P&S QFT - Chapter 2 problems

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2.1

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \tag{1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

(a) Euler-Lagrange equations are:

$$\frac{\partial L}{\partial A_\nu} - \partial_\mu \left(\frac{\partial L}{\partial \left(\partial_\mu A_\nu \right)} \right) = 0 \tag{3}$$

Thus:

$$\partial_\mu F^{\mu\nu} = 0 \tag{4}$$

Putting $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$:

$$0 = \partial_\mu F^{\mu 0} = \partial_i E^i = \nabla \cdot \mathbf{E} \tag{5}$$

$$0 = -\partial_\mu F^{\mu j} = -\partial_0 F^{0j} + \partial_i F^{ij} = \frac{\partial}{\partial t} \mathbf{E}^j - (\nabla \times \mathbf{B})^j \tag{6}$$

Putting it all together:

$$\nabla \cdot \mathbf{E} = 0 \tag{7}$$

$$\nabla \times \mathbf{B} = \frac{\partial}{\partial t} \mathbf{E} \tag{8}$$

(b) From (2.17):

$$T^{\mu}_{\nu} \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}\partial_{\nu}\phi - \mathcal{L}\delta^{\mu}_{\nu} \tag{9}$$

Now putting our Lagrangian to use:

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}A_{\nu}\right)} = -F^{\mu\nu} \tag{10}$$

Non-symmetric term of the tensor is:

$$-F^{\mu\rho}\partial_{\nu}A_{\rho} = -F^{\mu\rho}F_{\nu\rho} - F^{\mu\rho}\partial_{\rho}A_{\nu} \tag{11}$$

Symmetrize:

$$\partial_{\lambda}K^{\lambda\mu\nu} = \partial_{\lambda}\left(F^{\mu\lambda}A^{\nu}\right) = F^{\mu\lambda}\partial_{\lambda}A^{\nu} \tag{12}$$

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_{\lambda}K^{\lambda\mu\nu} = -g_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} - \mathcal{L}g^{\mu\nu} \tag{13}$$

Which is obviously symmetric now! Working out some particular components:

$$\hat{T}^{00} = -g_{\rho\sigma}F^{0\rho}F^{0\sigma} - \mathcal{L} = F^{0i}F^{0i} - \mathcal{L} = |\mathbf{E}|^2 - \mathcal{L} \tag{14}$$

Let's see what $\mathcal{L}$ is:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = +\frac{1}{2}F^{0i}F^{0i} - \frac{1}{2}F^{ij}F^{ij} = \frac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right) \tag{15}$$

Finally:

$$\hat{T}^{00} = \frac{1}{2}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right) \tag{16}$$

Now about spatial components:

$$\hat{T}^{0i} = -g_{\rho\sigma}F^{0\rho}F^{i\sigma} = F^{0j}F^{ij} = E^{j}\epsilon_{ijk}B^{k} = \mathbf{E}\times\mathbf{B} \tag{17}$$

2.2

$$S = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right) \tag{18}$$

(a) Conjugate momenta to $\phi(x)$ and $\phi^*(x)$ are of course:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \left(\partial_0 \phi(x) \right)} = \partial^0 \phi^*(x) \tag{19}$$

$$\pi^*(x) = \frac{\partial \mathcal{L}}{\partial \left(\partial_0 \phi^*(x) \right)} = \partial^0 \phi(x) \tag{20}$$

Canonical commutation relations:

$$[\phi(x), \pi(y)] \equiv i\delta^4(x-y) \tag{21}$$

$$[\phi^*(x), \pi^*(y)] \equiv i\delta^4(x-y) \tag{22}$$

(All other commutators between ϕ, $\phi*$, π, and $\pi*$ are zero).
Hamiltonian is:

$$\begin{aligned} H &= \int d^3x \; \pi \partial_0 \phi + \pi^* \partial_0 \phi^8 - \mathcal{L} \\ &= \int d^3x \; 2\pi\pi^* - \partial_0 \phi^* \partial^0 \phi + \partial_i \phi^* \partial^i \phi + m^2 \phi^* \phi \\ &= \int d^3x \; \pi\pi^* + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \end{aligned} \tag{23}$$

Heisenberg equation of motion:

$$\frac{\partial \phi}{\partial t} = i[H, \phi] = \pi^* \tag{24}$$

$$\begin{aligned} \frac{\partial^2 \phi}{\partial t^2} &= i[H, \frac{\partial \phi}{\partial t}] = i[H, \pi^*] \\ &= i \left[\int d^3x \; \nabla \phi^*(x) \cdot \nabla \phi(x) + m^2 \phi^*(x) \phi(x), \pi^*(y) \right] \\ &= i \left[\int d^3x \; -\phi^*(x) \cdot \nabla^2 \phi(x), \pi^*(y) \right] - m^2 \phi(y) \\ &= +\nabla^2 \phi(y) - m^2 \phi(y) \end{aligned} \tag{25}$$

Finally we get:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = 0 \tag{26}$$

Which is of course is the Klein-Gordon equation.

(b)