## Extensors

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#### Abstract

The work here is based on a laconic and tiny article by René Guitart about *extensors*<sup>1</sup>: http://www.numdam.org/item/DIA\_1980\_\_3\_\_A3\_\_0.pdf. The pages below can be considered a long exercise.

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### References

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- [Gol06] R. Goldblatt. *Topoi: The Categorial Analysis of Logic.* Dover Books on Mathematics. Dover Publications, 2006. ISBN: 9780486450261.
- [Mik22] C. J. Mikkelsen. Lattice Theoretic and Logical Aspects of Elementary Topoi. Reprints in Theory and Applications of Categories, 2022. URL: http://www.tac.mta.ca/tac/reprints/articles/29/tr29. pdf.

## 1 Preamble

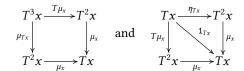
[This section is important for me. It may be deleted in the final version. Let me recall some words here, and then (try to) set a strategy to attack the paper.]

# **Definition 1.1.** A *monad* for a category $\mathcal C$ consists of:

- one functor  $T: \mathcal{C} \to \mathcal{C}$
- one natural transformation  $\eta: 1_{\mathcal{C}} \to T$
- one natural transformation  $\mu: T^2 \to T$ , where  $T^2 := T \circ T$

<sup>1 &#</sup>x27;Unofficial' translation. In French it is extenseurs.

such that the diagrams



commute for every object x of C; let us write such monad as the triple  $(T, \eta, \mu)$ .

In the sketched proof, the author talks about some 'monade des parties du topos'. **Set** is a topos and

**Sandbox 1.2** (The 'power set' monad). Consider the functor  $2^{\bullet}$ : **Set**  $\rightarrow$  **Set** where, for X a set,  $2^{\bullet}(X) := 2^{X}$  and, for  $f: X \rightarrow Y$ , define  $2^{f} := 2^{\bullet}(f)$  to be the function  $2^{A} \rightarrow 2^{B}$  that takes subsets E of A to fE.

Now, we need functions  $\xi_X : X \to 2^X$ , one for each set X, forming a natural transformation  $\xi : 1_{Set} \to 2^{\bullet}$ ; we consider that one introduced as  $\xi_X(x) := \{x\}$ . To conclude the structure of monad, consider for X set, the function

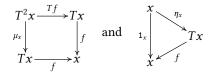
$$\delta_X: 2^{2^X} \to 2^X, \ \delta_X(F) := \bigcup_{E \in F} E.$$

The family of such functions determine a natural transformation  $\delta: 2^{\bullet} \circ 2^{\bullet} \to 2^{\bullet}$ .

**Definition 1.3.** If you are given a monad  $(T, \eta, \mu)$  for a category C, you can introduce an *algebra* for this monad, consisting of:

- one object x in C
- one morphism  $f: Tx \to x$

all this making commute the following diagrams



[I've just written a couple of definitions, but for the next days here is what to do for me: find more examples motivating monads and algebras for monads; you must learn something about *Kleisli* and *Eilenberg-Moore categories...* Not only for the sake of these pages.]

[Now understand this: 'un treillis complet est une 2°-algèbre'.]

Let us try now to bring this to any topoi, if possible.

## 2 One monad for topoi

#### 2.1 (Covariant) Power object functor

One of the possible definitions of (elementary) topos is:

a category with finite limits and power objects.

For future reference and notation, here is the definition of power object:

**Definition 2.1.** For if C is a category with finite products and  $a \in |C|$ , a *power object* in C consists of

- one object 'of the parts of a', we write  $\Omega^a$
- one monomorphism  $\epsilon_a : \epsilon_a \to \Omega^a \times a$ , the 'membership relation'

with the property: for every  $b \in |\mathcal{C}|$  and monomorphism  $\rho : r \to b \times a$  there exists one and only one morphism  $f : b \to \Omega^a$  for which there is a pullback square

$$r \xrightarrow{\rho} b \times a \qquad \downarrow f \times 1_a \\ \xi_a \xrightarrow{f} \Omega^a \times a$$

in C.

[A finer notation may be  $\wp(a)$ . Do I really need to refer so explicitly to subobject classifiers here?] Now if we want to replicate the above phenomenon inside any topos  $\mathcal{E}$ , we need a suitable functor  $\Omega^{\bullet}: \mathcal{E} \to \mathcal{E}$ . For  $a \in |\mathcal{E}|$ ,

$$\Omega^{\bullet}(a) := \Omega^{a}$$
.

It remains to understand what  $\Omega^f := \Omega^{\bullet}(f)$ , for  $f : a \to b$  in  $\mathcal{E}$ , should be to make a functor. Consider the power objects of a and b, that is  $\Omega^a$  and  $\Omega^b$  together with their respective membership relations  $\epsilon_a : \epsilon_a \to \Omega^a \times a$  and  $\epsilon_b : \epsilon_b \to \Omega^b \times b$ .

$$\begin{array}{ccc}
\epsilon_{a} & \xrightarrow{\epsilon_{a}} & \Omega^{a} \times a \xrightarrow{1_{\Omega^{a}} \times f} & \Omega^{a} \times b \\
& \downarrow & & \downarrow \\
\epsilon_{b} & \xrightarrow{\epsilon_{b}} & \Omega^{b} \times b
\end{array}$$

Here, the dotted morphism is of the form  $\Omega^f \times \mathbf{1}_b$ , where  $\Omega^f : \Omega^a \to \Omega^b$  is that morphism for which there is a pullback square

$$\begin{array}{c}
\epsilon_{a} \xrightarrow{\epsilon_{a}} \Omega^{a} \times a \xrightarrow{1_{\Omega^{a}} \times f} \Omega^{a} \times b \\
\downarrow & \downarrow & \downarrow \\ \Omega^{f} \times 1_{b} \\
\epsilon_{b} \xrightarrow{\epsilon_{a}} \Omega^{b} \times b
\end{array}$$

Let us verify functoriality now. Take

$$a \xrightarrow{f} b \xrightarrow{g} c$$

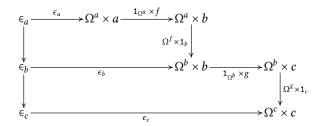
so that we have also

$$\Omega^a \xrightarrow{\Omega^f} \Omega^b \xrightarrow{\Omega^g} \Omega^c$$

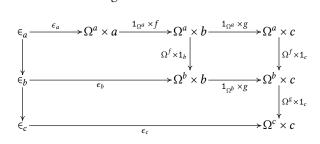
Let us study  $\Omega^{gf}: a \to c$ . By definition,  $\Omega^{gf}$  is the morphism for which there is a pullback square

$$\begin{array}{ccc}
\epsilon_{a} & \xrightarrow{\epsilon_{a}} & \Omega^{a} \times a \xrightarrow{\mathbf{1}_{\Omega^{a}} \times (gf)} & \Omega^{a} \times c \\
\downarrow & & \downarrow & & \downarrow \\
\epsilon_{c} & \xrightarrow{\epsilon_{b}} & \Omega^{c} \times c
\end{array}$$
(2.1)

We have then two pullback squares glued like this:



The first one gives the morphism  $\Omega^f$ , whereas the second one  $\Omega^g$ . Let us add two arrows to make a rectangle:

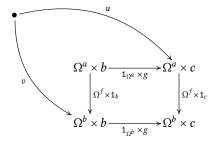


The plan is: if we can show

$$\begin{array}{c}
\Omega^{a} \times b \xrightarrow{1_{\Omega^{a}} \times g} \Omega^{a} \times c \\
\Omega^{f} \times 1_{b} \downarrow & \downarrow \Omega^{f} \times 1_{c} \\
\Omega^{b} \times b \xrightarrow{1_{cb} \times g} \Omega^{b} \times c
\end{array}$$

is a pullback square, then applying the *Pullback Lemma* twice will allow us to conclude the rectangle (2.1) is a pullback square too.

Mere brute force is employed here. In



assume  $(\Omega^f \times 1_c) u = (1_{\Omega^b} \times g) v$ . Now a lot of notation is needed, to write the products involved: we write them as

$$\Omega^{a} \stackrel{\alpha_{1}}{\longleftarrow} \Omega^{a} \times b \stackrel{\alpha_{2}}{\longrightarrow} b$$

$$\Omega^{a} \stackrel{\beta_{1}}{\longleftarrow} \Omega^{a} \times c \stackrel{\beta_{2}}{\longrightarrow} c$$

$$\Omega^{b} \stackrel{\gamma_{1}}{\longleftarrow} \Omega^{b} \times b \stackrel{\gamma_{2}}{\longrightarrow} b$$

$$\Omega^{b} \stackrel{\delta_{1}}{\longleftarrow} \Omega^{b} \times c \stackrel{\delta_{2}}{\longrightarrow} c$$

Here are the equations defining the products of morphisms involved

$$\beta_{1} (\mathbf{1}_{\Omega^{a}} \times \mathbf{g}) = \alpha_{1}$$

$$\beta_{2} (\mathbf{1}_{\Omega^{a}} \times \mathbf{g}) = \mathbf{g}\alpha_{2}$$

$$\gamma_{1} (\Omega^{f} \times \mathbf{1}_{b}) = \Omega^{f}\alpha_{1}$$

$$\gamma_{2} (\Omega^{f} \times \mathbf{1}_{b}) = \alpha_{2}$$

$$\delta_{1} (\Omega^{f} \times \mathbf{1}_{c}) = \Omega^{f}\beta_{1}$$

$$\delta_{2} (\Omega^{f} \times \mathbf{1}_{c}) = \beta_{2}$$

$$\delta_{1} (\mathbf{1}_{\Omega^{b}} \times \mathbf{g}) = \gamma_{1}$$

$$\delta_{2} (\mathbf{1}_{\Omega^{b}} \times \mathbf{g}) = \mathbf{g}\gamma_{2}$$

[We do not use all of them!] By universal property of product, we have one  $\chi: \bullet \to \Omega^a \times b$  for which  $\beta_1 u = \alpha_1 \chi$  and  $\gamma_2 v = \alpha_2 \chi$ . We show now that  $\chi$  is what we are looking for using the universal property of the the products of the square.

$$\begin{split} \gamma_1 v &= \delta_1 \left( \mathbf{1}_{\Omega^b} \times g \right) v = \delta_1 \left( \Omega^f \times \mathbf{1}_c \right) u = \Omega^f \beta_1 u = \Omega^f \alpha_1 \chi = \gamma_1 \left( \Omega^f \times \mathbf{1}_b \right) \chi \\ \gamma_2 v &= \alpha_1 \chi = \gamma_2 \left( \Omega^f \times \mathbf{1}_b \right) \chi \end{split}$$

which implies  $v = (\Omega^f \times 1_b) \chi$ ; moreover,

$$\begin{split} \beta_1 v &= \alpha_1 \chi = \beta_1 \left( \mathbf{1}_{\Omega^a} \times g \right) \chi \\ \beta_2 u &= \delta_2 \left( \Omega^f \times \mathbf{1}_c \right) u = \delta_2 \left( \mathbf{1}_{\Omega^b} \times g \right) v = g \gamma_2 v = g \alpha_2 \chi = \beta_2 \left( \mathbf{1}_{\Omega^a} \times g \right) \chi \end{split}$$

that is  $u = (\mathbf{1}_{\Omega^a} \times g) \chi$ . Finally, we can conclude now that  $\Omega^{gf} = \Omega^g \Omega^f$ .

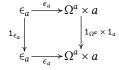
It only remains to prove  $\Omega^{1_a}$   $1_{\Omega^a}$  for  $a \in |\mathcal{E}|$ . By definition,  $\Omega^{1_a}$  is the morphism for which there is a pullback square

$$\begin{array}{c}
\epsilon_a \xrightarrow{\epsilon_a} \Omega^a \times a \\
\downarrow \qquad \qquad \downarrow \Omega^{1_a} \times 1_a \\
\epsilon_a \xrightarrow{\epsilon_a} \Omega^a \times a
\end{array}$$

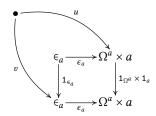
Consequently, if we show there is a pullback square

$$\begin{array}{c} \boldsymbol{\epsilon}_{a} \overset{\boldsymbol{\epsilon}_{a}}{\longrightarrow} \boldsymbol{\Omega}^{a} \times \boldsymbol{a} \\ \boldsymbol{\downarrow} & \boldsymbol{\downarrow}^{\mathbf{1}_{\Omega^{a}} \times \mathbf{1}_{a}} \\ \boldsymbol{\epsilon}_{a} \overset{\boldsymbol{\epsilon}_{a}}{\longrightarrow} \boldsymbol{\Omega}^{a} \times \boldsymbol{a} \end{array}$$

then we have concluded the verification of the functoriality. For that scope, consider the commutative square



Consider



with the outer square being commutative, that is  $u = \epsilon_a v$ . Hence, the morphism  $v : \bullet \to \epsilon_a$  works.

#### 2.2 One monadic structure

Let us employ the definition of power object again: for  $a \in |\mathcal{E}|$ , the monomorphism

$$1_a \times 1_a : a \times a \rightarrow a \times a$$

has the morphism  $\eta_a:a\to\Omega^a$  for which there is a unique pullback square

$$\begin{array}{c}
a \times a \xrightarrow{1_a \times 1_a} a \times a \\
\downarrow \\
\downarrow \\
\in_a \xrightarrow{e} \Omega^a \times a
\end{array}$$

We have to verify the morphisms  $\eta_a$  for  $a \in |\mathcal{E}|$  form a natural transformation  $1_{\mathcal{E}} \to \Omega^{\bullet}$ , we shall write  $\eta$ . We demonstrate

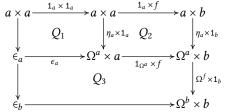
$$\begin{array}{ccc}
a & \xrightarrow{\eta_a} & \Omega^a \\
f & & \downarrow & \downarrow \\
b & \xrightarrow{\eta_b} & \Omega^b
\end{array}$$

commutes for every  $f: a \to b$  in  $\mathcal{E}$ . [Definitive strategy yet to be found.] If we prove there exist two pullback squares

then we have to conclude the naturality. The proof the first square is a pullback one is relatively simple. Consider

$$\begin{array}{c}
a \times a \xrightarrow{\mathbf{1}_{a} \times f} a \times b \\
f \times f \downarrow \qquad \qquad \downarrow f \times \mathbf{1}_{b} \\
b \times b \xrightarrow{\mathbf{1}_{b} \times \mathbf{1}_{b}} b \times b \\
\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \eta_{b} \times \mathbf{1}_{b} \\
\in b \xrightarrow{\epsilon_{b}} \Omega^{b} \times b
\end{array}$$

Here, the lower square is of pullback by how  $\eta_b$  is defined. The upper one is a pullback square too. [Proof here!] Thus so is the perimetric rectangle because of the pullback lemma. For the second square consider



Again, using twice of the pullback lemma yields the perimetric rectangle is a pullback square: in fact,  $Q_1$  and  $Q_3$  are pullback squares by definition of  $\eta_a$  and  $\Omega^f$ , whereas one can prove  $Q_2$  is a pullback square too. [Proof here!]

### 2.3 One algebra for the monad above

# 3 The actual article and a proof

#### [The symbolism here is weird...]

An *extensor* [in french, it is 'extenseur'] for a category  $\mathcal C$  amounts at having one object a of  $\mathcal C$  and for every span

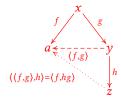


of C one morphism  $y \to a$  of C we denote  $\langle f, g \rangle$ ; all this complying with the following rules:

1. For any consecutive morphisms  $x \xrightarrow{g} y \xrightarrow{h} z$  we have

$$\langle\langle f,g\rangle,h\rangle=\langle f,hg\rangle.$$

[Here is a drawing of all this:

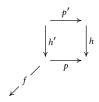


]

2. If



is a pullback square [in french, it is 'produit fibré', that is fibred product], then for every f as in



we have

$$\langle f, p \rangle h = \langle fh', p' \rangle.$$

3. If in

$$\stackrel{h}{\longleftarrow} \stackrel{h}{\stackrel{h'}{\longleftarrow}} \stackrel{q}{\longrightarrow}$$

we have ph'h = p and qhh' = q, then for if f is as in

$$\leftarrow f \leftarrow p \leftarrow h' \qquad q \rightarrow h'$$

then

$$\langle aph', q \rangle = \langle ap, qh \rangle$$
.

[This definition is a mess.] [Rewrite?]
[TEX the unique result of the paper. What is a 'sup-treillis'? A 'treillis' is a lattice. Maybe a lattice with products?]