P&S QFT - Chapter 3 problems

Fedor Indutny

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3.1

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right) \tag{1}$$

(a)

$$L^i = \tfrac{1}{2}\epsilon^{ijk}J^{jk},\ K^i = J^{0i}$$

$$\begin{aligned}
[L^i, L^j] &= \frac{1}{4}\epsilon^{ikl}\epsilon^{jmn}[J^{kl}, J^{mn}] \\
&= -\frac{i}{4}\epsilon^{ikl}\epsilon^{jmn}\left(\delta^{lm}J^{kn} - \delta^{km}J^{ln} - \delta^{ln}J^{km} + \delta^{kn}J^{lm}\right) \\
&= \textit{(...renaming summation variables...)} \\
&= -\frac{i}{4}\left(\epsilon^{ikl}\epsilon^{jln} - \epsilon^{ilk}\epsilon^{jln} - \epsilon^{ikl}\epsilon^{jnl} + \epsilon^{ilk}\epsilon^{jnl}\right)J^{kn} \\
&= -i\epsilon^{ikl}\epsilon^{jln}J^{kn} = -i\epsilon^{lik}\epsilon^{lnj}J^{kn} = -i\left(\delta^{in}\delta^{kj} - \delta^{ij}\delta^{kn}\right)J^{kn} \\
&= -i\left(J^{ji} - \delta^{ij}J^{kk}\right) = iJ^{ij}
\end{aligned} \tag{2}$$

Now:

$$\epsilon^{ijk}L^k = \frac{1}{2}\epsilon^{ijk}\epsilon^{klm}J^{lm} = \frac{1}{2}\left(\delta^{il}\delta^{jm} - \delta^{im}\delta^{lj}\right)J^{lm} = J^{ij} \tag{3}$$

Thus:

$$[L^i, L^j] = i\epsilon_{ijk}L^k \tag{4}$$

Commutator of boost and rotation:

$$
\begin{aligned}
[L^i, K^j] &= \frac{1}{2}\epsilon^{ikl}[J^{kl}, J^{0j}] = \frac{i}{2}\epsilon^{ikl}\left(g^{l0}J^{kj} - g^{k0}J^{lj} - g^{lj}J^{k0} + g^{kj}J^{l0}\right) \\
&= \frac{i}{2}\epsilon^{ikl}\left(g^{lj}K^k - g^{kj}K^l\right) = -\frac{i}{2}\left(\epsilon^{ikj}K^k - \epsilon^{ijl}K^l\right) \\
&= i\epsilon^{ijk}K^k
\end{aligned}
\tag{5}
$$

Commutator of two boosts:

$$
[K^i, K^j] = [J^{0i}, J^{0j}] = -iJ^{ij} = -i\epsilon^{ijk}L^k \tag{6}
$$