P&S QFT - Problem 5.1 / part 2

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Work out cross section for electron-muon scattering, in the muon rest frame, retaining the electron mass but sending $m_\mu \to \infty$.

1 Expected result

$$\frac{d\sigma}{d\Omega} = \frac{\mathcal{Z}^2\alpha^2}{4|\mathbf{p}|^2\beta^2\sin^4(\theta/2)}\Big(1 - \beta^2\sin^2\frac{\theta}{2}\Big) \tag{1}$$

2 Definitions

- E - energy of electron
- β - speed of electron
- p, p' - 4-momentum of ingoing/outgoing electrons
- k, k' - 4-momentum of ingoing/outgoing muon
- $t = (p - p')^2$ - Photon momentum (Mandelstam variable)

3 Solution

Scattering amplitude *(spinor indices omitted)*:

$$\mathcal{M} = \overline{u}(p')(-ie\gamma^\mu)u(p)\Big(\frac{-ig_{\mu\nu}}{t + i\varepsilon}\Big)\overline{u}(k')(-ie\gamma^\nu)u(k) \tag{2}$$

Averaging $|\mathcal{M}|^2$ over all spin configurations and dividing by 4 (for input spins):

$$|\mathcal{M}|^2 = \frac{e^4}{4t^2} tr\Big[(\not{p}' + m)\gamma^\mu(\not{p} + m)\gamma^\nu\Big] \cdot \\ tr\Big[(\not{k}' + m_\mu)\gamma_\mu(\not{k} + m_\mu)\gamma_\nu\Big] \tag{3}$$

Working out traces separately:

$$\begin{aligned} tr\Big[(\not{p}' + m)\gamma^\mu(\not{p} + m)\gamma^\nu\Big] &= tr\Big[\not{p}'\gamma^\mu\not{p}\gamma^\nu\Big] + m^2 tr\Big[\gamma^\mu\gamma^\nu\Big] \\ &= p'^\rho p^\sigma tr\Big[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\Big] + 4m^2 g^{\mu\nu} \\ &= 4p'^\rho p^\sigma \Big(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}\Big) + 4m^2 g^{\mu\nu} \\ &= 4\Big((m^2 - p\cdot p')g^{\mu\nu} + (p^\mu p'^\nu + p^\nu p'^\mu)\Big) \end{aligned} \tag{4}$$

Other trace:

$$tr\Big[(\not{k}' + m_\mu)\gamma_\mu(\not{k} + m_\mu)\gamma_\nu\Big] = 4\Big((m_\mu^2 - k\cdot k')g_{\mu\nu} + (k_\mu k'_\nu + k_\nu k'_\mu)\Big) \tag{5}$$

We're going to split $|\mathcal{M}|^2$ into four parts:

$$|\mathcal{M}|^2 = \frac{e^4}{4t^2}\Big[(I) + (II) + (III) + (IV)\Big] \tag{6}$$

Doing (I):

$$\begin{aligned} (I) &= 16(m^2 - p\cdot p')g^{\mu\nu}(m_\mu^2 - k\cdot k')g_{\mu\nu} \\ &= 64(m^2 - p\cdot p')(m_\mu^2 - k\cdot k') \end{aligned} \tag{7}$$

Doing (II):

$$\begin{aligned} (II) &= 16(p^\mu p'^\nu + p^\nu p'^\mu)(m_\mu^2 - k\cdot k')g_{\mu\nu} \\ &= 32p\cdot p'(m_\mu^2 - k\cdot k') \end{aligned} \tag{8}$$

Naturally (III) is:

$$(III) = 32k\cdot k'(m^2 - p\cdot p') \tag{9}$$

(IV):

$$\begin{aligned} (IV) &= 16(p^\mu p'^\nu + p^\nu p'^\mu)(k_\mu k'_\nu + k_\nu k'_\mu) \\ &= 32(p \cdot k p' \cdot k' + p \cdot k' p' \cdot k) \end{aligned} \tag{10}$$

Summing all together:

$$\begin{aligned} |\mathcal{M}|^2 = \frac{8e^4}{t^2}\Big[&2(m^2 - p \cdot p')(m_\mu^2 - k \cdot k') + p \cdot p'(m_\mu^2 - k \cdot k') + \\ &k \cdot k'(m^2 - p \cdot p') + p \cdot k p' \cdot k' + p \cdot k' p' \cdot k\Big] \\ = \frac{8e^4}{t^2}\Big[&2m^2 m_\mu^2 - 2m^2 k \cdot k' - 2m_\mu^2 p \cdot p' + 2p \cdot p' k \cdot k' + \\ &m_\mu^2 p \cdot p' - p \cdot p' k \cdot k' + m^2 k \cdot k' - p \cdot p' k \cdot k' + \\ &p \cdot k p' \cdot k' + p \cdot k' p' \cdot k\Big] \\ = \frac{8e^4}{t^2}\Big[&2m^2 m_\mu^2 - m^2 k \cdot k' - m_\mu^2 p \cdot p' + p \cdot k p' \cdot k' + p \cdot k' p' \cdot k\Big] \end{aligned} \tag{11}$$

Specialize p, k, p', k':

$$\begin{aligned} p &= (E, 0, 0, |\mathbf{p}|) \\ k &= (m_\mu, 0, 0, 0) \\ p' &= (E, 0, 0, \sin\theta|\vec{p'}|, \cos\theta|\vec{p'}|) \\ k' &= (m_\mu, 0, -\sin\theta|\vec{p'}|, |\mathbf{p}| - \cos\theta|\vec{p'}|) \end{aligned} \tag{12}$$

Time to do differential cross section ($u = |\vec{p'}|$):

$$d\sigma = \frac{1}{2E 2m_\mu \beta}\frac{d^3p'}{(2\pi)^3}\frac{1}{2E'}\frac{d^3k'}{(2\pi)^3}\frac{1}{2m_\mu}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(p + k - p' - k') \tag{13}$$

We can use 3 of 4 delta functions to eliminate d^3k':

$$\begin{aligned} \ldots &= \frac{1}{4(4\pi)^2 E m_\mu^2 \beta}\frac{d^3p'}{E'}|\mathcal{M}|^2\delta(E - E') \\ &= \frac{1}{4(4\pi)^2 E m_\mu^2 \beta}\frac{d\Omega\, du}{E'}u^2|\mathcal{M}|^2\delta(E - E') \end{aligned} \tag{14}$$

When in doubt - integrate! *(and use $E' = \sqrt{m^2 + u^2}$)*:

$$\int_0^\infty \frac{du}{E'} u^2 |\mathcal{M}|^2 \delta(E-E') = \int_0^\infty dE' u |\mathcal{M}|^2 \delta(E-E') = |\mathbf{p}||\mathcal{M}|^2 \tag{15}$$

Thus:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4(4\pi)^2 E m_\mu^2 \beta} |\mathbf{p}||\mathcal{M}|^2 \tag{16}$$

Given that $E\beta = |\mathbf{p}|$:

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{4(4\pi)^2 m_\mu^2} \tag{17}$$

Few useful dot products:

$$\begin{aligned}
p \cdot k &= E m_\mu \\
p \cdot k' &= E m_\mu - |\mathbf{p}|^2 (1-\cos\theta) \\
p' \cdot k &= E m_\mu \\
p' \cdot k' &= E m_\mu + \sin^2\theta |\mathbf{p}|^2 + \cos^2\theta |\mathbf{p}|^2 - \cos\theta |\mathbf{p}|^2 \\
&= E m_\mu + |\mathbf{p}|^2 (1-\cos\theta) \\
p \cdot p' &= E^2 - \cos\theta |\mathbf{p}| u \\
k \cdot k' &= m_\mu^2 \\
t &= (p-p')^2 = -\sin^2\theta |\mathbf{p}|^2 - (\cos\theta |\mathbf{p}| - |\mathbf{p}|)^2 \\
&= -2|\mathbf{p}|^2 (1-\cos\theta) \\
&= -4|\mathbf{p}|^2 \sin^2\frac{\theta}{2} \\
t^2 &= 16|\mathbf{p}|^4 \sin^4 \frac{\theta}{2}
\end{aligned} \tag{18}$$

Let's put some things together:

$$\begin{aligned}
|\mathcal{M}|^2 = \frac{8e^4}{t^2} \Big[& 2m^2 m_\mu^2 - m^2 m_\mu^2 - m_\mu^2 (E^2 - \cos\theta |\mathbf{p}|^2) + \\
& E m_\mu (E m_\mu + |\mathbf{p}|^2 (1-\cos\theta)) + \\
& (E m_\mu - |\mathbf{p}|^2 (1-\cos\theta)) E m_\mu \Big] \\
= \frac{8e^4}{t^2} \Big[& m^2 m_\mu^2 + \cos\theta |\mathbf{p}|^2 m_\mu^2 + E^2 m_\mu^2 \Big]
\end{aligned} \tag{19}$$

$$
\begin{aligned}
\frac{|\mathcal{M}|^2}{m_\mu^2} &= \frac{8e^4}{t^2}\Big[m^2 + (1 - 2\sin^2\frac{\theta}{2})|\mathbf{p}|^2 + E^2\Big] \\
&= \frac{16e^4}{t^2}\Big(E^2 - \sin^2\frac{\theta}{2}|\mathbf{p}|^2\Big) \\
&= \frac{e^4}{|\mathbf{p}|^4\sin^4\frac{\theta}{2}}\Big(E^2 - \sin^2\frac{\theta}{2}|\mathbf{p}|^2\Big) \\
&= \frac{e^4}{|\mathbf{p}|^4\sin^4\frac{\theta}{2}}E^2\Big(1 - \sin^2\frac{\theta}{2}\beta^2\Big) \\
&= \frac{e^4}{|\mathbf{p}|^2\beta^2\sin^4\frac{\theta}{2}}\Big(1 - \sin^2\frac{\theta}{2}\beta^2\Big)
\end{aligned} \tag{20}
$$

And final:

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{|\mathcal{M}|^2}{4(4\pi)^2 m_\mu^2} \\
&= \frac{\alpha^2}{4|\mathbf{p}|^2\beta^2\sin^4\frac{\theta}{2}}\Big(1 - \beta^2\sin^2\frac{\theta}{2}\Big) \implies \quad \square
\end{aligned} \tag{21}
$$