

## P&S QFT - Chapter 2 problems

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### 2.1

$$S = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \quad (1)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (2)$$

(a) Euler-Lagrange equations are:

$$\frac{\partial L}{\partial A_\nu} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu A_\nu)} \right) = 0 \quad (3)$$

Thus:

$$\partial_\mu F^{\mu\nu} = 0 \quad (4)$$

Putting  $E^i = -F^{0i}$  and  $\epsilon^{ijk} B^k = -F^{ij}$ :

$$0 = \partial_\mu F^{\mu 0} = \partial_i E^i = \nabla \cdot \mathbf{E} \quad (5)$$

$$0 = -\partial_\mu F^{\mu j} = -\partial_0 F^{0j} + \partial_i F^{ij} = \frac{\partial}{\partial t} \mathbf{E}^j - (\nabla \times \mathbf{B})^j \quad (6)$$

Putting it all together:

$$\nabla \cdot \mathbf{E} = 0 \quad (7)$$

$$\nabla \times \mathbf{B} = \frac{\partial}{\partial t} \mathbf{E} \quad (8)$$

(b) From (2.17):

$$T_\nu^\mu \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta_\nu^\mu \quad (9)$$

Now putting our Lagrangian to use:

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} = -F^{\mu\nu} \quad (10)$$

Non-symmetric term of the tensor is:

$$-F^{\mu\rho} \partial_\nu A_\rho = -F^{\mu\rho} F_{\nu\rho} - F^{\mu\rho} \partial_\rho A_\nu \quad (11)$$

Symmetrize:

$$\partial_\lambda K^{\lambda\mu\nu} = \partial_\lambda (F^{\mu\lambda} A^\nu) = F^{\mu\lambda} \partial_\lambda A^\nu \quad (12)$$

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} = -g_{\rho\sigma} F^{\mu\rho} F^{\nu\sigma} - \mathcal{L} g^{\mu\nu} \quad (13)$$

Which is obviously symmetric now! Working out some particular components:

$$\hat{T}^{00} = -g_{\rho\sigma} F^{0\rho} F^{0\sigma} - \mathcal{L} = F^{0i} F^{0i} - \mathcal{L} = |\mathbf{E}|^2 - \mathcal{L} \quad (14)$$

Let's see what  $\mathcal{L}$  is:

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = +\frac{1}{2} F^{0i} F^{0i} - \frac{1}{2} F^{ij} F^{ij} = \frac{1}{2} (|\mathbf{E}|^2 - |\mathbf{B}|^2) \quad (15)$$

Finally:

$$\hat{T}^{00} = \frac{1}{2} (|\mathbf{E}|^2 + |\mathbf{B}|^2) \quad (16)$$

Now about spatial components:

$$\hat{T}^{0i} = -g_{\rho\sigma} F^{0\rho} F^{i\sigma} = F^{0j} F^{ij} = E^j \epsilon_{ijk} B^k = \mathbf{E} \times \mathbf{B} \quad (17)$$

## 2.2

$$S = \int d^4x (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi) \quad (18)$$

(a) Conjugate momenta to  $\phi(x)$  and  $\phi^*(x)$  are of course:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi(x))} = \partial^0 \phi^*(x) \quad (19)$$

$$\pi^*(x) = \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi^*(x))} = \partial^0 \phi(x) \quad (20)$$

Canonical commutation relations:

$$[\phi(x), \pi(y)] \equiv i\delta^4(x - y) \quad (21)$$

$$[\phi^*(x), \pi^*(y)] \equiv i\delta^4(x - y) \quad (22)$$

(All other commutators between  $\phi$ ,  $\phi^*$ ,  $\pi$ , and  $\pi^*$  are zero).

Hamiltonian is:

$$\begin{aligned} H &= \int d^3x \pi \partial_0 \phi + \pi^* \partial_0 \phi^* - \mathcal{L} \\ &= \int d^3x 2\pi\pi^* - \partial_0 \phi^* \partial^0 \phi + \partial_i \phi^* \partial^i \phi + m^2 \phi^* \phi \\ &= \int d^3x \pi \pi^* + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \end{aligned} \quad (23)$$

Heisenberg equation of motion:

$$\frac{\partial \phi}{\partial t} = i[H, \phi] = \pi^* \quad (24)$$

$$\begin{aligned} \frac{\partial^2 \phi}{\partial t^2} &= i[H, \frac{\partial \phi}{\partial t}] = i[H, \pi^*] \\ &= i \left[ \int d^3x \nabla \phi^*(x) \cdot \nabla \phi(x) + m^2 \phi^*(x) \phi(x), \pi^*(y) \right] \\ &= i \left[ \int d^3x -\phi^*(x) \cdot \nabla^2 \phi(x), \pi^*(y) \right] - m^2 \phi(y) \\ &= +\nabla^2 \phi(y) - m^2 \phi(y) \end{aligned} \quad (25)$$

Finally we get:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = 0 \quad (26)$$

Which is of course is the Klein-Gordon equation.

(b) Introduce creation and annihilation operators expansion for the fields:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left( a_{\mathbf{p}} e^{-ip \cdot x} + b_{\mathbf{p}}^\dagger e^{ip \cdot x} \right) \Big|_{p^0=E_{\mathbf{p}}} \quad (27)$$

$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left( b_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x} \right) \Big|_{p^0=E_{\mathbf{p}}} \quad (28)$$

$$\pi^*(x) = \frac{\partial}{\partial t} \phi(x); \quad \pi(x) = \frac{\partial}{\partial t} \phi^*(x) \quad (29)$$

With usual commutation relations:

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = [b_{\mathbf{p}}, b_{\mathbf{p}'}^\dagger] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \quad (30)$$

(The rest of commutators are zero).

Let's check canonical commutation relations between momenta and the fields (skipping the resolution of the momentum delta function, and assuming  $x^0 = y^0$ ):

$$\begin{aligned} [\phi(\mathbf{x}), \pi(\mathbf{y})] &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} iE_p \left( e^{ip \cdot (x-y)} + e^{ip \cdot (y-x)} \right) \\ &= i\delta^3(\mathbf{x} - \mathbf{y}) \end{aligned} \quad (31)$$

(Similar result holds for  $\phi^*$  and  $\pi^*$  too).

(c) Skipping few infinities and canceling the terms where  $p' = -p$ :

$$\begin{aligned} Q &= \int d^3x \frac{i}{2} (\phi^* \pi^* - \pi \phi) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \left( -b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \right) \end{aligned} \quad (32)$$

The charges are  $\frac{1}{2}$  and  $-\frac{1}{2}$  for  $a$  and  $b$  respectively.

(d) To see if the charge  $Q^i$  is conserved, let's compute the commutator with  $H$ :

$$H = \sum_{a=1,2} \int d^3x \pi_a \pi_a^* + \nabla \phi_a^* \cdot \nabla \phi_a + m^2 \phi_a^* \phi_a \quad (33)$$

$$Q^i = \int d^3x \frac{i}{2} (\phi_a^* (\sigma^i)_{ab} \pi_b^* - \pi_a (\sigma^i)_{ab} \phi_b) \quad (34)$$

$$\frac{\partial}{\partial t} Q^i = i[H, Q^i] \quad (35)$$

Now for commutators:

$$\int [\pi_c \pi_c^*, \phi_a^* (\sigma^i)_{ab} \pi_b^*] = \delta^{ac} \pi_c (\sigma^i)_{ab} \pi_b^* = -i \pi_a (\sigma^i)_{ab} \pi_b^* \quad (36)$$

$$\int [\pi_c \pi_c^*, \pi_a (\sigma^i)_{ab} \phi_b] = -i \pi_a (\sigma^i)_{ab} \pi_b^* \quad (37)$$

These are precisely the same and thus cancel each other. Next:

$$[\nabla \phi_c^* \cdot \nabla \phi_c, \phi_a^* (\sigma^i)_{ab} \pi_b^*] = -i \phi_a^* (\sigma^i)_{ab} \nabla^2 \phi_b \quad (38)$$

(...and the same term for conjugated part of  $Q$ ).

Finally:

$$[m^2 \phi_c^* \phi_c, \phi_a^* (\sigma^i)_{ab} \pi_b^*] = i m^2 \phi_a^* (\sigma^i)_{ab} \phi_b \quad (39)$$

(...and the same term for conjugated part of  $Q$ ).

Everything cancels out perfectly and we get:

$$\frac{\partial}{\partial t} Q^i = 0 \quad (40)$$

Some intermediate steps before working out commutators of charges:

$$\begin{aligned} & [\phi_a^* (\sigma^i)_{ab} \pi_b^*, \phi_c^* (\sigma^j)_{cd} \pi_d^*] = \\ & \phi_a^* (\sigma^i)_{ab} \pi_b^* \phi_c^* (\sigma^j)_{cd} \pi_d^* - \phi_c^* (\sigma^j)_{cd} \pi_d^* \phi_a^* (\sigma^i)_{ab} \pi_b^* = \\ & \phi_a^* (\sigma^i)_{ab} \pi_b^* \phi_c^* (\sigma^j)_{cd} \pi_d^* + i \phi_c^* (\sigma^j \sigma^i)_{cb} \pi_b^* - \\ & i \phi_a^* (\sigma^j)_{cd} \phi_c^* \pi_b^* (\sigma^i)_{ab} \pi_d^* = \\ & \phi_c^* ([\sigma^j, \sigma^i])_{cb} \pi_b^* = 2 \epsilon_{ijk} \phi_a^* (\sigma^k)_{ab} \phi_b^* \end{aligned} \quad (41)$$

The other term is a conjugate of this. Thus we get:

$$[Q^i, Q^j] = 2\epsilon_{ijk}Q^k \quad (42)$$

Which means that the commutation relations of  $Q^i$  are the same as of generators of Lie Algebra  $su(2)$ !