

Natural Transformations

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Last revision: April 10, 2022.

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1 DEFINITION

For \mathcal{C} and \mathcal{D} categories and $F, G : \mathcal{C} \rightarrow \mathcal{D}$ functors, a *transformation* from F to G amounts at having for every $x \in |\mathcal{C}|$ one morphism $F(x) \rightarrow G(x)$ of \mathcal{D} . In other words, a transformation is aimed to measure the difference of two parallel functor by the unique means we have, viz morphisms.

In general, we stick to the following convention: if η is the name of a transformation from F to G , then η_x indicates the component $F(x) \rightarrow G(x)$ of the transformation.

We are not interested in all transformations, of course.

Definition 1.1 (Natural transformations). A transformation η from a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ to a functor $G : \mathcal{C} \rightarrow \mathcal{D}$ is said to be *natural* whenever for every $a, b \in |\mathcal{C}|$ and $f \in \mathcal{C}(a, b)$ the square

$$\begin{array}{ccc} F(a) & \xrightarrow{\eta_a} & G(a) \\ F(f) \downarrow & & \downarrow G(f) \\ F(b) & \xrightarrow{\eta_b} & G(b) \end{array}$$

commutes. This property is the “naturality” of η .

There are some notations for referring to natural transformations: one may write $\eta : F \Rightarrow G$ or even

$$\begin{array}{ccc} & F & \\ \mathcal{C} & \begin{array}{c} \curvearrowright \\ \Downarrow \eta \\ \curvearrowleft \end{array} & \mathcal{D} \\ & G & \end{array}$$

if they want to explicit also categories.

Natural transformations can be composed: taken two consecutive natural transformations

$$\begin{array}{ccc} & F & \\ \mathcal{C} & \begin{array}{c} \curvearrowright \\ \Downarrow \eta \\ \Downarrow \theta \\ \curvearrowleft \end{array} & \mathcal{D} \\ & G & \\ & H & \end{array}$$

2. Equivalent categories, again

the transformation $\theta\eta$ that have the components $\theta_x\eta_x : F(x) \rightarrow H(x)$, for $x \in |\mathcal{C}|$ of \mathcal{D} is natural. Such composition is associative. Moreover, for every functor $F : \mathcal{C} \rightarrow \mathcal{D}$ there is the natural transformation $1_F : F \Rightarrow F$ with components $1_{F(x)} : F(x) \rightarrow F(x)$, for $x \in |\mathcal{C}|$; they are identities in categorical sense:

$$\begin{aligned}\eta 1_F &= \eta \text{ for every natural transformation } \eta : F \Rightarrow G \\ 1_F \mu &= \mu \text{ for every natural transformation } \mu : H \Rightarrow F.\end{aligned}$$

All this suggests to, given two categories \mathcal{C} and \mathcal{D} , form a category with functors $\mathcal{C} \rightarrow \mathcal{D}$ as objects and natural transformations as morphism, them being composable as explained above. [...]

[Consider <https://mathoverflow.net/q/39073...>]

2 EQUIVALENT CATEGORIES, AGAIN

3 THE YONEDA LEMMA

[Maybe, I should stick to *small* categories...]

We have the *evaluation functor*

$$\text{ev}_{\mathcal{C}} : \mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \rightarrow \mathbf{Set}$$

that on objects

$$\text{ev}_{\mathcal{C}}(x, F) := F(x)$$

and on morphisms

$$\text{ev} \left(\begin{array}{c} a \\ f \downarrow \\ b \end{array}, \begin{array}{c} F \\ \eta \downarrow \\ G \end{array} \right) := \eta_b F(f) = G(f) \eta_a.$$

Lemma 3.1 (A lemma for the Yoneda Lemma). Let \mathcal{C} be a locally small category. Then for every $x \in |\mathcal{C}|$ and functor $F : \mathcal{C} \rightarrow \mathbf{Set}$,

$$[\mathcal{C}, \mathbf{Set}](\mathcal{C}(x, -), F) \cong F(x).$$

In particular, the classes $[\mathcal{C}, \mathbf{Set}](\mathcal{C}(x, -), F)$ are actual sets.

Proof. For x and F as in the hypothesis, take functions

$$\lambda_{x,F} : [\mathcal{C}, \mathbf{Set}](\mathcal{C}(x, -), F) \rightarrow F(x), \lambda_{x,F}(\alpha) := \alpha_x(1_x).$$

Now, for every $a \in F(x)$ we have the transformation $\mu_{x,F}(a)$ from $\mathcal{C}(x, \bullet)$ to F which has the components

$$\mathcal{C}(x, c) \rightarrow F(c), f \rightarrow (F(f))(a);$$

it is immediate to show that it is natural. Thus we have functions

$$\mu_{x,F} : F(x) \rightarrow [\mathcal{C}, \mathbf{Set}](\mathcal{C}(x, -), F).$$

We prove

$$\begin{aligned}\lambda_{x,F} \mu_{x,F} &= 1_{F(x)} \\ \mu_{x,F} \lambda_{x,F} &= 1_{[\mathcal{C}, \mathbf{Set}](\mathcal{C}(x, -), F)}.\end{aligned}$$

3. The Yoneda Lemma

In fact, for $a \in F(x)$ we have $\lambda_{x,F}(\mu_{x,F}(a))$ is the component $\mathcal{C}(x, x) \rightarrow F(x)$ of $\mu_{x,F}(a)$ evaluated at 1_x , viz $1_{F(x)}(a) = a$. Besides, for if $\alpha : \mathcal{C}(x, \bullet) \Rightarrow F$ natural transformation we have $\mu_{x,F}(\lambda_{x,F}(\alpha)) = \mu_{x,F}(\alpha_x(1_x))$ is the natural transformation $\mathcal{C}(x, \bullet) \Rightarrow F$ with components

$$\mathcal{C}(x, c) \rightarrow F(c), f \mapsto (F(f))(\alpha_x(1_x)) = \alpha_c(f)$$

for $c \in |\mathcal{C}|$; that is $\mu_{x,F}\lambda_{x,F}(\alpha) = \alpha$. The proof is complete now. \square

Let \mathcal{C} be a locally small category. We have the functor

$$\mathfrak{Y}_{\mathcal{C}} : \mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \rightarrow \mathbf{Set}$$

given on objects as follows

$$\mathfrak{Y}_{\mathcal{C}}(x, F) := [\mathcal{C}, \mathbf{Set}](\mathcal{C}(x, -), F)$$

and on morphisms

$$\left[\mathfrak{Y}_{\mathcal{C}} \left(\begin{array}{c} a \\ f \downarrow \\ b \end{array}, \begin{array}{c} F \\ \eta \Downarrow \\ G \end{array} \right) \right] \left(\begin{array}{c} \mathcal{C}(a, -) \\ \Downarrow \alpha \\ F \end{array} \right) := \left\{ \mathcal{C}(b, c) \xrightarrow{\eta_c \alpha_c(-f)} G(c) \mid c \in |\mathcal{C}| \right\}.$$

Observe that Lemma 3.1 solves annoying size issues in the definition of $\mathfrak{Y}_{\mathcal{C}}$ on objects. While the statement of this lemma is important for technical reasons, its proof guides us to the following completion.

Proposition 3.2 (Yoneda Lemma). For \mathcal{C} locally small category, $\mathfrak{Y}_{\mathcal{C}} \cong \text{ev}_{\mathcal{C}}$.

Proof. The transformation $\lambda : \mathfrak{Y}_{\mathcal{C}} \Rightarrow \text{ev}_{\mathcal{C}}$ having as components the functions $\lambda_{x,F}$ of the proof of Lemma 3.1 is natural, that is

$$\begin{array}{ccc} \mathfrak{Y}_{\mathcal{C}}(a, F) & \xrightarrow{\lambda_{a,F}} & \text{ev}_{\mathcal{C}}(a, F) \\ \mathfrak{Y}_{\mathcal{C}}(f, \eta) \downarrow & & \downarrow \text{ev}_{\mathcal{C}}(f, \eta) \\ \mathfrak{Y}_{\mathcal{C}}(b, G) & \xrightarrow{\lambda_{b,G}} & \text{ev}_{\mathcal{C}}(b, G) \end{array}$$

commutes for every $f \in \mathcal{C}(a, b)$ and $\eta \in [\mathcal{C}, \mathbf{Set}](F, G)$. In fact, for every natural transformation $\eta \in \mathfrak{Y}_{\mathcal{C}}(a, F)$ we have

$$\text{ev}_{\mathcal{C}}(f, \eta)(\lambda_{a,F}(\alpha)) = \eta_b \alpha_b \mathcal{C}(a, f)(1_a) = \eta_b \alpha_b f;$$

besides,

$$\lambda_{b,G}(\mathfrak{Y}_{\mathcal{C}}(f, \eta)(\alpha)) = \eta_b \alpha_b(-f)(1_b) = \eta_b \alpha_b f.$$

We can conclude λ is an isomorphism, as the proof of Lemma 3.1 tells us its components are isomorphisms. \square