Extensors

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Abstract

The work here is based on a laconic and tiny article by René Guitart about *extensors*[1]: `http://www.numdam.org/item/DIA_1980__3__A3_0.pdf`. The pages below can be considered a long exercise.

Contents

References

[BW05] M. Barr and C. Wells. *Toposes, Triples and Theories.* Reprints in Theory and Applications of Categories, 2005. URL: `http://www.tac.mta.ca/tac/reprints/articles/12/tr12.pdf`.

[Gol06] R. Goldblatt. *Topoi: The Categorial Analysis of Logic.* Dover Books on Mathematics. Dover Publications, 2006. ISBN: 9780486450261.

[Mik22] C. J. Mikkelsen. *Lattice Theoretic and Logical Aspects of Elementary Topoi.* Reprints in Theory and Applications of Categories, 2022. URL: `http://www.tac.mta.ca/tac/reprints/articles/29/tr29.pdf`.

1 Preamble

[This section is important for me. It may be deleted in the final version. Let me recall some words here, and then (try to) set a strategy to attack the paper.]

Definition 1.1. A *monad* for a category $\mathcal{C}$ consists of:

- one functor $T : \mathcal{C} \to \mathcal{C}$
- one natural transformation $\eta : 1_{\mathcal{C}} \to T$
- one natural transformation $\mu : T^2 \to T$, where $T^2 := T \circ T$

1 'Unofficial' translation. In French it is extenseurs.

such that the diagrams

$$\begin{tikzcd} T^3x \arrow[r, "T\mu_x"] \arrow[d, "\mu_{Tx}"'] & T^2x \arrow[d, "\mu_x"] \\ T^2x \arrow[r, "\mu_x"'] & Tx \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Tx \arrow[r, "\eta_{Tx}"] \arrow[d, "T\mu_x"'] \arrow[dr, "1_{Tx}"] & T^2x \arrow[d, "\mu_x"] \\ T^2x \arrow[r, "\mu_x"'] & Tx \end{tikzcd}$$

commute for every object x of $\mathcal{C}$; let us write such monad as the triple (T, η, μ).

In the sketched proof, the author talks about some 'monade des parties du topos'. **Set** is a topos and

Sandbox 1.2 (The 'power set' monad)**.** Consider the functor $2^\bullet : \mathbf{Set} \to \mathbf{Set}$ where, for X a set, $2^\bullet(X) := 2^X$ and, for $f : X \to Y$, define $2^f := 2^\bullet(f)$ to be the function $2^A \to 2^B$ that takes subsets E of A to fE.
Now, we need functions $\xi_X : X \to 2^X$, one for each set X, forming a natural transformation $\xi : 1_{\mathbf{Set}} \to 2^\bullet$; we consider that one introduced as $\xi_X(x) := \{x\}$.
To conclude the structure of monad, consider for X set, the function

$$\delta_X : 2^{2^X} \to 2^X \, , \; \delta_X(F) := \bigcup_{E \in F} E.$$

The family of such functions determine a natural transformation $\delta : 2^\bullet \circ 2^\bullet \to 2^\bullet$.

Definition 1.3. If you are given a monad (T, η, μ) for a category $\mathcal{C}$, you can introduce an *algebra* for this monad, consisting of:

- one object x in $\mathcal{C}$
- one morphism $f : Tx \to x$

all this making commute the following diagrams

$$\begin{tikzcd} T^2x \arrow[r, "Tf"] \arrow[d, "\mu_x"'] & Tx \arrow[d, "f"] \\ Tx \arrow[r, "f"'] & x \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} x \arrow[dr, "\eta_x"] \arrow[dd, "1_x"'] & \\ & Tx \arrow[dl, "f"] \\ x & \end{tikzcd}$$

[I've just written a couple of definitions, but for the next days here is what to do for me: find more examples motivating monads and algebras for monads; you must learn something about *Kleisli* and *Eilenberg-Moore categories*... Not only for the sake of these pages.]

[Now understand this: 'un treillis complet est une $2^\bullet$-algèbre'.]

Let us try now to bring this to any topoi, if possible.

2 One monad for topoi

2.1 (Covariant) Power object functor

One of the possible definitions of (elementary) topos is:

a category with finite limits and power objects.

For future reference and notation, here is the definition of power object:

Definition 2.1. For if $\mathcal{C}$ is a category with finite products and $a \in |\mathcal{C}|$, a *power object* in $\mathcal{C}$ consists of

- one object 'of the parts of a', we write Ω^a
- one monomorphism $\epsilon_a : \in_a \rightarrowtail \Omega^a \times a$, the 'membership relation'

with the property: for every $b \in |\mathcal{C}|$ and monomorphism $\rho : r \to b \times a$ there exists one and only one morphism $f : b \to \Omega^a$ for which there is a pullback square

$$\begin{tikzcd} r \arrow[r, "\rho"] \arrow[d] & b \times a \arrow[d, "f \times 1_a"] \\ \in_a \arrow[r, "\epsilon_a"'] & \Omega^a \times a \end{tikzcd}$$

in $\mathcal{C}$.

[A finer notation may be $\wp(a)$. Do I really need to refer so explicitly to subobject classifiers here?] Now if we want to replicate the above phenomenon inside any topos $\mathcal{E}$, we need a suitable functor $\Omega^\bullet : \mathcal{E} \to \mathcal{E}$. For $a \in |\mathcal{E}|$,

$$\Omega^\bullet(a) := \Omega^a.$$

It remains to understand what $\Omega^f := \Omega^\bullet(f)$, for $f : a \to b$ in $\mathcal{E}$, should be to make a functor. Consider the power objects of a and b, that is Ω^a and Ω^b together with their respective membership relations $\epsilon_a : \in_a \rightarrowtail \Omega^a \times a$ and $\epsilon_b : \in_b \rightarrowtail \Omega^b \times b$.

$$\begin{tikzcd} \in_a \arrow[r, "\epsilon_a"] & \Omega^a \times a \arrow[r, "1_{\Omega^a} \times f"] & \Omega^a \times b \arrow[d, dashed] \\ \in_b \arrow[rr, "\epsilon_b"'] & & \Omega^b \times b \end{tikzcd}$$

Here, the dotted morphism is of the form $\Omega^f \times 1_b$, where $\Omega^f : \Omega^a \to \Omega^b$ is that morphism for which there is a pullback square

$$\begin{tikzcd} \in_a \arrow[r, "\epsilon_a"] \arrow[d] & \Omega^a \times a \arrow[r, "1_{\Omega^a} \times f"] & \Omega^a \times b \arrow[d, dashed, "\Omega^f \times 1_b"] \\ \in_b \arrow[rr, "\epsilon_a"'] & & \Omega^b \times b \end{tikzcd}$$

Let us verify functoriality now. Take

$$\begin{tikzcd} a \arrow[r, "f"] & b \arrow[r, "g"] & c \end{tikzcd}$$

so that we have also

$$\begin{tikzcd} \Omega^a \arrow[r, "\Omega^f"] & \Omega^b \arrow[r, "\Omega^g"] & \Omega^c. \end{tikzcd}$$

Let us study $\Omega^{gf} : a \to c$. By definition, Ω^{gf} is the morphism for which there is a pullback square

$$\begin{tikzcd} \in_a \arrow[r, "\epsilon_a"] \arrow[d] & \Omega^a \times a \arrow[r, "1_{\Omega^a} \times (gf)"] & \Omega^a \times c \arrow[d, "\Omega^{gf}"] \\ \in_c \arrow[rr, "\epsilon_b"'] & & \Omega^c \times c \end{tikzcd} \tag{2.1}$$

We have then two pullback squares glued like this:

$$
\begin{tikzcd}
\in_a \arrow[r, "\epsilon_a"] \arrow[d] & \Omega^a \times a \arrow[r, "1_{\Omega^a} \times f"] & \Omega^a \times b \arrow[d, "\Omega^f \times 1_b"'] & \\
\in_b \arrow[rr, "\epsilon_b"'] \arrow[d] & & \Omega^b \times b \arrow[r, "1_{\Omega^b} \times g"'] & \Omega^b \times c \arrow[d, "\Omega^g \times 1_c"] \\
\in_c \arrow[rrr, "\epsilon_c"'] & & & \Omega^c \times c
\end{tikzcd}
$$

The first one gives the morphism Ω^f, whereas the second one Ω^g. Let us add two arrows to make a rectangle:

$$
\begin{tikzcd}
\in_a \arrow[r, "\epsilon_a"] \arrow[d] & \Omega^a \times a \arrow[r, "1_{\Omega^a} \times f"] & \Omega^a \times b \arrow[d, "\Omega^f \times 1_b"'] \arrow[r, "1_{\Omega^a} \times g"] & \Omega^a \times c \arrow[d, "\Omega^f \times 1_c"] \\
\in_b \arrow[rr, "\epsilon_b"'] \arrow[d] & & \Omega^b \times b \arrow[r, "1_{\Omega^b} \times g"'] & \Omega^b \times c \arrow[d, "\Omega^g \times 1_c"] \\
\in_c \arrow[rrr, "\epsilon_c"'] & & & \Omega^c \times c
\end{tikzcd}
$$

The plan is: if we can show

$$
\begin{tikzcd}
\Omega^a \times b \arrow[r, "1_{\Omega^a} \times g"] \arrow[d, "\Omega^f \times 1_b"'] & \Omega^a \times c \arrow[d, "\Omega^f \times 1_c"] \\
\Omega^b \times b \arrow[r, "1_{\Omega^b} \times g"'] & \Omega^b \times c
\end{tikzcd}
$$

is a pullback square, then applying the *Pullback Lemma* twice will allow us to conclude the rectangle (2.1) is a pullback square too.

Mere brute force is employed here. In

$$
\begin{tikzcd}
\bullet \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\
& \Omega^a \times b \arrow[r, "1_{\Omega^a} \times g"'] \arrow[d, "\Omega^f \times 1_b"] & \Omega^a \times c \arrow[d, "\Omega^f \times 1_c"] \\
& \Omega^b \times b \arrow[r, "1_{\Omega^b} \times g"'] & \Omega^b \times c
\end{tikzcd}
$$

assume $\left(\Omega^f \times 1_c\right) u = \left(1_{\Omega^b} \times g\right) v$. Now a lot of notation is needed, to write the products involved: we write them as

$$\Omega^a \xleftarrow{\alpha_1} \Omega^a \times b \xrightarrow{\alpha_2} b$$

$$\Omega^a \xleftarrow{\beta_1} \Omega^a \times c \xrightarrow{\beta_2} c$$

$$\Omega^b \xleftarrow{\gamma_1} \Omega^b \times b \xrightarrow{\gamma_2} b$$

$$\Omega^b \xleftarrow{\delta_1} \Omega^b \times c \xrightarrow{\delta_2} c$$

Here are the equations defining the products of morphisms involved

$$\begin{aligned}
&\beta_1\left(1_{\Omega^a} \times g\right) = \alpha_1\\
&\beta_2\left(1_{\Omega^a} \times g\right) = g\alpha_2\\
&\gamma_1\left(\Omega^f \times 1_b\right) = \Omega^f\alpha_1\\
&\gamma_2\left(\Omega^f \times 1_b\right) = \alpha_2\\
&\delta_1\left(\Omega^f \times 1_c\right) = \Omega^f\beta_1\\
&\delta_2\left(\Omega^f \times 1_c\right) = \beta_2\\
&\delta_1\left(1_{\Omega^b} \times g\right) = \gamma_1\\
&\delta_2\left(1_{\Omega^b} \times g\right) = g\gamma_2
\end{aligned}$$

[We do not use all of them!] By universal property of product, we have one $\chi : \bullet \to \Omega^a \times b$ for which $\beta_1 u = \alpha_1\chi$ and $\gamma_2 v = \alpha_2\chi$. We show now that χ is what we are looking for using the universal property of the the products of the square.

$$\begin{aligned}
&\gamma_1 v = \delta_1\left(1_{\Omega^b} \times g\right) v = \delta_1\left(\Omega^f \times 1_c\right) u = \Omega^f\beta_1 u = \Omega^f\alpha_1\chi = \gamma_1\left(\Omega^f \times 1_b\right)\chi\\
&\gamma_2 v = \alpha_1\chi = \gamma_2\left(\Omega^f \times 1_b\right)\chi
\end{aligned}$$

which implies $v = \left(\Omega^f \times 1_b\right)\chi$; moreover,

$$\begin{aligned}
&\beta_1 v = \alpha_1\chi = \beta_1\left(1_{\Omega^a} \times g\right)\chi\\
&\beta_2 u = \delta_2\left(\Omega^f \times 1_c\right) u = \delta_2\left(1_{\Omega^b} \times g\right) v = g\gamma_2 v = g\alpha_2\chi = \beta_2\left(1_{\Omega^a} \times g\right)\chi
\end{aligned}$$

that is $u = \left(1_{\Omega^a} \times g\right)\chi$. Finally, we can conclude now that $\Omega^{gf} = \Omega^g\Omega^f$.

It only remains to prove $\Omega^{1_a} 1_{\Omega^a}$ for $a \in |\mathcal{E}|$. By definition, Ω^{1_a} is the morphism for which there is a pullback square

$$\begin{tikzcd}
\in_a \arrow[r, "\epsilon_a"] \arrow[d] & \Omega^a \times a \arrow[d, "\Omega^{1_a} \times 1_a"] \\
\in_a \arrow[r, "\epsilon_a"'] & \Omega^a \times a
\end{tikzcd} .$$

Consequently, if we show there is a pullback square

$$\begin{tikzcd}
\in_a \arrow[r, "\epsilon_a"] \arrow[d] & \Omega^a \times a \arrow[d, "1_{\Omega^a} \times 1_a"] \\
\in_a \arrow[r, "\epsilon_a"'] & \Omega^a \times a
\end{tikzcd}$$

then we have concluded the verification of the functoriality. For that scope, consider the commutative square

$$\begin{tikzcd}
\in_a \arrow[r, "\epsilon_a"] \arrow[d, "1_{\epsilon_a}"'] & \Omega^a \times a \arrow[d, "1_{\Omega^a} \times 1_a"] \\
\in_a \arrow[r, "\epsilon_a"'] & \Omega^a \times a
\end{tikzcd}$$

Consider

$$\begin{tikzcd}
\bullet \arrow[drr, bend left, "u"] \arrow[ddr, bend right, "v"'] & & \\
& \in_a \arrow[r, "\epsilon_a"'] \arrow[d, "1_{\epsilon_a}"] & \Omega^a \times a \arrow[d, "1_{\Omega^a} \times 1_a"] \\
& \in_a \arrow[r, "\epsilon_a"'] & \Omega^a \times a
\end{tikzcd}$$

with the outer square being commutative, that is $u = \epsilon_a v$. Hence, the morphism $v : \bullet \to \in_a$ works.

2.2 One monadic structure

Let us employ the definition of power object again: for $a \in |\mathcal{E}|$, the monomorphism

$$1_a \times 1_a : a \times a \to a \times a$$

has the morphism $\eta_a : a \to \Omega^a$ for which there is a unique pullback square

$$\begin{tikzcd}
a \times a \arrow[r, "1_a \times 1_a"] \arrow[d] & a \times a \arrow[d, "\eta_a \times 1_a"] \\
\in_a \arrow[r, "\epsilon_a"'] & \Omega^a \times a
\end{tikzcd}$$

We have to verify the morphisms η_a for $a \in |\mathcal{E}|$ form a natural transformation $1_{\mathcal{E}} \to \Omega^\bullet$, we shall write η. We demonstrate

$$\begin{tikzcd}
a \arrow[r, "\eta_a"] \arrow[d, "f"'] & \Omega^a \arrow[d, "\Omega^f"] \\
b \arrow[r, "\eta_b"'] & \Omega^b
\end{tikzcd}$$

commutes for every $f : a \to b$ in $\mathcal{E}$. [Definitive strategy yet to be found.]

If we prove there exist two pullback squares

$$\begin{tikzcd}
a \times a \arrow[r, "1_a \times f"] \arrow[d] & a \times b \arrow[d, "(\eta_b f) \times 1_b"] \\
\in_b \arrow[r, "\epsilon_b"'] & \Omega^b \times b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \times a \arrow[r, "1_a \times f"] \arrow[d] & a \times b \arrow[d, "(\Omega^f \eta_a) \times 1_b"] \\
\in_b \arrow[r, "\epsilon_b"'] & \Omega^b \times b
\end{tikzcd}$$

then we have to conclude the naturality. The proof the first square is a pullback one is relatively simple. Consider

$$\begin{tikzcd}
a \times a \arrow[r, "1_a \times f"] \arrow[d, "f \times f"'] & a \times b \arrow[d, "f \times 1_b"] \\
b \times b \arrow[r, "1_b \times 1_b"] \arrow[d] & b \times b \arrow[d, "\eta_b \times 1_b"] \\
\in_b \arrow[r, "\epsilon_b"'] & \Omega^b \times b
\end{tikzcd}$$

Here, the lower square is of pullback by how η_b is defined. The upper one is a pullback square too. [Proof here!] Thus so is the perimetric rectangle because of the pullback lemma. For the second square consider

$$\begin{tikzcd}
a \times a \arrow[r, "1_a \times 1_a"] \arrow[d] \arrow[dr, phantom, "Q_1"] & a \times a \arrow[r, "1_a \times f"] \arrow[d, "\eta_a \times 1_a"] \arrow[dr, phantom, "Q_2"] & a \times b \arrow[d, "\eta_a \times 1_b"] \\
\in_a \arrow[r, "\epsilon_a"'] \arrow[d] & \Omega^a \times a \arrow[r, "1_{\Omega^a} \times f"'] \arrow[d, phantom, "Q_3"] & \Omega^a \times b \arrow[d, "\Omega^f \times 1_b"] \\
\in_b \arrow[rr] & & \Omega^b \times b
\end{tikzcd}$$

Again, using twice of the pullback lemma yields the perimetric rectangle is a pullback square: in fact, Q_1 and Q_3 are pullback squares by definition of η_a and Ω^f, whereas one can prove Q_2 is a pullback square too. [Proof here!]

2.3 One algebra for the monad above

3 The actual article and a proof

[The symbolism here is weird...]

An *extensor* [in french, it is 'extenseur'] for a category $\mathcal{C}$ amounts at having one object a of $\mathcal{C}$ and for every *span*

$$\begin{tikzcd} & x \arrow[dl, "f"'] \arrow[dr, "g"] & \\ a & & y \end{tikzcd}$$

of $\mathcal{C}$ one morphism $y \to a$ of $\mathcal{C}$ we denote $\langle f, g\rangle$; all this complying with the following rules:

1. For any consecutive morphisms $x \xrightarrow{g} y \xrightarrow{h} z$ we have

$$\langle\langle f, g\rangle, h\rangle = \langle f, hg\rangle\,.$$

[Here is a drawing of all this:

$$\begin{tikzcd} & x \arrow[dl, "f"'] \arrow[dr, "g"] & \\ a & & y \arrow[ll, dashed, "\langle f,g\rangle"] \arrow[d, "h"] \\ & & z \arrow[ull, dotted, "\langle\langle f,g\rangle,h\rangle=\langle f,hg\rangle"] \end{tikzcd}$$

]

2. If

$$\begin{tikzcd} {} \arrow[r, "p'"] \arrow[d, "h'"'] & {} \arrow[d, "h"] \\ {} \arrow[r, "p"'] & {} \end{tikzcd}$$

is a pullback square [in french, it is 'produit fibré', that is fibred product], then for every f as in

$$\begin{tikzcd} & {} \arrow[r, "p'"] \arrow[d, "h'"] & {} \arrow[d, "h"] \\ & {} \arrow[r, "p"'] \arrow[dl, "f"] & {} \\ {} & & \end{tikzcd}$$

we have

$$\langle f, p\rangle h = \langle fh', p'\rangle\,.$$

3. If in

$$\begin{tikzcd} {} & {} \arrow[l, "p"] \arrow[r, bend left, "h"] & {} \arrow[l, bend left, "h'"] \arrow[r, "q"] & {} \end{tikzcd}$$

we have $ph'h = p$ and $qhh' = q$, then for if f is as in

$$\begin{tikzcd} {} & {} \arrow[l, "f"] & {} \arrow[l, "p"] \arrow[r, bend left, "h"] & {} \arrow[l, bend left, "h'"] \arrow[r, "q"] & {} \end{tikzcd}$$

then

$$\langle aph', q\rangle = \langle ap, qh\rangle\,.$$

[This definition is a mess.] [Rewrite?]

[TeX the unique result of the paper. What is a ‘sup-treillis’? A ‘treillis’ is a lattice. Maybe a lattice with products?]