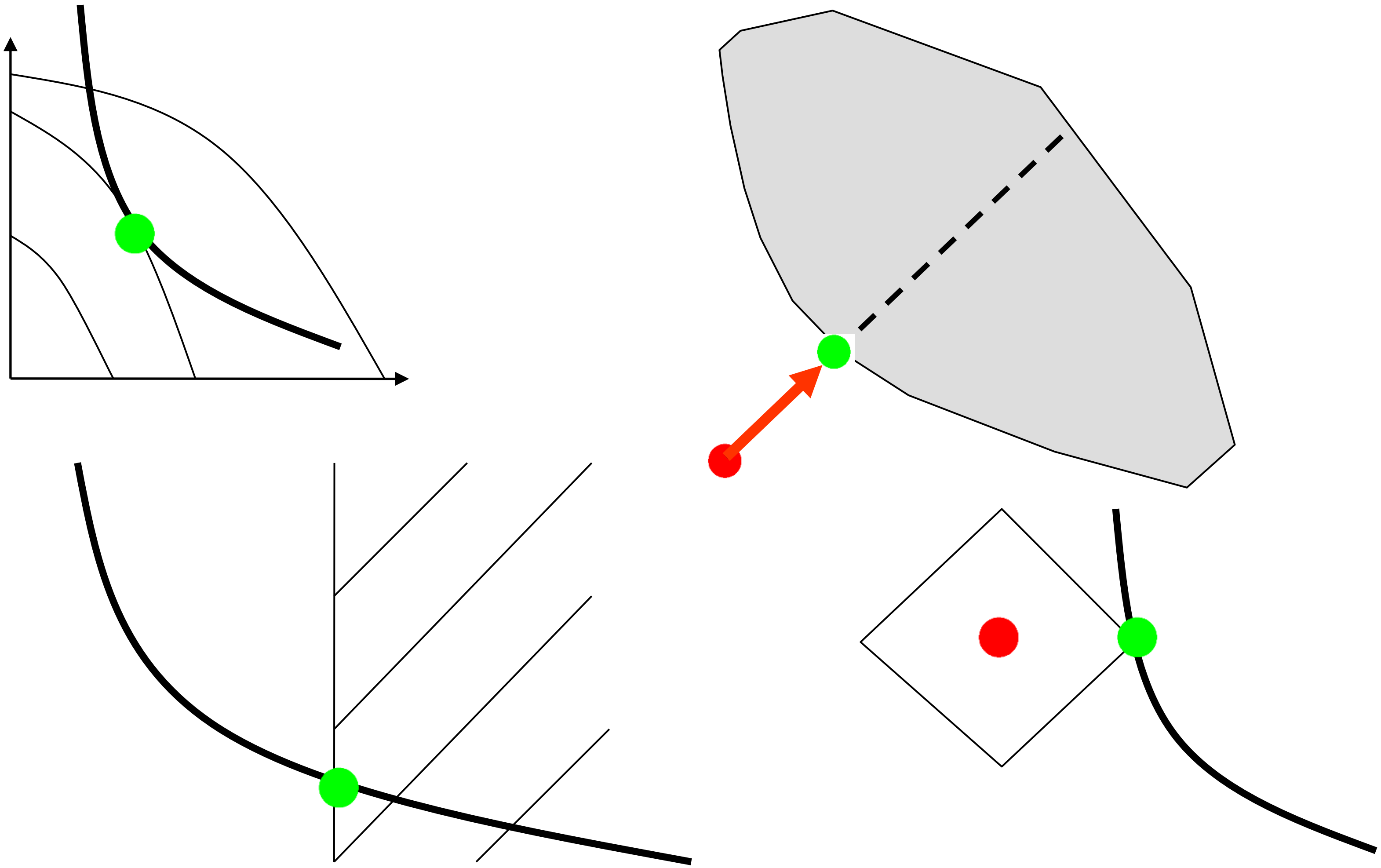
# Single point methods for finding the Pareto front

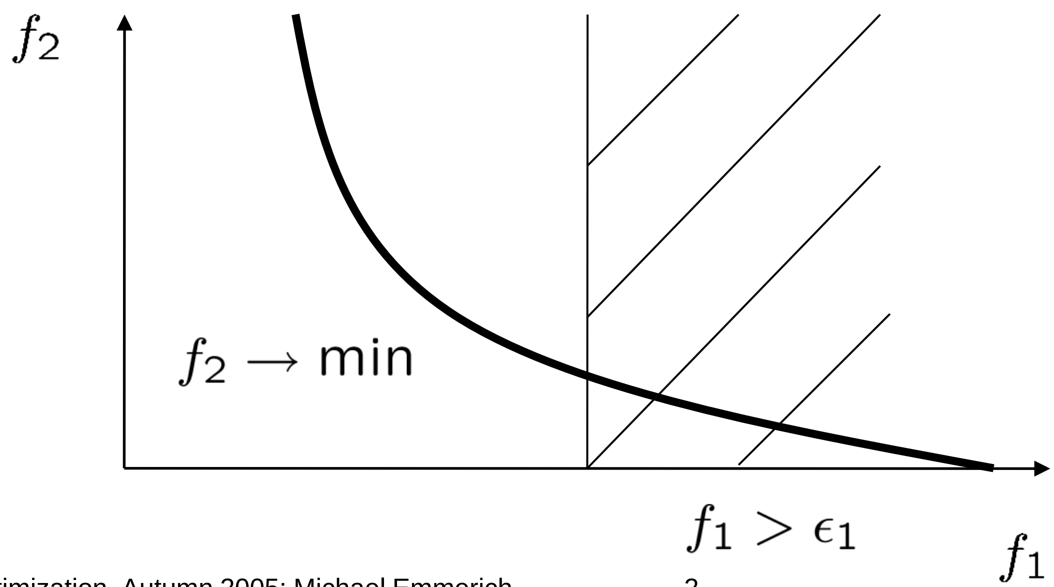


# Single point methods

Strategy 1: Sum up function values by means of an utility function

$$f_{eq}(\mathbf{x}) = \sum_{i=1}^{m} w_i f_i(\mathbf{x})$$

Strategy 2: Add constraints to the problem (e.g. transform objectives into constraints)



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# Learning goals

- What are different ways to solve multiobjective optimization problems by formulating them as single objective optimization problems (with constraints)?
- Can we use linear weighting functions to find all Pareto optimal points?
- Which points (on the Pareto front) do we find for different scalarization functions?
- How and when can we use single point methods to find all points on a Pareto front?

# Weighted sum scalarization

$f_{eq}(\mathbf{x}) = \sum_{i=1}^{m} w_i f_i(\mathbf{x}) \quad w_i > 0, i = 1, \dots, m \quad \sum_{i=1}^{m} w_i = 1$

Regardless of the choice of weights, the weighted sum minimization will always result in an efficient solution $\mathbf{x}^*$

Proof: $f_i(\mathbf{x}) \leq f_i(\mathbf{x}')$ for $i = 1, \dots, m$ and $f_j(\mathbf{x}) < f_j(\mathbf{x}')$ for some $j \in \{1, \dots, m\} \Rightarrow \sum w_i f_i(\mathbf{x}) < \sum w_i f_i(\mathbf{x}')$

Not all Pareto optimal solutions can be obtained with the weighted sum scalarization

(1) Efficient solutions obtained with the weighted sum approach are Pareto optimal in the Geoffrion sense.

(2) Solutions that belong to concave parts of the Pareto front cannot be obtained

# Proper efficiency

**Definition:** Domination in the Geoffrion sense:
A solution $\mathbf{x}^* \in S$ is called a proper Pareto optimal solution iff:

(a) it is efficient

(b) there exists a number $M > 0$ such that $\forall i = 1, \ldots, m$ and $\forall x \in \mathcal{X}$ satisfying $f_i(\mathbf{x}) < f_i(\mathbf{x}^*)$, there exists an index $j$ such that $f_j(\mathbf{x}^*) < f_j(\mathbf{x})$ and:

$$\frac{f_i(\mathbf{x}^*) - f_i(\mathbf{x})}{f_j(\mathbf{x}) - f_j(\mathbf{x}^*)} \leq M$$

**Definition:** *Proper efficient solutions* are optimal due to Geoffrion's domination criterion. They have a bounded tradeoff considering their objectives.



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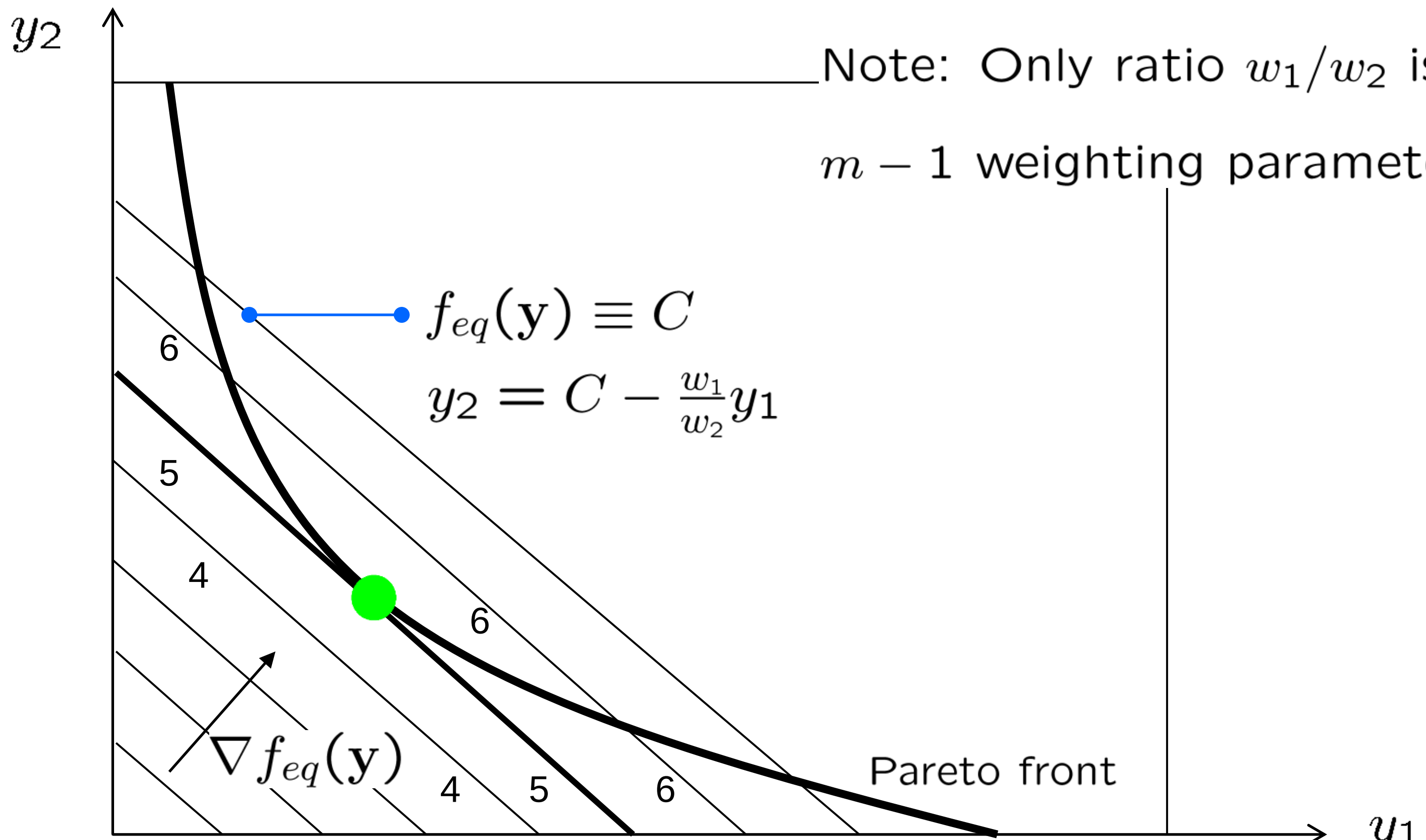
# Convex Pareto front

Linear scalarization $f_{eq}(\mathbf{y}) = w_1 y_1 + w_2 y_2, w_1 > 0, w_2 > 0$

● : $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{Y}$ with minimal value for $w_1 y_1(\mathbf{x}) + w_2 y_2(\mathbf{x})$

Note: Only ratio $w_1/w_2$ is important!

$m-1$ weighting parameters suffice!



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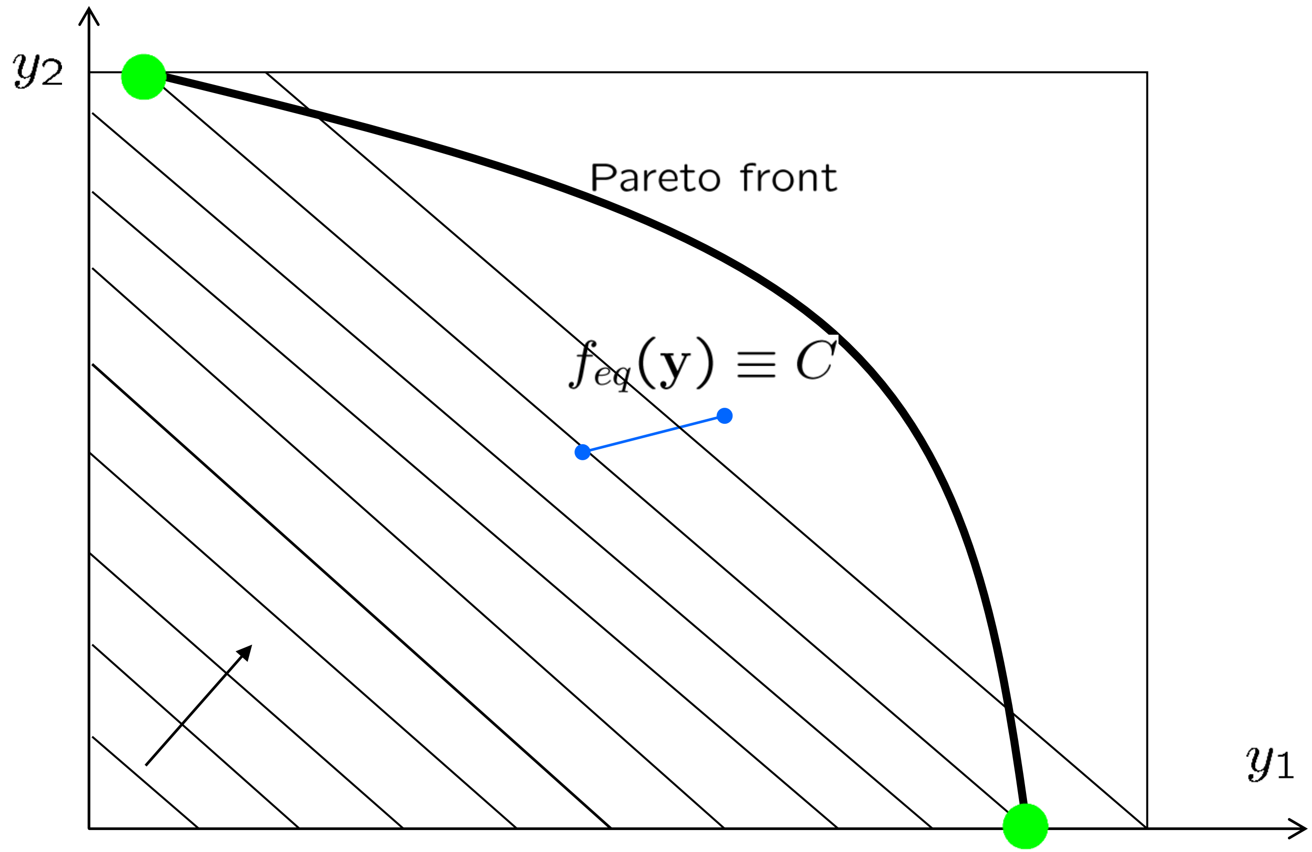
# Concave Pareto front




Only extremal points can be obtained in case of concave Pareto fronts

# Example: Schaffer problem

$$f_1(x) = x^2,\ f_2(x) = (x-2)^2 \qquad x \in \mathcal{X} = [0,2]$$

$$f_{eq} = w_1 x^2 + w_2 (x-2)^2 \to \min$$

Ansatz: Find all $x$ with $\frac{\partial f_{eq}}{\partial x} = 0,\ \frac{\partial^2 f_{eq}}{\partial x^2} > 0$

$$\frac{\partial f_{eq}}{\partial x} = 2x(w_1 + w_2) - 4w_2 = 0 \text{ (a)},\ \frac{\partial^2 f_{eq}}{\partial x^2} = 2(w_1 + w_2) > 0$$

(b) is always fulfilled, since $w_i > 0$

(a) $\Leftrightarrow x^* = 2w_2/(w_1 + w_2) \underbrace{=}_{w_1 + w_2 = 1} 2w_2$



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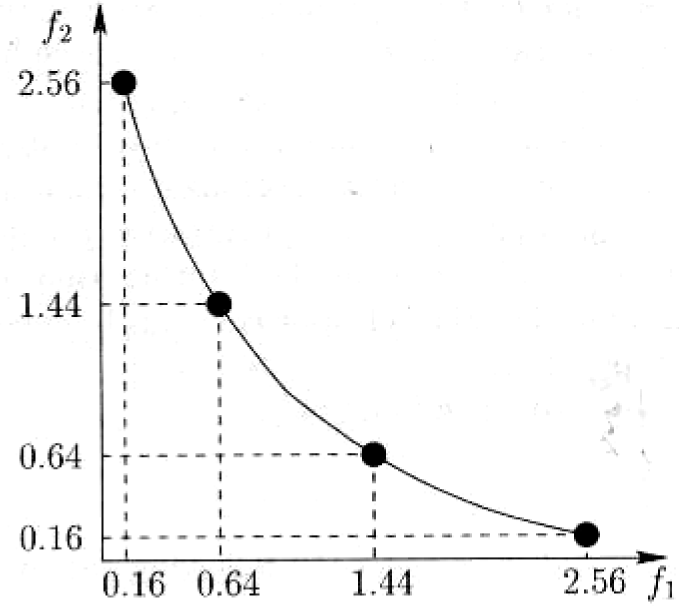
# Example: Schaffer problem

**Table 2.1.** Recapitulatory table.

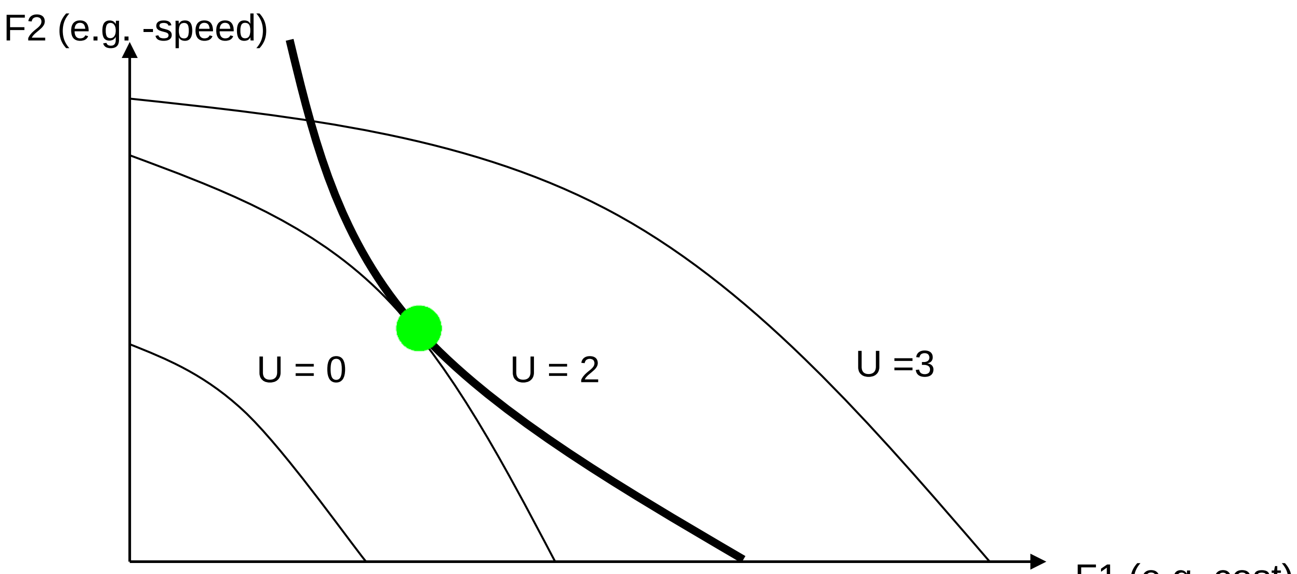
| | | | | |
|---|---|---|---|---|
| $w_1$ | 0.2 | 0.4 | 0.6 | 0.8 |
| $w_2$ | 0.8 | 0.6 | 0.4 | 0.2 |
| $x^*$ | 1.6 | 1.2 | 0.8 | 0.4 |
| $f_1(x^*)$ | 2.56 | 1.44 | 0.64 | 0.16 |
| $f_2(x^*)$ | 0.16 | 0.64 | 1.44 | 2.56 |




Source: Siarry et al. Multiobjective Optimization, Springer, Berlin

# Utility functions

Once a (proper) utility function is given the tangential points of the iso-utility curves with the Pareto front are the obtained non-dominated points.




Keeney and Raiffa: Decisions with Multiple Objectives: Preferences and Value Tradeoff, Cambridge Univ. Press, 1993



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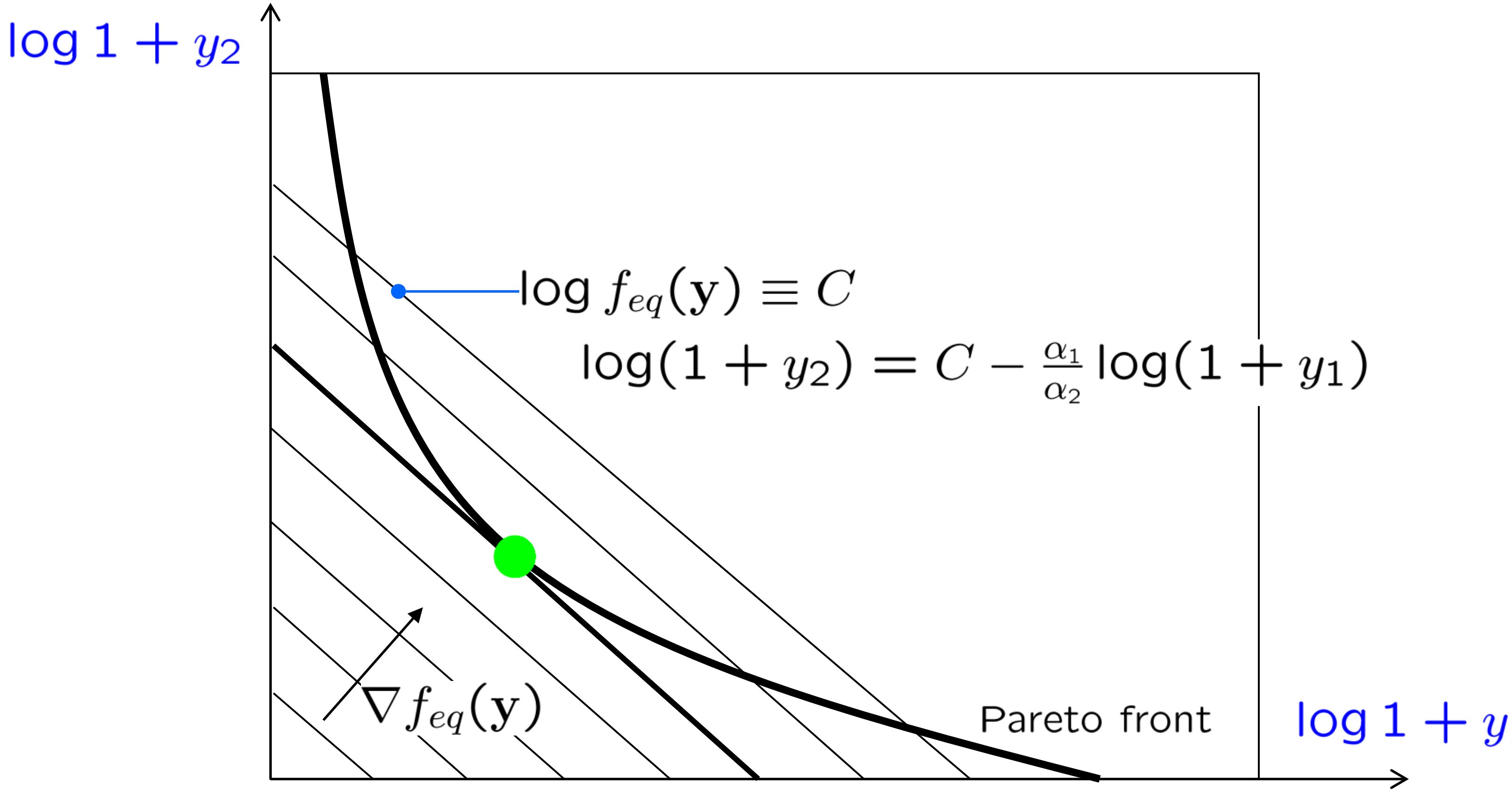
# Cobbs Douglas utility functions

Cobbs Douglas Utility: $f_{eq}(\mathbf{x}) = \prod_{\mathbf{i}=1}^{\mathbf{m}} (\mathbf{1} + \mathbf{f_i}(\mathbf{x}))^{\alpha_\mathbf{i}}, \alpha_\mathbf{i} > \mathbf{0}$

Note: $\log f_{eq}(\mathbf{x}) = \sum_{\mathbf{i}=1}^{\mathbf{m}} \alpha_\mathbf{i} \log \mathbf{f_i}(\mathbf{x}), \alpha_\mathbf{i} > \mathbf{0}$ can be used for drawing iso utility lines!



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# Distance to a reference point (DRP) method

- These methods aim for minimizing the distance to an ideal point
- The ideal point has multiple components and these are the objective function values to be minimized
- Examples:
  - In a machine learning problem the false positive rate fp and false negative fn rate should be simultaneously minimized. The ideal point is fn=$(0,0)^T$
  - In a control problem the pressure should be kept close to p* and the temperature close to T*. The ideal point is $(T,p)^T=(p^*, T^*)^T$.
  - In an building optimization problem the fuel consumption EC should be ideally 0 and the annual operation cost AC and investment cost IC, too. The ideal point is $(EC,AC,IC)^T=(0,0,0)^T$

# Minkowsky distance functions

General distance to reference point $\mathbf{f}^* \in \mathbb{R}^m$

$$f_{eq}(\mathbf{x}) = \left( \sum_{i=1}^{m} |f_i(\mathbf{x}) - f_i^*|^p \right)^{1/p}$$

Example $p = 1$:

$$f_{eq}(\mathbf{x}) = \sum_{i=1}^{m} |f_i(\mathbf{x}) - f_i^*|$$

Example $p = 2$:

$$f_{eq}(\mathbf{x}) = \left( \sum_{i=1}^{m} |f_i(\mathbf{x}) - f_i^*|^2 \right)^{1/2}$$

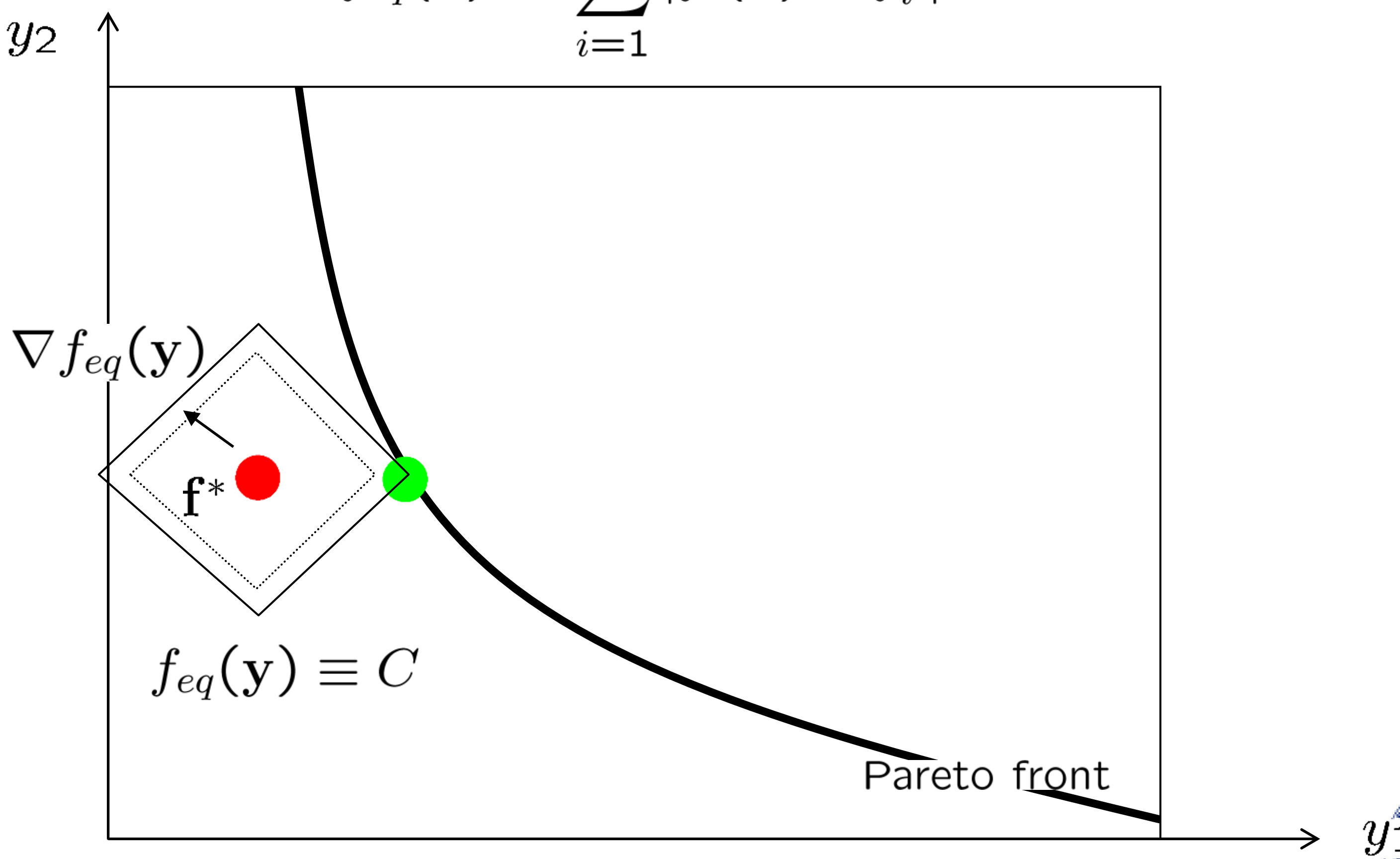
Example $p = \infty$: Tschebyscheff Distance

$$f_{eq}(\mathbf{x}) = \max_{i=1,\ldots,m} |f_i(\mathbf{x}) - f_i^*|$$

# View of DRP as a utility function:

Example $p = 1$:

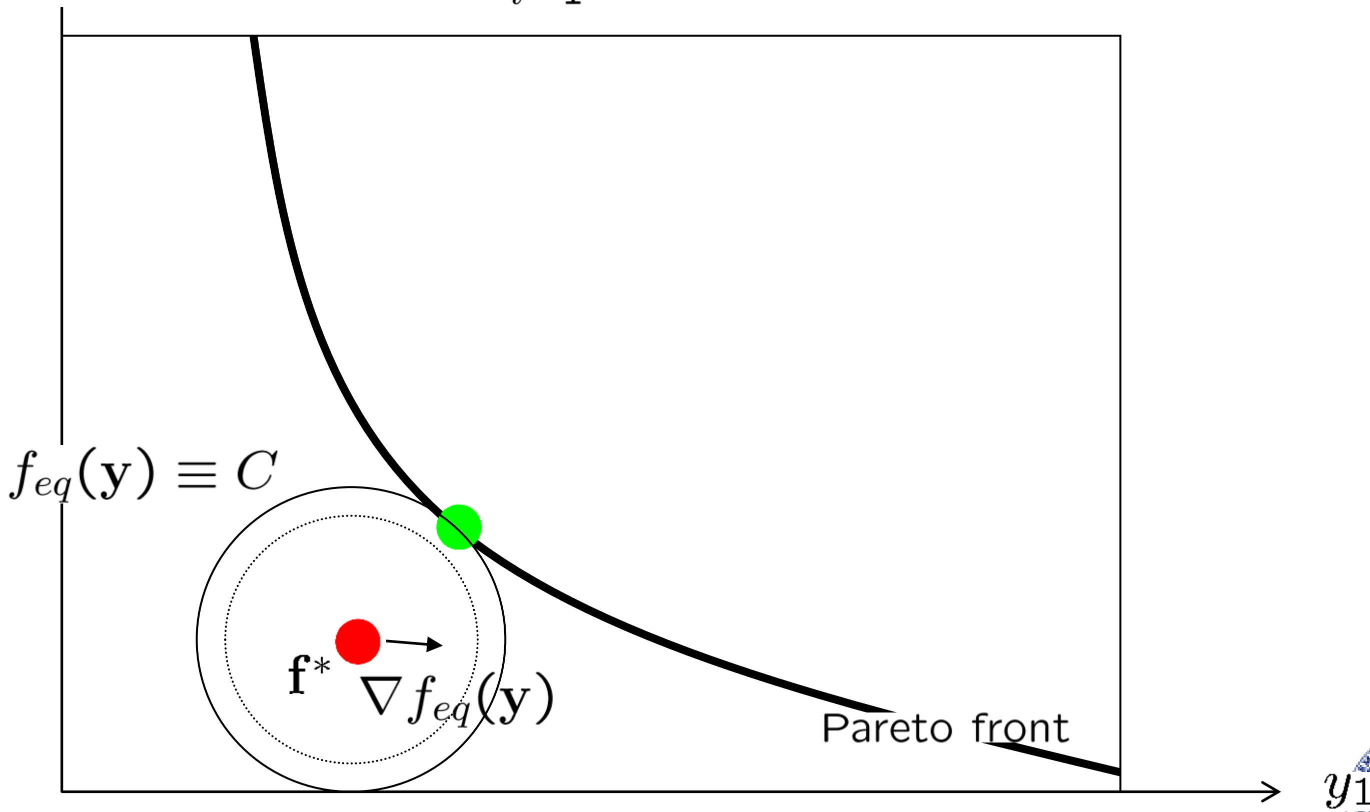
$$f_{eq}(\mathbf{x}) = \sum_{i=1}^{m} |f_i(\mathbf{x}) - f_i^*|$$



# View of DRP as a utility function:

Example $p = 1$:

$$f_{eq}(\mathbf{x}) = \sum_{i=1}^{m} |f_i(\mathbf{x}) - f_i^*|^2$$



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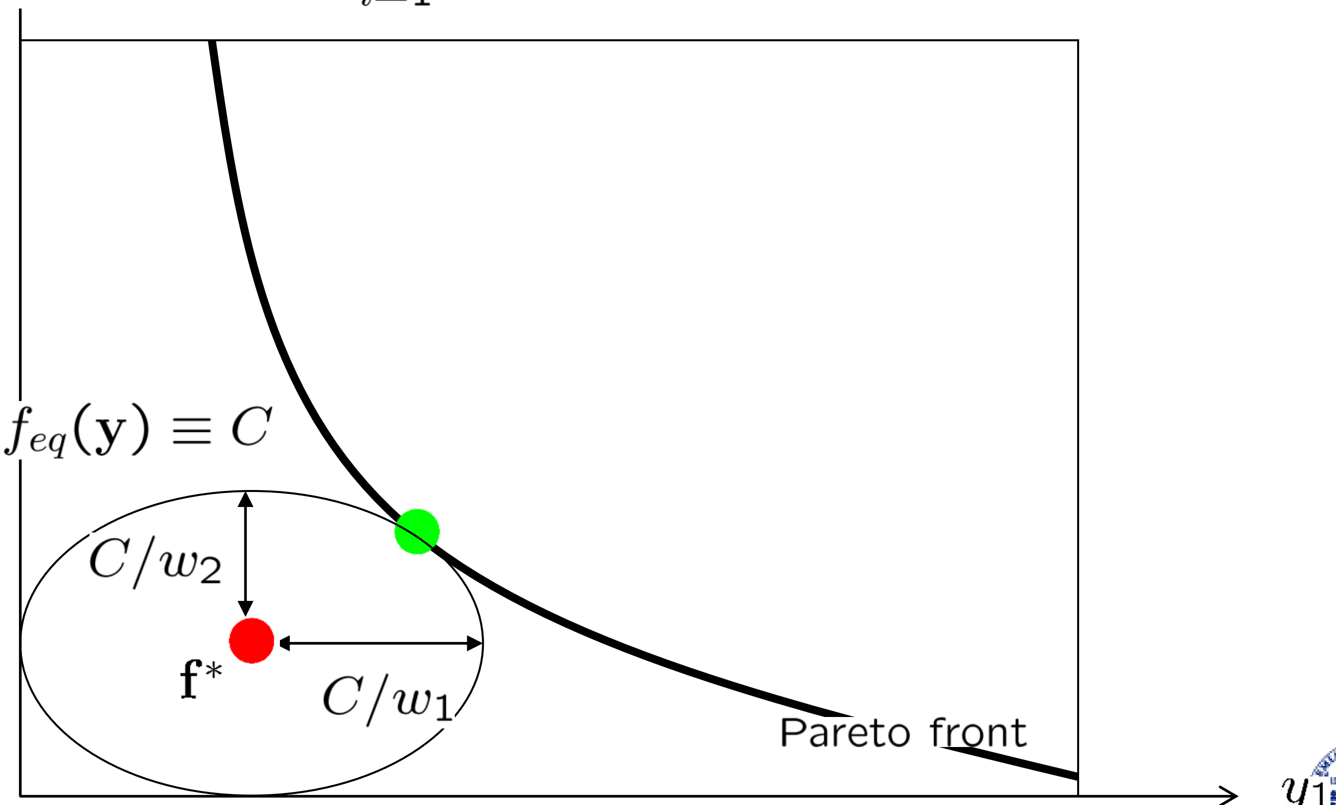
# View of DRP as a utility function: Weighted euclidian distance function

Example $p = 1$:

$$f_{eq}(\mathbf{x}) = (\sum_{i=1}^{m} w_i |f_i(\mathbf{x}) - f_i^*|^2)^{1/2}$$



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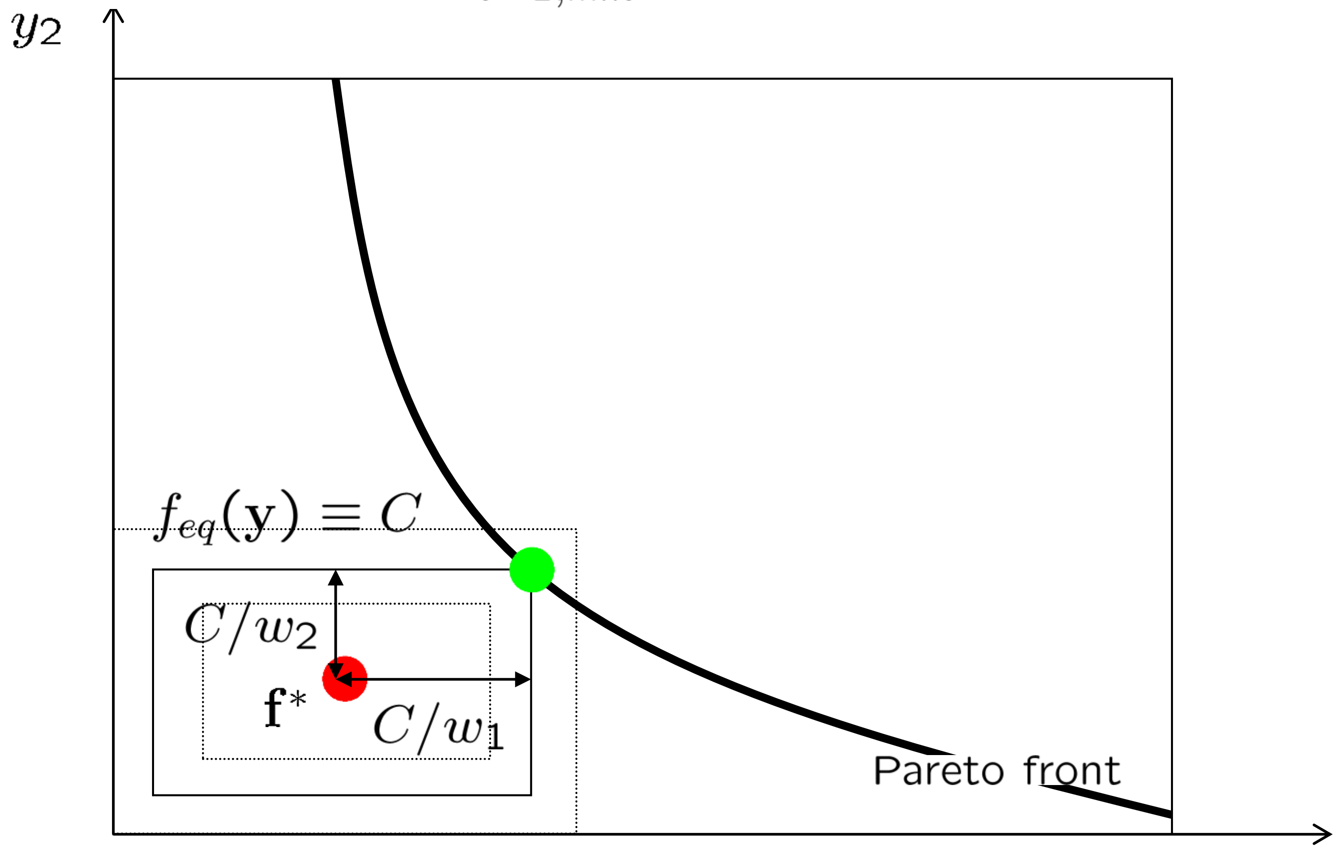
# Tschebychev DRP:

Example $p = 1$:

$$f_{eq}(\mathbf{x}) = \max_{i=1,\dots m} w_i |f_i(\mathbf{x}) - f_i^*|$$



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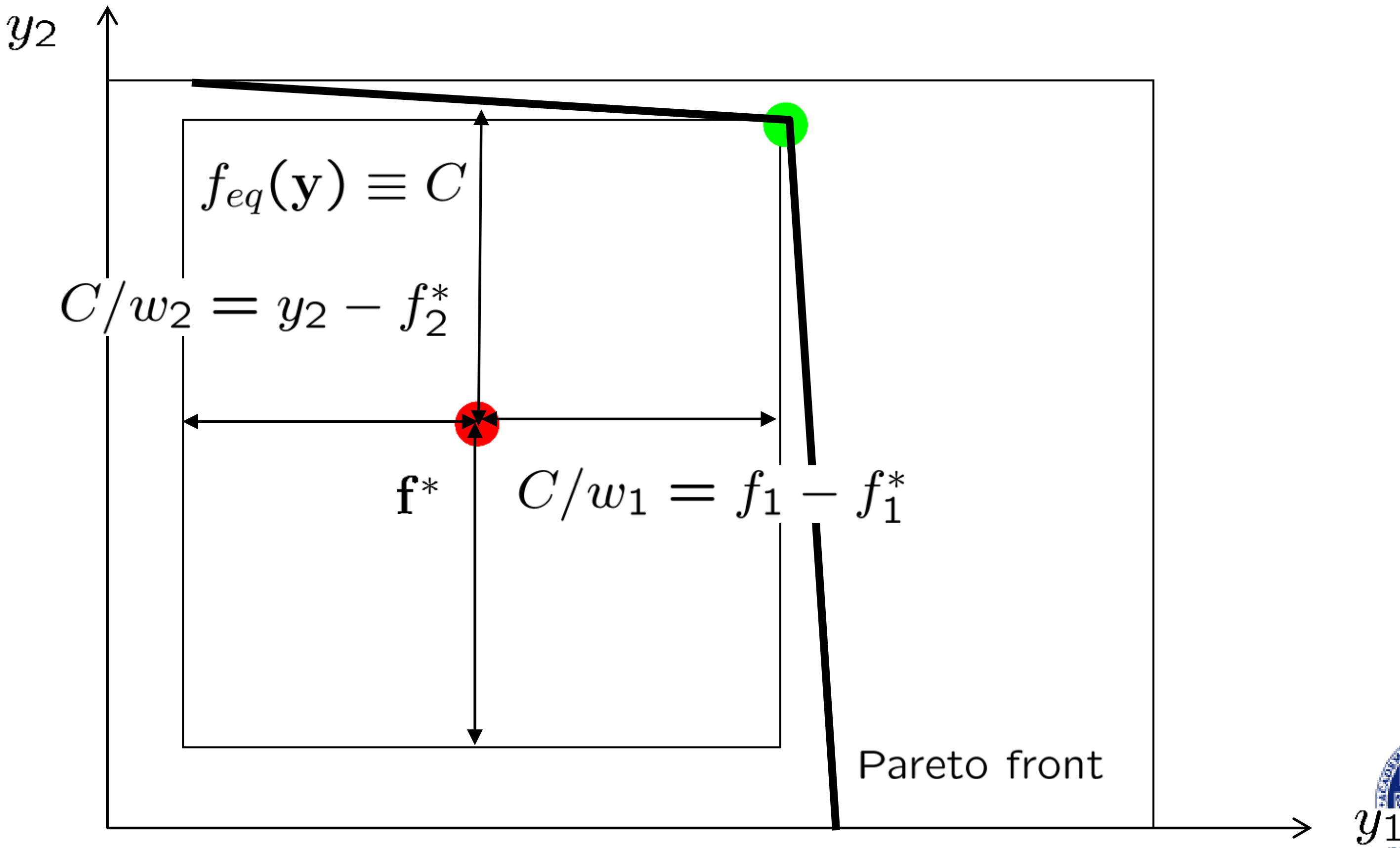
# Tschebychev DRP:

Let $\mathbf{f}^* \preceq \mathbf{f}^I$ (reference point is dominated by ideal point). For every properly efficient point $\mathbf{y} \in \mathcal{Y}_N$ we can find a combination of weights, such that the minimization of Tschebytscheff utility $f_{eq}$ leads to $\mathbf{x}$ with $\mathbf{f}(\mathbf{x}) = \mathbf{y}$.



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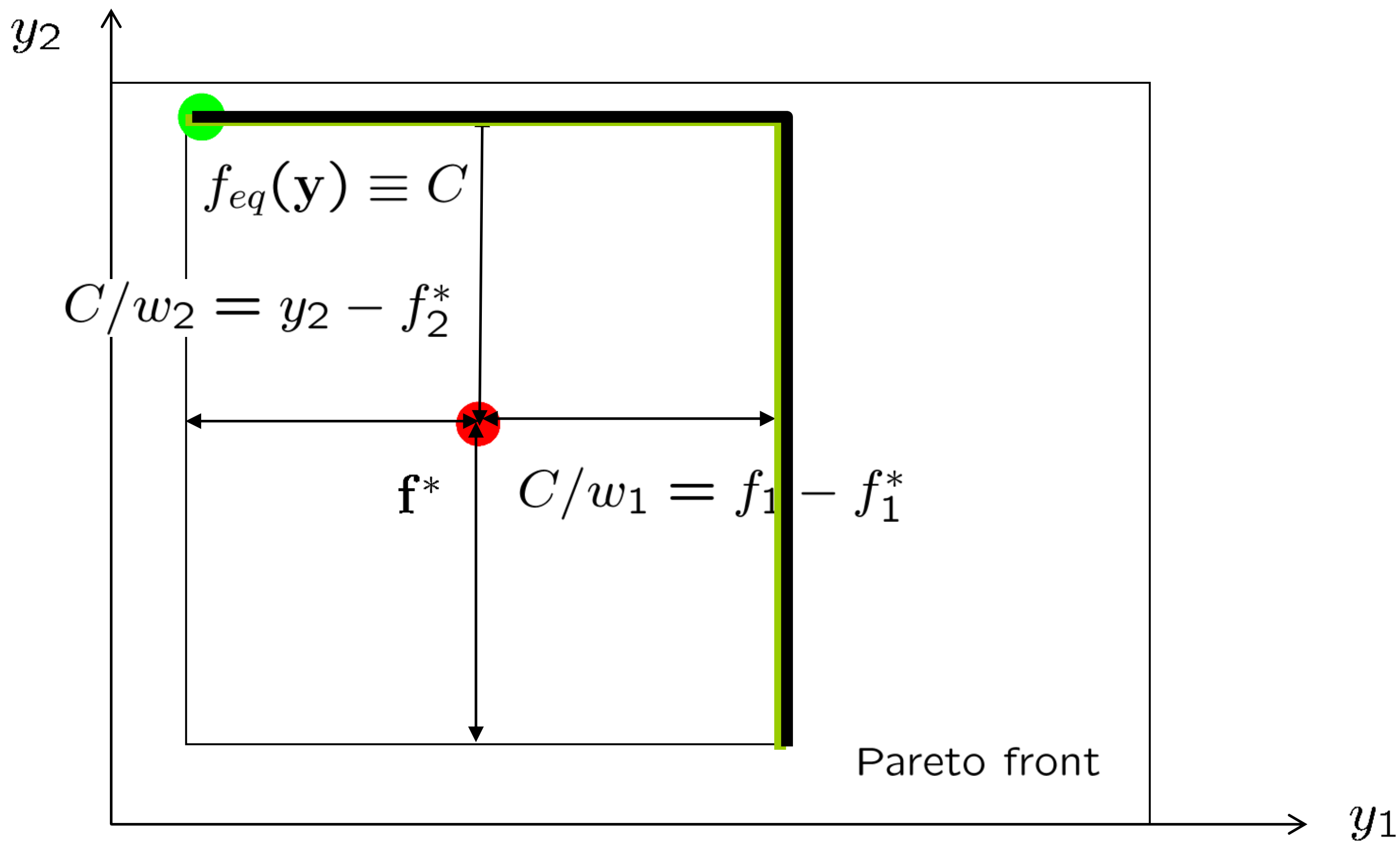
# Tschebychev DRP:

Weakly efficient solutions may be among the results!



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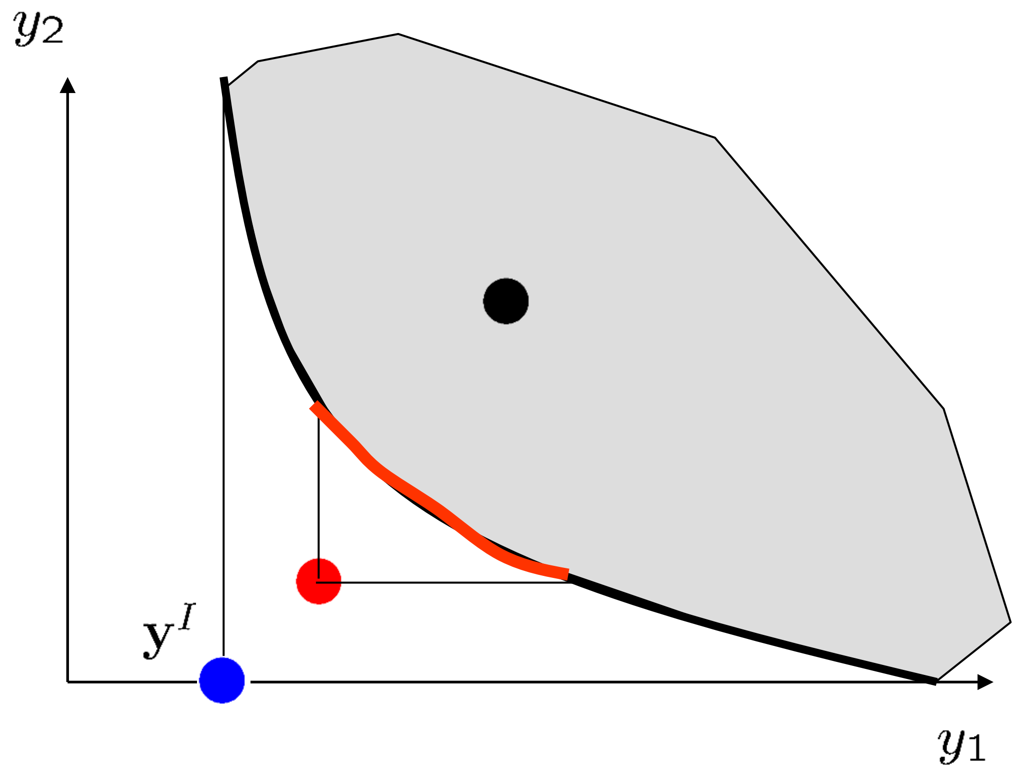
# Choice of reference point

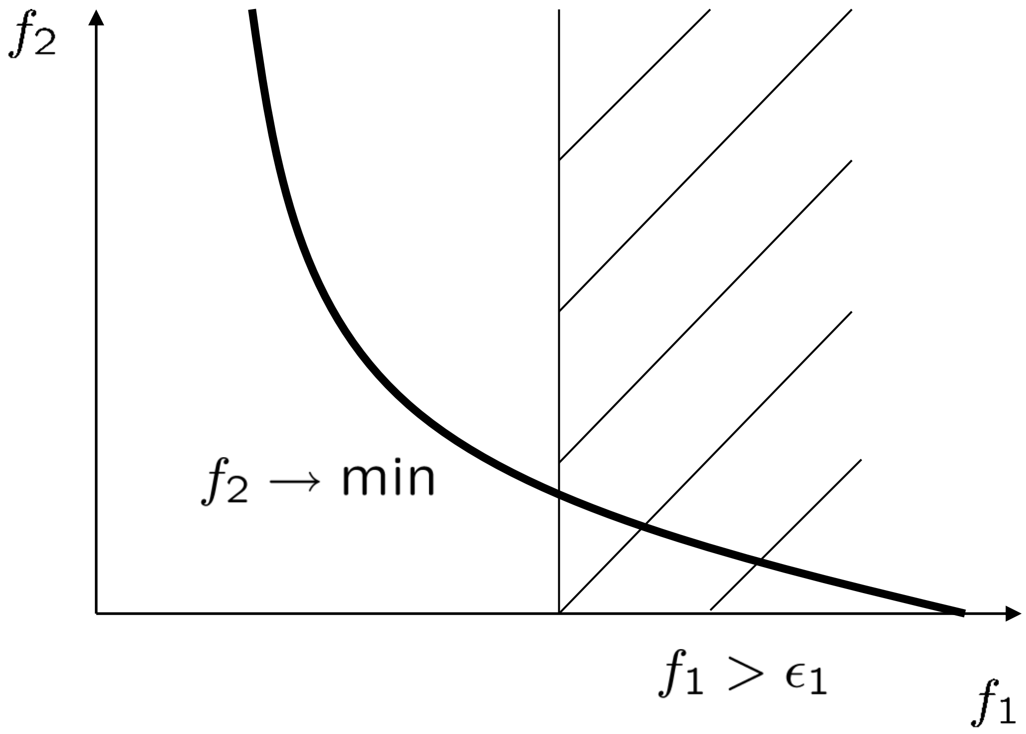
Reference point should be dominated by ideal point $\mathbf{y}^I$!




Pessimistic choice of $\mathbf{f}^*$ $\Rightarrow$ not all points on Pareto surface can be obtained, or even dominated point result in the minimization of $f_{eq}$.

# ε-Constraint method

$$f_m(\mathbf{x}) \to \min, \ \text{s.t.} f_i(\mathbf{x}) \le \epsilon_i, i = 1, \ldots, m$$



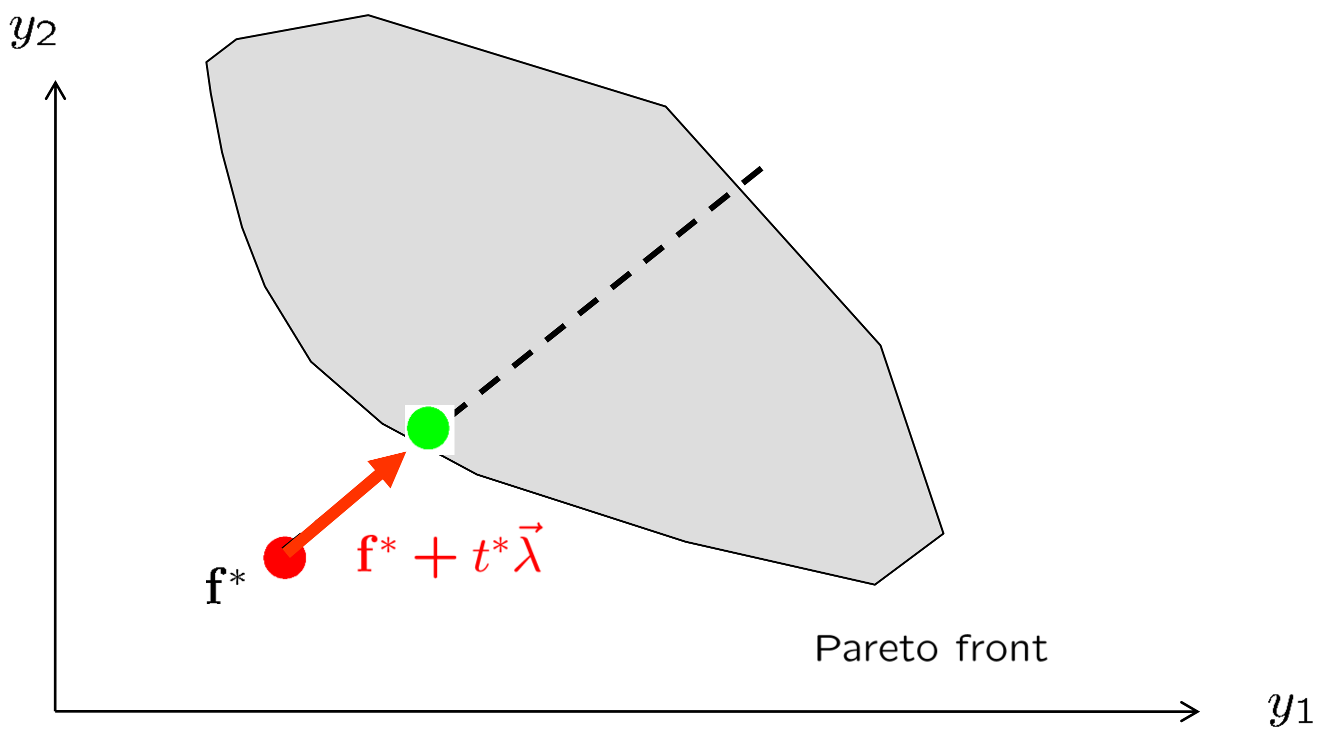

With the dimension the number of $\epsilon$ combinations grows exponentially.

# Goal programming

(1) Choose reference point!

(2) Choose positive direction $\lambda \in \mathbf{R}^m_>$!

(3) Find minimal $t$ s.t. $\mathbf{f}^* + t\vec{\lambda} \in \mathcal{Y}$



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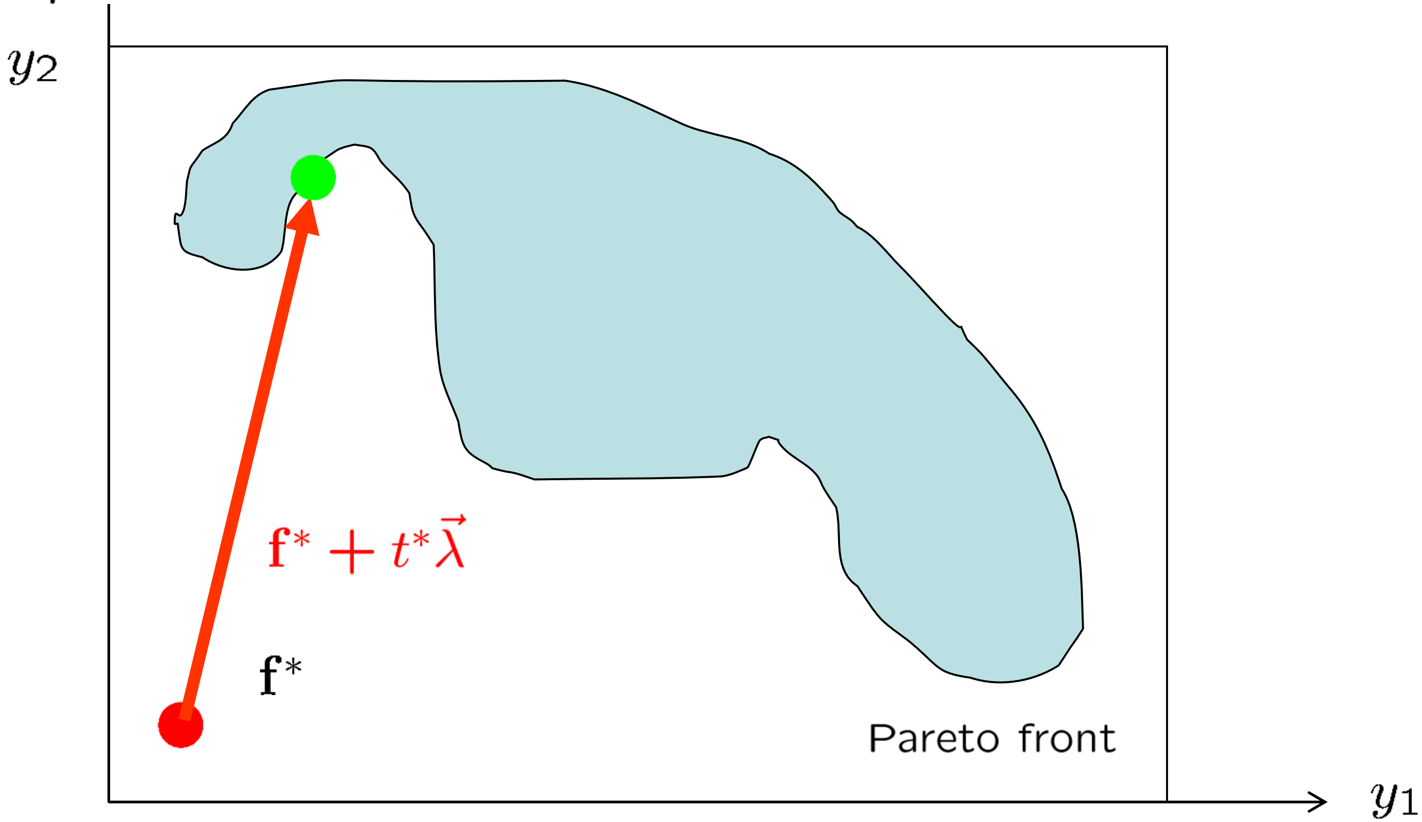
# Goal programming

If $\lambda_i > 0$ goal programming can obtain all properly efficient points!




Goal programming might result in dominated solutions!

This is also possible if reference point dominates ideal point!



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# Summary:Scalarization methods

Scalarization methods can obtain Pareto optimal solutions.

Except Tschebyscheff scalarization the methods cannot find all proper efficient solutions, , in particular concave parts are easily overseen!

The goal attainment method may even find non-efficient points

The weights $w_i$ and $\epsilon$-constants have different meaning, the understanding of which is essential to the understanding of the respective method.

Finding tangential points of the $f_{eq} \equiv C$ isolines gives us a practical means for geometrically determining the solution of the monocriterial functions.



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