A MATHEMATICAL INTRODUCTION TO QUANTUM ENTANGLEMENT

5 lectures. Plan:

1. Entanglement of pure states
2. Mixed states. Ent. criteria
3. Duality of cones. Positive maps
4. Ent. in GPTs general probabilistic theories
5. Tensor norms

① Entanglement of pure states

Entanglement : quantum phenomenon where the state of each particle of a group cannot be described independently of the others

Pure states   Hilbert spaces $H \simeq \mathbb{C}^d$

- $\psi \in H$ $\|\psi\| = 1$ $|\psi\rangle$ pure state
- qubits $d=2$ $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
  $\alpha, \beta \in \mathbb{C}$ $|\alpha|^2 + |\beta|^2 = 1$

  examples : $|0\rangle, |1\rangle$ classical states

  $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

  $|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
- quantum gates : unitary matrices $U \in \mathcal{U}(d)$

  Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

  $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

  $H|1\rangle = |-\rangle$

Two quantum systems H_A , H_B $H_{AB} = H_A \otimes H_B$

$|\psi_{AB}\rangle \in H_{AB}$ ↗ $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ separable
↘ $|\psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$ entangled

Examples
- $|0\rangle \otimes |0\rangle = |00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$

  $\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) = |0\rangle \otimes \underbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}_{|+\rangle}$ are separable

$$|\Omega\rangle := \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \in \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$$

2-qubit state

maximally entangled state, Bell state, singlet state ...

Claim $|\Omega\rangle$ is entangled.

$$|\Omega\rangle = \begin{pmatrix} x \\ y \end{pmatrix}\begin{matrix} {\scriptstyle|0\rangle} \\ {\scriptstyle|1\rangle} \end{matrix} \otimes \begin{pmatrix} a \\ b \end{pmatrix}\begin{matrix} {\scriptstyle|0\rangle} \\ {\scriptstyle|1\rangle} \end{matrix} \Leftrightarrow$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\begin{matrix} \leftarrow|00\rangle \\ \leftarrow|01\rangle \\ \leftarrow|10\rangle \\ \leftarrow|11\rangle \end{matrix} = \begin{bmatrix} xa \\ xb \\ ya \\ yb \end{bmatrix}$$

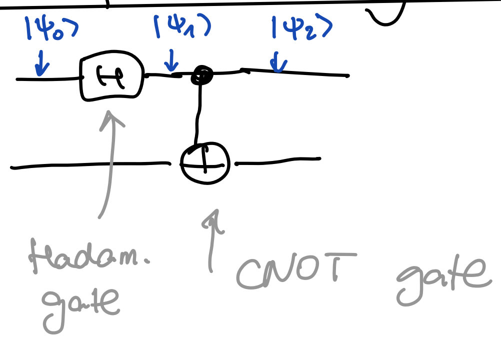
$$\begin{aligned} xa &= 1 \\ xb &= 0 \\ ya &= 0 \\ yb &= 1 \end{aligned}$$

$0 = (xb)(ya)$, $* = 1$

$* = xyab$

impossible!

Circuit for building $|\Omega\rangle$



CNOT gate (2-qubit)

$$\begin{aligned} \mathrm{CNOT}\,|00\rangle &= |00\rangle \\ |01\rangle &= |01\rangle \\ |10\rangle &= |11\rangle \\ |11\rangle &= |10\rangle \end{aligned}$$

control qubit — target qubit

$$\mathrm{CNOT} = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ \begin{matrix} 00 & 01 & 10 & 11 \end{matrix} \end{array} \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix}$$

CNOT : NOT on the target qubit if the control $= 1$

circuit

$|\psi_0\rangle$: initial state $= |00\rangle = |0\rangle \otimes |0\rangle$

$$|\psi_1\rangle = (H \otimes I)\,|\psi_0\rangle = \underbrace{H|0\rangle}_{=|+\rangle} \otimes |0\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$$

$$|\psi_2\rangle = \mathrm{CNOT}\cdot|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(\mathrm{CNOT}\,|00\rangle + \mathrm{CNOT}\,|10\rangle\right) = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = |\Omega\rangle$$

Main theoretical tool : Schmidt decomposition

Fact : Any state $|\psi_{AB}\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ can be written as

$$|\psi_{AB}\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |a_i\rangle \otimes |b_i\rangle$$

where :
- $\lambda_i \geq 0$, $\sum \lambda_i = 1$
- $\{a_i\}_{i=1}^{r}$ is an orthonormal family in $\mathbb{C}^{d_A}$
- $\{b_i\}_{i=1}^{r}$ is an $\perp_n$ fam. in $\mathbb{C}^{d_B}$

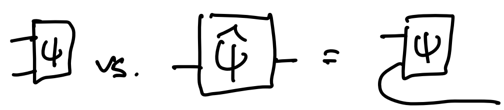
→ $\{\lambda_i\}$ are called the Schmidt coefficients of $|\psi_{AB}\rangle$

→ r is called the Schmidt rank of $|\psi_{AB}\rangle$

Remark Schmidt dec. of $|\psi_{AB}\rangle \iff$ SVD of $\hat{\psi}$

$\hat{\psi} \in \mathcal{M}_{d_A \times d_B}(\mathbb{C})$

$\hat{\psi}_{ij} = \langle ij|\psi_{AB}\rangle$



Entanglement for multipartite pure states

$$H_{ABC} = H_A \otimes H_B \otimes H_C$$

→ separable states : $|\psi_{ABC}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle$

→ entangled states $|\psi_{ABC}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle$

Examples of entangled states

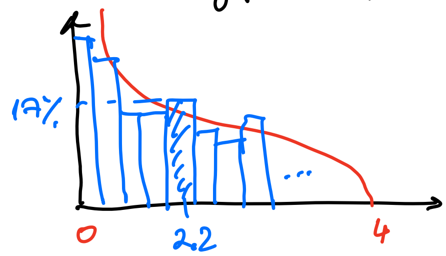
- $\frac{1}{\sqrt{2}}\left(|000\rangle + |011\rangle\right) = |0\rangle_A \otimes |\Omega\rangle_{BC}$
- $|GHZ\rangle := \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$
- $|W\rangle := \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$

$|\psi_{AB}\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ random $d \approx 500$

$\rightsquigarrow$

$\vec{\lambda} = (\lambda_1, \dots, \lambda_{500})$ = schmidt coeffs of $|\psi_{AB}\rangle$

histogram $(d \cdot \vec{\lambda})$ =



② Mixed states

$$d_A := \dim H_A \ll \dim H_B =: d_B$$

Syst of interest — environment

syst in contact with environment $\rightsquigarrow$ $|\psi_{AB}\rangle$ is entangled

$\rightsquigarrow$ it does not make sense to talk about the state of "A" system alone ?!

→ say we want to measure an observable X_A on A alone

$$\langle \psi_{AB} | X_A \otimes I_B | \psi_{AB}\rangle = \mathrm{Tr}\big(X_A \otimes I_B \cdot \underbrace{|\psi_{AB}\rangle\langle\psi_{AB}|}_{\rho_{AB}}\big)$$

$|\psi_{AB}\rangle\langle\psi_{AB}|$: rank-1 proj on $|\psi_{AB}\rangle$ vector $\in \mathcal{M}_{d_A \cdot d_B}$

$$= \langle X_A \otimes I_B, \rho_{AB}\rangle_{HS}$$

$\langle Y, Z\rangle_{HS} := \mathrm{Tr}(Y^* \cdot Z)$

$$= \langle F(X_A), \rho_{AB}\rangle_{HS}$$

$F(Y) = Y \otimes I_B$

$$= \langle X_A, F^*(\rho_{AB})\rangle_{HS}$$

$F^* = \mathrm{Tr}_B$ partial trace

$$= \langle X_A, \underbrace{\mathrm{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}|}_{\rho_A}\rangle_{HS}$$

$$= \langle X_A, \rho_A\rangle_{HS}$$

$\rho_A \in \mathcal{M}_{d_A}(\mathbb{C})$

Def A density matrix of size d is $\rho \in \mathcal{M}_d(\mathbb{C})$

$\rho \geq 0$ and $\operatorname{Tr} \rho = 1$

positive semidefinite (PSD)
spectrum $\subseteq [0, \infty)$

$\rho \in \mathcal{M}_d^{1,+}(\mathbb{C})$

Fact The set $\mathcal{M}_d^{1,+}$ of density matrices is a convex body, having extreme points

$$\operatorname{ext} \mathcal{M}_d^{1,+} = \{ |\varphi\rangle\langle\varphi| : \varphi \in \mathbb{C}^d, \|\varphi\| = 1 \}$$

$=$ "pure states"

The partial trace operation

$$\operatorname{Tr}_B : \underbrace{\mathcal{M}_{d_A \cdot d_B}}_{\simeq\, \mathcal{M}_{d_A} \otimes \mathcal{M}_{d_B}} \longrightarrow \mathcal{M}_{d_A}$$

$$A \otimes B \longmapsto (\operatorname{Tr} B) \cdot A$$

Eigenvalues vs Schmidt coefficients

Consider $|\psi_{AB}\rangle$ having S.D. $|\psi_{AB}\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |a_i\rangle |b_i\rangle$

$$\rho_A = \operatorname{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}| = \operatorname{Tr}_B \sum_{ij} \sqrt{\lambda_i \lambda_j}\, |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|$$

$$= \sum_{ij} \sqrt{\lambda_i \lambda_j}\, |a_i\rangle\langle a_j| \cdot \underbrace{\operatorname{Tr} |b_i\rangle\langle b_j|}_{= \langle b_j, b_i \rangle = \delta_{ij}}$$

$$\boxed{\rho_A = \sum_i \lambda_i\, |a_i\rangle\langle a_i|}$$

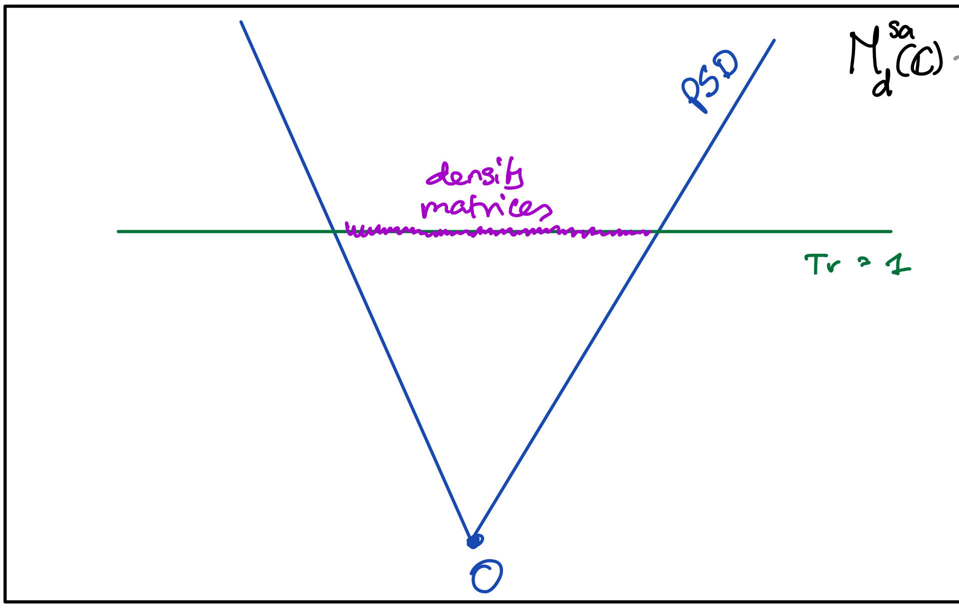
is a spectral decomp of ρ_A, since $\{a_i\}$ ON

So: [the schmidt coeffs of $|\psi_{AB}\rangle$]^2

$\parallel$

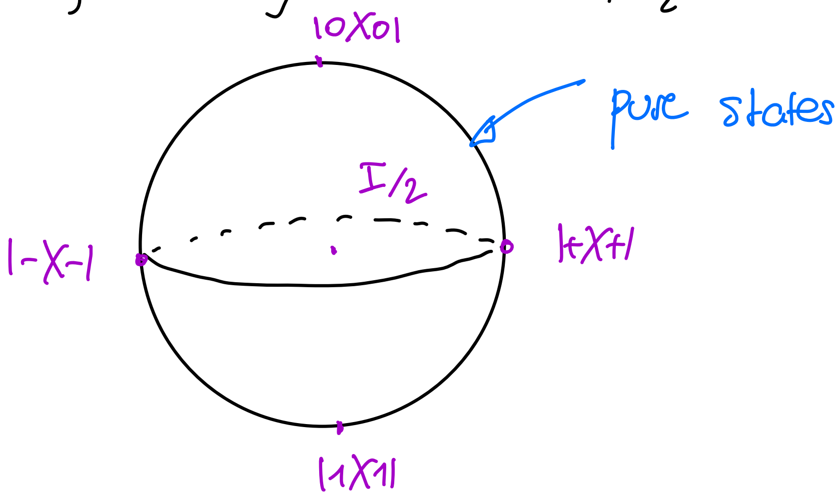
eigenvalues of $\rho_A = \operatorname{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}|$

Density matrices $= \{ \rho \in M_d^{sa}(\mathbb{C}) : \rho \geq 0 \ , \mathrm{Tr}\, \rho = 1 \}$



PSD_d = cone of positive, semidefinite matrices

- in dim $d=2$, density matrices $M_2^{1,+}$ = Bloch ball



$$\rho_{\vec{a}} = \frac{1}{2} (I + a_x \sigma_X + a_y \sigma_Y + a_z \sigma_Z)$$

$\rho_{\vec{a}}$ is a density matrix $\iff \|\vec{a}\| \leq 1$

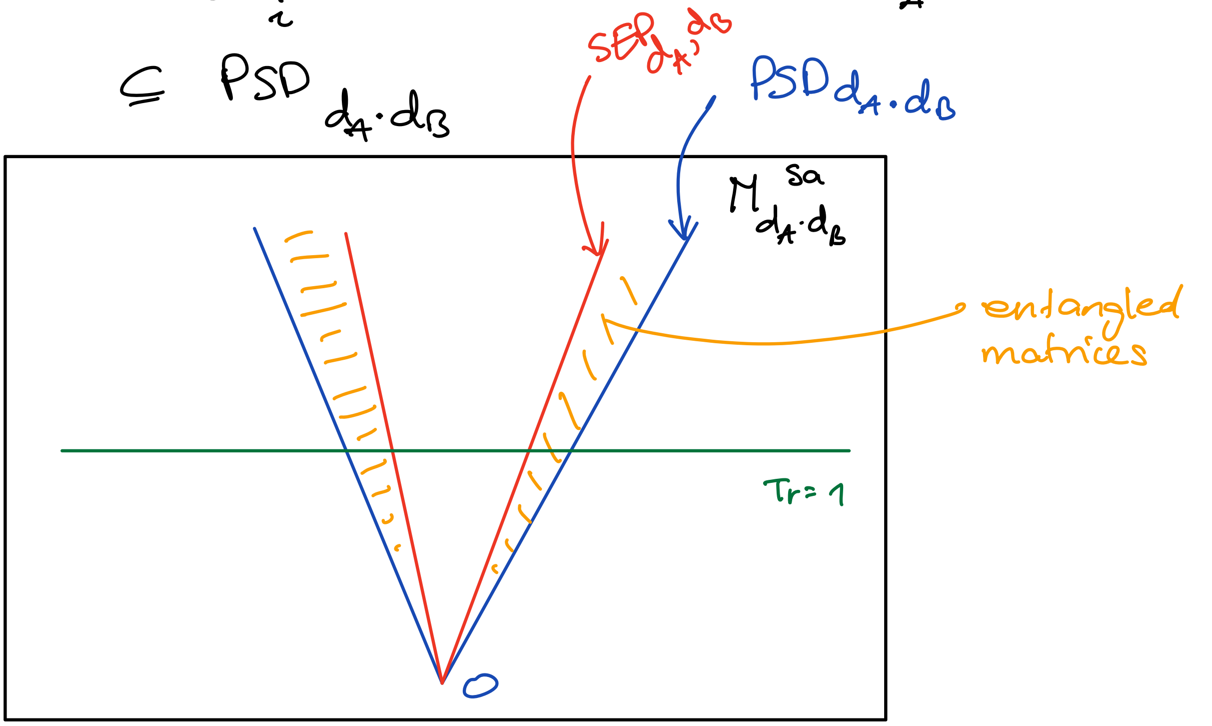
<u>Bipartite density matrices</u> $\quad M_{d_A d_B}^{1,+} \ni \rho$

$$\rho = \sum_{i=1}^{m} p_i \, \alpha_i \otimes \beta_i \quad \text{where} \quad p_1, \dots, p_m \geq 0 \ \sum p_i = 1$$

$\alpha_1, \dots, \alpha_m \in M_{d_A}^{1,+} \ , \beta_1, \dots, \beta_m \in M_{d_B}^{1,+}$

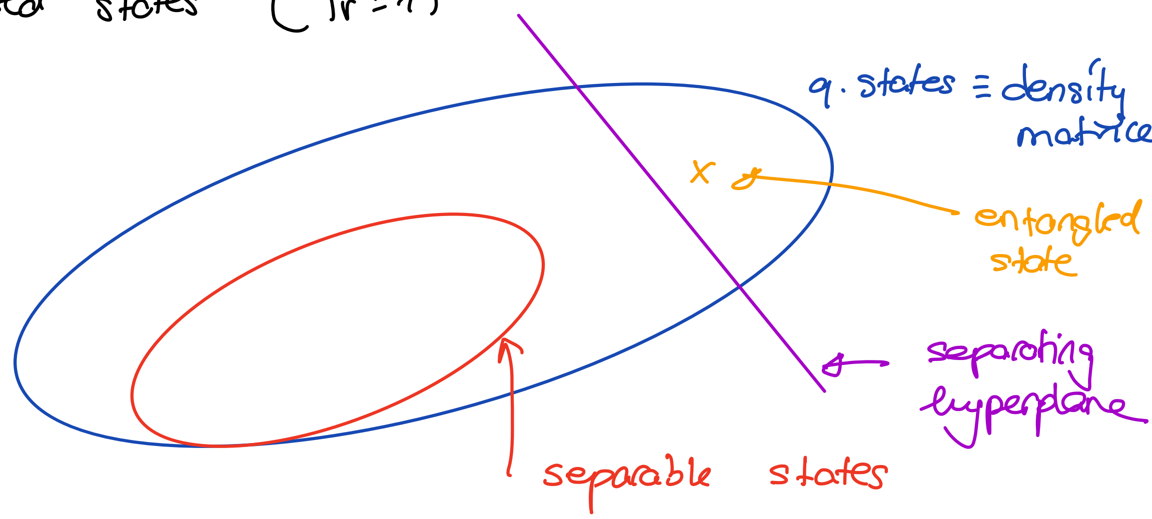
is called separable. Otherwise, it is entangled.

$$\begin{aligned} SEP_{d_A, d_B} &= \text{cone of separable (not normalized) states} \\ &= \Big\{ \sum_i A_i \otimes B_i \;:\; A_i \in PSD_{d_A},\ B_i \in PSD_{d_B} \Big\} \\ &\subsetneq PSD_{d_A \cdot d_B} \end{aligned}$$



<u>Fact</u> Deciding membership inside SEP_{d_A, d_B} is an NP-hard problem

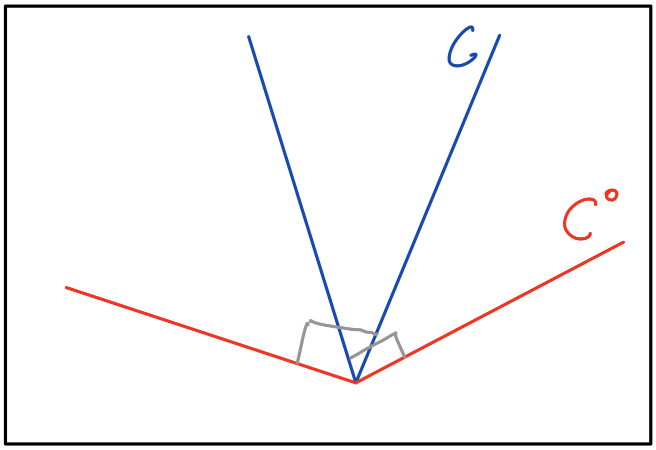
$\to$ Normalized states ($Tr = 1$)



$\rightsquigarrow$ Hahn - Banach separation theorem.

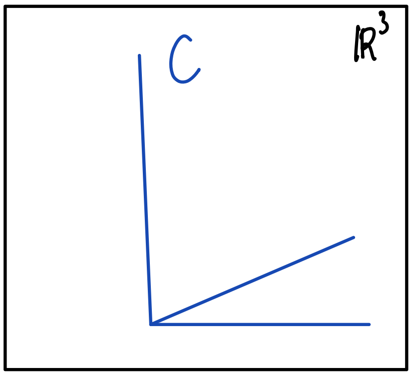
<u>Dual cones</u> Given a cone C, define

$$C^\circ = \{ v : \langle v, x \rangle \geq 0 \quad \forall x \in C \}$$



<u>Examples</u>

- $C = \mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_i \geq 0\}$



<u>Remark</u> Let $\mathbb{1} = (1, \dots, 1) \in \mathbb{R}^d$

$$\mathbb{R}^d_+ \cap \underbrace{\{x : \langle x, \mathbb{1} \rangle = 1\}}_{x : \sum x_i = 1} = $$

$$= \Delta_d = \{\text{prob. dist.}\}$$

$\dim_{\mathbb{R}} = d$

$$(\mathbb{R}^d_+)^\circ = \mathbb{R}^d_+ \quad \rightsquigarrow \text{ self-dual.}$$

- $PSD_d = \{X \in M_d^{sa}(\mathbb{C}) : \text{spec}(X) \subseteq [0, \infty)\}$

$$\Updownarrow$$

$$X = YY^*$$

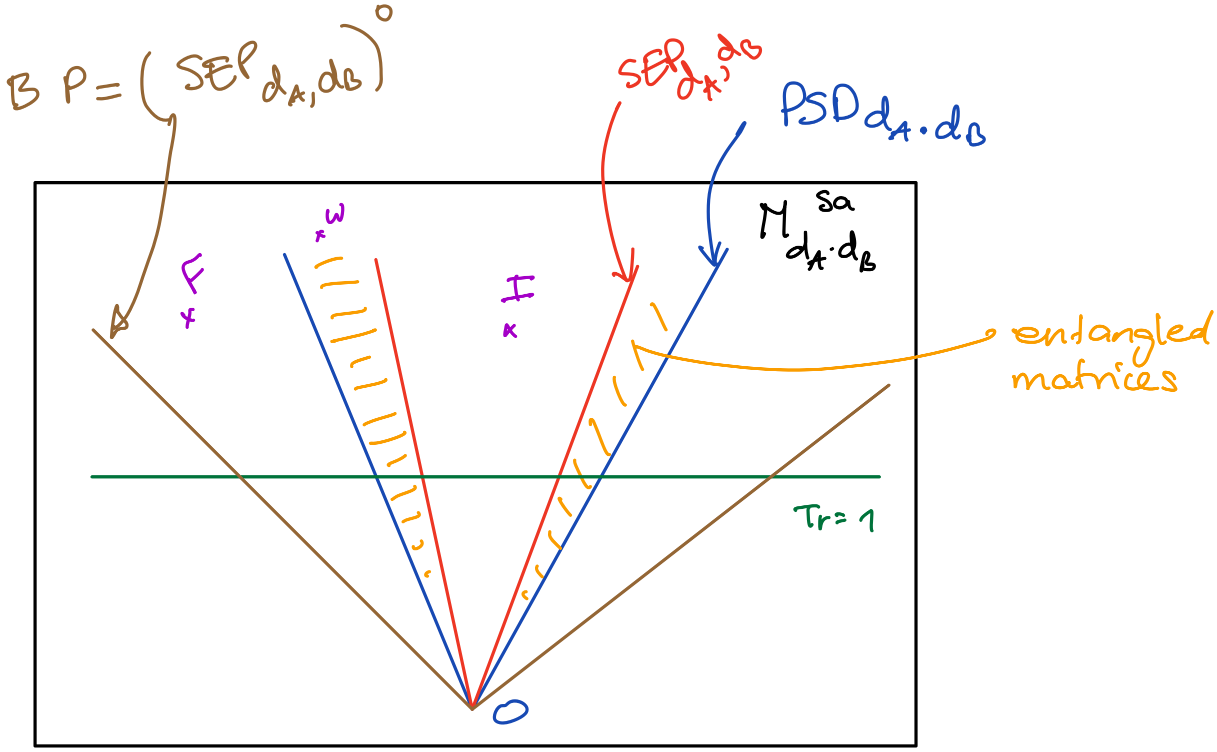
$\dim_{\mathbb{R}} = d^2$

$$(PSD_d)^\circ = \{Y : \langle Y, X \rangle_{HS} \geq 0 \ \forall X \in PSD_d\}$$

$$= PSD_d \quad \rightsquigarrow \text{ self-dual}$$

- $C = SEP_{d_A, d_B}$ $\qquad SEP^\circ_{d_A, d_B} = \{W : \langle W, X \rangle \geq 0 \ \forall X \in SEP\}$

<u>Fact</u> Each element $W \in SEP^\circ$ gives an entanglement criterion : if $\langle W, X \rangle \not\geq 0 \Rightarrow X \notin SEP$
$(X \in PSD)$ (i.e. X entangled)



if $\quad C_1 \subseteq C_2 \implies C_1^\circ \supseteq C_2^\circ$

so: $\quad SEP \subseteq PSD \implies SEP^\circ \supseteq PSD^\circ = PSD$

<u>fact</u> $\quad SEP$ is generated by $|x\rangle\langle x| \otimes |y\rangle\langle y|$

$SEP = \{ \sum A_i \otimes B_i \}$

do spectral decomp of A_i, B_i

$$SEP^\circ = \{ W : \langle W, S \rangle \geq 0 \quad \forall S \in SEP \}$$

$$= \{ W : \langle W, |x\rangle\langle x| \otimes |y\rangle\langle y| \rangle \geq 0 \quad \forall \begin{smallmatrix} x \in \mathbb{C}^{d_A} \\ y \in \mathbb{C}^{d_B} \end{smallmatrix} \}$$

$$= \{ W : \langle x \otimes y | W | x \otimes y \rangle \geq 0 \quad \forall x, y \}$$

$$= \{ \text{block-positive matrices} \} =: BP_{d_A, d_B}$$

<u>Remark</u> $\quad PSD_{d_A d_B} = \{ P : \langle z | P | z \rangle \geq 0 \quad \forall z \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \}$

<u>Example</u> $\quad SEP \subseteq PSD \subseteq BP$

- $\omega := |\Omega\rangle\langle\Omega| \in PSD \setminus SEP$ (i.e. it is entangled)
- Flip operator $F \in M_d \otimes M_d$ $\qquad F \in \mathcal{U}(d^2)$

$$F\, x \otimes y = y \otimes x \quad \forall x, y$$

$$F = P_s - P_a \qquad P_s = V^2(\mathbb{C}^d) \qquad d_A = d_B = d$$
$$P_a = \Lambda^2(\mathbb{C}^d)$$

$$\Downarrow$$

$$F \notin PSD \quad (\text{it has eigenvalue } -1)$$

However, $F \in BP$:

$$\langle x \otimes y | F | x \otimes y \rangle = \langle x \otimes y, y \otimes x \rangle$$
$$= \langle x, y \rangle \langle y, x \rangle = |\langle x, y \rangle|^2 \geq 0$$

$$\Rightarrow F \in BP \setminus PSD.$$

Relation to maps (Benoit's talk)

$$\begin{array}{cccccc} (M_{d_A} \otimes M_{d_B})^{sa} & SEP & \subseteq & PSD & \subseteq & BP \\ L^{Her}(M_{d_A} \to M_{d_B}) & EB & \subseteq & CP & \subseteq & Pos \\ & \text{entangl. breaking} & & \text{completely positive} & & \text{positive maps} \end{array}$$

$C \updownarrow J$

linear maps $\Phi : M_{d_A}(\mathbb{C}) \longrightarrow M_{d_B}(\mathbb{C})$ which preserve hermiticity: $\Phi(M_{d_A}^{sa}) \subseteq M_{d_B}^{sa}$

$$(M_{d_A} \otimes M_{d_B})^{sa} \simeq L^{Her}(M_{d_A} \to M_{d_B}) \simeq M_{d_A}^* \otimes M_{d_B}$$
$$M_{d_A}^* \cong M_{d_A}$$

Choi-Jamiołkowski isomorphism

Def $\Phi : M_{d_A} \to M_{d_B}$ is entanglement breaking if

$\forall n \quad (\Phi \otimes id_n)(X) \in SEP_{d_B, n}$

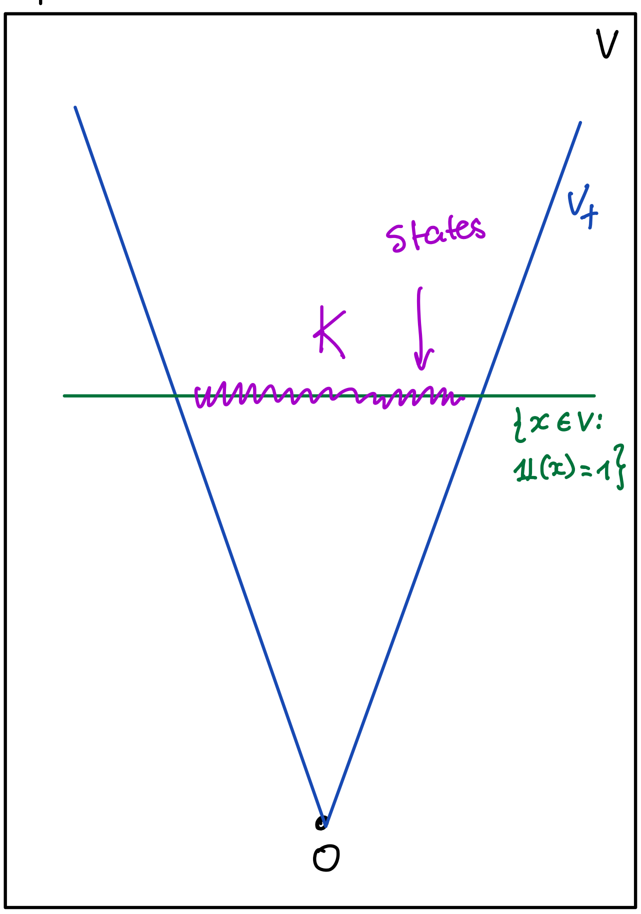
$\forall \; X \in PSD_{d_A \cdot n}$

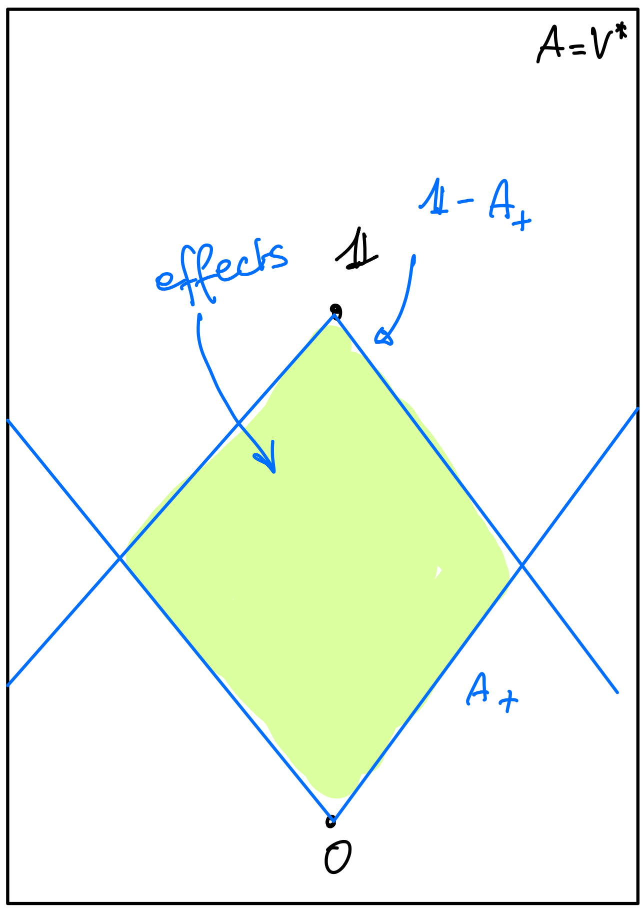
④ Entanglement in GPTs

Def A generalized probabilistic theory (GPT) is a triple $(V, V_+, \mathbb{1})$ where

- V is a real vector space
- V_+ is a proper cone in V
- $\mathbb{1}$ is a linear form on V

$$\mathbb{1} : V \to \mathbb{R}$$





→ $A := V^*$ is the dual vector space

→ $K := \{ x \in V : x \in V_+ \text{ and } \mathbb{1}(x) = 1 \}$

state space = set of states of the GPT

! K is a convex set   ! $\dim K = \dim V - 1$

→ A_+ = dual cone of V_+ in A : $A_+ = V_+^{\circ}$

$$A_+ = \{ \alpha \in A : \forall x \in V_+ , \underbrace{\alpha(x)}_{= \langle \alpha , x \rangle} \geq 0 \}$$

duality bracket

→ $\mathbb{1} - A_+ = \{ \mathbb{1} - \alpha : \alpha \in A_+ \}$

Definition

An effect in a GPT is an element $\alpha \in A$ s.t. $0 \leq \alpha \leq \mathbb{1}$

ordering given by A_+ :
$\alpha \leq \beta \Leftrightarrow \beta - \alpha \in A_+$

A measurement in a GPT is a collection $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ of effects s.t. $\sum \alpha_i = \mathbb{1}$

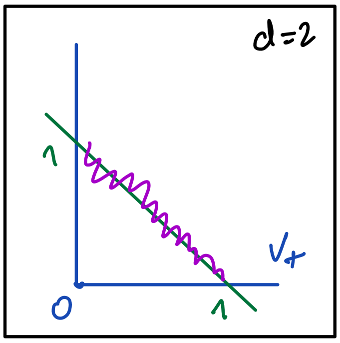
Born's rule in GPTs

When measuring a state $x \in K$ using a measurement $\underline{\alpha}$, we get outcome $i \in \{1, \dots, n\}$ with probability

$$\mathbb{P}(i) = \alpha_i(x) = \langle \alpha_i , x \rangle$$

Example 1 Classical theory $\mathcal{Cl}_d = \{ \mathbb{R}^d , \mathbb{R}^d_+ , \mathbb{1} \}$

$\mathbb{1}(x) = \sum x_i$



states: $K = \{x \in \mathbb{R}^d : x_i \geq 0, \sum x_i = 1\}$

$= \{$ probability vectors $\}$

effects: $A_+ = V_+$ (self-dual)

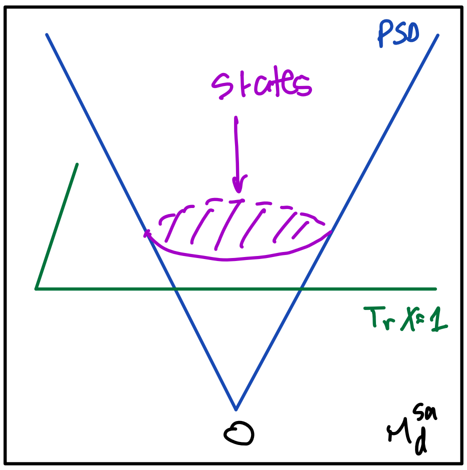
! $\text{ext}(K) = \{e_i\}_{i=1}^d$ Dirac masses

K is a simplex

(every point in K has a unique decomp in ext pts: $x = \sum_{i=1}^d x_i e_i$)

Example 2 Quantum mechanics

$QM_d = (M_d^{sa}(\mathbb{C}), PSD_d, \mathrm{Tr})$



states $= \{X \in M_d^{sa} : X \in PSD_d, \mathrm{Tr}\, X = 1\}$

$=$ density matrices

PSD self-dual $\Rightarrow$ effects A

$$0 \leq A \leq I$$

$A \in PSD$ " $A \leq \mathrm{Tr}$ "

duality bracket

$\langle A, X \rangle \leq \mathrm{Tr}\, X \quad \forall X \in PSD$

$\Leftrightarrow \langle I - A, X \rangle \geq 0 \quad \forall X \in PSD$

$\Leftrightarrow I - A \geq 0 \Leftrightarrow I - A \in PSD$

! $\text{ext}(K) = \{$ pure states $\cancel{|\varphi\rangle\langle\varphi|, \varphi \in \mathbb{C}^d}\}$

or rank 1 projections

K is not a simplex:

$d=2$ $\quad \frac{I}{2} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}(|+\rangle\langle +| + |-\rangle\langle -|)$

state space of QM_2 is the Bloch ball

Example 3 Ice cream GPT $IC_d = \{ \mathbb{R}^{d+1}, V_+, \mathbb{1} \}$

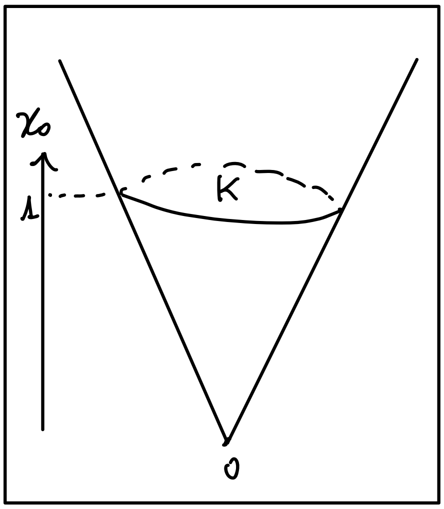
$x \in \mathbb{R}^{d+1}$ $x = (x_0, x_1, \ldots, x_d)$

$V_+ := \{ x \in \mathbb{R}^{d+1} : x_0 \geq x_1^2 + x_2^2 + \cdots + x_d^2 \}$

$\mathbb{1}(x) = x_0$

$K = \{ x \in V_+ , \mathbb{1}(x) = 1 \} = \{ (1, \underline{x}) : \|\underline{x}\| \leq 1 \}$

$\underline{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$



K = euclidean unit ball of $\mathbb{R}^d$

Bloch ball $IC_3 = QM_2$

Entanglement in GPTs What is a joint GPT ?

$(V^{(1)}, V_+^{(1)}, \mathbb{1}^{(1)})$ and $(V^{(2)}, V_+^{(2)}, \mathbb{1}^{(2)})$

$\rightsquigarrow$ how to define $V^{(12)}$, $V_+^{(12)}$, $\mathbb{1}^{(12)}$?

$\rightarrow$ $V^{(12)} := V^{(1)} \otimes V^{(2)}$

$\rightarrow$ $\mathbb{1}^{(12)} = \mathbb{1}^{(1)} \otimes \mathbb{1}^{(2)}$

$$\mathbb{1}^{(12)}(x \otimes y) = \mathbb{1}^{(1)}(x) \cdot \mathbb{1}^{(2)}(y)$$

$\rightarrow$ how about the cone ?

We would like:

- $x \in V_+^{(1)}, y \in V_+^{(2)} \Rightarrow x \otimes y \in V_+^{(12)}$
- $\alpha \in A_+^{(1)}, \beta \in A_+^{(2)} \Rightarrow \alpha \otimes \beta \in A_+^{(12)}$

$\hookrightarrow$ yield a LB and an UB for $V_+^{(12)}$

Def C_1, C_2 two cones

$$C_1 \otimes_{\min} C_2 = \operatorname{Cone}\{x \otimes y : x \in C_1, y \in C_2\}$$
$$= \{\sum x_i \otimes y_i : x_i \in C_1, y_i \in C_2\}$$

$$C_1 \otimes_{\max} C_2 = (C_1^\circ \otimes_{\min} C_2^\circ)^\circ$$
$$= \{z \in V_1 \otimes V_2 : \langle \alpha \otimes \beta, z \rangle \geq 0 \quad \forall \alpha \in C_1^\circ, \beta \in C_2^\circ\}$$

Theorem Any cone $V_+^{(12)}$ satisfying (*) is s.t.

$$V_+^{(1)} \otimes_{\min} V_+^{(2)} \subseteq V_+^{(12)} \subseteq V_+^{(1)} \otimes_{\max} V_+^{(2)}$$

Example QM_{d_A}, QM_{d_B}

$$QM_{d_A} \otimes_{\min} QM_{d_B} \subsetneq QM_{d_A d_B} \subsetneq QM_{d_A} \otimes_{\max} QM_{d_B}$$

$$QM_{d_A} \otimes_{\min} QM_{d_B} = SEP_{d_A, d_B}, \qquad QM_{d_A} \otimes_{\max} QM_{d_B} = BP_{d_A, d_B}$$

Thm (ALPP) arXiv: 2109.04446

$$C_1 \otimes_{\min} C_2 = C_1 \otimes_{\max} C_2$$

if and only if C_1 or C_2 is classical ($\mathcal{Cl}_d$)