A MATHEMATICAL INTRODUCTION TO QUANTUM ENTANGLEMENT

5 lectures. Plan:

1. Entanglement of pure states
2. Mixed states. Ent. criteria
3. Duality of cones. Positive maps
4. Tensor norms
5. Ent. in GPTs
   general probabilistic theories

① Entanglement of pure states

Entanglement : quantum phenomenon where the state of each particle of a group cannot be described independently of the others

Pure states Hilbert spaces $H \simeq \mathbb{C}^d$

- $\psi \in H$ $\|\psi\| = 1$ $|\psi\rangle$ pure state
- qubits $d=2$ $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\alpha, \beta \in \mathbb{C}$ $|\alpha|^2 + |\beta|^2 = 1$

examples : $|0\rangle, |1\rangle$ classical states

$$|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

- quantum gates : unitary matrices $U \in \mathcal{U}(d)$

Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$$

$$H|1\rangle = |-\rangle$$

Two quantum systems H_A , H_B $H_{AB} = H_A \otimes H_B$

$$|\psi_{AB}\rangle \in H_{AB} \begin{cases} |\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle & \text{separable} \\ |\psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle & \text{entangled} \end{cases}$$

Examples • $|0\rangle \otimes |0\rangle = |00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$

$$\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) = |0\rangle \otimes \underbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}_{|+\rangle}$$ are separable

• $|\Omega\rangle := \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \in \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$

2-qubit state

maximally entangled state, Bell state, singlet state ...

Claim $|\Omega\rangle$ is entangled.

$$|\Omega\rangle = \begin{pmatrix} x \\ y \end{pmatrix}\begin{matrix}|0\rangle \\ |1\rangle\end{matrix} \otimes \begin{pmatrix} a \\ b \end{pmatrix}\begin{matrix}|0\rangle \\ |1\rangle\end{matrix} \iff$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\begin{matrix}\leftarrow|00\rangle \\ \leftarrow|01\rangle \\ \leftarrow|10\rangle \\ \leftarrow|11\rangle\end{matrix} = \begin{bmatrix} xa \\ xb \\ ya \\ yb \end{bmatrix}$$

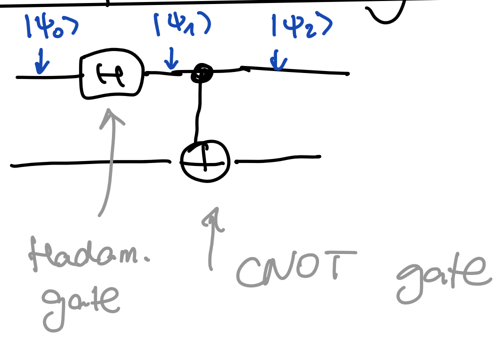
$$\begin{aligned} xa &= 1 \\ xb &= 0 \\ ya &= 0 \\ yb &= 1 \end{aligned}$$

$0 = xb \cdot ya$, $\ast = 1$

$\ast = xyab$

impossible!

Circuit for building $|\Omega\rangle$



CNOT gate (2-qubit)

$$\begin{aligned} \mathrm{CNOT}\ |00\rangle &= |00\rangle \\ |01\rangle &= |01\rangle \\ |10\rangle &= |11\rangle \\ |11\rangle &= |10\rangle \end{aligned}$$

control qubit, target qubit

$$\mathrm{CNOT} = \begin{array}{c} \left[\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & \multicolumn{2}{c}{0} \\ \hline & & 0 & 1 \\ \multicolumn{2}{c|}{0} & 1 & 0 \end{array}\right] \begin{array}{l} 00 \\ 01 \\ 10 \\ 11 \end{array} \\ \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \end{array}$$

CNOT : NOT on the target qubit if the control $= I$

circuit

$|\psi_0\rangle$: initial state $= |00\rangle = |0\rangle \otimes |0\rangle$

$$|\psi_1\rangle = (H \otimes I)\,|\psi_0\rangle = \underbrace{H|0\rangle}_{=|+\rangle} \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$$

$$|\psi_2\rangle = \mathrm{CNOT}\cdot|\psi_1\rangle =$$

$$= \frac{1}{\sqrt{2}}\left(\mathrm{CNOT}\,|00\rangle + \mathrm{CNOT}\,|10\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = |\Omega\rangle$$

Main theoretical tool : Schmidt decomposition

Fact: Any state $|\psi_{AB}\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ can be written as

$$|\psi_{AB}\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |a_i\rangle \otimes |b_i\rangle$$

where :
- $\lambda_i \geq 0$, $\sum \lambda_i = 1$
- $\{a_i\}_{i=1}^{r}$ is an orthonormal family in $\mathbb{C}^{d_A}$
- $\{b_i\}_{i=1}^{r}$ is an $\perp_n$ fam. in $\mathbb{C}^{d_B}$

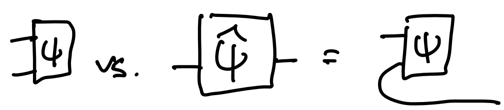
→ $\{\lambda_i\}$ are called the Schmidt coefficients of $|\psi_{AB}\rangle$

→ r is called the Schmidt rank of $|\psi_{AB}\rangle$

Remark Schmidt dec. of $|\psi_{AB}\rangle \Longleftrightarrow$ SVD of $\hat{\psi}$

$\hat{\psi} \in \mathcal{M}_{d_A \times d_B}(\mathbb{C})$

$\hat{\psi}_{ij} = \langle ij|\psi_{AB}\rangle$



Entanglement for multipartite pure states

$$H_{ABC} = H_A \otimes H_B \otimes H_C$$

→ separable states: $|\psi_{ABC}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle$

→ entangled states $|\psi_{ABC}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle$

Examples of entangled states

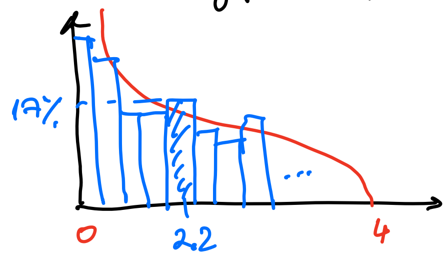
- $\frac{1}{\sqrt{2}}\left(|000\rangle + |011\rangle\right) = |0\rangle_A \otimes |\Omega\rangle_{BC}$
- $|GHZ\rangle := \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$
- $|W\rangle := \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$

$|\psi_{AB}\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ random $\quad d \approx 500$

$\rightsquigarrow$

$\vec{\lambda} = (\lambda_1, \ldots, \lambda_{500})$ = schmidt coeffs of $|\psi_{AB}\rangle$

histogram $(d \cdot \vec{\lambda})$ =



② Mixed states

$d_A := \dim H_A \ll \dim H_B =: d_B$

$\uparrow$ Syst of interest $\qquad \uparrow$ environment

syst in contact with environment $\rightsquigarrow$ $|\psi_{AB}\rangle$ is entangled

$\rightsquigarrow$ it does not make sense to talk about the state of "A" system alone ?!

→ say we want to measure an observable X_A on A alone

$$\langle \psi_{AB} | X_A \otimes I_B | \psi_{AB} \rangle = \mathrm{Tr}\big(X_A \otimes I_B \cdot \underbrace{|\psi_{AB}\rangle\langle\psi_{AB}|}_{\text{rank-1 proj on } |\psi_{AB}\rangle \text{ vector}} \big)$$

ρ_{AB}

$\in \mathcal{M}_{d_A \cdot d_B}$

$$= \langle X_A \otimes I_B , \rho_{AB} \rangle_{HS}$$

$\langle Y, Z \rangle_{HS} := \mathrm{Tr}(Y^* \cdot Z)$

$$= \langle F(X_A), \rho_{AB} \rangle_{HS}$$

$F(Y) = Y \otimes I_B$

$$= \langle X_A , F^*(\rho_{AB}) \rangle_{HS}$$

$F^* = \mathrm{Tr}_B$ partial trace

$$= \langle X_A , \underbrace{\mathrm{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}|}_{\rho_A} \rangle_{HS}$$

$$= \langle X_A, \rho_A \rangle_{HS}$$

$\rho_A \in \mathcal{M}_{d_A}(\mathbb{C})$

Def A density matrix of size d is $\rho \in \mathcal{M}_d(\mathbb{C})$

$\rho \geq 0$ and $\operatorname{Tr} \rho = 1$

positive semidefinite (PSD)
spectrum $\subseteq [0,\infty)$

$\rho \in \mathcal{M}_d^{1,+}(\mathbb{C})$

Fact The set $\mathcal{M}_d^{1,+}$ of density matrices is a convex body, having extreme points

$$\operatorname{ext} \mathcal{M}_d^{1,+} = \{ |\varphi\rangle\langle\varphi| : \varphi \in \mathbb{C}^d, \|\varphi\| = 1 \}$$

$=$ "pure states"

The partial trace operation

$$\operatorname{Tr}_B : \mathcal{M}_{d_A \cdot d_B} \simeq \mathcal{M}_{d_A} \otimes \mathcal{M}_{d_B} \longrightarrow \mathcal{M}_{d_A}$$

$$A \otimes B \longmapsto (\operatorname{Tr} B) \cdot A$$

Eigenvalues vs Schmidt coefficients

Consider $|\psi_{AB}\rangle$ having S.D. $|\psi_{AB}\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |a_i\rangle |b_i\rangle$

$$\rho_A = \operatorname{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}| = \operatorname{Tr}_B \sum_{ij} \sqrt{\lambda_i \lambda_j}\, |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|$$

$$= \sum_{ij} \sqrt{\lambda_i \lambda_j}\, |a_i\rangle\langle a_j| \cdot \underbrace{\operatorname{Tr} |b_i\rangle\langle b_j|}_{= \langle b_j, b_i\rangle = \delta_{ij}}$$

$$\boxed{\rho_A = \sum_i \lambda_i |a_i\rangle\langle a_i|}$$

is a spectral decomp of ρ_A, since $\{a_i\}$ ON

So: [the schmidt coeffs of $|\psi_{AB}\rangle$]^2

$\shortparallel$

eigenvalues of $\rho_A = \operatorname{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}|$