Combined Mathematics I – 2014 Advanced Level Examination

Part A

1. Using the Principle of Mathematical Induction, prove that $\sum_{r=1}^{n} r(3r-1) = n^2(n+1)$ for all $n \in \mathbb{Z}^+$.

2. Using a graphical method or otherwise, find all values of x satisfying the inequality $|x+1| > 3x+7$.

3. Sketch the loci of the points represented by the complex numbers z satisfying

   (i) $\operatorname{Arg}(z+1) = \frac{\pi}{3}$,

   (ii) $\operatorname{Arg}(z-1) = \frac{5\pi}{6}$

   on the same Argand diagram and find the complex number represented by their point of intersection.

4. Let $n \in \mathbb{Z}^+$ The coefficient of x^{n-2} in the expansion of $\left(2+\frac{3}{x}\right)(1+x)^n$ is 120. Find the value of n.

5. Show that $\lim_{x\to 0} \frac{tan^2 2x}{x(1-\sqrt{1+x})} = -8$.

6. Find the area of the region enclosed by the straight line $y = 2x$ and the curve $y = x^2$.

7. Let C be the curve given by $x = e^t + e^{-t}$, $y = e^t - e^{-t}$, where t is a real parameter. Find $\frac{dy}{dx}$ in terms of t and show that the equation of the tangent line at the point on C corresponding to $t = \ln 2$ is $5x - 3y - 8 = 0$.

8. Let $\lambda \in \mathbb{R}$ and $\lambda \neq \pm 1$. The area of the region enclosed by the coordinate axes and the straight line $(1+\lambda)x - 2(1-\lambda)y - 2(1-\lambda) = 0$ is 4 square units. Find the values of λ.

9. Find the equation of the circle which touches the y-axis at the point $(0,3)$ and intersects the circle $x^2 + y^2 - 8x + 4y - 5 = 0$ orthogonally.

10. Let $\tan\alpha = -1$ and $\sin\beta = \frac{1}{\sqrt{5}}$, where $\frac{3\pi}{2} < \alpha < 2\pi$ and $\frac{\pi}{2} < \beta < \pi$. Find the value of $\cos(\alpha+\beta)$.

Part B

- Answer five questions only.

11. (a) Let $a \in \mathbb{R}$ and let $f(x) = 3x^3 + 5x^2 + ax - 1$. It is given that $(3x - 1)$ is a factor of $f(x)$. Find the value of a. Express $f(x)$ in the form $(3x-1)(x+k)^2$ where k is a constant.

By writing $3x - 1$ in the above expression in the form $b(x+1)+c$, where b and c are constants, find the remainder when $f(x)$ is divided by $(x+1)^3$.

(b) Let $a, b, c \in \mathbb{R}$ and $ac \neq 0$. Show that zero is not a root of the equation $ax^2 + bx + c = 0$. Let α and β be the roots of this equation, and let $\lambda = \frac{\alpha}{\beta}$ Show that $ac(\lambda+1)^2 = b^2\lambda$.

Let $p, q, r \in \mathbb{R}$ and $pr \neq 0$. Also, let γ and δ be the roots of the equation $px^2 + qx + r = 0$, and let $\mu = \frac{\gamma}{\delta}$. Show that $\lambda = \mu$ or $\lambda = \frac{1}{\mu}$ holds if and only if $acq^2 = prb^2$.

It is given that the roots of the equations $kx^2 - 3x + 2 = 0$ and $8x^2 + 6kx + 1 = 0$ are in the same ratio, where $k \in \mathbb{R}$. Find the value of k.

12. (a) Six schools participate in a Youth Sports Conference and each school is represented by three players comprising a cricketer, a soccer player and a hockey player. It is required to select a committee of six members from among these players. Find the number of different ways in which this committee can be formed.

(i) if two players from each sport must be included.

(ii) if two players from each sport must be included, is such a way that all six schools are represented,

(iii)if two players each from two schools and one player each from two of the remaining schools must be included.

(b) Let $U_r = \frac{r^2-r-5}{r(r+1)(r+4)(r+5)}$ for $r \in \mathbb{Z}^+$.

By comparing coefficients of r^n for $n = 0,1,2,3$ show that there exist constants A and B such that

$r^2 - r - 5 = A(r^2-1)(r+5) - Br^2(r+4)$ for $r \in \mathbb{Z}^+$.

Find $f(r)$ such that $U_r = f(r) - f(r+1)$ for $r \in \mathbb{Z}^+$.

Show that $\sum_{r=1}^{n} U_r = \frac{n}{(n+1)(n+5)}$ for $n \in \mathbb{Z}^+$.

Show further that the infinite series $\sum_{r=1}^{\infty} U_r$ is convergent and find its sum.

Hence, find $\sum_{r=3}^{\infty} 3U_r$

13. (a) Let $a, b \in \mathbb{R}$ and let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & a \\ 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b & 1 \\ 1 & 1 \end{pmatrix}$.

Find the values of a and b such that $\mathbf{A}^T\mathbf{A} = \mathbf{B}$, where $\mathbf{A}^T$ denotes the transpose of the matrix **A.**

Let $\mathbf{C} = \begin{pmatrix} 7 & 5 \\ 5 & 3 \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} u \\ u+1 \end{pmatrix}$, where $u \in \mathbb{R}$.

Also, let $\mathbf{CX} = \lambda\mathbf{BX}$, where $\lambda \in \mathbb{R}$. Find the value of λ and the value of u.

For this value of λ, find the matrix $\mathbf{C} - \lambda\mathbf{B}$ and show that its inverse **does not exist**.

(b) Let $z \in$ ₵. Show that

(i) $|1-z|^2 = 1 - 2\mathrm{Re}z + |z|^2$ and

(ii) $\mathrm{Re}\left(\frac{1}{1-z}\right) = \frac{1-\mathrm{Re}\, z}{|1-z|^2}$ for $z \neq 1$.

Deduce that $\mathrm{Re}\left(\frac{1}{1-z}\right) = \frac{1}{2}$ if and only if $|z| = 1$ and $z \neq 1$.

Let S be the set consisting of complex numbers z satisfying both conditions $\mathrm{Re}\left(\frac{1}{1-z}\right) = \frac{1}{2}$ and $-\frac{\pi}{3} < Arg\, z < \frac{\pi}{3}$.

Plot the points that present the complex numbers in S in an Argand diagram.

Show that if z is in S and $\mathrm{Re}\, z + \mathrm{Im}\, z = \frac{\pi}{\sqrt{2}}$, then $z = \cos\left(\frac{\pi}{12}\right) - i\sin\left(\frac{\pi}{12}\right)$.

14. (a) Let $f(x) = \frac{8x}{(x+1)(x^2+3)}$ for $x \neq -1$.

Show that $f'(x) = \frac{8(1-x)(2x^2+3x+3)}{(x+1)^2(x^2+3)^2}$ for $x \neq -1$.

Sketch the graph of $y = f(x)$ indicating the turning point and the asymptotes.

Using the graph of $y = f(x)$ find the number of solutions of the equation $(x+1)(x^2+3) = 16x$.

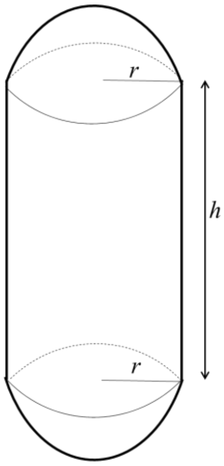
(b) A hollow composite body is to be formed by rigidly joining two hollow hemispheres of radius r metres to a right circular hollow cylinder with the same radius and height h metres, as shown in the figure. The total volume of the composite body is $36\pi m^3$.

Show that $h = \frac{108-4r^3}{3r^2}$.

The cost of material for the cylindrical surface is 300 rupees per square metre and that for hemispherical surfaces is 1000 rupees per square metre. Show that the total cost of material C rupees required to make this composite body is given by,

$C = 800\pi\left(4r^2 + \frac{27}{r}\right)$ for $0 < r < 3$

Find the value of r such that C is minimum.



15. (a) Find $\displaystyle\int \frac{3x+2}{x^2+2x+5}\,dx$

(b) Using integration by parts, show that $\displaystyle\int_1^{e^x} \cos(\ln x)\,dx = -\frac{1}{2}(e^{\pi}+1)$

(c) Establish the formula $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx,$ where a is a constant.

Let $p(x) = (x-\pi)(2x+\pi)$ and $\displaystyle I = \int_0^{\frac{\pi}{2}} \frac{sin^2 x}{p(x)}\,dx$,

Using the above result, show that $\displaystyle I = \int_0^{\frac{\pi}{2}} \frac{cos^2 x}{p(x)}\,dx$.

Using the above two integrals for ***I***, **deduce** that $\displaystyle I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{p(x)}\,dx$.

Hence, show that $I = \frac{1}{6\pi}\ln\left(\frac{1}{4}\right)$.

16. Let l_1 and l_2 be the straight lines given by $y = 5$ and $x + 2y = 4$ respectively. Show that the acute angle between l_1 and l_2 is $\tan^{-1}\left(\frac{3}{4}\right)$, and find the equation of the bisector of this angle.

Let A be the point of intersection of l_1 and l_2 and let $R = \{(x,y): x + 2y \leq 4\}$ and $2x + y \geq 5$.

Find the coordinates of the point A and shade the region R in the xy-plane.

Show that the equation of the circle S of radius $\sqrt{5}$ which lies in the region R and which touches both lines l_1 and l_2 is $x^2 + y^2 - 14x + 8y + 60 = 0$.

Using the usual formula for the chord of contact, show that the equation of the chord of contact of the tangents drawn from the point A to the circle S is $x - y = 10$.

Find the equation of the circle passing through the point A and the points of contact of S with l_1 and l_2.

17. (a) Let $f(x) = \frac{1-\tan x}{1+\tan^2 x}$ for $-\frac{\pi}{2} < x \ \frac{\pi}{2}$.

Express $f(x)$ in the form $A\cos(2x + \alpha) + B$, where $A(> 0), B$ and $\alpha \left(0 < \alpha < \frac{\alpha}{2}\right)$ are constants to be determined.

Hence, solve the equation $f(x) = \frac{2+\sqrt{2}}{4}$.

Using the first expression given for $f(x)$, show that $f(x) = \frac{2+\sqrt{2}}{4}$ can be written as $2\tan^2 x + 4k\tan x - k^2 = 0$, where $k = 2 - \sqrt{2}$.

Deduce that $\tan\frac{\pi}{24} = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$.

Also, sketch the graph of $y = 2f(x)$ for $-\frac{\pi}{2} < x \ \frac{\pi}{2}$.

(b) In the usual notation, state the **Sine Rule** for a triangle.

Let ABC be a triangle. In the usual notation, it is given that $a : b : c = 1 : \lambda : \mu$, where λ and μ are constants.

Show that $\mu^2(\sin 2A + \sin 2B + \sin 2C) = 4\lambda\, sin^3 C$.

Combined Mathematics II – 2014 Advanced Level Examination

<u>Part A</u>

1. A particle is projected under gravity with speed u in a direction making an angle $\frac{\pi}{4}$ with the horizontal, from a point O on a horizontal ground towards a vertical wall of height a which is at a horizontal distance $2a$ from O. Show that if $u > 2ga$, then the particle will go over the wall.

2. A vehicle of mass $M\ kg$ pulls a trailer of the same mass by a light inextensible cable along a straight horizontal road. The resistance to the motion of the vehicle and to the motion of the trailer are R and $2R$ newtons respectively. Show that at the instant when the engine of the vehicle is working at power $P\ kW$ and the vehicle is moving with speed $V\ ms^{-1}$, the tension in the cable is $\frac{1}{2}\left(R + \frac{1000P}{V}\right)$ newtons.

3. A particle P of mass m moves with speed u on a smooth horizontal floor towards a vertical wall, in a straight line perpendicular to the wall. Before hitting the wall, the particle P collides directly with another particle Q of the same mass lying at rest on its path and the particle Q in turn strikes the wall and rebounds. The coefficient of restitution for both collisions is $e(0 < e < 1)$.

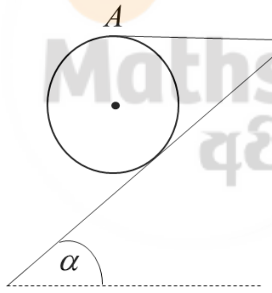
   Show that the impulse on the particle Q by the wall is $\frac{1}{2}(1+e)^2mu$.

4. One end of a light elastic string of natural length a and modulus of elasticity $4\,mg$ is tied to a fixed point O and the other end is attached to a particle of mass m. The particle is released under gravity, from rest at O. Using the Principle of Conservation of Energy, find the maximum length of the string in the subsequent motion.

5. In the usual notation, let $i + 2j$ and $3i + 3j$ be the position vectors of two points A and B respectively, with respect to a fixed origin O. Also, let C be the point such that $OABC$ is a parallelogram. Show that $\overrightarrow{OC} = 2i + j$.

   Let $A\hat{O}C = \theta$ By considering $\overrightarrow{OA}.\overrightarrow{OC}$ show that $\cos\theta = \frac{4}{5}$.

6. A uniform solid sphere of weight W rests on a rough plane, inclined at an angle α to the horizontal, being supported by a light inextensible string attached to the highest point A of the sphere and to a point B on the inclined plane, as shown in the figure. The sphere is in limiting equilibrium when the string AB is horizontal. Show that the angle friction is $\frac{\alpha}{2}$ and find the tension in the string.



7. Let A and B be two events of a sample space Ω. In the usual notation, show that

   $P\big((A \cup B) \cap A' \cup B'\big) = P(A) + P(B) - 2P(A \cap B)$.

8. A bag contains 6 red balls and 4 white balls of the same size. Three balls are drawn, one at a time, from the bag at random, without replacement. Find the probability that the third ball is red, given that the second ball is white.

9. The mean and the median of five observations are 7 and 9 respectively. The only mode of the observations is 11. Assuming that all observations are positive integers, find the largest observation and the smallest observation.

10. The mean of the following frequency distribution of 100 observations is 31.8.

5-15	15-25	25-35	35-45	45-55
16	x	30	y	20

Find the values of x and y, and estimate the median of the distribution.

Part B

❖ Answer Five questions only.

11. (a) Two particles P and Q are placed at a point O on a fixed smooth plane inclined at an angle $\alpha\left(0 < \alpha < \frac{\pi}{2}\right)$ to the horizontal. The particle P is given a velocity u upwards along the line of greatest slope through O, and at the same instant the particle Q is released from rest. Assuming that the two particles do not leave the inclined plane, sketch the velocity-time graphs for the motion of P and Q on the same diagram.

Using these graphs, show that, at the instant the particle P returns to the point O, the particle Q is at a distance $\frac{2u^2}{g\sin\alpha}$ from O.

(b) A river with parallel straight banks flows with uniform velocity $\mathbf{u}$. Two points A and B on either bank are situated such that $\overrightarrow{AB}$ makes an acute angle α with $\mathbf{u}$. A boy starts at A and reaches B, swimming in a fixed direction with a constant velocity of magnitude $2u$ relative to water, where $u = |\mathbf{u}|$. He then stars at B and swims in such a fixed direction with a velocity of the same magnitude $2u$ relative to water to return to A. Sketch the velocity triangles for the motion from A to B and for the motion from B to A, in the same diagram.

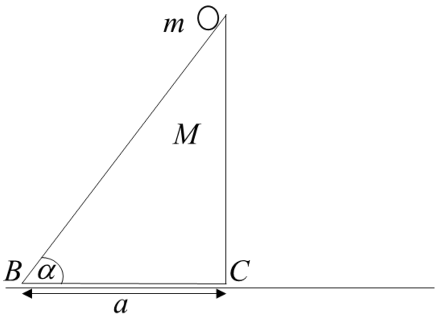
Hence, show that for the motion from A to B and for the motion from B to A, his velocity relative to water must make the same angle θ with $\overrightarrow{AB}$ and $\overrightarrow{BA}$ respectively, where $\sin\theta = \frac{1}{2}\sin\alpha$.

If the time taken to swim from B to A is $k(1 < k < 3)$ times the time taken to swim from A to B, show that $\cos\theta = \frac{1}{2}\left(\frac{k+1}{k-1}\right)\cos\alpha$.

Using the above expressions for $\sin\theta$ and $\cos\theta$ show also that $\cos\alpha = \frac{(k-1)}{2}\sqrt{\frac{3}{k}}$.

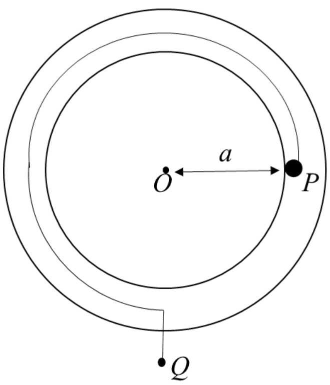
12. (a) The triangle ABC in the given figure represents a vertical cross-section through the centre of gravity of a uniform smooth wedge of mass M. The line AB is a line of greatest slope of the face contain it, $A\hat{B}C = \alpha, A\hat{C}B = \frac{\pi}{2}$ and $BC = a$. The wedge is placed with the face containing BC on

a smooth horizontal floor. A particle of mass m is gently placed on the line AB at the point. A and released from rest. Show that until the particle leaves the wedge, the acceleration of the wedge is $\frac{mg\sin\alpha\cos\alpha}{M+msin^2\alpha}$, and find the acceleration of the particle relative to the wedge.



Now, suppose that $\alpha = \frac{\pi}{4}$ and $M = \frac{5m}{2}$. Show that the speed of the wedge at the instant when the particle leaves the wedge is $\sqrt{\frac{2ag}{21}}$.

(b) A narrow smooth circular tube of radius a and centre O is fixed in a vertical plane. One end of a light inextensible string of length greater than $\frac{3\pi a}{2}$ is attached to a particle P of mass m which is held inside the tube with OP horizontal. The string passes through the tube and though a small smooth hole at the lowest point of the tube as shown in the figure, and carries a particle O of mass $2m$ at the other end. The particle P is released from rest from the above position with the string taut. By applying the Principle of Conservation of Energy, show that the speed v of the particle P when OP has turned through an angle $\theta \left(0 < \theta < \frac{3\pi}{2}\right)$ is given by $v^2 = \frac{2ga}{3}(2\theta - \sin\theta)$, and find the reaction on the particle P from the tube.



13. A thin light elastic spring of natural length $4a$ and modulus of elasticity $8\,mg$ stands vertically with its lower end O fixed. A particle P of mass m is attached to its upper end. The particle P is in equilibrium at a point a vertically above O. Show that $OA = \frac{7a}{2}$.

Now, another particle Q of the same mass m is gently attached to P, and the composite particle begins to move from rest at A. Show that the equation of motion of the composite particle is

$\ddot{x} = -\frac{g}{a}x$, where x is the displacement of the composite particle from the point B vertically above O such that $OB = 3a$.

Let C be the lowest point reached by the composite particle. Find the length OC and the time taken by the composite particle to move from A to C.

At the instant when the composite particle is at C, the particle Q is gently removed. Show that, for the subsequent motion of the particle P, the equation of motion is $\ddot{y} = -\frac{2g}{a}y$ where y is the displacement of the particle P from the point A.

Assuming a solution for this equation in the form $y = \alpha\cos\omega t + \beta\sin\omega t$, find the values of the constants α, β and ω.

Hence, show that the time taken by the particle P to move from C to D is $\frac{\pi}{3}\sqrt{\frac{2a}{g}}$, where D is the point vertically above O such that $OD = 4a$.

Find also the speed of the particle P when it reaches D.

14. (a) Let $ABCD$ be a trapezium such that $\overrightarrow{DC} = \frac{1}{2}\overrightarrow{AB}$. Also, let $\overrightarrow{AB} = p$ and $\overrightarrow{AD} = q$. The point E lies on BC such that $\overrightarrow{BF} = \frac{1}{3}\overrightarrow{BC}$. The point of intersection F of AE and BD satisfies $\overrightarrow{BF} = \lambda\overrightarrow{BD}$, where $\lambda (0 < \lambda < 1)$ is a constant. Show that $\overrightarrow{AE} = \frac{5}{6}p + \frac{1}{3}q$ and that $\overrightarrow{AF} = (1-\lambda)p + \pi q$.

Hence, find the value of λ.

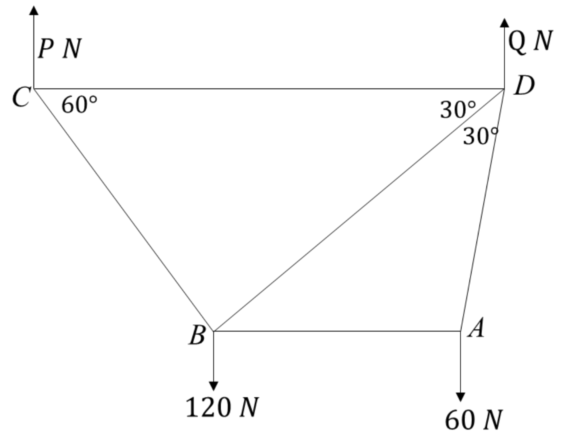
(b) Let $ABCD$ be a square of side a metres. Forces of magnitudes $4, 6\sqrt{2}, 8, 10, X$ and Y newtons act along AD, CD, AC, BD, AB and CB respectively, in the directions indicated by the order of the letters. The system reduces to a single resultant acting along $\overrightarrow{OE}$, where O and E are the mid-points of AC and CD respectively. Find the values of X and Y, and show that the magnitude of the resultant is $4K$ newtons, where $K = 2 - \sqrt{2}$.

Let F be the point such that $OAFD$ is a square. Find the two forces, one along $\overrightarrow{AD}$ and the other through the point F, which are equivalent to the above system of forces.

A couple of moment $6\ ka$ newton metres acting in the sense $ABCD$, in the plane of the forces, is added to the original system. Find the line of action of the resultant of the new system.

15. (a) Four uniform rods AB, BC, CD and DA, each of weight w per unit length, with $AB = AD = l\sqrt{3}$ and $BC = DC = l$ are smoothly jointed at their ends so as to form a framework $ABCD$. The joints A and C are connected by a light inextensible string of length $2l$. The framework suspended from the joint A hangs in equilibrium in a vertical plane. Show that the tension in the string is $\frac{wl}{4}(5 + \sqrt{3})$.

(b)



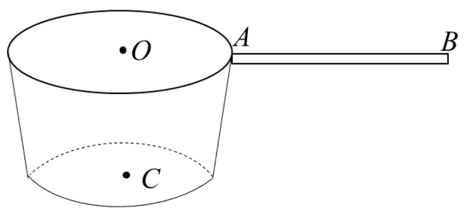
The given figure represents a framework of five light rods AB, AD, BC, BD and CD smoothly jointed at the ends. The framework carries loads $60\ N$ and $120\ N$ at A and B respectively, and is kept in equilibrium with the rods AB and CD horizontal, by two vertical forces $P\ N$ and $Q\ N$ applied at C and D respectively. Draw a stress diagram using Bow's notation.

Hence, find the stresses in all five rods stating whether they are tensions or thrusts.

16. Show by integration that the centre of gravity of the frustum obtained by cutting a uniform hollow hemisphere shell of radius a and surface density σ by a plane parallel to its circular rim and at a distance $a\cos\alpha$ from the centre O is at the mid-point of OC, where C is the centre of the smaller circular rim.

A bowl is made by rigidly fixing the edge of a thin uniform circular plate of radius $a\sin\alpha$ having the same surface density σ to the smaller circular rim of the above frustum. Show that the centre of gravity of this bowl is on OC at a distance $\left(\frac{1+\cos\alpha+cos^2\alpha}{1+2\cos\alpha-cos^2\alpha}\right) a\cos\alpha$ from O.

Let $\alpha = \frac{\pi}{3}$ and let W be the weight of the bowl. A saucepan is made by rigidly fixing a thin uniform rod AB of length b and weight $\frac{W}{4}$ to the rim of the bowl as a handle such that the points O, A and B are collinear as shown in the figure. Find the position of the centre of gravity of the saucepan.



The saucepan is freely suspended from the end B of the handle and hangs in equilibrium with the handle making an angle $\tan^{-1}\left(\frac{1}{7}\right)$ with the downward vertical. Show that $3b = 4a$.

17. (a) Let A and B be two events of a sample space Ω with $P(B > 0)$. Define $P(A|B)$, the conditional probability of A given B.

Show that $P(A) = P(B)P(A|B) + P(B')P(A|B')$, where $0 < P(B) < 1$ and B' denotes the complementary event of B.

In a large company 80% of the employees are male and 20% are female, The highest educational qualification of 57% of the employees is $G.C.E.(O/L)$ and that of 32% is $G.C.E\ (A/L)$. All the other employees are graduates, of the female employees in the company, the highest educational qualification of 40% is $G.C.E\ (O/L)$ and that of 45% is $G.C.E\ (A/L)$. An employee is selected at random from the employees of the company. Find the probability of each of the events that this employee is

(i) a female with $G.C.E\ (O/L)$ as the highest educational qualification,

(ii) a male with $G.C.E\ (O/L)$ as the highest educational qualification,

(iii)a graduate, given that the employee is a male,

(iv)a female, given that the employee is not a graduate.

(b) Let the mean and the variance of the set of data $\{x_1, x_2, \dots \dots . . x_n\}$ be $\overline{x}$ and ${\sigma_x}^2$ respectively.

(i) Show that $\quad {\sigma_x}^2 = \frac{1}{n}\sum_{i=1}^{n} {x_i}^2 - \overline{x}^2$

(ii) Let α and β be real constants. Show that $\sum_{i=1}^{n} (\alpha x_i + \beta)^2 = n\alpha^2{\sigma_x}^2 + n(\alpha\overline{x} + \beta)^2$

Let $y_i = \alpha x_i + \beta$ for $i = 1,2, \dots . . , n$. Show that $\overline{y} = \alpha\overline{x} + \beta$ and using (i) and (ii) above, deduce that ${\sigma_y}^2 = \alpha^2{\sigma_x}^2$, where $\overline{y}$ and ${\sigma_y}^2$ are the mean and the variance to the set $\{y_1, y_2, \dots \dots . . y_n\}$ respectively.

The mean of the marks obtained by candidates in a certain examination is 45. These marks are to be scaled linearly to give a mean of 50 and a standard deviation of 15. It is given that the scaled mark 68 corresponds to the original mark 60. Calculate the standard deviation of the original marks.

It is given further that the original mark m obtained by a candidate is not lowered by the above scaling.

Show that $m \leq 20$.