A MATHEMATICAL INTRODUCTION TO QUANTUM ENTANGLEMENT

5 lectures. Plan:

1. Entanglement of pure states
2. Mixed states. Ent. criteria
3. Duality of cones. Positive maps
4. Ent. in GPTs general probabilistic theories
5. Tensor norms

① Entanglement of pure states

Entanglement : quantum phenomenon where the state of each particle of a group cannot be described independently of the others

Pure states

Hilbert spaces $H \simeq \mathbb{C}^d$

- $\psi \in H \quad \|\psi\| = 1 \qquad |\psi\rangle$ pure state
- qubits $d=2 \qquad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

  $\alpha, \beta \in \mathbb{C} \quad |\alpha|^2 + |\beta|^2 = 1$

  examples : $|0\rangle, |1\rangle$ classical states

  $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

  $|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

- quantum gates : unitary matrices $U \in \mathcal{U}(d)$

  Hadamard gate $\quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

  $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

  $H|1\rangle = |-\rangle$

Two quantum systems

$H_A \, , \, H_B \qquad H_{AB} = H_A \otimes H_B$

$|\psi_{AB}\rangle \in H_{AB}$

- $\nearrow \; |\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ separable
- $\searrow \; |\psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$ entangled

Examples

- $|0\rangle \otimes |0\rangle = |00\rangle, \; |01\rangle, |10\rangle, |11\rangle$

  $\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) = |0\rangle \otimes \underbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}_{|+\rangle}$ are separable

- $|\Omega\rangle := \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \in \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$

2-qubit state

maximally entangled state, Bell state, singlet state ...

Claim $|\Omega\rangle$ is entangled.

$$|\Omega\rangle = \begin{pmatrix} x \\ y \end{pmatrix}\begin{matrix} {\scriptstyle |0\rangle} \\ {\scriptstyle |1\rangle} \end{matrix} \otimes \begin{pmatrix} a \\ b \end{pmatrix}\begin{matrix} {\scriptstyle |0\rangle} \\ {\scriptstyle |1\rangle} \end{matrix} \Leftrightarrow$$

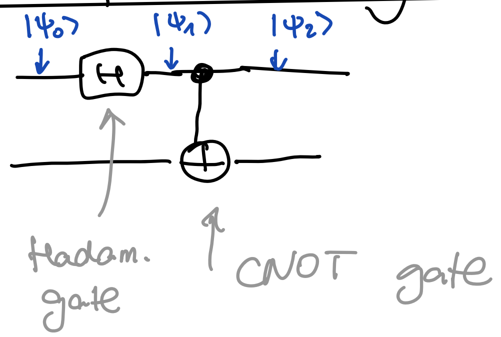
$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\begin{matrix} \leftarrow |00\rangle \\ \leftarrow |01\rangle \\ \leftarrow |10\rangle \\ \leftarrow |11\rangle \end{matrix} = \begin{bmatrix} xa \\ xb \\ ya \\ yb \end{bmatrix}$$

$$\left.\begin{matrix} xa = 1 \\ 0 = (xb = 0 \\ ya = 0 \\ yb = 1 \end{matrix}\right\} x = 1$$

$x = xyab$

impossible!

Circuit for building $|\Omega\rangle$



CNOT gate (2-qubit)

$CNOT\,|00\rangle = |00\rangle$

$|01\rangle = |01\rangle$

$|10\rangle = |11\rangle$

$|11\rangle = |10\rangle$

control qubit — target qubit

$$CNOT = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & 0 & 0 & 0 \\ 01 & 0 & 1 & 0 & 0 \\ 10 & 0 & 0 & 0 & 1 \\ 11 & 0 & 0 & 1 & 0 \end{array}$$

CNOT : NOT on the target qubit if the control $= I$

circuit

$|\psi_0\rangle$: initial state $= |00\rangle = |0\rangle \otimes |0\rangle$

$|\psi_1\rangle = (H \otimes I)\,|\psi_0\rangle = \underbrace{H|0\rangle}_{=|+\rangle} \otimes |0\rangle$

$= \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$

$|\psi_2\rangle = CNOT \cdot |\psi_1\rangle =$

$= \frac{1}{\sqrt{2}}\left(CNOT\,|00\rangle + CNOT\,|10\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = |\Omega\rangle$

Main theoretical tool : Schmidt decomposition

Fact : Any state $|\psi_{AB}\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ can be written as

$$|\psi_{AB}\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |a_i\rangle \otimes |b_i\rangle$$

where :
- $\lambda_i \geq 0$, $\sum \lambda_i = 1$
- $\{a_i\}_{i=1}^{r}$ is an orthonormal family in $\mathbb{C}^{d_A}$
- $\{b_i\}_{i=1}^{r}$ is an $\perp_n$ fam. in $\mathbb{C}^{d_B}$

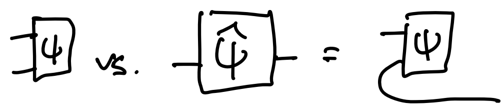
→ $\{\lambda_i\}$ are called the Schmidt coefficients of $|\psi_{AB}\rangle$

→ r is called the Schmidt rank of $|\psi_{AB}\rangle$

Remark Schmidt dec. of $|\psi_{AB}\rangle \iff$ SVD of $\hat{\psi}$

$\hat{\psi} \in \mathcal{M}_{d_A \times d_B}(\mathbb{C})$

$\hat{\psi}_{ij} = \langle ij | \psi_{AB}\rangle$



Entanglement for multipartite pure states

$$H_{ABC} = H_A \otimes H_B \otimes H_C$$

→ separable states : $|\psi_{ABC}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle$

→ entangled states $|\psi_{ABC}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle$

Examples of entangled states

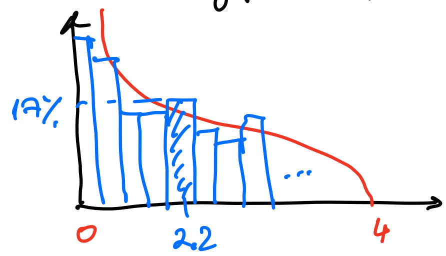
- $\frac{1}{\sqrt{2}}\left(|000\rangle + |011\rangle\right) = |0\rangle_A \otimes |\Omega\rangle_{BC}$
- $|GHZ\rangle := \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$
- $|W\rangle := \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$

$|\psi_{AB}\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ random $d \simeq 500$

$\rightsquigarrow$

$\vec{\lambda} = (\lambda_1, \dots, \lambda_{500})$ = schmidt coeffs of $|\psi_{AB}\rangle$

histogram $(d \cdot \vec{\lambda})$ =



② Mixed states

$d_A := \dim H_A \ll \dim H_B =: d_B$

Syst of interest — environment

syst in contact with environment $\rightsquigarrow$ $|\psi_{AB}\rangle$ is entangled
$\rightsquigarrow$ it does not make sense to talk about the state of "A" system alone ?!

→ say we want to measure an observable X_A on A alone

$$\langle \psi_{AB} | X_A \otimes I_B | \psi_{AB} \rangle = \mathrm{Tr}\Big(X_A \otimes I_B \cdot \underbrace{|\psi_{AB}\rangle\langle\psi_{AB}|}_{\text{rank-1 proj on } |\psi_{AB}\rangle \text{ vector}} \Big) \quad \rho_{AB}$$

$\in \mathcal{M}_{d_A \cdot d_B}$

$$= \langle X_A \otimes I_B , \rho_{AB} \rangle_{HS}$$

$\langle Y, Z \rangle_{HS} := \mathrm{Tr}(Y^* \cdot Z)$

$$= \langle F(X_A), \rho_{AB} \rangle_{HS}$$

$F(Y) = Y \otimes I_B$

$$= \langle X_A , F^*(\rho_{AB}) \rangle_{HS}$$

$F^* = \mathrm{Tr}_B$ partial trace

$$= \langle X_A , \underbrace{\mathrm{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}|}_{\rho_A} \rangle_{HS}$$

$$= \langle X_A, \rho_A \rangle_{HS}$$

$\rho_A \in \mathcal{M}_{d_A}(\mathbb{C})$

Def A density matrix of size d is $\rho \in \mathcal{M}_d(\mathbb{C})$

$\rho \geq 0$ and $\mathrm{Tr}\,\rho = 1$

positive semidefinite (PSD)
spectrum $\subseteq [0,\infty)$

$\rho \in \mathcal{M}_d^{1,+}(\mathbb{C})$

Fact The set $\mathcal{M}_d^{1,+}$ of density matrices is a convex body, having extreme points

$$\mathrm{ext}\,\mathcal{M}_d^{1,+} = \{ |\psi\rangle\langle\psi| : \psi \in \mathbb{C}^d, \|\psi\| = 1 \}$$

$=$ "pure states"

__The partial trace operation__

$$\mathrm{Tr}_B : \mathcal{M}_{d_A \cdot d_B} \simeq \mathcal{M}_{d_A} \otimes \mathcal{M}_{d_B} \longrightarrow \mathcal{M}_{d_A}$$

$$A \otimes B \longmapsto (\mathrm{Tr}\,B) \cdot A$$

__Eigenvalues vs Schmidt coefficients__

Consider $|\psi_{AB}\rangle$ having S.D. $|\psi_{AB}\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |a_i\rangle |b_i\rangle$

$$\rho_A = \mathrm{Tr}_B\, |\psi_{AB}\rangle\langle\psi_{AB}| = \mathrm{Tr}_B \sum_{ij} \sqrt{\lambda_i \lambda_j}\, |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|$$

$$= \sum_{ij} \sqrt{\lambda_i \lambda_j}\, |a_i\rangle\langle a_j| \cdot \underbrace{\mathrm{Tr}\, |b_i\rangle\langle b_j|}_{= \langle b_j, b_i \rangle = \delta_{ij}}$$

$$\boxed{\rho_A = \sum_i \lambda_i\, |a_i\rangle\langle a_i|}$$

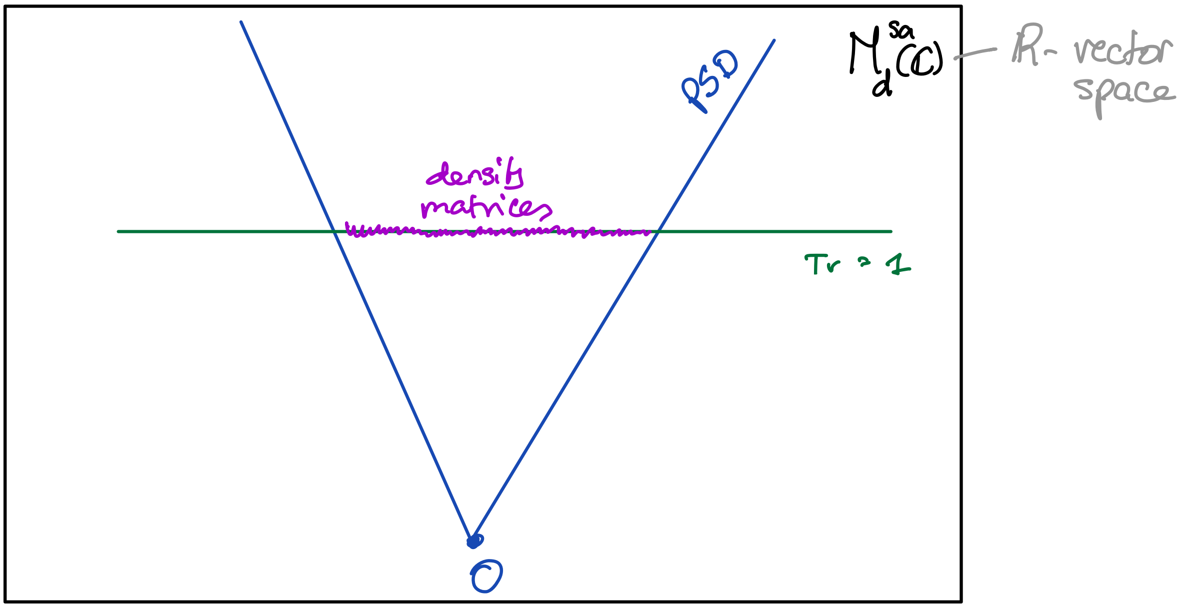
is a spectral decomp of ρ_A, since $\{a_i\}$ ONB

So: [the schmidt coeffs of $|\psi_{AB}\rangle$]^2

|||

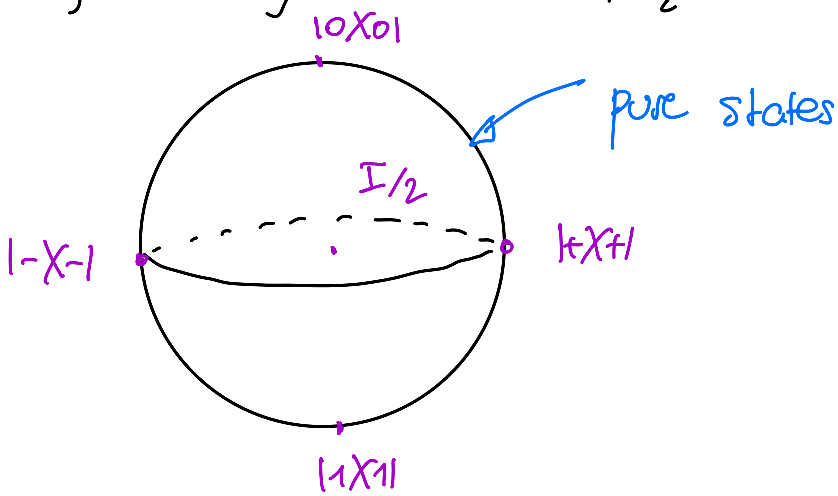
eigenvalues of $\rho_A = \mathrm{Tr}_B\, |\psi_{AB}\rangle\langle\psi_{AB}|$

Density matrices $= \{ \rho \in M_d^{sa}(\mathbb{C}) : \rho \geq 0 \ , \mathrm{Tr}\, \rho = 1 \}$



PSD_d = cone of positive, semidefinite matrices

· in dim $d=2$, density matrices $M_2^{1,+}$ = Bloch ball



$$\rho_{\vec{a}} = \frac{1}{2} \left(I + a_x \sigma_X + a_y \sigma_Y + a_z \sigma_Z \right)$$

$\rho_{\vec{a}}$ is a density matrix $\Longleftrightarrow \|\vec{a}\| \leq 1$

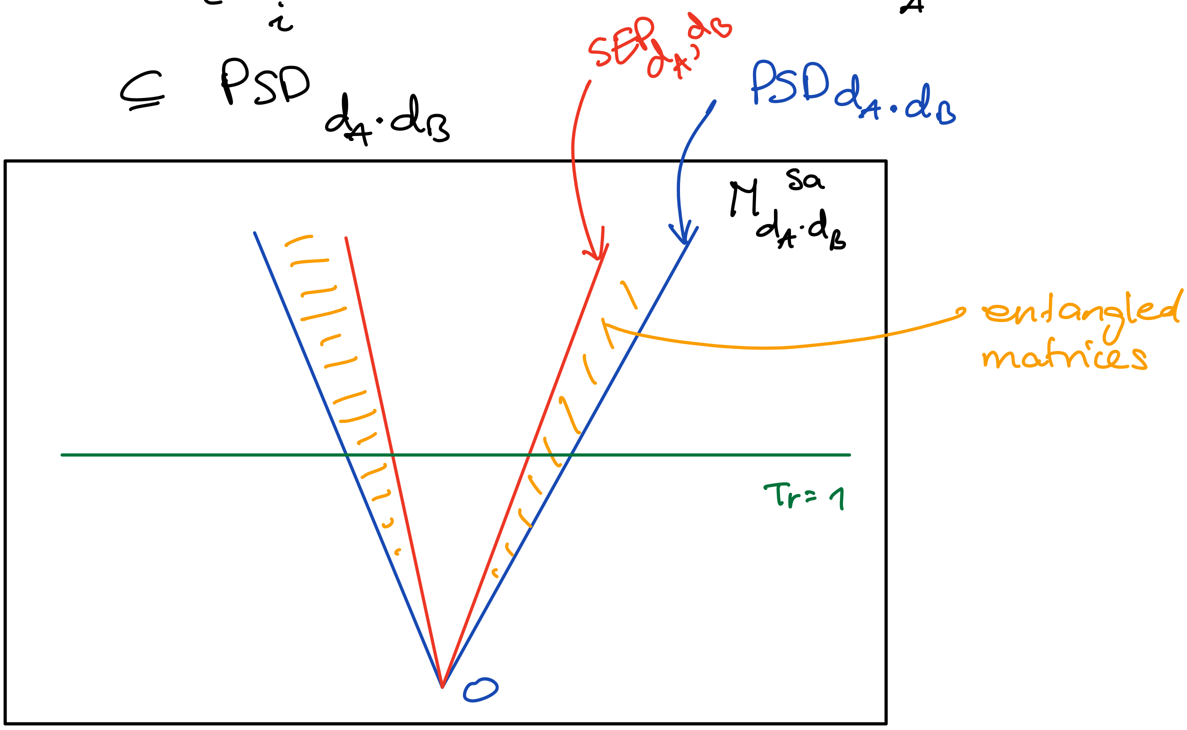
Bipartite density matrices $\quad M_{d_A d_B}^{1,+} \ni \rho$

$$\rho = \sum_{i=1}^{m} p_i \, \alpha_i \otimes \beta_i \qquad \text{where} \qquad p_1, \dots, p_m \geq 0 \ \sum p_i = 1$$

$\alpha_1, \dots, \alpha_m \in M_{d_A}^{1,+}, \ \beta_1, \dots, \beta_m \in M_{d_B}^{1,+}$

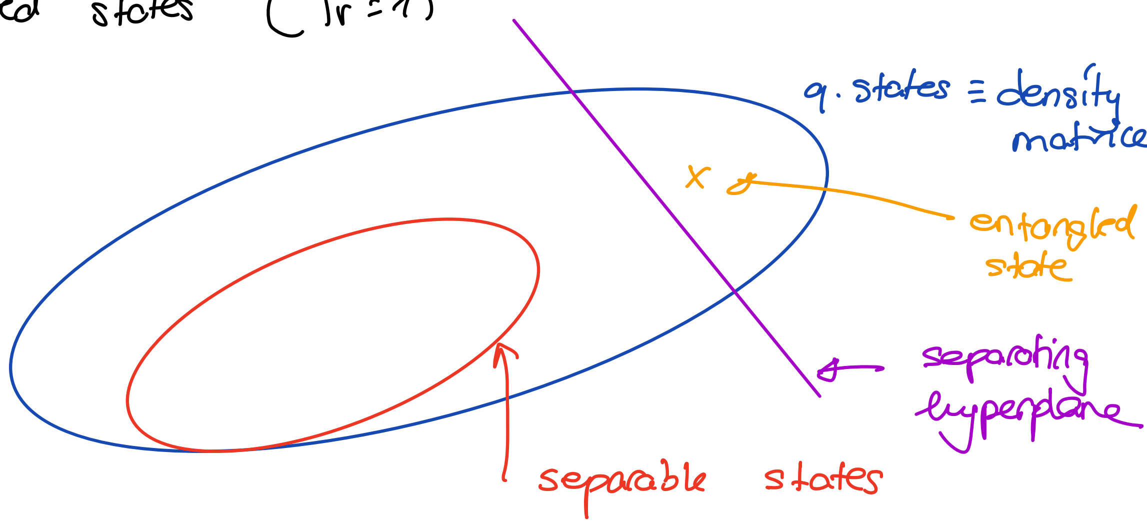
is called separable. Otherwise, it is entangled.

$$SEP_{d_A, d_B} = \text{cone of separable (not normalized) states}$$
$$= \left\{ \sum_i A_i \otimes B_i \ : \ A_i \in PSD_{d_A} , B_i \in PSD_{d_B} \right\}$$
$$\subsetneq PSD_{d_A \cdot d_B}$$



<u>Fact</u> Deciding membership inside SEP_{d_A, d_B} is an NP-hard problem

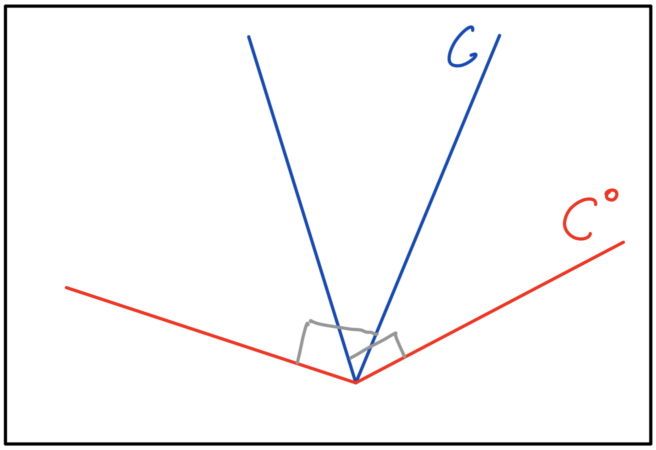
→ Normalized states (Tr = 1)



⇝ Hahn - Banach separation theorem.

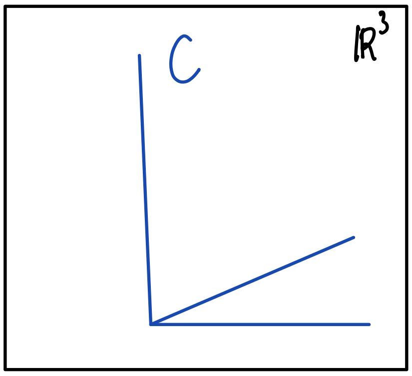
<u>Dual cones</u> Given a cone C, define

$$C^\circ = \{ v \ : \ \langle v, x \rangle \geq 0 \quad \forall x \in C \}$$



Examples

- $C = \mathbb{R}^d_+ = \{ x \in \mathbb{R}^d : \ x_i \geq 0 \}$



Remark Let $\mathbb{1} = (1, \dots, 1) \in \mathbb{R}^d$

$$\mathbb{R}^d_+ \cap \underbrace{\{ x : \langle x, \mathbb{1} \rangle = 1 \}}_{x : \ \sum x_i = 1} =$$

$$= \Delta_d = \{ \text{prob. dist.} \}$$

$\dim_{\mathbb{R}} = d$

$$(\mathbb{R}^d_+)^\circ = \mathbb{R}^d_+ \quad \rightsquigarrow \text{ self-dual.}$$

- $PSD_d = \{ X \in M_d^{sa}(\mathbb{C}) : \ \text{spec}(X) \subseteq [0, \infty) \}$

$$\Updownarrow$$

$$X = Y Y^*$$

$\dim_{\mathbb{R}} = d^2$

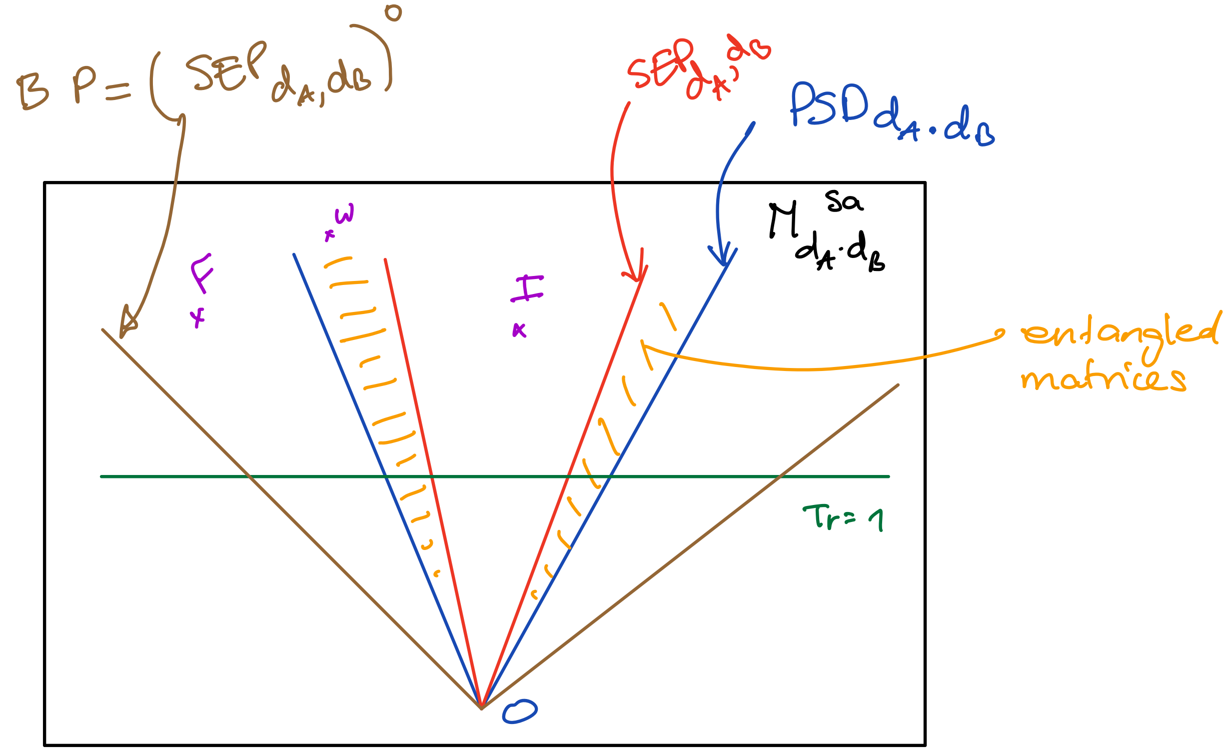
$$(PSD_d)^\circ = \{ Y : \ \langle Y, X \rangle_{HS} \geq 0 \ \ \forall X \in PSD_d \}$$

$$= PSD_d \quad \rightsquigarrow \text{ self-dual}$$

- $C = SEP_{d_A, d_B}$ $\qquad SEP^\circ_{d_A, d_B} = \{ W : \langle W, X \rangle \geq 0 \ \ \forall X \in SEP \}$

Fact Each element $W \in SEP^\circ$ gives an entanglement criterion : if $\langle W, X \rangle \not\geq 0 \Rightarrow X \notin SEP$

$(X \in PSD)$ $\qquad$ (i.e. X entangled)



if $\quad C_1 \subseteq C_2 \implies C_1^\circ \supseteq C_2^\circ$

so: $\quad SEP \subseteq PSD \implies SEP^\circ \supseteq PSD^\circ = PSD$

<u>fact</u> $\quad SEP$ is generated by $|x\rangle\langle x| \otimes |y\rangle\langle y|$

$SEP = \{ \sum A_i \otimes B_i \}$

do spectral decomp of A_i, B_i

$$
\begin{aligned}
SEP^\circ &= \{ W : \langle W, S \rangle \geq 0 \ \ \forall S \in SEP \} \\
&= \{ W : \langle W, |x\rangle\langle x| \otimes |y\rangle\langle y| \rangle \geq 0 \ \ \forall \begin{smallmatrix} x \in \mathbb{C}^{d_A} \\ y \in \mathbb{C}^{d_B} \end{smallmatrix} \} \\
&= \{ W : \langle x \otimes y | W | x \otimes y \rangle \geq 0 \ \ \forall x, y \} \\
&= \{ \text{block - positive matrices} \} =: BP_{d_A, d_B}
\end{aligned}
$$

<u>Remark</u> $\quad PSD_{d_A d_B} = \{ P : \langle z | P | z \rangle \geq 0 \ \ \forall z \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \}$

<u>Example</u> $\quad SEP \subseteq PSD \subseteq BP$

- $\omega := |\Omega\rangle\langle\Omega| \in PSD \setminus SEP$ (i.e. it is entangled)
- Flip operator $F \in M_d \otimes M_d$ $\quad F \in \mathcal{U}(d^2)$

$$F\, x\otimes y = y \otimes x \quad \forall x, y$$

$$F = P_s - P_a \qquad P_s = V^2(\mathbb{C}^d) \qquad d_A = d_B = d$$
$$P_a = \Lambda^2(\mathbb{C}^d)$$

$$\rightsquigarrow F \notin PSD \quad (\text{it has eigenvalue } -1)$$

However, $F \in BP$:

$$\langle x\otimes y | F | x \otimes y\rangle = \langle x\otimes y, y\otimes x\rangle$$
$$= \langle x, y\rangle\langle y, x\rangle = |\langle x, y\rangle|^2 \geq 0$$

$$\Rightarrow F \in BP \setminus PSD.$$

Relation to maps (Benoit's talk)

$(M_{d_A} \otimes M_{d_B})^{sa}$ $\quad SEP \subsetneq PSD \subsetneq BP$

$\updownarrow C, J$

$L^{Her}(M_{d_A} \to M_{d_B})$ $\quad EB \subsetneq CP \subsetneq Pos$

entangl. breaking $\quad$ completely positive $\quad$ positive maps

$\rightsquigarrow$ linear maps $\Phi : M_{d_A}(\mathbb{C}) \longrightarrow M_{d_B}(\mathbb{C})$ which preserve hermiticity: $\Phi(M_{d_A}^{sa}) \subseteq M_{d_B}^{sa}$

$$(M_{d_A} \otimes M_{d_B})^{sa} \simeq L^{Her}(M_{d_A} \to M_{d_B}) \simeq M_{d_A}^* \otimes M_{d_B}$$
$$M_{d_A}^* \cong M_{d_A}$$

$\uparrow$ Choi - Jamiołkowski isomorphism

Def $\Phi : M_{d_A} \to M_{d_B}$ is entanglement breaking if

$\forall n \quad (\Phi \otimes id_n)(X) \in SEP_{d_B, n}$

$\forall \; X \in PSD_{d_A \cdot n}$