P&S QFT - Problem 5.2

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1 Solution

There are two diagrams that contribute at the first order: t-channel, and s-channel. Due to fermion exchange the signs of t-channel diagram is different from s-'channel's. Scattering amplitude is thus *(spinor indices omitted)*:

$$\begin{aligned} \mathcal{M} = &\frac{ie^2}{t}\overline{u}(p')\gamma^\mu u(p)\overline{v}(k)\gamma_\mu v(k') \\ &- \frac{ie^2}{s}\overline{u}(p')\gamma^\mu v(k')\overline{v}(k)\gamma_\mu u(p) \end{aligned} \tag{1}$$

Squaring $\mathcal{M}$ and averaging/summing over spin indices we will get four terms:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{4}\Big(\frac{(I)}{t^2} - \frac{(II)}{ts} - \frac{(III)}{ts} + \frac{(IV)}{s^2}\Big) \tag{2}$$

We'll express this in terms of Mandelstam variables, so it should be useful to evaluate them first (taking mass of electron to be 0):

$$\begin{aligned} s &= (p+k)^2 = (p'+k')^2 \approx 2p\cdot k = 2p'\cdot k' \\ t &= (p'-p)^2 = (k'-k)^2 \approx -2p\cdot p' = -2k\cdot k' \\ u &= (k'-p)^2 = (p'-k)^2 \approx -2p\cdot k' = -2p'\cdot k \end{aligned} \tag{3}$$

First term *(t-channel squared)*:

$$
\begin{aligned}
(I) &= \sum \overline{u}(p')\gamma^\mu u(p)\overline{v}(k)\gamma_\mu v(k') \cdot \overline{v}(k')\gamma^\nu v(k)\overline{u}(p)\gamma_\nu u(p') \\
&= \sum \Big(\overline{u}(p')\gamma^\mu u(p)\overline{u}(p)\gamma_\nu u(p')\Big) \cdot \Big(\overline{v}(k)\gamma_\mu v(k')\overline{v}(k')\gamma^\nu v(k)\Big) \\
&= tr\Big[\not{p}'\gamma^\mu\not{p}\gamma^\nu\Big] tr\Big[\not{k}\gamma_\mu\not{k}'\gamma_\nu\Big]
\end{aligned}
\tag{4}
$$

This trace is going to be used for (IV), let's compute it once and for both:

$$
\begin{aligned}
tr\Big[\not{p}'\gamma^\mu\not{p}\gamma^\nu\Big] &= p'_\rho p_\sigma tr(\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu) \\
&= 4p'_\rho p_\sigma(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}) \\
&= 4(p^\mu p'^\nu + p^\nu p'^\mu - p\cdot p' g^{\mu\nu})
\end{aligned}
\tag{5}
$$

Thus (I) becomes:

$$
\begin{aligned}
(I) &= 16(p^\mu p'^\nu + p^\nu p'^\mu - p\cdot p' g^{\mu\nu})(k_\mu k'_\nu + k_\nu k'_\mu - k\cdot k' g_{\mu\nu}) \\
&= 16(2p\cdot k p'\cdot k' + 2p\cdot k' p'\cdot k - 4p\cdot p' k\cdot k' + 4p\cdot p' k\cdot k') \\
&= 32(p\cdot k p'\cdot k' + p\cdot k' p'\cdot k) \\
&= 8(s^2 + u^2)
\end{aligned}
\tag{6}
$$

Let's expand (II) *(t- and s- channels)*:

$$
\begin{aligned}
(II) &= \sum \overline{u}(p')\gamma^\mu u(p)\overline{v}(k)\gamma_\mu v(k') \cdot \overline{u}(p)\gamma^\nu v(k)\overline{v}(k')\gamma_\nu u(p') \\
&= \sum \overline{u}(p')\gamma^\mu u(p)\overline{u}(p)\gamma^\nu v(k)\overline{v}(k)\gamma_\mu v(k')\overline{v}(k')\gamma_\nu u(p') \\
&= tr\Big[\not{p}'\gamma^\mu\not{p}\gamma^\nu\not{k}\gamma_\mu\not{k}'\gamma_\nu\Big] \\
&= \textit{(After symbolic computation)} - 32p\cdot k' p'\cdot k \\
&= -8u^2
\end{aligned}
\tag{7}
$$

s- and t- channels:

$$\begin{aligned}(III) &= \sum \overline{u}(p')\gamma^\mu v(k')\overline{v}(k)\gamma_\mu u(p) \cdot \overline{v}(k')\gamma^\nu v(k)\overline{u}(p)\gamma_\nu u(p') \\ &= \sum \overline{u}(p')\gamma^\mu v(k')\overline{v}(k')\gamma^\nu v(k)\overline{v}(k)\gamma_\mu u(p)\overline{u}(p)\gamma_\nu u(p') \\ &= tr\Big[\not{p}'\gamma^\mu \not{k}'\gamma^\nu \not{k}\gamma_\mu \not{p}\gamma_\nu\Big] \\ &= \textit{(After symbolic computation)} - 32p\cdot k'p'\cdot k \\ &= -8u^2\end{aligned} \tag{8}$$

s- channel squared:

$$\begin{aligned}(IV) &= \sum \overline{u}(p')\gamma^\mu v(k')\overline{v}(k)\gamma_\mu u(p) \cdot \overline{u}(p)\gamma^\nu v(k)\overline{v}(k')\gamma_\nu u(p') \\ &= \sum \Big(\overline{u}(p')\gamma^\mu v(k')\overline{v}(k')\gamma_\nu u(p')\Big) \cdot \Big(\overline{v}(k)\gamma_\mu u(p)\overline{u}(p)\gamma^\nu v(k)\Big) \\ &= tr\Big[\not{p}'\gamma^\mu \not{k}'\gamma_\nu\Big] tr\Big[\not{k}\gamma_\mu \not{p}\gamma^\nu\Big] \\ &= 32(p\cdot p'k\cdot k' + p\cdot k'p'\cdot k) \\ &= 8(t^2+u^2)\end{aligned} \tag{9}$$

Combining all four:

$$\begin{aligned}\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= 2e^4\Big(\frac{(s^2+u^2)}{t^2} + 2\frac{u^2}{ts} + \frac{t^2+u^2}{s^2}\Big) \\ &= 2e^4\left[u^2\Big(\frac{1}{t^2}+\frac{2}{ts}+\frac{1}{s^2}\Big) + \frac{s^2}{t^2} + \frac{t^2}{s^2}\right] \\ &= 2e^4\left[u^2\Big(\frac{1}{t}+\frac{1}{s}\Big)^2 + \Big(\frac{t}{s}\Big)^2 + \Big(\frac{s}{t}\Big)^2\right]\end{aligned} \tag{10}$$

Differential cross-section is defined in (4.85) in P&S:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{|\mathcal{M}|^2}{64\pi^2 E_{cm}^2} = \frac{|\mathcal{M}|^2}{16\pi^2 4s} \\ &= \frac{2e^4}{16\pi^2 4s}\left[u^2\Big(\frac{1}{t}+\frac{1}{s}\Big)^2 + \Big(\frac{t}{s}\Big)^2 + \Big(\frac{s}{t}\Big)^2\right] \\ &= \frac{\alpha^2}{2s}\left[u^2\Big(\frac{1}{t}+\frac{1}{s}\Big)^2 + \Big(\frac{t}{s}\Big)^2 + \Big(\frac{s}{t}\Big)^2\right]\end{aligned} \tag{11}$$

Integrating it over azimuthal angle ϕ:

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left[u^2\left(\frac{1}{t}+\frac{1}{s}\right)^2+\left(\frac{t}{s}\right)^2+\left(\frac{s}{t}\right)^2\right] \tag{12}$$

2 $\cos\theta$ dependence

To explore angular dependence we need to specialize the 4-vectors:

$$\begin{aligned} p &= (E,0,0,|\mathbf{p}|) \\ k &= (E,0,0,-|\mathbf{p}|) \\ p' &= (E,0,\sin\theta|\mathbf{p}|,\cos\theta|\mathbf{p}|) \\ k' &= (E,0,-\sin\theta|\mathbf{p}|,-\cos\theta|\mathbf{p}|) \end{aligned} \tag{13}$$

This time let's expand Mandelstam variables in terms of $|\mathbf{p}|$ and $\cos\theta$ (assuming $s \approx 2|\mathbf{p}|^2$):

$$\begin{aligned} t &= (p'-p)^2 = -2|\mathbf{p}|^2(1-\cos\theta) \\ u &= (k'-p)^2 = -2|\mathbf{p}|^2(1+\cos\theta) \\ t/s &= -1+\cos\theta \\ u/s &= -1-\cos\theta \\ u/t &= \frac{1+\cos\theta}{1-\cos\theta} \end{aligned} \tag{14}$$

Only t and u are dependent on $\cos\theta$. Putting them into differential cross section:

$$\begin{aligned} \frac{d\sigma}{d\cos\theta} &= \frac{\pi\alpha^2}{s}\left[\frac{u^2}{t^2}(t/s+1)^2+\left(\frac{t}{s}\right)^2+\left(\frac{s}{t}\right)^2\right] \\ &= \frac{\pi\alpha^2}{s}\left[\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^2\cos^2\theta+(1-\cos\theta)^2+\left(\frac{1}{1-\cos\theta}\right)^2\right] \\ &= \frac{\pi\alpha^2}{s}\left[\frac{(1+\cos\theta)^2\cos^2\theta+(1-\cos\theta)^4+1}{(1-\cos\theta)^2}\right] \end{aligned} \tag{15}$$