P&S QFT - Chapter 3 problems

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3.1

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right) \tag{1}$$

(a)

$$L^i = \tfrac{1}{2}\epsilon^{ijk}J^{jk},\; K^i = J^{0i}$$

$$\begin{aligned}
[L^i, L^j] &= \frac{1}{4}\epsilon^{ikl}\epsilon^{jmn}[J^{kl}, J^{mn}] \\
&= -\frac{i}{4}\epsilon^{ikl}\epsilon^{jmn}\left(\delta^{lm}J^{kn} - \delta^{km}J^{ln} - \delta^{ln}J^{km} + \delta^{kn}J^{lm}\right) \\
&= \textit{(...renaming summation variables...)} \\
&= -\frac{i}{4}\left(\epsilon^{ikl}\epsilon^{jln} - \epsilon^{ilk}\epsilon^{jln} - \epsilon^{ikl}\epsilon^{jnl} + \epsilon^{ilk}\epsilon^{jnl}\right)J^{kn} \\
&= -i\epsilon^{ikl}\epsilon^{jln}J^{kn} = -i\epsilon^{lik}\epsilon^{lnj}J^{kn} = -i\left(\delta^{in}\delta^{kj} - \delta^{ij}\delta^{kn}\right)J^{kn} \\
&= -i\left(J^{ji} - \delta^{ij}J^{kk}\right) = iJ^{ij}
\end{aligned} \tag{2}$$

Now:

$$\epsilon^{ijk}L^k = \frac{1}{2}\epsilon^{ijk}\epsilon^{klm}J^{lm} = \frac{1}{2}\left(\delta^{il}\delta^{jm} - \delta^{im}\delta^{lj}\right)J^{lm} = J^{ij} \tag{3}$$

Thus:

$$[L^i, L^j] = i\epsilon^{ijk}L^k \tag{4}$$

Commutator of boost and rotation:

$$
\begin{aligned}
[L^i, K^j] &= \frac{1}{2}\epsilon^{ikl}[J^{kl}, J^{0j}] = \frac{i}{2}\epsilon^{ikl}\left(g^{l0}J^{kj} - g^{k0}J^{lj} - g^{lj}J^{k0} + g^{kj}J^{l0}\right) \\
&= \frac{i}{2}\epsilon^{ikl}\left(g^{lj}K^k - g^{kj}K^l\right) = -\frac{i}{2}\left(\epsilon^{ikj}K^k - \epsilon^{ijl}K^l\right) \\
&= i\epsilon^{ijk}K^k
\end{aligned} \tag{5}
$$

Commutator of two boosts:

$$
[K^i, K^j] = [J^{0i}, J^{0j}] = -iJ^{ij} = -i\epsilon^{ijk}L^k \tag{6}
$$

Combinations:

$$
\mathbf{J}_+ = \frac{1}{2}\left(\mathbf{L} + i\mathbf{K}\right) \tag{7}
$$

$$
\mathbf{J}_- = \frac{1}{2}\left(\mathbf{L} - i\mathbf{K}\right) \tag{8}
$$

Commutator of combinations:

$$
\begin{aligned}
[J_+^i, J_-^j] &= \frac{1}{4}[L^i + iK^i, L^j - iK^j] \\
&= \frac{1}{4}\left([L^i, L^j] + [K^i, K^j] + i[K^i, L^j] + i[K^j, L^i]\right) \\
&= \frac{1}{4}\left(i\epsilon^{ijk}L^k - i\epsilon^{ijk}L^k + i\epsilon^{ijk}K^k - i\epsilon^{ijk}K^k\right) = 0
\end{aligned} \tag{9}
$$

Commutator of the components of J_+:

$$
\begin{aligned}
[J_+^i, J_+^j] &= \frac{1}{4}\left([L^i, L^j] - [K^i, K^j] + i[K^i, L^j] - i[K^j, L^i]\right) \\
&= \frac{i}{4}\left(\epsilon^{ijk}L^k + \epsilon^{ijk}L^k + i\epsilon^{ijk}K^k + i\epsilon^{ijk}K^k\right) \\
&= i\epsilon^{ijk}J_+^k
\end{aligned} \tag{10}
$$

Similar result holds for $\mathbf{J}_-$.

(b) For spin $\frac{1}{2}$ representations $\mathbf{J} = \frac{\boldsymbol{\sigma}}{2}$. Since $\mathbf{J}_+$ and $\mathbf{J}_-$ commute the states in the Lorentz group representation naturally decouple into a tensor product of eigenstates of $\mathbf{J}_+$ and $\mathbf{J}_-$.

For $(j_+, j_-) = (\frac{1}{2}, 0)$ representation:

$$\mathbf{L} = \mathbf{J}_+ + \mathbf{J}_- = \frac{\boldsymbol{\sigma}}{2} \tag{11}$$

$$\mathbf{K} = \frac{1}{i}\left(\mathbf{J}_+ - \mathbf{J}_-\right) = \frac{\boldsymbol{\sigma}}{2i} \tag{12}$$

Thus $(\frac{1}{2}, 0)$ state transforms as:

$$\psi_L \to \left(1 - i\boldsymbol{\theta} \cdot \frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta} \cdot \frac{\boldsymbol{\sigma}}{2}\right) \psi_L \tag{13}$$

While $(0, \frac{1}{2})$ state transforms as:

$$\psi_R \to \left(1 - i\boldsymbol{\theta} \cdot \frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta} \cdot \frac{\boldsymbol{\sigma}}{2}\right) \psi_R \tag{14}$$

(c)

$$\boldsymbol{\sigma}^T = -\sigma^2 \boldsymbol{\sigma} \sigma^2 \tag{15}$$

The $(\frac{1}{2}, \frac{1}{2})$ matrix representation is essentially a dot product of the vector and σ:

$$\begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix} = V^0 + \mathbf{V} \cdot \boldsymbol{\sigma} \tag{16}$$

To ease calculations define:

$$\boldsymbol{\tau} \equiv \frac{1}{2}\left(\boldsymbol{\beta} - i\boldsymbol{\theta}\right) \tag{17}$$

Applying right transformation on the left:

$$\left(1 + \boldsymbol{\tau} \cdot \boldsymbol{\sigma}\right)\left(V^0 + \mathbf{V} \cdot \boldsymbol{\sigma}\right) = \left(V^0 + \boldsymbol{\tau} \cdot \mathbf{V}\right) + \left(V^0 \boldsymbol{\tau} + \mathbf{V} + i\boldsymbol{\tau} \times \mathbf{V}\right) \cdot \boldsymbol{\sigma} \tag{18}$$

Applying transposed left transformation on the right:

$$\begin{aligned} &\left(\left(V^0 + \boldsymbol{\tau} \cdot \mathbf{V}\right) + \left(V^0 \boldsymbol{\tau} + \mathbf{V} + i\boldsymbol{\tau} \times \mathbf{V}\right) \cdot \boldsymbol{\sigma}\right)\left(1 + \sigma^2 \boldsymbol{\tau}^* \cdot \boldsymbol{\sigma} \sigma^2\right) = \\ &\left(V^0 + \boldsymbol{\tau} \cdot \mathbf{V}\right) \end{aligned} \tag{19}$$