NOTES ON POINT-SET TOPOLOGY

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0. Elements of Set Theory

0.1. **Sets.** We assume you are quite acquainted with Set Theory (at least with the naïve version[1]), which covers *sets* and *functions* among other things. This section exists only to recall some notations. Given two sets X and Y, a function f from X to Y is written as $f : X \to Y$. The expression $x \in X$ means x is an element of the set X. $\varnothing$ is the empty set, the one who has not any element. $A \subseteq B$ signifies that every $x \in A$ is an element of B too, and there exists the *power set* of any set X, viz the set

$$\wp X := \{E \mid E \subseteq X\} .$$

Delimiting your discourse into an environment is a good habit: when we talk about sets, we are tacitly moving inside a *universe of discourse*, a large set which contains all the things we need. The notation

$$\{x \in U \mid p(x)\} ,$$

where U is a prefixed universe of discourse and p is a predicate, denotes the set of all and only the elements of U for which $p(x)$ is true. Inside any universe of discourse one can perform some basic operations:

$$\begin{aligned} A \cup B &:= \{x \in U \mid x \in A \text{ or } x \in B\} \\ A \cap B &:= \{x \in U \mid x \in A \text{ and } x \in B\} \\ A - B &:= \{x \in A \mid x \notin B\} . \end{aligned}$$

[1] Here, ‘naïve’ stands for ‘non axiomatic’ (you may have heard something about ZFC or NBG). For most of our uses the innate idea one has of sets and membership relation is enough.

The expression (x, y) indicates an *ordered pair* and the product of two sets gathers ordered pairs: for A and B sets

$$A \times B := \{(x, y) \mid x \in A, x \in B\}.$$

There exist *families* too, that is sets whose elements are themselves sets. Sometimes families may be *indexed*, that is labels are attached to distinguish its member: in that case, you already may have met expressions as $\{X_i\}_{i \in I}$ or $\{X_i \mid i \in I\}$ before now.[2]

We want unions and intersections of all the elements of a family to be defined as well: let $\mathcal{E}$ be any family of sets coming from an universe of discourse U:

$$\bigcup \mathcal{E} := \bigcup_{A \in \mathcal{E}} A := \{x \in U \mid \exists A \in \mathcal{E} : x \in A\}$$
$$\bigcap \mathcal{E} := \bigcap_{A \in \mathcal{E}} A := \{x \in U \mid \forall A \in \mathcal{E} : x \in A\}$$

In particular, $A \cup B = \bigcup \{A, B\} = \bigcup_{I \in \{A,B\}} I$ and $A \cap B = \bigcap \{A, B\} = \bigcap_{I \in \{A,B\}} I$. With indexed families it's what you would expect:

$$\bigcup_{i \in I} A_i := \{x \in U \mid \exists i \in I : x \in A_i\} \ , \quad \bigcap_{i \in I} A_i := \{x \in U \mid \forall i \in I : x \in A_i\}$$

The following facts are known as the 'de Morgan's Laws'

$$X - \bigcup_{I \in \mathcal{E}} I = \bigcap_{I \in \mathcal{E}} (X - I) \ , \quad X - \bigcap_{I \in \mathcal{E}} I = \bigcup_{I \in \mathcal{E}} (X - I)$$

Another thing deserves to be remarked:

$$\bigcup_{A \in \mathcal{E}_1, B \in \mathcal{E}_2} F(A, B) \quad \text{and} \quad \bigcap_{A \in \mathcal{E}_1, B \in \mathcal{E}_2} F(A, B),$$

where $F : \wp U \times \wp U \to \wp U$ for some universe U, are useful shorthands for

$$\bigcup_{A \in \mathcal{E}_1} \left(\bigcup_{B \in \mathcal{E}_2} F(A, B) \right) \quad \text{and} \quad \bigcap_{A \in \mathcal{E}_1} \left(\bigcap_{B \in \mathcal{E}_2} F(A, B) \right)$$

respectively.

We have the *product* of sets of a family too: given a family $\{X_i\}_{i \in I}$

$$\prod_{i \in I} X_i := \left\{ f : I \to \bigcup_{i \in I} X_i \mid f(i) \in X_i \text{ for every } i \in I \right\}.$$

When $\{X_i\}_{i \in I}$ is finite and $I = \{1, \dots, n\}$, the definition is quite simplified

$$\prod_{i \in I} X_i := \{(x_1, \dots, x_n) \mid x_i \in X_i \text{ for every } i \in I\}.$$

[2]Formally, an indexed set X is any surjection $I \to X$ for some I which provides 'indexes'. But what exactly is a index set? For instance, for a family of three sets the set $\{\mathrm{do}, \mathrm{re}, \mathrm{mi}\}$ fits well. Usually, natural, integers or real numbers are used.

0.2. **Functions.** If f is a function $X \to Y$, the *image* via f of an $A \subseteq X$ is the set

$$fA := \{y \in Y \mid \exists x \in A : f(x) = y\},$$

while the *preimage* of a $B \subseteq Y$ via f the set

$$f^{-1}B := \{x \in X \mid f(x) \in B\}.$$

The preimage of singletons, due to their relevance, have a dedicated name: for $y \in Y$ the *fibre* of y via f is the set $f^{-1}\{y\}$.

The preimage function has a regular behaviour:

$$f^{-1}\bigcup_{I\in\mathcal{E}} I = \bigcup_{I\in\mathcal{E}} f^{-1}I$$
$$f^{-1}\bigcap_{I\in\mathcal{E}} I = \bigcap_{I\in\mathcal{E}} f^{-1}I.$$

The image function doesn't: whilst

$$f\bigcup_{E\in\mathcal{E}} E = \bigcup_{E\in\mathcal{E}} fE$$

is always true, the only thing one can say of f in combination with $\bigcap$ in general is

$$f\bigcap_{E\in\mathcal{E}} E \subseteq \bigcap_{E\in\mathcal{E}} fE.$$

1. The Category of Topological Spaces

1.1. **Topologies, open/closed sets and base.** Let us start in medias res, by giving a definition which will set up our discourse.

Definition 1.1 (Topological spaces). Given any set X, a *topology* for X is a any collection $\mathcal{O}$ of subsets of X which satisfies:

(1) $\varnothing, X \in \mathcal{O}$;
(2) for every $H \subseteq \mathcal{O}$ we have $\bigcup_{A\in H} A \in \mathcal{O}$;
(3) for every couple $A, B \in \mathcal{O}$ also $A \cap B \in \mathcal{O}$.

A set X equipped with a topology $\mathcal{O}$ for it becomes a *topological space*, its elements are called *points* and the elements of $\mathcal{O}$ *open sets*.

A topological space is nothing of unseen or weird: for every set X there exists its power-set; consider there is $\{\varnothing, X\}$ too. As you may quickly check, both those sets are topologies for X. The former one is the *discrete topology* and the latter is the *indiscrete topology*.

Example 1.1. More likely, one is more acquainted with $\mathbb{R}$, who is said to have a topological structure since first courses. In fact one is told that inside $\mathbb{R}$ there are neighbourhoods of its elements, that is open intervals of the shape $(x_0 - \varepsilon, x_0 + \varepsilon)$, for some $\varepsilon > 0$. Clearly, we must fix all that, and make precise the claim '$\mathbb{R}$ has a topological structure' (we will do so later, as soon as we can). In fact the intersection of two neighbourhoods does not need to be a neighbourhood (it may be empty!), and the same holds for unions.

Definition 1.2 (Closed sets). For X topological space, a set $C \subseteq X$ is said *closed* when $X - C$ is an open set.

Theorem 1.1. For X topological space, $\varnothing$ and X are closed sets. Furthermore, given any family $\mathcal{F}$ of closed sets, the intersection $\bigcap_{C \in \mathcal{F}} C$ is closed and the union of two closed sets is closed.

Proof. $X = X - \varnothing$, so X is closed since $\varnothing$ is open. Analogously, cause $\varnothing = X - X$ and X is open, $\varnothing$ is closed.
For every $C \in \mathcal{F}$, the set $X - C$ is open. Thus, by definition, the union

$$\bigcup_{C \in \mathcal{F}} (X - C) = X - \bigcap_{C \in \mathcal{F}} C$$

is open: hence $\bigcap_{C \in \mathcal{F}} C$ is closed.
If $A, B \subseteq X$ are closed, $X - A$ and $X - B$ are open. Again by definition, $(X - A) \cap (X - B) = X - A \cup B$ is open; that is $A \cup B$ is closed. □

At this point, there is an important remark. A topological space is closed under the union of any collection of open sets; since we did not apply restrictions, such collections of open sets may be finite or infinite. Instead, we explicitly want a topological space to be closed under *binary* intersections (by induction, it is closed under finite intersection). To be honest, we don't care whether an intersection of an infinite amount of open sets is open or not.

Also, observe that saying what exactly are the closed sets in a topological space X thoroughly determines it. In fact, if you provide a family $\mathcal{C}$ which gathers precisely all its closed sets, then the set $\mathcal{O} := \{X - A \mid A \in \mathcal{C}\}$ is a topology for X. That is, you have two manners to define topological space: you can say what is 'open' or what is 'closed'.

Actually, there is another way to give a complete description of a topological space: providing a base.

Definition 1.3. Let X be a topological space. A *base* of X is any collection $\mathcal{B}$ of open sets of X made as follows: for every open set A of X there exists $\mathcal{E} \subseteq \mathcal{B}$ such that

$$A = \bigcup_{E \in \mathcal{E}} E\,.$$

Let us put that in other words: the collection $\mathcal{B}$ is said to be a base of X whenever for every A and $x \in A$ there is an $E \in \mathcal{B}$ such that $E \subseteq A$ and $x \in E$.

The role of a base is clear: in a topological space an open set is exactly an union of sets of its base. The question now is: when a collection of subsets of a given set is a topology. The following lemma has an enormous importance.

Theorem 1.2 (Base Lemma). Let X be a set, and consider a set $\mathcal{B} \subseteq \wp X$ as follows:

(1) for every $x \in X$ there is a $E \in \mathcal{B}$ such that $x \in E$;

(2) for every $A, B \in \mathcal{B}$ and for every $x \in A \cap B$ there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

Hence the family

$$\mathcal{O} := \left\{ E \subseteq X \mid \exists B \subseteq \mathcal{B} : E = \bigcup_{I \in B} I \right\}$$

is the unique topology for X with base $\mathcal{B}$.

Proof. We have to verify whether the three axioms of definition 1.1 hold. By definition of $\mathcal{O}$ himself, $\varnothing$ is an element of $\mathcal{O}$, while (1) is equivalent to

$$X = \bigcup_{E \in \mathcal{B}} E ,$$

so $X \in \mathcal{O}$ too. Consider now any $\mathcal{S} \subseteq \mathcal{O}$ and we ask ourselves whether $\bigcup_{E \in \mathcal{S}} E \in \mathcal{O}$ or not. By how $\mathcal{O}$ is made, there is an $F(E) \subseteq \mathcal{B}$, for $E \in \mathcal{S}$, such that $E = \bigcup_{I \in F(E)} I$. Thus we have

$$\bigcup_{E \in \mathcal{S}} E = \bigcup_{E \in \mathcal{S}} \left(\bigcup_{I \in F(E)} I \right) = \bigcup_{I \in \bigcup_{E \in \mathcal{S}} F(E)} I$$

which lies in $\mathcal{O}$, since it is regarded as an union of elements of $\mathcal{B}$. Let $A, B \in \mathcal{O}$ and study $A \cap B$. Either of them can be rewritten as unions of elements of $\mathcal{B}$:

$$A = \bigcup_{I \in H_1} I , \quad B = \bigcup_{J \in H_2} J$$

for some $H_1, H_2 \subseteq \mathcal{B}$. Then

$$A \cap B = \bigcup_{I \in H_1} \left(\bigcup_{J \in H_2} I \cap J \right) .$$

The condition (2) says that the intersection of two any $A, B \in \mathcal{B}$ is a suitable union of elements of $\mathcal{B}$. Take $F(I, J) \subseteq \mathcal{B}$ such that $I \cap J = \bigcup_{E \in F(I,J)} E$, and so

$$A \cap B = \bigcup_{I \in H_1} \left(\bigcup_{J \in H_2} \left(\bigcup_{E \in F(I,J)} E \right) \right) = \bigcup_{E \in \bigcup_{I \in H_1} \bigcup_{J \in H_2} F(I,J)} E :$$

we can conclude that $A \cap B \in \mathcal{O}$, because $A \cap B$ is written as union of elements of $\mathcal{B}$. □

Example 1.2 ($\mathbb{R}^n$ with Euclidean topology). This example explains in which sense $\mathbb{R}$, or in general $\mathbb{R}^n$, has topological structure. A n-dimensional *ball* or *sphere* with centre $x_0 \in \mathbb{R}^n$ and radius $r \in \mathbb{R}$, $r > 0$ is the set

$$B(x_0, r) := \{ x \in \mathbb{R}^n \mid \| x - x_0 \| < r \} .$$

The family $\{ B(x, r) \mid x \in \mathbb{R}^n, r > 0 \}$ is a base for a topology over $\mathbb{R}^n$, as you can easily show. This topology, the *Euclidean topology*, is the one the entire Analysis is developed upon.

Theorem 1.3 (Equivalent bases). Let X be a set with topologies $\mathcal{O}_1$ and $\mathcal{O}_2$. If they have respectively $\mathcal{B}_1$ and $\mathcal{B}_2$ as bases and if:

(1) for every $B_1 \in \mathcal{B}_1$ and for every $x \in B_1$ there is a $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$ and
(2) for every $B_2 \in \mathcal{B}_2$ and for every $x \in B_2$ there is a $B_1 \in \mathcal{B}_1$ such that $x \in B_1 \subseteq B_2$

then $\mathcal{O}_1 = \mathcal{O}_2$.

Proof. (1) says every element of $\mathcal{B}_1$ is an union of elements of $\mathcal{B}_2$; conversely, (2) states every element of $\mathcal{B}_2$ is an union of elements of $\mathcal{B}_1$. The proof is quite formal but simple: as exercise, you may fill in with details. □

EXERCISE 1.1. Open intervals (that is those of shape (a, b) for some $a, b \in \mathbb{R}$, $a \leq b$) forms a base for $\mathbb{R}$. Show that the family of open boxes

$$\left\{ \prod_{k=1}^{n} I_k \mid I_1, \ldots, I_n \text{ open intervals of } \mathbb{R} \right\}$$

is a base for Euclidean topology over $\mathbb{R}^n$, as well.

CONSTRUCTION 1.1 (Boxed topology space). In general, for X and Y topological spaces, one can easily show the family

$$\{U \times V \mid U \subseteq X \text{ and } V \subseteq Y \text{ open sets}\}$$

satisfies the conditions (1) and (2) of the Base Lemma (theorem 1.2). The *product space* $X \times Y$ is precisely the one that has such base.

Since topologies are set of sets, they can be compared in an obvious way.

DEFINITION 1.4 (Coarser and finer topologies). Given a set X and two any topologies $\mathcal{O}_1$ and $\mathcal{O}_2$ over it, when $\mathcal{O}_1 \subseteq \mathcal{O}_2$ we say $\mathcal{O}_1$ is *coarser* than $\mathcal{O}_2$ or, equivalently, $\mathcal{O}_2$ is *finer* than $\mathcal{O}_1$. In any case we write $\mathcal{O}_2 \succcurlyeq \mathcal{O}_1$ or $\mathcal{O}_1 \preccurlyeq \mathcal{O}_2$.

Note that $\preccurlyeq$ is the relation $\subseteq$ restricted to topologies for the same set: thus $\preccurlyeq$ is a partially order relation.

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