VON NEUMANN-BERNAYS-GÖDEL SET THEORY

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1. Introduction

In a platonic sense, somehow, there exist things called *classes*; but not all classes are at the same level: some are said *proper*, others are called *sets*. Yet, this is a set theory, not a class one, since our aim is to correctly manipulate sets. You are probably aware of *Russell's Antinomy*: Von Neumann, Bernays and Gödel provided a solution[1] to that important issue. Along these pages we deal with their standpoint and axioms.

You certainly have an (at least vague) idea of what a set is; also, you should know that the smallest sensible formula is $x \in X$, which stands for 'x is a member of X' or 'x is an element of X'. Here, NBG poses the very first difference between sets and proper classes: uniquely sets are allowed to be members of something, whilst proper classes cannot. That is, you can write $x \in X$ when x is a name used for indicating a set; instead, classes which have something as element can be either sets or proper classes.

We remark that from such standpoint *every object is a set*: things named *urelements*, objects are not made of elements, are not contemplated in these pages. There is nothing of strange: for instance, a population of a city is the collection of its inhabitants, every person is an aggregate of cells, cells are made of molecules, molecules are composed of atoms, et cetera...

Another crucial difference pertains language: we want to quantify, that is use $\forall$ and $\exists$, uniquely on sets not on proper classes. When we write

$$\forall x : p(x) \quad \text{or} \quad \exists x : p(x),$$

for some predicate p, the symbol x is used for sets, not on proper classes. Why this? Do not we want to quantify over proper classes too? First of all, as we said, this is a set theory. Secondly, in some sense, quantifying over

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[1]One among many others. You may have heard about Zermelo and Fraenkel (ZFC axioms) or Tarski and Gröthendieck (TG axioms).

things requires you to have in mind an environment where those things live: if such things are classes, you need something can gather classes (in particular also proper classes), which NBG does not support. You may want to introduce 'collections of classes', and then? You will recursively need a 'collections of collections of classes', et cetera... So where do advantages lie? In general, in this axiomatization we also have *schemas*, which can be intended as 'sentences on demand': if a symbol denotes a proper class or a class you do not know whether it is a set or a proper class, whenever you replace it with the name of a class you generate a sentence telling you something about that class. Hence a schema is not a single sentence, but many sentences in one.

2. General axioms about classes

The very first axiom makes clear what is the right notion of *equality* for classes: two classes are equal whenever have the same members. Let us formally express that concept.

Axiom 2.1 (Axiom of extensionality). For A and B classes, we assume

$$A = B \Leftrightarrow (\forall x : x \in A \Leftrightarrow x \in B)$$

The previous axiom is a schema. As you would expect, there are some basilar properties.

Theorem 2.1. A class in equal to itself. If a class A is equal to another class B, then B equals A. For A, B and C classes, if $A = B$ and $B = C$, then $A = C$.

Proof. That is purely a question of Logic. $\forall x : x \in A \Leftrightarrow x \in A$ is always true. If we have $\forall x : x \in A \Leftrightarrow x \in B$, we have $\forall x : x \in B \Leftrightarrow x \in A$ too. Finally, if $\forall x : x \in A \Leftrightarrow x \in B$ and $\forall x : x \in B \Leftrightarrow x \in C$, then $\forall x : x \in A \Leftrightarrow x \in C$. □

Given two classes A and B, we say A is *subclass* of B and write $A \subseteq B$ whenever

$$\forall x : x \in A \Rightarrow x \in B.$$

The following axiom allows us to build any class by providing a predicate.

Axiom 2.2 (Abstraction axiom). Given a sensible predicate on sets p, there exists the class, written as $\{x \mid p(x)\}$, whose members are exactly the sets x for whom $p(x)$ is true:

$$\forall a : a \in \{x \mid p(x)\} \Leftrightarrow p(a).$$

By saying 'sensible predicate' we mean a predicate, built with syntactic rules of Logic, which makes sense for sets.

Construction 2.1 (The empty class). There exists the *empty class*

$$\varnothing := \{x \mid x \neq x\}.$$

The reason why that name is no element x is such that $x \neq x$, so no $x \in \varnothing$. Let X be a class with the property $\forall x : x \notin X$. Then we have $\forall x : x \in$

$X \Rightarrow x \in \varnothing$, hence $X \subseteq \varnothing$. In general, if X is a class, then $\varnothing \subseteq X$ since $\forall x : x \in \varnothing \Rightarrow x \in X$ is true. So, we can conclude that $X = \varnothing$.

Construction 2.2 (The class of all the sets). There exists the *class of all sets*: just pick the predicate $x = x$ and you have the class

$$\mathbf{V} := \{x \mid x = x\}.$$

Any class X is a subclass of $\mathbf{V}$: in fact, every $x \in X$ satisfies $x = x$, so $x \in \mathbf{V}$.

Theorem 2.2. The *Russell class* $\mathbf{R} := \{x \mid x \notin x\}$ is a proper class.

Proof. Assume $\mathbf{R}$ is a set instead. Hence, follows from how $\mathbf{R}$ is made, $\mathbf{R} \in \mathbf{R} \Leftrightarrow \mathbf{R} \notin \mathbf{R}$, an absurd. □

Here we are how NBG resolves *Russell's Antinomy*!

Construction 2.3 (Operations with classes). You are likely aware you can perform some operations. Formally, they are allowed by the the axiom 2.3: given two classes A and B we have the classes

$$\begin{aligned} A \cup B &:= \{x \mid x \in A \text{ or } x \in B\} \\ A \cap B &:= \{x \mid x \in A \text{ and } x \in B\} \\ \neg A &:= \{x \mid x \notin A\}. \end{aligned}$$

once one chooses suitable predicates.

With the following axiom we dive into sets.

Axiom 2.3 (Comprehension axiom). A subclass of a set is a set.

Imagine you have a class E you know it is a set and a predicate on elements: also the class

$$\{x \in E \mid p(x)\} := \{x \mid x \in E \text{ and } p(x)\}$$

is a set, because it is a subclass of the set E.

Theorem 2.3. $\mathbf{V}$ is a proper class.

Proof. Again, suppose $\mathbf{V}$ is a set. Since every class is a subclass of $\mathbf{V}$, also $\mathbf{R}$ is a set. Thus, because of the axiom 2.3, $\mathbf{R}$ must be a set, which is not by theorem 2.2. □

3. Empty, pair, union and power set

We already now there is the empty class (see construction 2.1), but we actually want it to be a set.

Axiom 3.1 (Empty set). $\varnothing$ is a set.

Now, we have the empty set, but we also want sets that are not empty.

Axiom 3.2 (Pair set). For every sets A and B the class $\{A, B\}$ is a set.

A *singleton* is a class which has a unique element. The previous axiom also implies singletons are sets too: if x is a set, so $\{x, x\}$ is; usually one avoids repetitions and simply writes $\{x\}$.

Axiom 3.3 (Union set). For every set X the class

$$\bigcup X := \{A \mid \exists B \in X : A \in B\}$$

is a set.

Sometimes, instead of $\bigcup X$, we write $\bigcup_{E \in X} E$.

Theorem 3.1. For every sets A and B the class $A \cup B$ is a set.

Proof. If A and B are sets, so $\{A, B\}$ is by theorem 3.2. Finally, because of theorem 3.3, $A \cup B = \bigcup \{A, B\}$ is a set. □

Axiom 3.4 (Power set). For every set X the class

$$\wp X := \{E \mid E \subseteq X\}$$

is a set, which we call *power set.*

4. Cartesian Product and Functions

Given two elements x and y, we know the class $\{\{x\}, \{x, y\}\}$ is a set.

Definition 4.1 (Ordered pairs). Given two objects x and y, the object

$$(x, y) := \{\{x\}, \{x, y\}\}$$

is an *ordered pair.*

Exercise 4.1. This exercise explains why 'ordered' pair. Demonstrate

$$\forall a, b, c, d : (a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d.$$

Definition 4.2 (Product of classes). Let A and B be two classes. The *product* of A and B is the class

$$A \times B := \{z \mid \exists a \in A, b \in B : z = (a, b)\}.$$

Theorem 4.1. The product of two sets is a set.

Proof. It is straightforward, once you notice $A \times B \subseteq \wp(\wp(A \cup B))$. If A and B are sets, so is $A \cup B$, because of axiom 3.3; using twice axiom 3.4, also $\wp(\wp(A \cup B))$ is a set; by axiom 2.3, we conclude $A \times B$ is a set. □

Definition 4.3 (Functions). Given two classes X and Y, a *function* from X to Y is a subclass $f \subseteq X \times Y$ such that

$$\forall x \in X \exists! y \in Y : (x, y) \in f.$$

Usually, one writes $f : X \to Y$ to tell f is a function from X to Y and $f(x) = y$ instead of $(x, y) \in f$. The classes X and Y are the *domain* and the *codomain* of f, respectively. For $A \subseteq X$ the *image* of A via f is the class

$$fA := \{y \in Y \mid \exists x \in A : f(x) = y\}$$

and for $B \subseteq Y$ the class

$$f^{-1}B := \{x \in X \mid f(x) \in B\}.$$

is the *preimage* of B via f.

Axiom 4.1 (Replacement axiom). Let X and Y be classes and $f : X \to Y$. If X is a set, then also fX is a set.