A MATHEMATICAL INTRODUCTION TO QUANTUM ENTANGLEMENT

5 lectures. Plan:

1. Entanglement of pure states
2. Mixed states. Ent. criteria
3. Duality of cones. Positive maps
4. Ent. in GPTs general probabilistic theories
5. Tensor norms

① Entanglement of pure states

Entanglement: quantum phenomenon where the state of each particle of a group cannot be described independently of the others

Pure states Hilbert spaces $H \simeq \mathbb{C}^d$

- $\psi \in H$ $\|\psi\| = 1$ $|\psi\rangle$ pure state
- qubits $d=2$ $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

  $\alpha, \beta \in \mathbb{C}$ $|\alpha|^2 + |\beta|^2 = 1$

  examples: $|0\rangle, |1\rangle$ classical states

  $$|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
  $$|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

- quantum gates: unitary matrices $U \in \mathcal{U}(d)$

  Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

  $$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$$
  $$H|1\rangle = |-\rangle$$

Two quantum systems H_A, H_B $H_{AB} = H_A \otimes H_B$

$|\psi_{AB}\rangle \in H_{AB}$
- $\nearrow$ $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ separable
- $\searrow$ $|\psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$ entangled

Examples · $|0\rangle \otimes |0\rangle = |00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$

$\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) = |0\rangle \otimes \underbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}_{|+\rangle}$ are separable

- $|\Omega\rangle := \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \in \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$

2-qubit state

maximally entangled state, Bell state, singlet state ...

<u>Claim</u> $|\Omega\rangle$ is entangled.

$$|\Omega\rangle = \begin{array}{c}\\ \\ \end{array}\begin{pmatrix} x \\ y \end{pmatrix}\begin{array}{l}|0\rangle \\ |1\rangle\end{array} \otimes \begin{pmatrix} a \\ b \end{pmatrix}\begin{array}{l}|0\rangle \\ |1\rangle\end{array} \iff$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\begin{array}{l} \leftarrow |00\rangle \\ \leftarrow |01\rangle \\ \leftarrow |10\rangle \\ \leftarrow |11\rangle \end{array} = \begin{bmatrix} xa \\ xb \\ ya \\ yb \end{bmatrix}$$

$$\begin{array}{l} xa = 1 \\ xb = 0 \\ ya = 0 \\ yb = 1 \end{array}$$

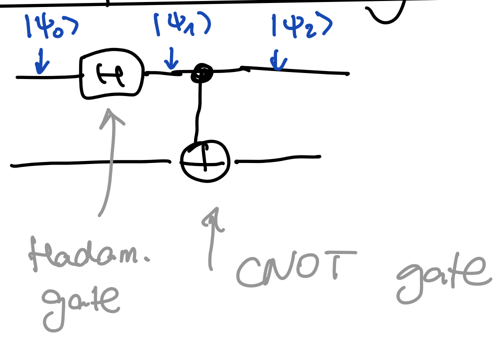
$0 = (xb)(ya)$

$* = 1$

$* = xyab$

impossible!

<u>Circuit for building $|\Omega\rangle$</u>



CNOT gate (2-qubit)

$\mathrm{CNOT}\ |00\rangle = |00\rangle$

$|01\rangle = |01\rangle$

$|10\rangle = |11\rangle$

$|11\rangle = |10\rangle$

control qubit   target qubit

$$\mathrm{CNOT} = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \end{array} \begin{array}{l} 00 \\ 01 \\ 10 \\ 11 \\ \\ \end{array}$$

CNOT : NOT on the target qubit if the control $= 1$

<u>circuit</u>

$|\psi_0\rangle$: initial state $= |00\rangle = |0\rangle \otimes |0\rangle$

$|\psi_1\rangle = (H \otimes I)\,|\psi_0\rangle = \underbrace{H|0\rangle}_{=|+\rangle} \otimes |0\rangle$

$= \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$

$|\psi_2\rangle = \mathrm{CNOT}\cdot|\psi_1\rangle =$

$= \frac{1}{\sqrt{2}}\left(\mathrm{CNOT}\,|00\rangle + \mathrm{CNOT}\,|10\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = |\Omega\rangle$

Main theoretical tool : Schmidt decomposition

Fact: Any state $|\psi_{AB}\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ can be written as

$$|\psi_{AB}\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |a_i\rangle \otimes |b_i\rangle$$

where :
- $\lambda_i \geq 0$, $\sum \lambda_i = 1$
- $\{a_i\}_{i=1}^{r}$ is an orthonormal family in $\mathbb{C}^{d_A}$
- $\{b_i\}_{i=1}^{r}$ is an $\perp_n$ fam. in $\mathbb{C}^{d_B}$

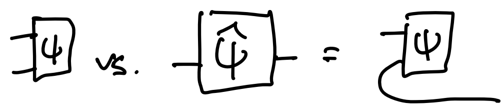
→ $\{\lambda_i\}$ are called the Schmidt coefficients of $|\psi_{AB}\rangle$

→ r is called the Schmidt rank of $|\psi_{AB}\rangle$

Remark Schmidt dec. of $|\psi_{AB}\rangle \Longleftrightarrow$ SVD of $\hat{\psi}$

$\hat{\psi} \in \mathcal{M}_{d_A \times d_B}(\mathbb{C})$

$\hat{\psi}_{ij} = \langle ij | \psi_{AB}\rangle$



Entanglement for multipartite pure states

$$H_{ABC} = H_A \otimes H_B \otimes H_C$$

→ separable states: $|\psi_{ABC}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle$

→ entangled states $|\psi_{ABC}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle$

Examples of entangled states

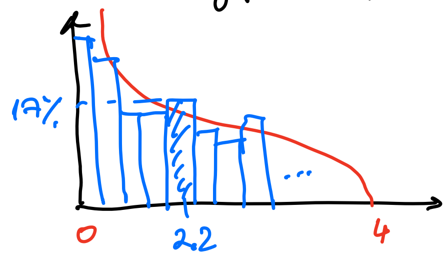
- $\frac{1}{\sqrt{2}}\left(|000\rangle + |011\rangle\right) = |0\rangle_A \otimes |\Omega\rangle_{BC}$
- $|GHZ\rangle := \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$
- $|W\rangle := \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$

$|\psi_{AB}\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ random $d \simeq 500$

$\rightsquigarrow$

$\vec{\lambda} = (\lambda_1, \ldots, \lambda_{500})$ = schmidt coeffs of $|\psi_{AB}\rangle$

histogram $(d \cdot \vec{\lambda})$ =



② Mixed states

$d_A := \dim H_A \ll \dim H_B =: d_B$

Syst of interest (↑ H_A) environment (↑ H_B)

syst in contact with environment $\rightsquigarrow$ $|\psi_{AB}\rangle$ is entangled

$\rightsquigarrow$ it does not make sense to talk about the state of "A" system alone ?!

→ Say we want to measure an observable X_A on A alone

$$\langle \psi_{AB} | X_A \otimes I_B | \psi_{AB} \rangle = \mathrm{Tr}\Big(X_A \otimes I_B \cdot \underbrace{|\psi_{AB}\rangle\langle\psi_{AB}|}_{\substack{\text{rank-1 proj on} \\ |\psi_{AB}\rangle \text{ vector} \\ \in \mathcal{M}_{d_A \cdot d_B}}} \Big) \quad \rho_{AB}$$

$$= \langle X_A \otimes I_B , \rho_{AB} \rangle_{HS}$$

$\langle Y, Z \rangle_{HS} := \mathrm{Tr}(Y^* \cdot Z)$

$$= \langle F(X_A), \rho_{AB} \rangle_{HS} \qquad F(Y) = Y \otimes I_B$$

$$= \langle X_A, F^*(\rho_{AB}) \rangle_{HS} \qquad F^* = \mathrm{Tr}_B \text{ partial trace}$$

$$= \langle X_A, \underbrace{\mathrm{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}|}_{\rho_A} \rangle_{HS}$$

$$= \langle X_A, \rho_A \rangle_{HS} \qquad \rho_A \in \mathcal{M}_{d_A}(\mathbb{C})$$

Def A density matrix of size d is $\rho \in \mathcal{M}_d(\mathbb{C})$

$\rho \geq 0$ and $\operatorname{Tr} \rho = 1$

positive semidefinite (PSD)
spectrum $\subseteq [0,\infty)$

$$\rho \in \mathcal{M}_d^{1,+}(\mathbb{C})$$

Fact The set $\mathcal{M}_d^{1,+}$ of density matrices is a convex body, having extreme points

$$\operatorname{ext} \mathcal{M}_d^{1,+} = \{ |\psi\rangle\langle\psi| : \psi \in \mathbb{C}^d, \|\psi\| = 1 \}$$
$$= \text{"pure states"}$$

The partial trace operation

$$\mathcal{M}_{d_A} \otimes \mathcal{M}_{d_B}$$
$$\begin{aligned} \operatorname{Tr}_B : \mathcal{M}_{d_A \cdot d_B} &\longrightarrow \mathcal{M}_{d_A} \\ A \otimes B &\longmapsto (\operatorname{Tr} B) \cdot A \end{aligned}$$

Eigenvalues vs Schmidt coefficients

Consider $|\psi_{AB}\rangle$ having S.D. $|\psi_{AB}\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |a_i\rangle |b_i\rangle$

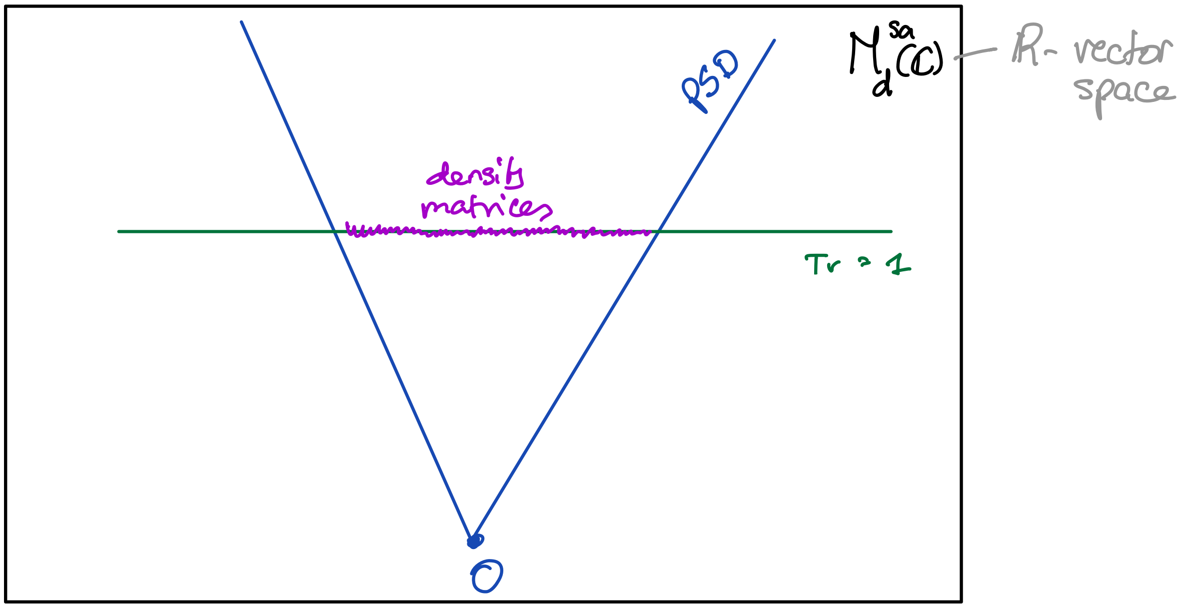
$$\rho_A = \operatorname{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}| = \operatorname{Tr}_B \sum_{ij} \sqrt{\lambda_i \lambda_j}\, |a_i\rangle\langle a_j| \otimes |b_i\rangle\langle b_j|$$
$$= \sum_{ij} \sqrt{\lambda_i \lambda_j}\, |a_i\rangle\langle a_j| \cdot \underbrace{\operatorname{Tr} |b_i\rangle\langle b_j|}_{= \langle b_j, b_i\rangle = \delta_{ij}}$$

$$\boxed{\rho_A = \sum_i \lambda_i |a_i\rangle\langle a_i|}$$

is a spectral decomp of ρ_A, since $\{a_i\}$ ON

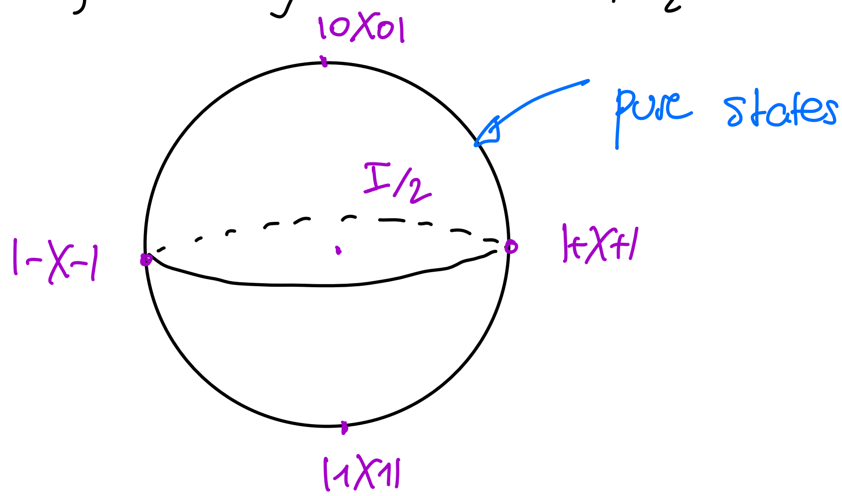
So: [the schmidt coeffs of $|\psi_{AB}\rangle$]^2
$=$
eigenvalues of $\rho_A = \operatorname{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}|$

Density matrices $= \{\rho \in M_d^{sa}(\mathbb{C}) : \rho \geq 0, \mathrm{Tr}\,\rho = 1\}$



PSD_d = cone of positive, semidefinite matrices

- in dim $d=2$, density matrices $M_2^{1,+}$ = Bloch ball



$$\rho_{\vec{a}} = \frac{1}{2}\left(I + a_x \sigma_X + a_y \sigma_Y + a_z \sigma_Z \right)$$

$\rho_{\vec{a}}$ is a density matrix $\iff \|\vec{a}\| \leq 1$

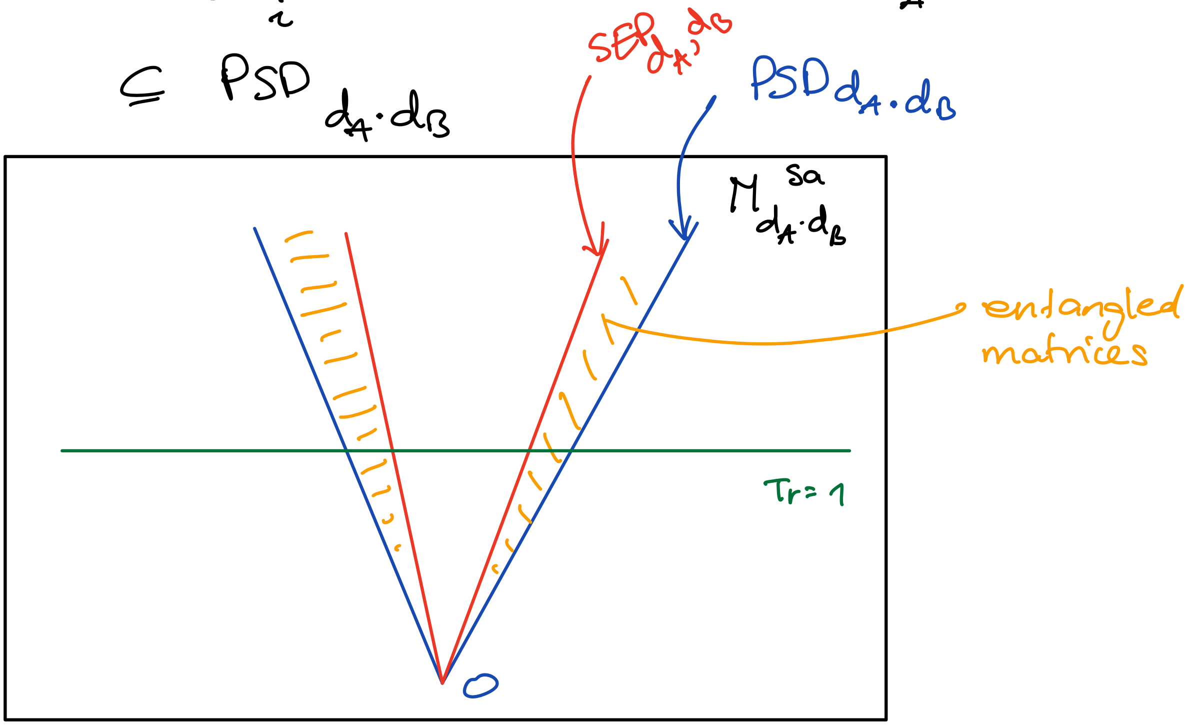
Bipartite density matrices $\quad M_{d_A d_B}^{1,+} \ni \rho$

$$\rho = \sum_{i=1}^{n} p_i \, \alpha_i \otimes \beta_i \quad \text{where} \quad p_1, \dots, p_n \geq 0 \quad \sum p_i = 1$$

$$\alpha_1, \dots, \alpha_n \in M_{d_A}^{1,+}, \quad \beta_1, \dots, \beta_n \in M_{d_B}^{1,+}$$

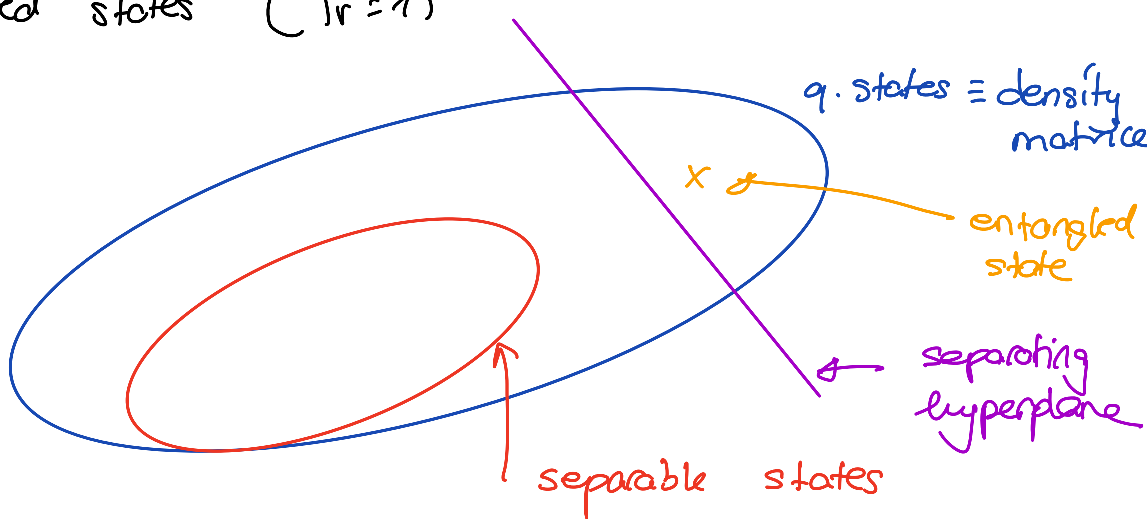
is called separable. Otherwise, it is entangled.

$$\mathrm{SEP}_{d_A, d_B} = \text{cone of separable (not normalized) states}$$
$$= \left\{ \sum_i A_i \otimes B_i \ : \ A_i \in \mathrm{PSD}_{d_A}, B_i \in \mathrm{PSD}_{d_B} \right\}$$
$$\subsetneq \mathrm{PSD}_{d_A \cdot d_B}$$



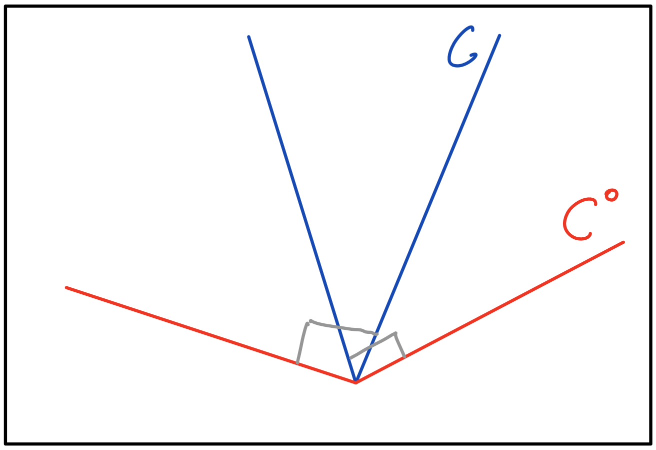
Fact: Deciding membership inside SEP_{d_A, d_B} is an NP-hard problem

→ Normalized states (Tr = 1)

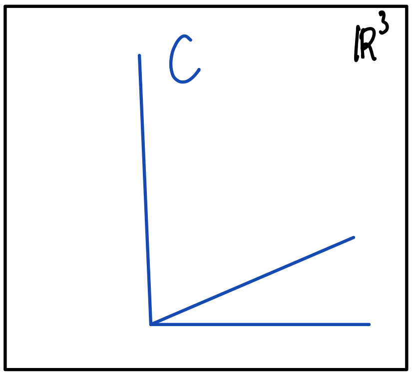


⇝ Hahn-Banach separation theorem.

Dual cones: Given a cone C, define
$$C^\circ = \{ v : \langle v, x \rangle \geq 0 \quad \forall x \in C \}$$



__Examples__ • $C = \mathbb{R}_+^d = \{x \in \mathbb{R}^d : x_i \geq 0\}$



__Remark__ Let $\mathbb{1} = (1, \dots, 1) \in \mathbb{R}^d$

$$\mathbb{R}_+^d \cap \underbrace{\{x : \langle x, \mathbb{1}\rangle = 1\}}_{x : \sum x_i = 1} =$$

$$= \Delta_d = \{\text{prob. dist.}\}$$

$\dim_{\mathbb{R}} = d$

$$(\mathbb{R}_+^d)^\circ = \mathbb{R}_+^d \quad \rightsquigarrow \text{self-dual.}$$

• $\mathrm{PSD}_d = \{X \in M_d^{sa}(\mathbb{C}) : \mathrm{spec}(X) \subseteq [0, \infty)\}$

$$\Updownarrow$$

$$X = YY^*$$

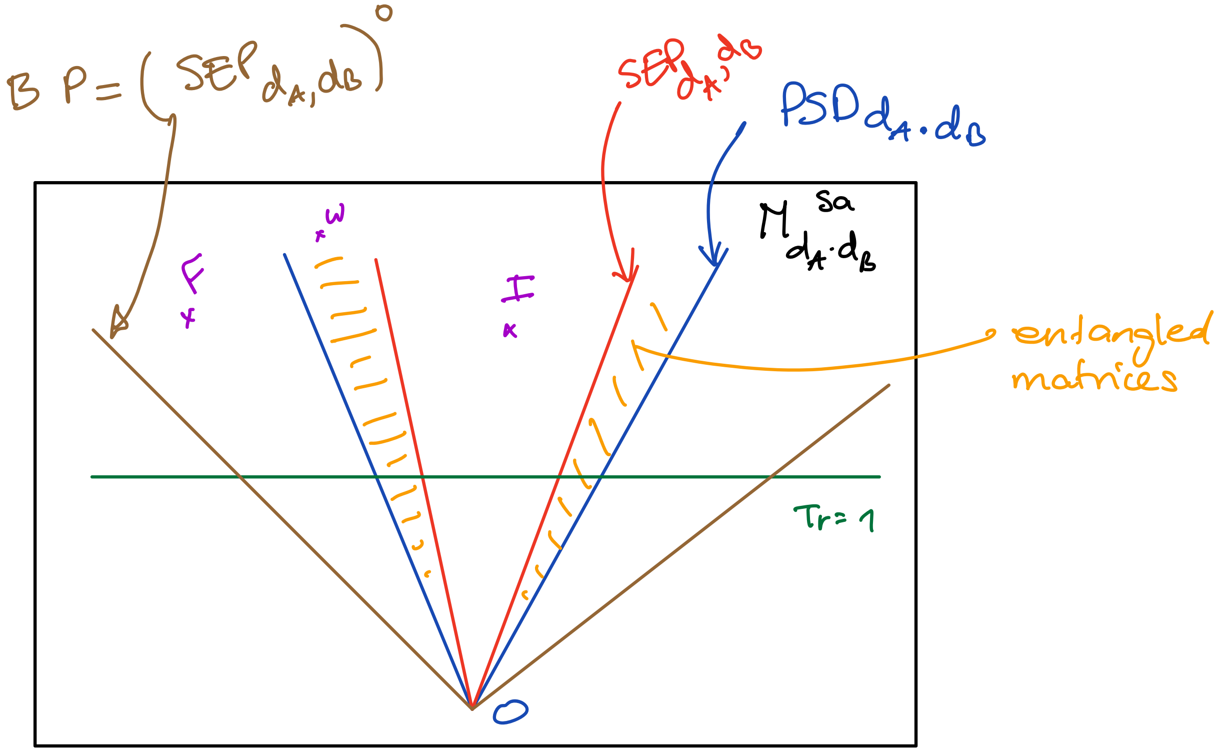
$\dim_{\mathbb{R}} = d^2$

$$(\mathrm{PSD}_d)^\circ = \{Y : \langle Y, X\rangle_{HS} \geq 0 \ \forall X \in \mathrm{PSD}_d\}$$

$$= \mathrm{PSD}_d \quad \rightsquigarrow \text{self-dual}$$

• $C = \mathrm{SEP}_{d_A, d_B}$ $\qquad \mathrm{SEP}_{d_A, d_B}^\circ = \{W : \langle W, X\rangle \geq 0 \ \forall X \in \mathrm{SEP}\}$

__Fact__ Each element $W \in \mathrm{SEP}^\circ$ gives an entanglement criterion $\underset{(X \in \mathrm{PSD})}{:}$ if $\langle W, X\rangle \not\geq 0 \Rightarrow X \notin \mathrm{SEP}$ (i.e. X entangled)



if $\quad C_1 \subseteq C_2 \implies C_1^\circ \supseteq C_2^\circ$

so: $\quad SEP \subseteq PSD \implies SEP^\circ \supseteq PSD^\circ = PSD$

<u>fact</u> $\quad$ SEP is generated by $|x\rangle\langle x| \otimes |y\rangle\langle y|$

$SEP = \{ \sum A_i \otimes B_i \}$

do spectral decomp of A_i, B_i

$$
\begin{aligned}
SEP^\circ &= \{ W : \langle W, S \rangle \geq 0 \quad \forall S \in SEP \} \\
&= \{ W : \langle W, |x\rangle\langle x| \otimes |y\rangle\langle y| \rangle \geq 0 \quad \forall \begin{smallmatrix} x \in \mathbb{C}^{d_A} \\ y \in \mathbb{C}^{d_B} \end{smallmatrix} \} \\
&= \{ W : \langle x \otimes y | W | x \otimes y \rangle \geq 0 \quad \forall x, y \} \\
&= \{ \text{block-positive matrices} \} =: BP_{d_A, d_B}
\end{aligned}
$$

<u>Remark</u> $\quad PSD_{d_A d_B} = \{ P : \langle z | P | z \rangle \geq 0 \quad \forall z \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \}$

<u>Example</u> $\quad SEP \subseteq PSD \subseteq BP$

- $\omega := |\Omega\rangle\langle\Omega| \in PSD \setminus SEP$ (i.e. it is entangled)
- Flip operator $F \in M_d \otimes M_d$ $\quad F \in \mathcal{U}(d^2)$

$$F\, x\otimes y = y \otimes x \quad \forall x, y$$

$$F = P_s - P_a \qquad P_s = \vee^2(\mathbb{C}^d) \qquad d_A = d_B = d$$
$$P_a = \Lambda^2(\mathbb{C}^d)$$

$$\Downarrow$$

$$F \notin PSD \quad (\text{it has eigenvalue } -1)$$

However, $F \in BP$:

$$\langle x\otimes y | F | x \otimes y\rangle = \langle x\otimes y, y \otimes x\rangle$$
$$= \langle x, y\rangle \langle y, x\rangle = |\langle x, y\rangle|^2 \geq 0$$

$$\Rightarrow F \in BP \setminus PSD.$$

Relation to maps (Benoit's talk)

$$(M_{d_A} \otimes M_{d_B})^{sa} \qquad SEP \subseteq PSD \subseteq BP$$

$$L^{Her}(M_{d_A} \to M_{d_B}) \qquad EB \subseteq CP \subseteq Pos$$

EB: entangl. breaking; CP: completely positive; Pos: positive maps

$\Downarrow$ linear maps $\Phi : M_{d_A}(\mathbb{C}) \to M_{d_B}(\mathbb{C})$ which preserve hermiticity: $\Phi(M_{d_A}^{sa}) \subseteq M_{d_B}^{sa}$

$$(M_{d_A} \otimes M_{d_B})^{sa} \simeq L^{Her}(M_{d_A} \to M_{d_B}) \simeq M_{d_A}^{*} \otimes M_{d_B}$$

$$M_{d_A}^{*} \cong M_{d_A}$$

$\uparrow$ Choi-Jamiołkowski isomorphism

Def $\Phi : M_{d_A} \to M_{d_B}$ is entanglement breaking if

$\forall m \quad (\Phi \otimes id_m)(X) \in SEP_{d_B, m}$

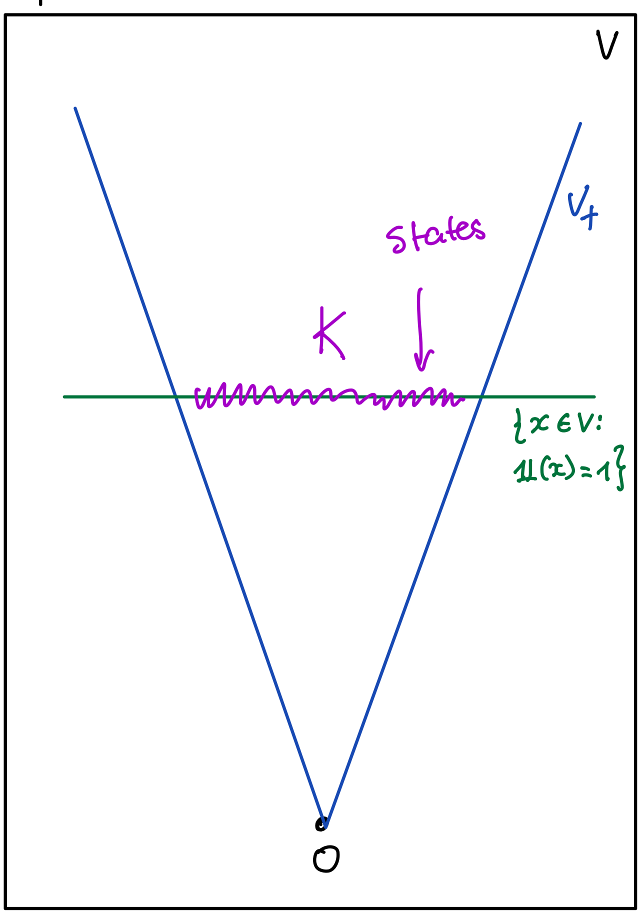
$\forall \; X \in PSD_{d_A \cdot m}$

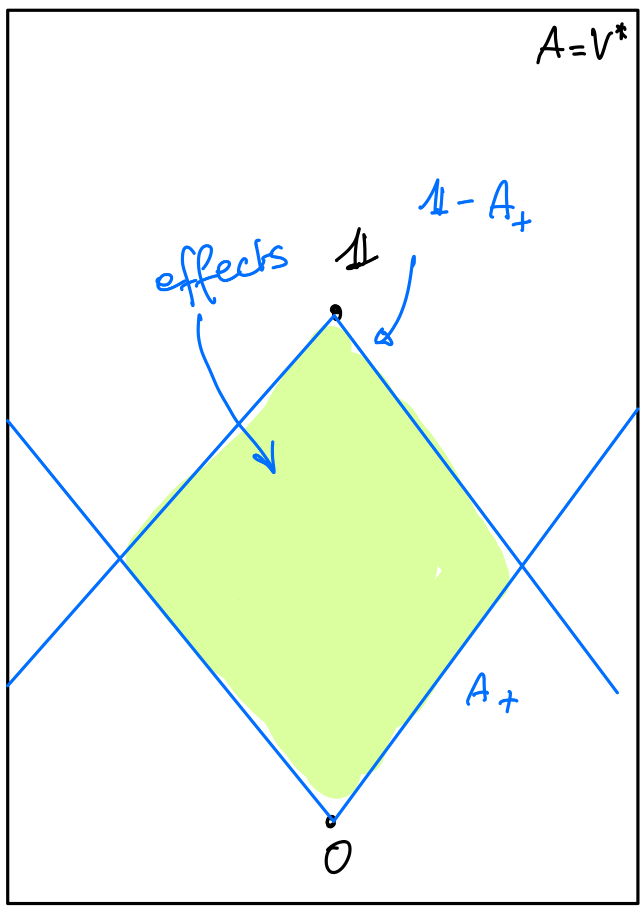
④ Entanglement in GPTs

Def A generalized probabilistic theory (GPT) is a triple $(V, V_+, \mathbb{1})$ where

- V is a real vector space
- V_+ is a proper cone in V
- $\mathbb{1}$ is a linear form on V

$$\mathbb{1} : V \to \mathbb{R}$$





→ $A := V^*$ is the dual vector space

→ $K := \{ x \in V : x \in V_+ \text{ and } \mathbb{1}(x) = 1 \}$

state space ≡ set of states of the GPT

! K is a convex set   ! $\dim K = \dim V - 1$

→ A_+ = dual cone of V_+ in A : $A_+ = V_+^\circ$

$$A_+ = \{ \alpha \in A : \forall x \in V_+ , \underbrace{\alpha(x)}_{\langle \alpha, x \rangle} \geq 0 \}$$

duality bracket

→ $\mathbb{1} - A_+ = \{ \mathbb{1} - \alpha : \alpha \in A_+ \}$

<u>Definition</u> An effect in a GPT is an element $\alpha \in A$ s.t. $0 \leq \alpha \leq \mathbb{1}$

ordering given by A_+ : $\alpha \leq \beta \Leftrightarrow \beta - \alpha \in A_+$

A measurement in a GPT is a collection $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ of effects s.t. $\sum \alpha_i = \mathbb{1}$

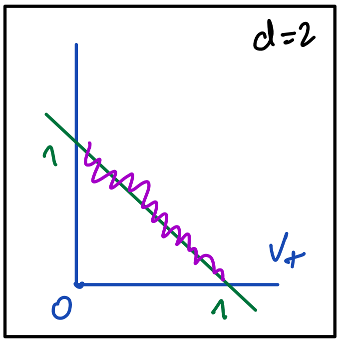
Born's rule in GPTs

When measuring a state $x \in K$ using a measurement $\underline{\alpha}$, we get outcome $i \in \{1, \dots, n\}$ with probability

$$\mathbb{P}(i) = \alpha_i(x) = \langle \alpha_i, x \rangle$$

<u>Example 1</u> Classical theory $\mathcal{C}\ell_d = \{ \mathbb{R}^d, \mathbb{R}^d_+, \mathbb{1} \}$

$\mathbb{1}(x) = \sum x_i$



states: $K = \{x \in \mathbb{R}^d : x_i \geq 0, \sum x_i = 1\}$
$= \{$probability vectors$\}$

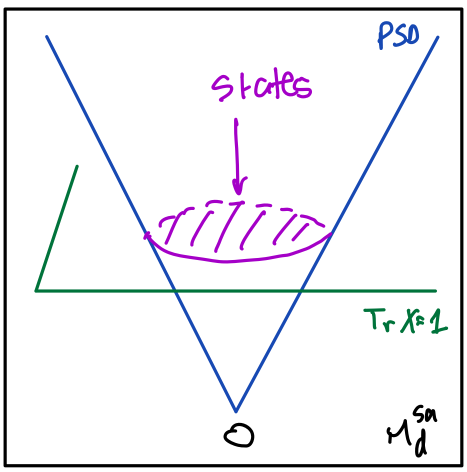
effects: $A_+ = V_+$ (self-dual)

! $\mathrm{ext}(K) = \{e_i\}_{i=1}^d$ Dirac masses

K is a simplex

(every point in K has a unique decomp in ext pts:
$x = \sum_{i=1}^d x_i e_i$)

Example 2 Quantum mechanics $QM_d = (M_d^{sa}(\mathbb{C}), PSD_d, \mathrm{Tr})$



states = $\{X \in M_d^{sa} : X \in PSD_d, \mathrm{Tr}X = 1\}$
= density matrices

PSD self-dual $\Rightarrow$ effects A

$$0 \leq A \leq I$$

$A \in PSD$ "$A \leq \mathrm{Tr}$"

duality bracket

$\langle A, X\rangle \leq \mathrm{Tr}X \quad \forall X \in PSD$

$\Leftrightarrow \langle I - A, X\rangle \geq 0 \quad \forall X \in PSD$

$\Leftrightarrow I - A \geq 0 \Leftrightarrow I - A \in PSD$

! $\mathrm{ext}(K) = \{$pure states ~~$|\varphi\rangle\langle\varphi|, \varphi \in \mathbb{C}^d$~~ $\}$
or rank 1 projections

K is not a simplex:

$d=2$ $\quad \frac{I}{2} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}(|+\rangle\langle +| + |-\rangle\langle -|)$

state space of QM_2 is the Bloch ball

Example 3 Ice cream GPT $IC_d = \{\mathbb{R}^{d+1}, V_+, \mathbb{1}\}$

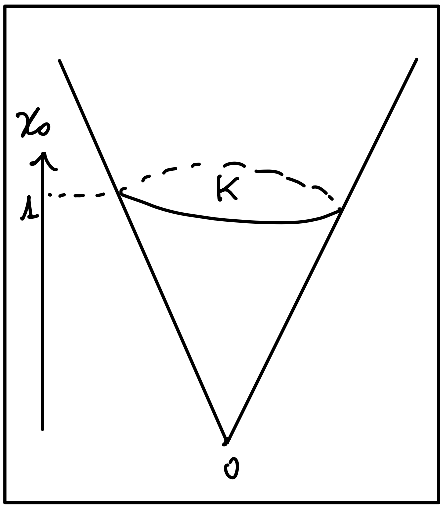
$x \in \mathbb{R}^{d+1}$ $\quad x = (x_0, x_1, \dots, x_d)$

$$V_+ := \{x \in \mathbb{R}^{d+1} : x_0 \geq x_1^2 + x_2^2 + \dots + x_d^2\}$$

$$\mathbb{1}(x) = x_0$$

$$K = \{x \in V_+, \mathbb{1}(x) = 1\} = \{(1, \underline{x}) : \|\underline{x}\| \leq 1\}$$

$$\underline{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$$



K = euclidean unit ball of $\mathbb{R}^d$

Bloch ball $IC_3 = QM_2$

Entanglement in GPTs

What is a joint GPT?

$$(V^{(1)}, V_+^{(1)}, \mathbb{1}^{(1)}) \quad \text{and} \quad (V^{(2)}, V_+^{(2)}, \mathbb{1}^{(2)})$$

$\leadsto$ how to define $V^{(12)}$, $V_+^{(12)}$, $\mathbb{1}^{(12)}$?

$\rightarrow$ $V^{(12)} := V^{(1)} \otimes V^{(2)}$

$\rightarrow$ $\mathbb{1}^{(12)} = \mathbb{1}^{(1)} \otimes \mathbb{1}^{(2)}$

$$\mathbb{1}^{(12)}(x \otimes y) = \mathbb{1}^{(1)}(x) \cdot \mathbb{1}^{(2)}(y)$$

$\rightarrow$ how about the cone?

We would like:

K
- $x \in V_+^{(1)}, y \in V_+^{(2)} \Rightarrow x \otimes y \in V_+^{(12)}$
- $\alpha \in A_+^{(1)}, \beta \in A_+^{(2)} \Rightarrow \alpha \otimes \beta \in A_+^{(12)}$

$\hookrightarrow$ yield a LB and an UB for $V_+^{(12)}$

Def C_1, C_2 two cones

$$C_1 \underset{\min}{\otimes} C_2 = \operatorname{Cone}\{x \otimes y : x \in C_1, y \in C_2\}$$
$$= \{\textstyle\sum x_i \otimes y_i : x_i \in C_1, y_i \in C_2\}$$

$$C_1 \underset{\max}{\otimes} C_2 = \left(C_1^\circ \underset{\min}{\otimes} C_2^\circ\right)^\circ$$
$$= \{z \in V_1 \otimes V_2 : \langle \alpha \otimes \beta, z\rangle \geq 0 \quad \forall \alpha \in C_1^\circ, \beta \in C_2^\circ\}$$

Theorem Any cone $V_+^{(12)}$ satisfying $\circledast$ is s.t.

$$V_+^{(1)} \underset{\min}{\otimes} V_+^{(2)} \subseteq V_+^{(12)} \subseteq V_+^{(1)} \underset{\max}{\otimes} V_+^{(2)}$$

Example QM_{d_A}, QM_{d_B}

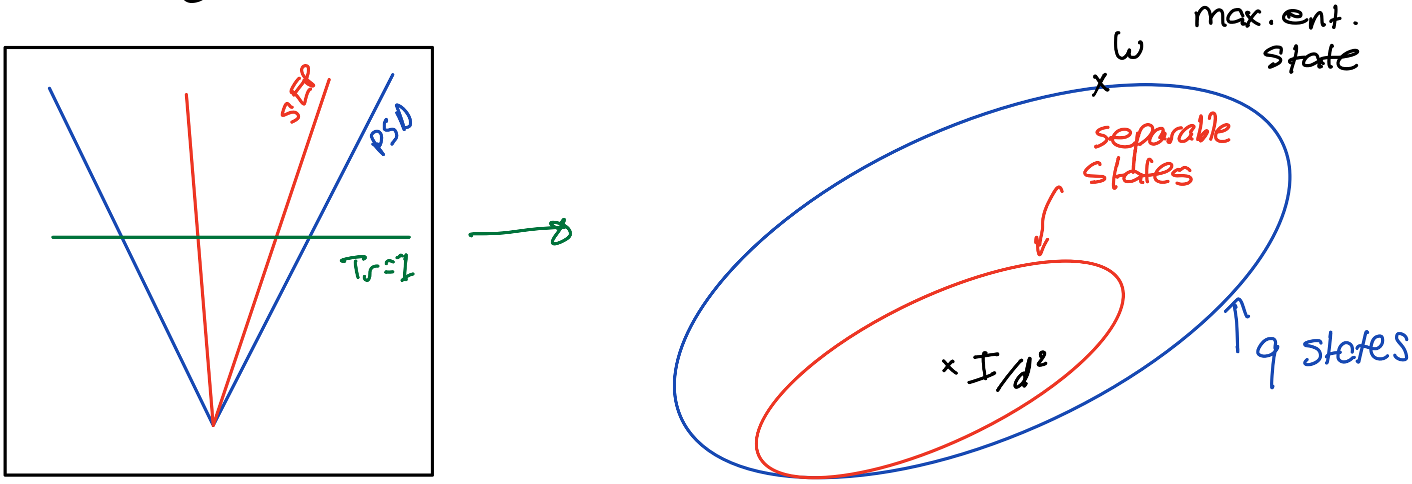
$$\underbrace{QM_{d_A} \underset{\min}{\otimes} QM_{d_B}}_{= SEP_{d_A, d_B}} \subsetneq QM_{d_A d_B} \subsetneq \underbrace{QM_{d_A} \underset{\max}{\otimes} QM_{d_B}}_{= BP_{d_A, d_B}}$$

Thm (ALPP) arXiv: 2109.04446

$$C_1 \underset{\min}{\otimes} C_2 = C_1 \underset{\max}{\otimes} C_2$$

if and only if C_1 or C_2 is classical ($\mathcal{C}\ell_d$)

⑤ Entanglement via tensor norms



Goal for today : · understand ↗ with norms

· use these norms to compare entanglement

Def A Banach space (f.d.) is $(X, \| \, \|_X)$.

v-space norm on X

The dual space $(X^*, \| \, \|_{X^*})$

$$X^* = \{ \alpha : X \to K \} \quad \text{linear forms}$$

$$\|\alpha\|_{X^*} = \sup_{\|x\|_X \leq 1} |\alpha(x)| \quad \text{''} \quad \langle \alpha, x \rangle$$

Examples

- ℓ_2^d (or euclidean space)
  $= (\mathbb{R}^d, \| \, \|_2)$
- $\ell_p^d \qquad \|x\| = \left(\sum |x_i|^p \right)^{1/p}$
- $\ell_\infty^d \qquad \|x\| = \max |x_i|$
- S_p^d Schatten classes $= (M_d(\mathbb{C}), \| \, \|_p)$

$$\|A\|_p = \left[\operatorname{Tr} (AA^*)^{p/2} \right]^{1/p}$$

$\|A\|_p = \|s\|_p$ $\quad$ s = vector of singular values
(S_p) $\quad$ (ℓ_p)

- S_∞^d $\quad$ $\|A\|_\infty$: usual op. norm
$$= \sup_{\|x\|,\|y\|\leq 1} |\langle x|A|y\rangle|$$

- S_1^d $\quad$ $\|A\|_1 = \|A\|_{tr}$, nuclear norm
$= \sum$ singular values of A.

__Fundamental question__ : what norm to put on $X \otimes Y$?

Want to have : $\quad$ $\circledast$ $\quad$ $\|x \otimes y\| = \|x\|_X \cdot \|y\|_Y \quad \forall x, y$

$\|\alpha \otimes \beta\|_* = \|\alpha\|_{X^*} \cdot \|\beta\|_{Y^*} \quad \forall \alpha, \beta$

→ A norm $\|\cdot\|$ on $X \otimes Y$ verifying $\circledast$ is called a tensor norm (or a cross norm)

__Theorem__ If $\|\cdot\|$ is a tensor norm on $X \otimes Y$, then

$$\underbrace{\|z\|_\varepsilon}_{\text{injective norm}} \leq \|z\| \leq \underbrace{\|z\|_\pi}_{\text{projective norm}} \qquad \forall z \in X \otimes Y$$

where

$$\|z\|_\varepsilon := \sup_{\substack{\|\alpha\|_{X^*}\leq 1 \\ \|\beta\|_{Y^*}\leq 1}} |\langle \alpha \otimes \beta, z\rangle|$$

$$\|z\|_\pi := \inf \Big\{ \sum_i \underbrace{\|x_i\|_X \cdot \|y_i\|_Y}_{=\|x_i \otimes y_i\|} : z = \sum_{fin} x_i \otimes y_i \Big\}$$

Examples $\to X = (\mathbb{R}^d, \|\ \|_2) = Y = \ell_2^d$

$$\|z\|_{\ell_2^d \otimes_\varepsilon \ell_2^d} = \sup_{\substack{\|\alpha\|_{\ell_2^*} \leq 1 \\ \|\beta\|_{\ell_2^*} \leq 1}} |\langle \alpha \otimes \beta, z \rangle|$$

duality bracket

$\ell_2^* = \ell_2$

euc. scalar product

$$= \sup_{\|a\|_2, \|b\|_2 \leq 1} |\langle a \otimes b, z \rangle|$$

$\mathbb{R}^d \otimes \mathbb{R}^d \simeq M_d(\mathbb{R}) \qquad z \longmapsto Z \in M_d(\mathbb{R})$

$$= \sup_{\|a\|, \|b\| \leq 1} |\langle a | Z | b \rangle|$$

$$= \|Z\|_{op} = \|Z\|_{S_\infty^d}$$

$$\ell_2^d \otimes_\varepsilon \ell_2^d = S_\infty^d$$

$\to \ell_2^d \otimes_\pi \ell_2^d = S_1^d$

$$\|Z\|_{\ell_2^d \otimes_\pi \ell_2^d} = \|Z\|_1 = \|Z\|_{tr}$$

(i.e. the best decomp in the def of $\|\ \|_\pi$ is the SVD:

$Z = \sum s_i |a_i\rangle\langle b_i| \rightsquigarrow \|Z\|_\pi = \sum s_i$)

Entanglement with tensor norms

A. Pure states $\qquad \psi \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}$

n-partite state

$$\|\psi\|_{\ell_2^{d_1} \otimes_\varepsilon \ell_2^{d_2} \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_2^{d_n}} = \sup_{\|\varphi_i\|_2 \le 1} |\langle \varphi_1 \otimes \cdots \otimes \varphi_n | \psi \rangle|$$

$$= \sup_{\substack{\|\varphi\| \le 1 \\ \varphi \text{ separable}}} |\langle \varphi, \psi \rangle|$$

= largest overlap with separable states

$-2 \log \|\psi\|_\varepsilon$ is known as the geometric measure of entanglement of ψ

→ $n=2$ $\|\psi\|_\varepsilon$ = largest Schmidt coeff of $\psi \in \mathbb{C}^d \otimes \mathbb{C}^d$

Fact Computing $\|\psi\|_{\ell_2 \otimes_\varepsilon \ell_2 \otimes_\varepsilon \ell_2}$ is NP-hard.

B. Density matrices

Goal: characterize $SEP_{d_A, d_B} = \{ \sum p_i \rho_i^A \otimes \rho_i^B \}$ using tensor norms

Claim $X \in M_d^{sa}(\mathbb{C})$ is a density matrix positive semidefinite ($X \in PSD$)

$\iff \|X\|_{S_d^1} = \operatorname{Tr} X = 1$

tr

Proof: $\|X\|_{S_d^1} = \sum s_i = \sum |\lambda_i| \overset{!}{=} \operatorname{Tr} X = \sum \lambda_i$

sing values

eigenvalues

→ Equivalently $X \in PSD \iff \|X\|_{\ell_2 \otimes_\pi \ell_2} = \operatorname{Tr} X$

Theorem Let $X \in M_{d_A}^{sa} \otimes M_{d_B}^{sa}$. $X \in SEP_{d_A, d_B} \iff$

$$\|X\|_{S_1^{d_A} \otimes_\pi S_1^{d_B}} = \mathrm{Tr}\, X = 1$$

Proof $\|x\|_\pi = \inf \{ \sum \|A_i\|_{tr} \|B_i\|_{tr} : X = \sum A_i \otimes B_i \}$

Since $\|A\|_{tr} = \mathrm{Tr}\,|A| \geq \mathrm{Tr}\, A$

$$\|X\|_\pi \geq \sum \mathrm{Tr}\, A_i \cdot \mathrm{Tr}\, B_i = \mathrm{Tr}\, X$$

idea: equality "$\iff$" $A_i, B_i \geq 0$

Conclusion $X \in (M_{d_A} \otimes M_{d_B})^{sa}$

- X is a density matrix $\iff$

$$\|X\|_{S_1^{d_A \cdot d_B}} = \mathrm{Tr}\, X = 1$$

$$\|X\|_{\underbrace{\ell_2^{d_A d_B} \otimes_\pi \ell_2^{d_A d_B}}_{= (\ell_2^{d_A} \otimes_{euc} \ell_2^{d_B}) \otimes_\pi (\ell_2^{d_A} \otimes_{euc} \ell_2^{d_B})}} = \mathrm{Tr}\, X = 1$$

- X is a separable density matrix $\iff$

$$\|X\|_{S_1^{d_A} \otimes_\pi S_1^{d_B}} = \mathrm{Tr}\, X = 1$$

$$\|X\|_{\ell_2^{d_A} \otimes_\pi \ell_2^{d_A} \otimes_\pi \ell_2^{d_B} \otimes_\pi \ell_2^{d_B}} = \mathrm{Tr}\, X = 1$$