$ the function

$$\mathcal{C}(a, d) \to \mathcal{C}(b, d)\,,\ h \to hf \tag{6.9}$$

is injective. Category theorists call the functions (6.8) *precompositions* with f and (6.9) *postcompositions* with f.