P&S QFT - Chapter 2 problems

Fedor Indutny

August 11, 2017

2.1

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \tag{1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

(a) Euler-Lagrange equations are:

$$\frac{\partial L}{\partial A_\nu} - \partial_\mu \left(\frac{\partial L}{\partial \left(\partial_\mu A_\nu \right)} \right) = 0 \tag{3}$$

Thus:

$$\partial_\mu F^{\mu\nu} = 0 \tag{4}$$

Putting $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$:

$$0 = \partial_\mu F^{\mu 0} = \partial_i E^i = \nabla \cdot \mathbf{E} \tag{5}$$

$$0 = -\partial_\mu F^{\mu j} = -\partial_0 F^{0j} + \partial_i F^{ij} = \frac{\partial}{\partial t} \mathbf{E}^j - (\nabla \times \mathbf{B})^j \tag{6}$$

Putting it all together:

$$\nabla \cdot \mathbf{E} = 0 \tag{7}$$

$$\nabla \times \mathbf{B} = \frac{\partial}{\partial t} \mathbf{E} \tag{8}$$

(b) From (2.17):

$$T^{\mu}_{\nu} \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\phi\right)}\partial_{\nu}\phi - \mathcal{L}\delta^{\mu}_{\nu} \tag{9}$$

Now putting our Lagrangian to use:

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}A_{\nu}\right)} = -F^{\mu\nu} \tag{10}$$

Non-symmetric term of the tensor is:

$$-F^{\mu\rho}\partial_{\nu}A_{\rho} = -F^{\mu\rho}F_{\nu\rho} - F^{\mu\rho}\partial_{\rho}A_{\nu} \tag{11}$$

Symmetrize:

$$\partial_{\lambda}K^{\lambda\mu\nu} = \partial_{\lambda}\left(F^{\mu\lambda}A^{\nu}\right) = F^{\mu\lambda}\partial_{\lambda}A^{\nu} \tag{12}$$

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_{\lambda}K^{\lambda\mu\nu} = -g_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} - \mathcal{L}g^{\mu\nu} \tag{13}$$

Which is obviously symmetric now! Working out some particular components:

$$\hat{T}^{00} = -g_{\rho\sigma}F^{0\rho}F^{0\sigma} - \mathcal{L} = F^{0i}F^{0i} - \mathcal{L} = |\mathbf{E}|^2 - \mathcal{L} \tag{14}$$

Let's see what $\mathcal{L}$ is:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = +\frac{1}{2}F^{0i}F^{0i} - \frac{1}{2}F^{ij}F^{ij} = \frac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right) \tag{15}$$

Finally:

$$\hat{T}^{00} = \frac{1}{2}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right) \tag{16}$$

Now about spatial components:

$$\hat{T}^{0i} = -g_{\rho\sigma}F^{0\rho}F^{i\sigma} = F^{0j}F^{ij} = E^{j}\epsilon_{ijk}B^{k} = \mathbf{E}\times\mathbf{B} \tag{17}$$

2.2

$$S = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi\right) \tag{18}$$

(a) Conjugate momenta to $\phi(x)$ and $\phi^*(x)$ are of course:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \left(\partial_0 \phi(x)\right)} = \partial^0 \phi^*(x) \tag{19}$$

$$\pi^*(x) = \frac{\partial \mathcal{L}}{\partial \left(\partial_0 \phi^*(x)\right)} = \partial^0 \phi(x) \tag{20}$$

Canonical commutation relations:

$$[\phi(x), \pi(y)] \equiv i\delta^4(x-y) \tag{21}$$

$$[\phi^*(x), \pi^*(y)] \equiv i\delta^4(x-y) \tag{22}$$

(All other commutators between ϕ, $\phi*$, π, and $\pi*$ are zero).

Hamiltonian is:

$$\begin{aligned} H &= \int d^3x \ \pi \partial_0 \phi + \pi^* \partial_0 \phi^8 - \mathcal{L} \\ &= \int d^3x \ 2\pi\pi^* - \partial_0 \phi^* \partial^0 \phi + \partial_i \phi^* \partial^i \phi + m^2 \phi^* \phi \\ &= \int d^3x \ \pi\pi^* + \nabla\phi^* \cdot \nabla\phi + m^2 \phi^* \phi \end{aligned} \tag{23}$$

Heisenberg equation of motion:

$$\frac{\partial \phi}{\partial t} = i[H, \phi] = \pi^* \tag{24}$$

$$\begin{aligned} \frac{\partial^2 \phi}{\partial t^2} &= i[H, \frac{\partial \phi}{\partial t}] = i[H, \pi^*] \\ &= i\left[\int d^3x \ \nabla\phi^*(x) \cdot \nabla\phi(x) + m^2 \phi^*(x)\phi(x), \pi^*(y)\right] \\ &= i\left[\int d^3x \ -\phi^*(x) \cdot \nabla^2 \phi(x), \pi^*(y)\right] - m^2 \phi(y) \\ &= +\nabla^2 \phi(y) - m^2 \phi(y) \end{aligned} \tag{25}$$

Finally we get:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = 0 \tag{26}$$

Which is of course is the Klein-Gordon equation.

(b) Introduce creation and annihilation operators expansion for the fields:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x} + b^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)\Big|_{p^0=E_{\mathbf{p}}} \tag{27}$$

$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(b_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)\Big|_{p^0=E_{\mathbf{p}}} \tag{28}$$

$$\pi^*(x) = \frac{\partial}{\partial t}\phi(x);\ \pi(x) = \frac{\partial}{\partial t}\phi^*(x) \tag{29}$$

With usual commutation relations:

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] = [b_{\mathbf{p}}, b^\dagger_{\mathbf{p}'}] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}') \tag{30}$$

(The rest of commutators are zero).

Let's check canonical commutation relations between momenta and the fields (skipping the resolution of the momentum delta function, and assuming $x^0 = y^0$):

$$\begin{aligned}[\phi(\mathbf{x}), \pi(\mathbf{y})] &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}iE_p\left(e^{ip\cdot(x-y)} + e^{ip\cdot(y-x)}\right) \\ &= i\delta^3(\mathbf{x}-\mathbf{y})\end{aligned} \tag{31}$$

(Similar result holds for ϕ^* and $\pi*$ too).

(c) Skipping few infinities and canceling the terms where $p' = -p$:

$$\begin{aligned}Q &= \int d^3x\frac{i}{2}\left(\phi^*\pi^* - \pi\phi\right) \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\left(-b^\dagger_{\mathbf{p}}b_{\mathbf{p}} + a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\right)\end{aligned} \tag{32}$$

The charges are $\frac{1}{2}$ and $-\frac{1}{2}$ for a and b respectively.

(d) To see if the charge Q^i is conserved, let's compute the commutator with H:

$$H = \sum_{a=1,2} \int d^3x \ \pi_a \pi_a^* + \nabla \phi_a^* \cdot \nabla \phi_a + m^2 \phi_a^* \phi_a \tag{33}$$

$$Q^i = \int d^3x \frac{i}{2} \left(\phi_a^* \left(\sigma^i\right)_{ab} \pi_b^* - \pi_a \left(\sigma^i\right)_{ab} \phi_b \right) \tag{34}$$

$$\frac{\partial}{\partial t} Q^i = i[H, Q^i] \tag{35}$$

Now for commutators:

$$\int [\pi_c \pi_c^*, \ \phi_a^* \left(\sigma^i\right)_{ab} \pi_b^*] = \delta^{ac} \pi_c \left(\sigma^i\right)_{ab} \pi_b^* = -i \pi_a \left(\sigma^i\right)_{ab} \pi_b^* \tag{36}$$

$$\int [\pi_c \pi_c^*, \ \pi_a \left(\sigma^i\right)_{ab} \phi_b] = -i \pi_a \left(\sigma^i\right)_{ab} \pi_b^* \tag{37}$$

These are precisely the same and thus cancel each other. Next:

$$[\nabla \phi_c^* \cdot \nabla \phi_c, \ \phi_a^* \left(\sigma^i\right)_{ab} \pi_b^*] = -i \phi_a^* \left(\sigma^i\right)_{ab} \nabla^2 \phi_b \tag{38}$$

(...and the same term for conjugated part of Q).

Finally:

$$[m^2 \phi_c^* \phi_c, \ \phi_a^* \left(\sigma^i\right)_{ab} \pi_b^*] = i m^2 \phi_a^* \left(\sigma^i\right)_{ab} \phi_b \tag{39}$$

(...and the same term for conjugated part of Q).

Everything cancels out perfectly and we get:

$$\frac{\partial}{\partial t} Q^i = 0 \tag{40}$$

Some intermediate steps before working out commutators of charges:

$$\begin{aligned}
&[\phi_a^* \left(\sigma^i\right)_{ab} \pi_b^*, \phi_c^* \left(\sigma^j\right)_{cd} \pi_d^*] = \\
&\quad \phi_a^* \left(\sigma^i\right)_{ab} \pi_b^* \phi_c^* \left(\sigma^j\right)_{cd} \pi_d^* - \phi_c^* \left(\sigma^j\right)_{cd} \pi_d^* \phi_a^* \left(\sigma^i\right)_{ab} \pi_b^* = \\
&\quad \phi_a^* \left(\sigma^i\right)_{ab} \pi_b^* \phi_c^* \left(\sigma^j\right)_{cd} \pi_d^* + i \phi_c^* \left(\sigma^j \sigma^i\right)_{cb} \pi_b^* - \\
&\qquad i \phi_a^* \left(\sigma^j\right)_{cd} \phi_c^* \pi_b^* \left(\sigma^i\right)_{ab} \pi_d^* = \\
&\quad \phi_c^* \left([\sigma^j, \sigma^i]\right)_{cb} \pi_b^* = 2\epsilon_{ijk} \phi_a^* \left(\sigma^k\right)_{ab} \phi_b^*
\end{aligned} \tag{41}$$

The other term is a conjugate of this. Thus we get:

$$[Q^i, Q^j] = 2\epsilon_{ijk} Q^k \tag{42}$$

Which means that the commutation relations of Q^i are the same as of generators of Lie Algebra $\mathfrak{su}(2)$!

Generalizing this to n complex scalar fields we see that one can form charges using the generators of $\mathfrak{su}(n)$ Lie algebra.

(2.3)

Given $(x-y)^2 = -r^2$ we can apply Lorentz transformation to make $x^3 - y^3 = r$, and the rest of the components zero. Thus:

$$\begin{aligned}
&\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\sqrt{\mathbf{p}^2 + m^2}} e^{ip^3 r} = \\
&\int_0^\pi d\theta \int_0^\infty dq\, \frac{q^2 \sin\theta}{(2\pi)^2} \frac{1}{2\sqrt{q^2+m^2}} e^{iqr\cos\theta} = \\
&\int_0^\infty dq\, \frac{q^2}{(2\pi)^2} \frac{1}{2\sqrt{q^2+m^2}} \int_{-1}^1 d\cos\theta\; e^{iqr\cos\theta} = \\
&\int_0^\infty dq\, \frac{q^2}{(2\pi)^2} \frac{1}{2\sqrt{q^2+m^2}} \frac{1}{iqr} \left(e^{iqr} - e^{-iqr}\right) = \\
&\frac{-i}{2(2\pi)^2 r} \int_{-\infty}^{+\infty} dq\, \frac{q e^{iqr}}{\sqrt{q^2+m^2}} = -\frac{m}{(2\pi)^2 r} K_0'(rm)
\end{aligned} \tag{43}$$