The functors π_0 and π_1

Last revision: 9th March 2021.

Contents

1. Set Theory prerequisites

Lemma 1.1. Let X and Y be two sets, $\mathcal{E}$ be a partition of X and $f : X \to Y$ be a function with the property: $f(E)$ is a singleton for $E \in \mathcal{C}$. Then there exists one and only one function $f^* : \mathcal{C} \to Y$ such that commutes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & \mathcal{C} \arrow[ur, "f^*"'] & \end{tikzcd}$$

where $p : X \to \mathcal{C}$ is the function that maps each $a \in X$ into the unique $E \in \mathcal{C}$ such that $a \in E$.

Proof. Consider the relation

$$f^* := \{(E, b) \in \mathcal{C} \times Y \mid \text{exists } a \in X \text{ such that } p(a) = E \text{ and } f(a) = b\} .$$

Picked any $E \in \mathcal{C}$, since it is not empty, we can have an $a \in E$ and then we have the element $f(a)$ of B: thus $(E, f(a)) \in f^*$. Furthermore, consider the pairs (E, b) and (E, c) of f^*: therefore $b = f(u)$ and $c = f(v)$ for some $u, v \in E$; but, since $f(E)$ in a singleton, we must conclude $b = c$. We have shown that the relation f^* actually is a function $\mathcal{C} \to Y$. Besides such function satisfies $f^*p = f$. It remains only the uniqueness part. Assume you have a function $g : \mathcal{C} \to Y$ such that $gp = f$. Now, for every $E \in \mathcal{C}$ we have at least an $a \in E$, and $g(E) = g(p(a)) = f^*(p(a)) = f^*(E)$. $\square$

Proposition 1.2. Let X and Y be two sets, ε_X and ε_Y equivalence relation on X and on Y respectively. Then for every function $f : X \to Y$ that satisfies

$$\text{for every } a, b \in X \text{ if } a \varepsilon_X b \text{ then } f(a) \varepsilon_Y f(b) \tag{1.1}$$

there exist one and only one function $f_\sharp : X/\varepsilon_X \to Y/\varepsilon_Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X/\varepsilon_X & \xrightarrow[f_\sharp]{} & Y/\varepsilon_X \end{array}$$

commutes, where p_X and p_Y are the canonical projections.

Proof. The function $p_Y f : X \to Y/\varepsilon_Y$ is constant on ε_X-equivalence classes: in fact, for every $a, b \in X$ we have

$$\underbrace{a\varepsilon_X b \implies f(a)\varepsilon_Y f(b)}_{\text{cause (1.1)}} \implies p_Y(f(a)) = p_Y(f(b)).$$

Apply the previous lemma, and the proof is done. □

2. Paths

We indicate with I the interval $[0, 1]$ equipped with the Euclidean topology inherited from $\mathbb{R}$.

Definition 2.1 (Paths, junction and inversion)**.** Let X be a topological space and $a, b \in X$. A *path* from a to b in X is a continuous function

$$\gamma : I \to X \text{ such that } \gamma(0) = a \text{ and } \gamma(1) = b\,.$$

$\Omega(X, a, b)$ denotes the set of the paths in X from a to b. There is a function called *junction*:

$$* : \Omega(X, a, b) \times \Omega(X, b, c) \to \Omega(X, a, c)$$

where

$$(\alpha * \beta)(t) := \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \text{otherwise} \end{cases}$$

and one named *inversion*:

$$\operatorname{inv} : \Omega(X, a, b) \to \Omega(X, b, a)$$

where

$$\operatorname{inv} \varphi(t) := \varphi(1-t)\,.$$

Note that the junction is not associative in a strict sense, that is not always $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$ holds. It is quite simple to demonstrate that

$$\operatorname{inv}(\alpha * \beta) = \operatorname{inv} \beta * \operatorname{inv} \alpha$$

for every $\alpha \in \Omega(X, a, b)$ and $\beta \in \Omega(X, b, c)$.

Lemma 2.2. Let X and Y be topological spaces, $a, b \in X$. For every continuous function $f : X \to Y$, $\alpha \in \Omega(X, a, b)$ and $\beta \in \Omega(Y, b, c)$ we have

$$f(\alpha * \beta) = f\alpha * f\beta$$
$$f(\operatorname{inv} \alpha) = \operatorname{inv}(f\alpha)\,.$$

Proof. Since f is continuous, then

$$f\alpha \in \Omega(Y, f(a), f(b)) \text{ and } f\beta \in \Omega(Y, f(b), f(c)).$$

In that case:

$$f((\alpha * \beta)(t)) := \begin{cases} f(\alpha(t)) & \text{if } 0 \leq t \leq \frac{1}{2} \\ f(\beta(t)) & \text{otherwise} \end{cases} = f(\alpha(t)) * f(\beta(t)) \, .$$

The proof of the other identity is quite similar. □

Definition 2.3 (Connected points)**.** Let X be a topological space and a and b two of its points. We write $a \sim_X b$ whenever $\Omega(X, a, b) \neq \varnothing$.

Proposition 2.4. For X topological space, $\sim_X$ is an equivalence relation.

Proof. For every $a \in X$ there exists the *constant path*

$$c_a : I \to X \, , \; c_a(t) := a \text{ for every } t \in I \; :$$

Hence $a \sim_X a$ for every $a \in X$. Besides, for every $a, b \in X$ and for every $\lambda \in \Omega(X, a, b)$ there is the inverse path, that is $\operatorname{inv} \lambda \in \Omega(X, b, a)$: so, if $a \sim_X b$, then $b \sim_X a$. Finally, for every $a, b, c \in X$ and $\alpha \in \Omega(X, a, b)$ and $\beta \in \Omega(X, b, c)$ there is the jointed path $\alpha * \beta \in \Omega(X, a, c)$: thus, if $a \sim_X b$ and $b \sim_X c$, then $a \sim_X c$. □

The writing $[x]_{\sim_X}$ indicates the $\sim_X$-equivalence class of $x \in X$.

3. The functor $\pi_0 : \mathbf{Top} \to \mathbf{Set}$

Proposition 3.1. There exists a functor

$$\pi_0 : \mathbf{Top} \to \mathbf{Set}$$

described as follows:

- maps each topological space X into the set $\pi_0(X) := X/\sim_X$;
- maps every continuous function $f : X \to Y$ into the set function

$$\pi_0(f) : X/\sim_X \to Y/\sim_Y \, , \; \pi_0([x]_{\sim_X}) := [f(x)]_{\sim_Y} \, .$$

Proof. Any topological space X has the set of $\sim_X$-equivalence classes of $X/\sim_X$, so it is natural to take into account the function

$$|\mathbf{Top}| \to |\mathbf{Set}| \, , \; X \to X/\sim_X \; .$$

Take two topological spaces X and Y and a continuous function $f : X \to Y$ and consider $a, b \in X$ such that $a \sim_X b$, that is there exists a path $\lambda : I \to X$ from a to b. Since f is a continuous function, so is $f\lambda : I \to Y$: this is a path in Y from $f(a)$ to $f(b)$, so $f(a) \sim_Y f(b)$ too. Thus, by Proposition 1.2, there exists a unique function $f_\sharp : X/\sim_X \to Y/\sim_Y$ such that

$$f_\sharp([a]_{\sim_X}) = [f(a)]_{\sim_Y} \text{ for every } a \in X \, .$$

In other words, for X and Y topological spaces we have the function

$$\mathbf{Top}(X, Y) \to \mathbf{Set}(X/\sim_X, Y/\sim_Y) \, , \; f \to f_\sharp \, .$$

We prove the functoriality. Taken two continuous maps $f : X \to Y$ and $g : Y \to Z$, we have to demonstrate that $(gf)_\sharp = g_\sharp f_\sharp$: for every $a \in X$

$$\begin{aligned}(gf)_\sharp([a]_{\sim_X}) &= [gf(a)]_{\sim_Z} = \\ &= g_\sharp([f(a)]_{\sim_Y}) = \\ &= g_\sharp(f_\sharp([a]_{\sim_X})) = \\ &= (g_\sharp f_\sharp)([a]_{\sim_X}) .\end{aligned}$$

It remains to investigate what $(\mathrm{id}_X)_\sharp$ actually is, for X topological space and id_X its identity: for each $a \in X$ we have

$$(\mathrm{id}_X)_\sharp([a]_{\sim_X}) = [\mathrm{id}_X(a)]_{\sim_X} = [a]_{\sim_X} ,$$

that is $(\mathrm{id}_X)_\sharp = \mathrm{id}_{X/\sim_X}$. Finally, we have a functor $\pi_0 : \mathbf{Top} \to \mathbf{Set}$ defined as follows:

$$\pi_0(X) = X/\sim_X \text{ and } \pi_0(f) = f_\sharp . \qquad \square$$

4. The functor $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$

Definition 4.1 (Homotopy)**.** Let X be a topological space, a and b be two of its points and $\alpha, \beta \in \Omega(X, a, b)$. A *homotopy* (plural: homotopies) from α to β is a continuous function

$$\Gamma : I \times I \to X$$

such that:

(1) $\Gamma(-, 0) = \alpha$ and $\Gamma(-, 1) = \beta$
(2) $\Gamma(0, t) = a$ and $\Gamma(1, t) = b$ for every $t \in I$.

We say that α is *homotopic* to β, and write $\alpha \simeq \beta$, whenever there is an homotopy from α to β; in that case, one may write

$$\Gamma : \alpha \simeq \beta$$

to specify that Γ is one of these homotopies.

Proposition 4.2. Paths in convex sets $E \subseteq \mathbb{R}^n$ are homotopic.

Proof. Since E is convex, take $a, b \in X$ and $\alpha, \beta \in \Omega(X, a, b)$ and consider

$$\Gamma : I \times I \to E\,,\ \Gamma(s, t) := (1 - t)\alpha(s) + t\beta(s).$$

This function is a homotopy from α to β. $\square$

Lemma 4.3. Let X be a topological space and $a, b \in X$. For every $\alpha \in \Omega(X, a, b)$ and $\varphi \in \Omega(I, 0, 1)$ we have

$$\alpha \simeq \alpha\varphi .$$

Proof. There is, in fact, the homotopy

$$\Lambda : I \times I \to X\,,\ \Lambda(s, t) := \alpha((1 - t)s + t\varphi(s)). \qquad \square$$

Proposition 4.4. $\simeq$ is an equivalence relation.

Proof. Let X be a topological space and $a, b \in X$.

(1) $\alpha \in \Omega(X, a, b)$ is homotopic to itself: one of these homotopies is

$$\Gamma : I \times I \to X\ ,\ \Gamma(s,t) := \alpha(s)\,.$$

(2) Take $\alpha, \beta \in \Omega(X, a, b)$ and a homotopy $\Lambda : \alpha \simeq \beta$. Then

$$\Lambda' : I \times I \to X\ ,\ \Lambda'(s,t) := \Lambda(s, 1-t)$$

is a homotopy $\beta \simeq \alpha$.

(3) Now take $\alpha, \beta, \gamma \in \Omega(X, a, b)$ and homotopies $\Delta_1 : \alpha \simeq \beta$, $\Delta_2 : \beta \simeq \gamma$. The function

$$\Delta_3 : I \times I \to X\ ,\ \Delta_3(s,t) := \begin{cases} \Delta_1(s, 2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \Delta_2(s, 2t-1) & \text{otherwise} \end{cases}$$

is a homotopy.

(Calculations are simple, and left to the reader.) □

Proposition 4.5. $*$ is compatible with $\simeq$. That is: given a topological space X, $a, b, c \in X$, $\alpha_1, \beta_1 \in \Omega(X, a, b)$ and $\alpha_2, \beta_2 \in \Omega(X, a, b)$ we have

$$\alpha_1 \simeq \beta_1 \text{ and } \alpha_2 \simeq \beta_2 \implies \alpha_1 * \alpha_2 \simeq \beta_1 * \beta_2\,.$$

Proof. Consider homotopies $\Gamma : \alpha_1 \simeq \beta_1$ and $\Delta : \alpha_2 \simeq \beta_2$. A homotopy $\alpha_1 * \alpha_2 \simeq \beta_1 * \beta_2$ is the following function

$$\Xi : I \times I \to X\ ,\ \Xi(s,t) := \begin{cases} \Gamma(2s, t) & \text{if } 0 \le s \le \frac{1}{2} \\ \Delta(2s-1, t) & \text{otherwise} \end{cases}\,.$$ □

The previous proposition is crucial for our purposes, because it permits a group structure upon a quotient set. Let's start with the magmatic base and then discover a bunch of properties which makes it a group.

Definition 4.6. Let X be a topological space and $a \in X$.

$$\pi_1(X, a) := \Omega(X, a, a)/\simeq$$

whose elements are written as $[\alpha]$, where $\alpha \in \Omega(X, a, a)$. Due to the Proposition 4.5, we can define the following operation:

$$\begin{array}{ccc} \pi_1(X,a) \times \pi_1(X,a) & \to & \pi_1(X,a) \\ ([f],[g]) & \to & [f][g] := [f * g]\,. \end{array}$$

Lemma 4.7. Consider a topological space X and an $a \in X$. Then:

(1) for every $f, g, h \in \Omega(X, a, a)$

$$(f * g) * h \simeq f * (g * h)$$

(2) the path $c_a : I \to X$ which maps every $t \in I$ into a has the property: for every $f \in \Omega(X, a, a)$

$$c_a * f \simeq f \simeq f * c_a$$

(3) for every $f \in \Omega(X, a, a)$ we have

$$f * \operatorname{inv} f \simeq \operatorname{inv} f * f \simeq c_a.$$

Idea for the proof. Use Lemma 4.3. (Proof yet to TEX-ify...) □

Proposition 4.8. Let X be a topological space and $a \in X$. The set $\pi_1(X, a)$ shipped with the multiplication defined above is a group. In particular, $[c_a]$ with $a \in X$ is the identity and $[\alpha]^{-1} = [\operatorname{inv} \alpha]$.

Proof. Direct consequence of the previous lemma. $\square$

The group $\pi_1(X, a)$ has some names: *fundamental group*, *first homotopy group*, or *Poincaré group*.

Proposition 4.9. For X topological space and $a, b \in X$, if $\Omega(X, a, b) \neq \varnothing$ then $\pi_1(X, a) \cong \pi_1(X, b)$.

Proof. So there exists $\alpha \in \Omega(X, a, b)$. Then consider the function

$$\lambda_\alpha : \pi_1(X, a) \to \pi_1(X, b)\,,\ \lambda_\alpha([f]) := [\alpha]^{-1}[f][\alpha]\,.$$

At a first glance, the function λ_α is a bijective since

$$\lambda_{\operatorname{inv} \alpha}\lambda_\alpha = \mathrm{id}_{\pi_1(X,a)} \text{ and } \lambda_\alpha\lambda_{\operatorname{inv} \alpha} = \mathrm{id}_{\pi_1(X,b)}\,.$$

It just remains to show λ_α is a homomorphism:

$$\begin{aligned} \lambda_\alpha([f][g]) &= [\alpha]^{-1}[f][g][\alpha] = \\ &= [\alpha]^{-1}[f][\alpha][\alpha]^{-1}[g][\alpha] = \\ &= \lambda_\alpha([f])\lambda_\alpha([g])\,. \end{aligned}$$

for $f, g \in \Omega(X, a, a)$. $\square$

The previous fact has important consequences concerning path-connected spaces: for any pair of its points a and b we have $\pi_1(X, a) \cong \pi_1(X, b)$. To put it in other words: no matter which point you choose, the corresponding fundamental groups are the same. In some sense, that said, we can use expressions as '*the* fundamental group of X' and write $\pi_1(X)$ to mean any of the groups $\pi_1(X, a)$, with $a \in X$.

Definition 4.10 (Simply connected space)**.** A path-connected topological space X is said to be *simply connected* whenever its fundamental group is banal.

Maybe, it is better to recall what the category $\mathbf{Top}_*$ is:

- its objects are the *pointed topological spaces*, viz topological spaces from which one point is highlighted; formally, a pointed topological space is a pair (X, a), where X is a topological space and a is any of its points;
- a morphism $f : (X, a) \to (Y, b)$ is precisely a continuous function $f : X \to Y$ such that $f(a) = b$;
- the composition occurs in the same way as it does in $\mathbf{Top}$.

Proposition 4.11 (The functor $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$)**.** There exists a functor

$$\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$$

so described:

- it maps a pointed topological space (X, x_0) into the group $\pi_1(X, x_0)$;
- for (X, a) and (Y, b) pointed topological spaces, it maps a continuous function $f : (X, a) \to (Y, b)$ into the homomorphism

$$f_* : \pi_1(X, a) \to \pi_1(X, b)\,,\ \ f_*([\alpha]) := [f\alpha]\,.$$

Proof. Consider a continuous function $f : (X, a) \to (Y, b)$ of pointed topological spaces: such a function induces the function

$$f' : \Omega(X, a, a) \to \Omega(Y, b, b)\,,\ \ f'(\alpha) := f\alpha\,.$$

Now, let $\alpha, \beta \in \Omega(X, a, a)$ with the homotopy $\Lambda : \alpha \simeq \beta$: the function $f\Lambda : I \times I \to Y$ is a homotopy from $f\alpha$ to $f\beta$ too. By Proposition 1.2, this implies that there exists one and only one function $f_* : \pi_1(X, a) \to \pi_1(Y, b)$ that does what we need. Now, we have to prove it is a homomorphism and the functoriality of $(\cdot)_*$. □