7.5

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1 Chapter Examples

1. To evaluate:

$$\int \frac{\tan^3 x}{\cos^3 x}$$

Solution:

$$\int \frac{\tan^3 x}{\cos^3 x} dx = \int \frac{\sin^3 x}{\cos^6 x} dx$$
$$\int \frac{1-\cos^2 x}{\cos^6 x} \sin x \quad dx$$
$$\textbf{Substituting u=} \cos x$$
$$\int \frac{u^2-1}{u^6} du = \int (u^{-4} - u^{-6}) du = \frac{u^{-3}}{4} + \frac{u^{-5}}{6}$$
$$= \frac{1}{4u^3} + \frac{1}{6u^5} = \frac{\sec^3}{4} + \frac{sec^5}{6}$$

2 Exercises

1.

$$\int \cos x(1+\sin^2 x) dx$$

$$= \int \cos x(2 - \cos^2 x)dx$$
$$= \int 2\cos x(2 - \cos^2 x)dx$$
$$= 2\int \cos x - \int \cos^3 x dx$$
$$= 2\sin x - \int \cos x(1 - \sin^2 x)dx$$
$$= 2\cos x - \int 1 - u^2 du$$
$$= 2\cos x - \int 1du + \int u^2 du$$
$$= 2\cos x - u + \frac{u^3}{3}$$
$$= 2\cos x - \sin x + \frac{1}{3}\sin^3 x$$

2. Evaluate:

$$\int (2x^2 - 1)e^{x^2}\ dx+$$

$$\int (2x^2 - 1)e^{x^2}\ dx$$
$$2x^2 e^{x^2}$$

Evaluate:

$$\int \frac{\ln x}{x\sqrt{1 + (\ln x)^2}}\ dx$$

$$\int \frac{\ln x}{x\sqrt{1 + (\ln x)^2}}\ dx$$
$$= \int \ln x \frac{1}{x\sqrt{1 + (\ln x)^2}}\ dx$$
$$= \frac{x\ln x - x}{x\sqrt{1 + (\ln x)^2}} - \int \frac{\ln x}{x^3\sqrt{1 + (\ln x)^2}(1 + (\ln x)^2)}\ dx$$

Let us now solve for the integral.

$$\int \frac{\ln x}{x^3\sqrt{1 + (\ln x)^2}(1 + (\ln x)^2)}\ dx$$
$$= \int \frac{\ln x}{x^3\sqrt{1 + (\ln x)^2}(1 + (\ln x)^2)}\ dx$$

Substituting $u = \frac{1}{x}$, we have,

$$\int \frac{u^3}{\sqrt{1 + (\ln \frac{1}{u})^2}(1 + (\ln \frac{1}{u})^2)}$$

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Evaluate:

$$\int \frac{x}{(2x+1)^3}\ dx$$

$$\frac{x}{(2x+1)^3}$$

$$= \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{(2x+1)^3}$$

$$\therefore x = A(4x^2 + 4x + 1) + B(2x + 1) + C$$

$$\therefore x = (4A)x^2 + (4A + 2B)x + A + B + C$$

$$\therefore 4A = 0$$

$$4A + 2B = 1$$

$$A + B + C = 0$$

$$\therefore A = 0B = \frac{1}{2};\ C = \frac{-1}{2}$$

$$\therefore \frac{x}{(2x+1)^3} = \frac{1}{2(2x+1)^2} - \frac{1}{2(2x+1)^3}$$

$$\therefore \int \frac{x}{(2x+1)^3}\ dx = \frac{1}{2}\int \frac{1}{(2x-1)^2}\ dx - \frac{1}{2}\int \frac{1}{(2x-1)^3}\ dx$$

$$\frac{1}{4}\int u^{-2}\ du - \frac{1}{4}\int u^{-3}\ du$$

$$= \frac{3}{4u^2} - \frac{2}{4u}$$

$$= \frac{1-4x}{16x^2 + 16x + 4}$$

To solve:

$$\int \frac{x-1}{x^2 - 4x + 5}\ dx$$

Solution:

$$\frac{x-1}{x^2 - 4x + 5}$$

$$= \frac{2x + 4(x+1)}{(x^2 - 4x + 5)^2} + \int \frac{2x+4}{(x^2 - 4x + 5)^2}\ dx$$

Taking $u = x^2 - 4x + 5$, $du = 2x + 4\ dx$. Therefore:

$$\int \frac{x-1}{x^2-4x+5}\ dx = \frac{2x+4(x+1)}{(x^2-4x+5)^2} + \int \frac{du}{u^2}$$
$$\frac{2x+4(x+1)}{(x^2-4x+5)^2} - \frac{1}{u}$$
$$= \frac{2x+4(x+1)-(x^2-4x+5)}{(x^2-4x+5)^2}$$
$$= \frac{x^2+10x-1}{(x^2-4x+5)^2}$$

To solve:

$$\int \frac{x}{x^4+x^2+1}\ dx$$

Solution:

$$\int \frac{x}{x^4+x^2+1}\ dx$$
$$= -\frac{x(4x^2+2x)}{(x^4+x^2+1)^2} - \int \frac{4x^2+2x}{(x^4+x^2+1)^2}\ dx$$

Taking $u = x^4 + x^2 + 1$ we have $du = (4x^2 + 2x)dx$. Therefore:

$$\int \frac{x}{x^4+x^2+1}\ dx$$
$$= -\frac{x(4x^2+2x)}{(x^4+x^2+1)^2} - \int \frac{du}{u^2}$$
$$= \frac{1}{u} - \frac{x(4x^2+2x)}{(x^4+x^2+1)^2}$$
$$= \frac{1}{x^4+x^2+1} - \frac{x(4x^2+2x)}{(x^4+x^2+1)^2} = \frac{(x^4+x^2+1)-(4x^3+2x^2)}{(x^4+x^2+1)^2}$$
$$= \frac{x^4-4x^3-x^2+1}{(x^4+x^2+1)^2}$$

To solve:

$$\int \sin^5 t \cos^4 t \; dt$$

$$\begin{aligned}\int \sin^5 t \cos^4 t \; dt &= \int (\sin^2)^2 \cos^2 t \sin t \; dt \\ &= \int (1-\cos^2 t) \cos t \sin t \; dt\end{aligned}$$

Substituting $u = cost$ we get $du = (sint)dt$

$$\begin{aligned}\int \sin^5 t \cos^4 t \; dt &= \int u(1-u^2)du \\ &= \int u \; du - \int u^3 \; du \\ &= \frac{u^2}{2} - \frac{u^4}{4} \\ &= \frac{\cos^2 t}{2} - \frac{\cos^4 t}{4}\end{aligned}$$

Evaluate:

$$\int \frac{x^3}{\sqrt{1+x^2}} \; dx$$

For this, we shall substitute $u = 1 + x^2$. Threfore $du = (2x)dx$ Now,

$$\begin{aligned}\int \frac{x^3}{\sqrt{1+x^2}} \; dx &= \int \frac{x^2}{\sqrt{1+x^2}} x \; dx \\ &= \frac{1}{2} \int \frac{u-1}{\sqrt{u}} \; du \\ &= \frac{1}{2} \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} \; du \\ &= \frac{1}{2} \int u^{\frac{1}{2}} \; du - \frac{1}{2} \int u^{-\frac{1}{2}} \; du \\ &= \frac{1}{2}\frac{2}{3} u^{\frac{3}{2}} - \frac{1}{2}(2)u^{\frac{1}{2}} + C \\ &= \frac{1}{3}u^{\frac{3}{2}} - u^{\frac{1}{2}} + C \\ &= \frac{\sqrt{x^2+1}^3}{3} - \frac{3\sqrt{x^2+1}}{3} + C \\ &= \frac{\sqrt{x^2+1}^3 - 3\sqrt{x^2+1}}{3} + C \\ &= \frac{(x^2-2)\sqrt{x^2+1}}{3} + C\end{aligned}$$

To solve:

$$\int \frac{dx}{(1-x^2)^{3/2}}\ dx$$

$$\int \frac{dx}{(1-x^2)^{3/2}} = \int (\frac{1}{\sqrt{1-x^2}})^3\ dx$$

Integrate with respect to t:

$$\frac{e^{\sqrt{t}}}{\sqrt{t}}$$

Solution: Substituting $\sqrt{t} = u$, $2du = dx(1/\sqrt{t})$

$$\therefore \int \frac{e^{\sqrt{t}}}{\sqrt{t}}\ dt$$
$$= 2\int e^u\ du$$
$$= 2e^u + C$$
$$= 2e^{\sqrt{t}} + C$$

Integrate with respect to x:

$$e^{x+e^x}$$

Solution The integral can be rewritten this way:

$$\int e^x \cdot e^{e^x}\ dx$$

Now we substitute $e^x = u$. Therefore $du = (e^x)dx$ Therefore:

$$\int e^{x+e^x}\ d = \int e^u\ du$$
$$= e^u + C$$
$$= e^{e^x} + C$$

Integrate with respect to x:

$$e^2$$

Substituting 2=x

$$e^2 = e^x$$
$$\therefore \int e^2\ dx = e^2 + C$$

You thought I was that dumb lol look at your face
Solution:

$$(e^2)x + C$$

Evaluate the integral with respect to x:

$$\arctan\sqrt{x}$$

Substitute $f(x) = \arctan\sqrt{x}$ and $g(x) = 1$ Therefore

$$\int \arctan\sqrt{x}\ dx$$

$$= x\arctan\sqrt{x} - \int \frac{x}{2\sqrt{x}(x+1)}\ dx$$

$$= x\arctan\sqrt{x} - \frac{1}{2}\int \frac{\sqrt{x}}{x+1}\ dx$$

Solving the integral term: Substituting $u = \sqrt{x}$, we have $du = (2\sqrt{x})dx$ Therefore:

$$\int \frac{\sqrt{x}}{x+1}\ dx = 2\int \frac{u^2}{u^2+1}du$$

$$= 2\int \frac{u^2+1-1}{u^2+1} = 2\int 1 - \frac{1}{u^2+1}du$$

$$= 2u - 2\arctan u$$

$$= 2\sqrt{x} - 2\arctan\sqrt{x}$$

$$= x\arctan\sqrt{x} + \arctan\sqrt{x} - \sqrt{x}$$

Evaluate the integral with respect to x:

$$\frac{\ln x}{x\sqrt{1+(\ln x)^2}}$$

We will substitute $u = 1 + (\ln x)^2$ Therefore $du/2 = (\ln x)/x)dx$. Therefore:

$$\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\ dx = \frac{1}{2}\int u^{-\frac{1}{2}}\ du = \frac{1}{2}2u^{\frac{1}{2}} = \sqrt{u}$$

$$= \sqrt{1+(\ln x)^2}$$

Evaluate the integral with respect to x:

$$\int_0^1 (1+\sqrt{x})^8\ dx$$

For this we substitute $u = \sqrt{x}$. Therefore we have $2du(\sqrt{x}) = dx$ Therefore:

$$\int_0^1 (1+\sqrt{x})^8\ dx = 2\int_1^2 u(1+u)^8du$$

Solving the antiderivative:

$$\int u(1+u)^8du = \frac{1}{2}u^2(1+u)^8 - \int u^2(1+u)^7du$$

Solving the integral:

$$\int u^2(1+u)^7du = \frac{1}{3}u^3(1+u)^7 - \frac{7}{3}\int u^3(1+u)^6du$$

Solving the integral:

$$\int u^3(1+u)^6du = \frac{1}{4}u^4(1+u)^6 - (1+\frac{1}{2})\int u^4(1+u)^5du$$

Solving the integral:

$$\int u^4(1+u)^5du = \frac{1}{5}u^5(1+u)^5 - \int u^5(1+u)^4$$

Solving the integral:

$$\int u^5(1+u)^4 = \frac{1}{6}u^6(1+u)^4\frac{2}{3} - \int u^6(1+u)^4du$$

Solving the integral:

$$\int u^6(1+u)^4du = \frac{1}{7}u^7(1+u)^4 - \frac{4}{7} - \int u^7(1+u)^3$$

Solving the integral:

$$\begin{aligned}\int u^7(1+u)^3 &= \int u^7(1+u^3+3u^2+3u)du\\ &= \int u^7du + \int u^{10}du + \int u^9du + \int u^8du\\ &= \frac{u^8}{8} + \frac{u^{11}}{11} + \frac{u^{10}}{10} + \frac{u^9}{9}\end{aligned}$$

Now we proceed to substitute the values and simplify the problem:

$$\int u^6(1+u)^4du = \frac{1}{7}u^7(1+u)^4 - \frac{4}{7} - (\frac{u^8}{8} + \frac{u^{11}}{11} + \frac{u^{10}}{10} + \frac{u^9}{9})$$

$$\int u^5(1+u)^4 = \frac{1}{6}u^6(1+u)^4\frac{2}{3} - (\frac{1}{7}u^7(1+u)^4 - \frac{4}{7} - (\frac{u^8}{8} + \frac{u^{11}}{11} + \frac{u^{10}}{10} + \frac{u^9}{9}))$$

$$\int u^4(1+u)^5du = \frac{1}{6}u^6(1+u)^4\frac{2}{3} - (\frac{1}{7}u^7(1+u)^4 - \frac{4}{7} - (\frac{u^8}{8} + \frac{u^{11}}{11} + \frac{u^{10}}{10} + \frac{u^9}{9})))$$

$$\int u^3(1+u)^6du = \frac{1}{4}u^4(1+u)^6 - (\frac{1}{6}u^6(1+u)^4\frac{2}{3} - (\frac{1}{7}u^7(1+u)^4 - \frac{4}{7} - (\frac{u^8}{8} + \frac{u^{11}}{11} + \frac{u^{10}}{10} + \frac{u^9}{9}))))$$

$$\int u^2(1+u)^7du$$

$$= \frac{1}{3}u^3(1+u)^7 - \frac{7}{3}(\frac{1}{4}u^4(1+u)^6 - (\frac{1}{6}u^6(1+u)^4\frac{2}{3} - (\frac{1}{7}u^7(1+u)^4 - \frac{4}{7} - (\frac{u^8}{8} + \frac{u^{11}}{11} + \frac{u^{10}}{10} + \frac{u^9}{9})))))$$

$$\int u(1+u)^8 du =$$

$$\frac{1}{2}u^2(1+u)^8-(\frac{1}{3}u^3(1+u)^7-\frac{7}{3}(\frac{1}{4}u^4(1+u)^6-(\frac{1}{6}u^6(1+u)^4\frac{2}{3}-(\frac{1}{7}u^7(1+u)^4-\frac{4}{7}-(\frac{u^8}{8}+\frac{u^{11}}{11}+\frac{u^{10}}{10}+\frac{u^9}{9}))))))$$

And now we have solved for the antiderivative. Since we need the complete integral, I am not going to bother simplifying the result stfu.

$$\frac{1}{2}u^2(1+u)^8-(\frac{1}{3}u^3(1+u)^7-\frac{7}{3}(\frac{1}{4}u^4(1+u)^6-(\frac{1}{6}u^6(1+u)^4\frac{2}{3}-(\frac{1}{7}u^7(1+u)^4-\frac{4}{7}-(\frac{u^8}{8}+\frac{u^{11}}{11}+\frac{u^{10}}{10}+\frac{u^9}{9}))))))$$

With limits of integration as 2 and 1 is equal to:

$$(\frac{1}{2}2^2(1+2)^8-(\frac{1}{3}2^3(1+2)^7-\frac{7}{3}(\frac{1}{4}2^4(1+2)^6-(\frac{1}{6}2^6(1+2)^4\frac{2}{3}-(\frac{1}{7}2^7(1+2)^4-\frac{4}{7}-(\frac{2^8}{8}+\frac{2^{11}}{11}+\frac{2^{10}}{10}+\frac{2^9}{9})))))))-$$

$$(\frac{1}{2}1^2(1+1)^8-(\frac{1}{3}1^3(1+1)^7-\frac{7}{3}(\frac{1}{4}1^4(1+1)^6-(\frac{1}{6}1^6(1+1)^4\frac{2}{3}-(\frac{1}{7}1^7(1+1)^4-\frac{4}{7}-(\frac{1^8}{8}+\frac{1^{11}}{11}+\frac{1^{10}}{10}+\frac{1^9}{9})))))))$$