Natural Transformations

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CONTENTS

1 DEFINITION

For $\mathcal{C}$ and $\mathcal{D}$ categories and $F, G : \mathcal{C} \to \mathcal{D}$ functors, a *transformation* from F to G amounts at having for every $x \in |\mathcal{C}|$ one morphism $F(x) \to G(x)$ of $\mathcal{D}$. In other words, a transformation is aimed to measure the difference of two parallel functor by the unique means we have, viz morphisms.

In general, we stick to the following convention: if η is the name of a transformation from F to G, then η_x indicates the component $F(x) \to G(x)$ of the transformation.

We are not interested in all transformations, of course.

Definition 1.1 (Natural transformations)**.** A transformation η from a functor $F : \mathcal{C} \to \mathcal{D}$ to a functor $G : \mathcal{C} \to \mathcal{D}$ is said to be *natural* whenever for every $a, b \in |\mathcal{C}|$ and $f \in \mathcal{C}(a, b)$ the square

$$\begin{tikzcd} F(a) \arrow[r, "\eta_a"] \arrow[d, "F(f)"'] & G(a) \arrow[d, "G(f)"] \\ F(b) \arrow[r, "\eta_b"'] & G(b) \end{tikzcd}$$

commutes. This property is the 'naturality' of η.

There are some notations for referring to natural transformations: one may write $\eta : F \Rightarrow G$ or even

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\eta"] \end{tikzcd}$$

if they want to explicit also categories.

Natural transformations can be composed: taken two consecutive natural transformations

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, "G"{name=M, description}] \arrow[r, bend right=60, "H"'{name=D}] & \mathcal{D} \arrow[from=U, to=M, Rightarrow, "\eta"] \arrow[from=M, to=D, Rightarrow, "\theta"] \end{tikzcd}$$

the transformation $\theta\eta$ that have the components $\theta_x \eta_x : F(x) \to H(x)$, for $x \in |\mathcal{C}|$ of $\mathcal{D}$ is natural. Such composition is associative. Moreover, for every functor $F : \mathcal{C} \to \mathcal{D}$ there is the natural transformation $1_F : F \Rightarrow F$ with components $1_{F(x)} : F(x) \to F(x)$, for $x \in |\mathcal{C}|$; they are identities in categorial sense:

$$\eta 1_F = \eta \text{ for every natural transformation } \eta : F \Rightarrow G$$
$$1_F \mu = \mu \text{ for every natural transformation } \mu : H \Rightarrow F.$$

All this suggests to, given two categories $\mathcal{C}$ and $\mathcal{D}$, form a category with functors $\mathcal{C} \to \mathcal{D}$ as objects and natural transformations as morphism, them being composable as explained above. [...]

[Consider `https://mathoverflow.net/q/39073`...]

2 Equivalent categories, again

3 The Yoneda lemma

We have the *evaluation functor*

$$\mathrm{ev}_{\mathcal{C}} : \mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$$

that on objects

$$\mathrm{ev}_{\mathcal{C}}(x, F) := F(x)$$

and on morphisms

$$\mathrm{ev}\left(\begin{array}{c} a \\ {\scriptstyle f}\big\downarrow \\ b \end{array} , \begin{array}{c} F \\ {\scriptstyle \eta}\big\Downarrow \\ G \end{array} \right) := \eta_b F(f) = G(f)\eta_a .$$

Lemma 3.1 (A lemma for the Yoneda Lemma)**.** Let $\mathcal{C}$ be a locally small category. Then for every $x \in |\mathcal{C}|$ and functor $F : \mathcal{C} \to \mathbf{Set}$,

$$[\mathcal{C}, \mathbf{Set}](\mathcal{C}(x, -), F) \cong F(x).$$

In particular, the classes $[\mathcal{C}, \mathbf{Set}](\mathcal{C}(x, -), F)$ are actual sets.

Proof. For x and F as in the hypothesis, take functions

$$\lambda_{x,F} : [\mathcal{C}, \mathbf{Set}](\mathcal{C}(x, -), F) \to F(x)\,,\ \lambda_{x,F}(\alpha) := \alpha_x(1_x).$$

Now, for every $a \in F(x)$ we have the transformation $\mu_{x,F}(a)$ from $\mathcal{C}(x, \bullet)$ to F which has the components

$$\mathcal{C}(x, c) \to F(c)\,,\ f \to \big(F(f)\big)(a);$$

it is immediate to show that it is natural. Thus we have functions

$$\mu_{x,F} : F(x) \to [\mathcal{C}, \mathbf{Set}](\mathcal{C}(x, -), F).$$

We prove

$$\lambda_{x,F}\mu_{x,F} = 1_{F(x)}$$
$$\mu_{x,F}\lambda_{x,F} = 1_{[\mathcal{C},\mathbf{Set}](\mathcal{C}(x,-),F)}.$$

In fact, for $a \in F(x)$ we have $\lambda_{x,F}\big(\mu_{x,F}(a)\big)$ is the component $\mathcal{C}(x, x) \to F(x)$ of $\mu_{x,F}(a)$ evaluated at 1_x, viz $1_{F(x)}(a) = a$. Besides, for if $\alpha : \mathcal{C}(x, \bullet) \Rightarrow F$

natural transformation we have $\mu_{x,F}\big(\lambda_{x,F}(\alpha)\big) = \mu_{x,F}\big(\alpha_x(1_x)\big)$ is the natural transformation $\mathcal{C}(x,\bullet) \Rightarrow F$ with components

$$\mathcal{C}(x,c) \to F(c)\,,\; f \to \big(F(f)\big)(\alpha_x(1_x)) = \alpha_c(f)$$

for $c \in |\mathcal{C}|$; that is $\mu_{x,F}\lambda_{x,F}(\alpha) = \alpha$. The proof is complete now. □

Let $\mathcal{C}$ be a locally small category. We have the functor

$$\text{よ}_{\mathcal{C}} : \mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$$

given on objects as follows

$$\text{よ}_{\mathcal{C}}(x,F) := [\mathcal{C}, \mathbf{Set}]\big(\mathcal{C}(x,-), F\big)$$

and on morphisms

$$\left[\text{よ}_{\mathcal{C}}\left(\begin{array}{cc} a & F \\ f\big\downarrow\;, & \eta\Downarrow \\ b & G \end{array}\right)\right]\left(\begin{array}{c} \mathcal{C}(a,-) \\ \Downarrow\alpha \\ F \end{array}\right) := \left\{\mathcal{C}(b,c) \xrightarrow{\eta_c\alpha_c(\_f)} G(c) \,\middle|\, c \in |\mathcal{C}| \right\}.$$

Observe that Lemma 3.1 solves annoying size issues in the definition of $\text{よ}_{\mathcal{C}}$ on objects. While the statement of this lemma is important for technical reasons, its proof guides us to the following completion.

Proposition 3.2 (Yoneda Lemma)**.** For $\mathcal{C}$ locally small category, $\text{よ}_{\mathcal{C}} \cong \mathrm{ev}_{\mathcal{C}}$.

Proof. The transformation $\lambda : \text{よ}_{\mathcal{C}} \Rightarrow \mathrm{ev}_{\mathcal{C}}$ having as components the functions $\lambda_{x,F}$ of the proof of Lemma 3.1 is natural, that is

$$\begin{tikzcd}
\text{よ}_{\mathcal{C}}(a,F) \arrow[r, "\lambda_{a,F}"] \arrow[d, "\text{よ}_{\mathcal{C}}(f{,}\eta)"'] & \mathrm{ev}_{\mathcal{C}}(a,F) \arrow[d, "\mathrm{ev}_{\mathcal{C}}(f{,}\eta)"] \\
\text{よ}_{\mathcal{C}}(b,G) \arrow[r, "\lambda_{b,G}"'] & \mathrm{ev}_{\mathcal{C}}(b,G)
\end{tikzcd}$$

commutes for every $f \in \mathcal{C}(a,b)$ and $\eta \in [\mathcal{C}, \mathbf{Set}](F,G)$. In fact, for every natural transformation $\eta \in \text{よ}_{\mathcal{C}}(a,F)$ we have

$$\mathrm{ev}_{\mathcal{C}}(f,\eta)\big(\lambda_{a,F}(\alpha)\big) = \eta_b\alpha_b\mathcal{C}(a,f)(1_a) = \eta_b\alpha_b f;$$

besides,

$$\lambda_{b,G}\big(\text{よ}_{\mathcal{C}}(f,\eta)(\alpha)\big) = \eta_b\alpha_b(\_f)(1_b) = \eta_b\alpha_b f.$$

Besides, λ is an isomorphism, as the proof of Lemma 3.1 tells us its components are isomorphisms. □