

Exs

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Problem 41

Prove the root test.

Solution

Firstly, I must clarify that the if the absolute value of a series is convergent then the series is, and this has been proven.

$$\begin{aligned}\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} &= L \\ \therefore \sqrt[n]{|a_n|} &< r \quad \text{when } n \geq N \\ \therefore a_N &< r^N \\ \therefore |a_{N+1}| &< r^N \cdot r \\ \therefore r < 1, \quad r^N \cdot r &< r^N \\ \therefore |a_N| r^k &> |a_N| r^{k+1} \\ \therefore n \text{ need not be greater than } N &\text{ for } r \text{ to be less than } 1 \\ \therefore \sum_{n=1}^{\infty} |a_N| r^n &\text{ is convergent.} \\ \therefore \sum_{n=1}^{\infty} |a_N| a_n = |a_N| \sum_{n=1}^{\infty} |a_n| &\text{ is also convergent} \\ \therefore \frac{|a_N|}{|a_N|} \sum_{n=1}^{\infty} |a_n| &\text{ is also convergent}\end{aligned}$$

Because the series a_n is absolutely convergent, it is convergent.

Problem 42

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n!)(1103 + 26390n)}{(n!)^4 396^{4n}}$$

Solution

Using the ratio test of convergence, we have:

$$\begin{aligned} & \frac{(n!)^4 396^{4n}}{(4n!)(1103 + 26390n)} \frac{(4n+4)!(1103 + 26390(n+1))}{396^4(n+1)(n!)^4 396^{4n}} \\ &= \frac{1}{(1103 + 26390n)} \frac{4(4n+1)(4n+2)(4n+3)(1103 + 26390n + 26390)}{396^{4n}} \end{aligned}$$

At this point if you don't realise that the fraction approaches 0 you have nothing useful in your cranium.

$$\pi = \frac{1}{\left(\frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n!)(1103+26390n)}{(n!)^4 396^{4n}}\right)}$$

Problem 43a

Waise yeh common sense wala sawal hai lekin mein apne jasbaaton ko qabu mein rakh kar iska jawaab dene ki imtehaan karunga is liye meri zabaan se kuch idhar udhar se nikla to bura mat maanna main jhoot nahi bolta. Abhi mudde ki baat shuru karte hain:

When the positive and negative series of an arbitrary a_n are persived as:

$$a_n^+ = \frac{a_n + |a_n|}{2} \text{ and } a_n^- = \frac{a_n - |a_n|}{2}$$

We can see that if a series is absolutely convergent both a_n and $|a_n|$ are convergent, thus the positive and negative series are convergent. If otherwise, than the sum of a_n and $|a_n|$ is not finite and thus the positive and negative series diverge.

Problem 45A

We have for a fact that the terms of a convergent series approach 0, and that the sequence n^2 constantly increases with the instant rate of change $2n$, obtained from differentiation. Therefore by the root test,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 b_{n+1}}{n^2 b_n}$$

Letting this fraction approach infinity we have, b_n and b_{n+1} will approximately be equal (engineering op), whereas the difference between n^2 and $(n+1)^2$ will keep on increasing, therefore the limit is greater than one, or perhaps the limit is even indeterminate. Therefore the sequence is indeterminate.

Bhai, isi liye dailogue bana hai: Parhai likhai mein dhyaan do, IAS YAS bano aur jake desh ko sambhalo.

Problem for fun lol

Test the following series for convergence.

$$\sum_{i=1}^{\infty} ne^{-n}$$

Solution

We shall use the integral test over here, thus we have to find the sum of the definite integral

$$\int_1^{\infty} xe^{-x} dx$$

Solving it:

$$\begin{aligned} \int xe^{-x} dx &= -xe^{-x} - \int -e^{-x} dx \\ &= -xe^{-x} + e^{-x} + C \\ &= -e^{-x}(x-1) + C \end{aligned}$$
$$\begin{aligned} \therefore \int_1^{\infty} xe^{-x} dx &= \lim_{t \rightarrow \infty} [-e^{-x}(x-1)]_1^t \\ &= \lim_{t \rightarrow \infty} [-te^{-t} + e^{-t} + 1] \\ &= 0 + 1 = 1 \end{aligned}$$

Therefore it can be concluded that the integral is convergent, and so the series is convergent.

Problem 2

$$\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$$

Solution

Using the ratio test:

$$\begin{aligned} \frac{x^{n+1}}{2n+1} \cdot \frac{2n-1}{x^n} \\ = x \frac{2n+1}{2n-1} \end{aligned}$$

Now assigning the limit:

$$\begin{aligned} \lim_{n \rightarrow \infty} x \frac{2n+1}{2n-1} &= x \lim_{n \rightarrow \infty} \frac{\frac{d}{dx} 2n+1}{\frac{d}{dx} 2n-1} \\ \therefore \lim_{n \rightarrow \infty} x \frac{2n+1}{2n-1} &= x \end{aligned}$$

\therefore The series is convergent whenever $|x| < 1$, or in other words when $-1 < x < 1$.
Therefore the interval of convergence is $(-1, 1)$ and the radius is 1.

Problem 9

For the following series determine the interval and radius of convergence:

$$(-1)^n \frac{n^2 x^n}{2^n}$$

Using the ratio test we have:

$$\frac{(-1)^{n+1}(n+1)^2 x^{n+1}}{2^{n+1}} \frac{(-1)^n 2^n}{n^2 x^n} = \frac{x(n+1)^2}{2n^2} = x \frac{n^2 + 2n + 1}{2n^2}$$

$$\begin{aligned} \text{Taking the limit: } \lim_{n \rightarrow \infty} -x \frac{n^2 + 2n + 1}{2n^2} &= x \lim_{n \rightarrow \infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{2} \\ &= -\frac{x}{2} \end{aligned}$$

$$|-\frac{x}{2}| < 1 \text{ if } |x| < 2 \text{ thus } -2 < x < 2$$

Therefore the interval of convergence is $(-2, 2)$ and the radius of convergence is 2.

11.9

Problem 1

If the radius of convergence of the series $\sum_{n=1}^{\infty} x^n$ is 10, then what is the radius of convergence of the series $\sum_{n=1}^{\infty} x^n$?

Solution

The radius of convergence will remain 10, as the radius of convergence remains the same between the derivative and the integral of a function.

Problem 4

Determine the infinite series and interval of convergence for the following function:

$$\frac{5}{1-4x^2}$$

Solution

$$\frac{5}{1-4x^2}$$

This fraction is already of the form

$$\frac{a}{1-r}$$

and therefore we can already craft the infinite series

$$\sum_{n=1}^{\infty} 5x^{n-1}$$

Now we use the root test to determine the interval of convergence:

$$\lim_{n \rightarrow \infty} \left| \frac{5x^n}{5x^{n-1}} \right| = \lim_{n \rightarrow \infty} |x|$$

\therefore For the series to converge $-1 < x < 1$.

Problem 10

$$f(x) = \frac{x^2}{a^3 - x^3}$$

Solution

$$f(x) = \frac{x^2}{a^3 - x^3} = x^2 \frac{1}{1 - (-a^3 - x^3 + 1)}$$

Problem 13A

Determine the radius of convergence:

$$\frac{1}{(1+x)^2}$$

Solution

Through differentiation:

$$\begin{aligned}\frac{1}{(1+x)^2} &= (1+x)^{-2} = f(x) \\ \therefore f'(x) &= \frac{-2}{1+x} = \sum_{n=0}^{\infty} (-2)x^n\end{aligned}$$

Using the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{-2x^{n+1}}{-2x^n} \right| = |x|$$

\therefore The series is convergent when $|x| < 1$

$$\therefore -1 < x < 1$$