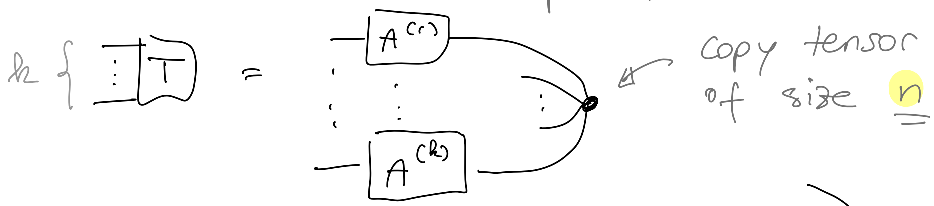
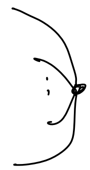
$T \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_k}$ k-partite tensor

$$T = \sum_{i=1}^{n} \underbrace{a_i^{(1)} \otimes a_i^{(2)} \otimes \cdots \otimes a_i^{(k)}}_{\text{simple tensors}}$$





$$= \sum_{i=1}^{n} |\underbrace{ii\cdots i}_{k \text{ times}}\rangle$$

The tensor rank of T is the smallest n for which such a decomposition exists. $R(T)$

<u>Example</u> In QiT $|\psi\rangle \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_k}$ a pure quantum state

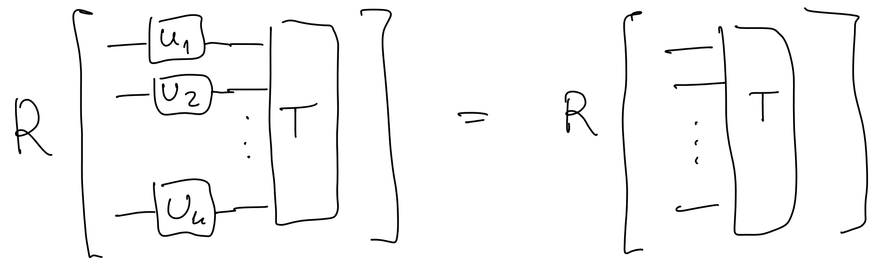
$$R(|\psi\rangle) = 1 \iff |\psi\rangle = |\varphi_1\rangle \otimes \cdots \otimes |\varphi_k\rangle$$

i.e. φ separable (or product)

$\Rightarrow$ R is an integer-valued entanglement measure for pure multipartite q. states.

$\rightarrow$ Note that R is invariant under local unitary operations. (LU)

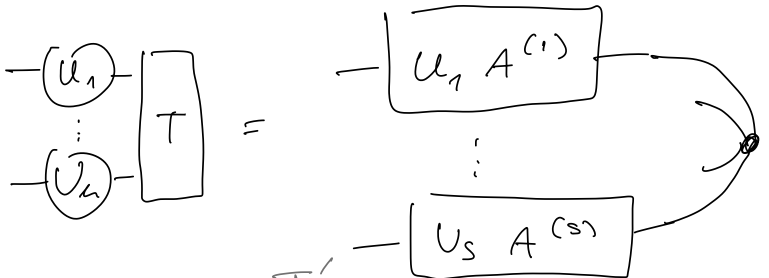
$$R((U_1 \otimes \cdots \otimes U_k) \cdot T) = R(T)$$



Proof if $r = R(T) \Rightarrow T = \sum_{i=1}^{r} \bigotimes_{s=1}^{k} a_i^{(s)}$

$$(U_1 \otimes \cdots \otimes U_k) T = \sum_{i=1}^{r} \bigotimes_{s=1}^{r} \left(U_s \, a_i^{(s)} \right)$$

 $\Rightarrow$



$$\Rightarrow R\Big(\underbrace{(U_1 \otimes \cdots \otimes U_k)\, T}_{T'} \Big) \leq R(T) = r$$

Also : $R\left(T = (U_1^{-1} \otimes \cdots \otimes U_k^{-1}) \cdot T' \right) \leq R(T')$

Actually $R(X_1 \otimes \cdots \otimes X_k T) = R(T)$

$\forall$ invertible X_s

if $T' = (X_1 \otimes \cdots \otimes X_k)\, T$

for invertible X_s then T' can be

obtained from T by SLOCC
(stochastic local operation and classical comm.)
with non-zero probability.

$\Longrightarrow$ tensor rank is a SLOCC invariant.

Example for matrices ($k=2$)

$$R(A) = \mathrm{rk}(A) = \#\text{ of non-zero singular values.}$$

$$A = \sum_{i=1}^{r} |a_i\rangle\langle b_i| \Leftrightarrow \sum_{i=1}^{r} |a_i\rangle \otimes |b_i\rangle$$

Examples

$k=2$

- $|00\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$

$$R(|00\rangle) = 1 \qquad |0\rangle \otimes |0\rangle$$

$$\mathrm{rank}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = 1 \quad \rightsquigarrow \text{separable}$$

- $|\Omega\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

$$R(|\Omega\rangle) = 2 \quad \Rightarrow \text{entangled}$$

$$\mathrm{rank}\left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \mathrm{rank}(I_2) = 2$$

$k=3$ $\quad |\varphi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ $\quad$ 3 qubits.
can be entangled in 2 different ways

[Dür, Vidal, Cirac '2000]

- $|000\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle$

  $R(|000\rangle) = 1 \implies$ separable

- $\frac{1}{\sqrt{2}}\left(|000\rangle + |011\rangle\right) = \underset{A}{|0\rangle} \otimes \underset{BC}{|\Omega\rangle}$

  $R(\uparrow) = 2$ entangled but bi-separable $A|BC$

- $|W\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$

- $|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$

  $R(|W\rangle) = 3$ and $R(|GHZ\rangle) = 2$

<u>But</u> : $|W\rangle$ and $|GHZ\rangle$ are entangled "differently"

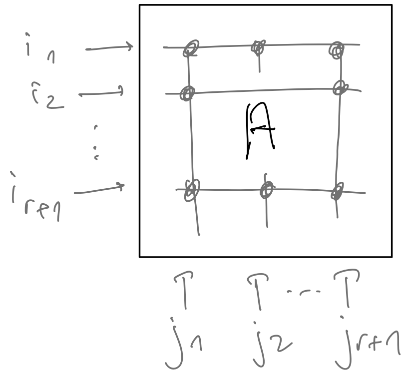
Interlude For matrices, the sets

$$\mathrm{Mat}_r := \{ A \in M_d(\mathbb{C}) : \mathrm{rk}(A) \le r \}$$

are closed

$A \in \mathrm{Mat}_r \iff$ all of its $(r+1)$ minors are 0

$\forall\, i_1 \cdots i_{r+1}$
$j_1 \cdots j_{r+1}$



$$\det\left(A\big|_{\substack{i_1 \cdots i_{r+1} \\ j_1 \cdots j_{r+1}}} \right) = 0$$

$\uparrow$
polynomial in the entries of A.

$A \in Mat_r \iff \left\{ \binom{d}{r+1}^2 \right.$ polynomials are zero at A ?

$\Rightarrow$ Mat_r is closed

For tensors, this is not true!

<u>Example</u> $R(|W\rangle) = 3$, but it is the limit

$$|W\rangle = \lim_{n\to\infty} T_n \quad \text{with} \quad R(T_n) = 2$$

$\left(\iff \{ T \in (\mathbb{C}^2)^{\otimes 3} : R(T) \le 2 \} \text{ is not closed} \right)$

$$|100\rangle + |010\rangle + |001\rangle = \lim_{\varepsilon \to 0} \underbrace{\frac{(|0\rangle + \varepsilon|1\rangle)^{\otimes 3} - |000\rangle}{\varepsilon}}_{R(\cdot) = 2}$$

$$(|0\rangle + \varepsilon|1\rangle)^{\otimes 3} = |000\rangle + \varepsilon \cdot |W\rangle + \varepsilon^2 (\cdots) + \varepsilon^3 |111\rangle$$

<u>Border rank</u>

$$\underline{R} = \inf \{ r : \exists T_n \longrightarrow T \text{ with } R(T_n) \le r \}$$

$$\underline{R}(|W\rangle) = 2 \quad \text{vs} \quad R(|W\rangle) = 3$$

<u>Theorem</u> $\{ T : \underline{R}(T) \le r \}$ is the zero set of some finite family of polynomials

In $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ there is only one such poly.

3-tangle $\quad \tau_3 = 4 \cdot$ Cayley hyperdet.
(1850's)

$\tau_3(|\psi\rangle)$ is a poly of degree 4 in the 8 variables.

- $\tau_3(W) = 0$.
- $\tau_3(GHZ) > 0$.
- τ_3 is a SLOCC invariant.

In particular:

$$GHZ \xrightarrow{SLOCC} W$$

$$W \xrightarrow{\cancel{SLOCC}} GHZ$$

In some sense, GHZ is more entangled than W.

<u>Fact</u> deciding $R(T) \le r$ is NP-hard (for $k \geq 3$)

3SAT: $\varphi = (x_1 \text{ OR } \overline{x_2} \text{ OR } x_5) \text{ AND } (x_2 \text{ OR } x_3 \text{ OR } \overline{x_1})$
AND $(\dots)$

Given such a statement, is it satisfiable?

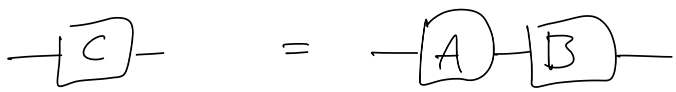
To any 3SAT problem $\varphi \rightsquigarrow T_\varphi \in \mathbb{C}^{\cdots} \otimes \mathbb{C}^{\cdots} \otimes \mathbb{C}^{\cdots}$
s.t. φ satisfiable $\Leftrightarrow R(T_\varphi) \le \dots$
where all the $\dots$ are polynomial in size of φ

The matrix multiplication tensor

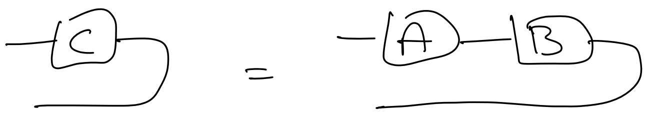
$$(\underset{m\times n}{A}, \underset{n\times p}{B}) \longrightarrow \underset{m\times p}{C} = A\cdot B.$$

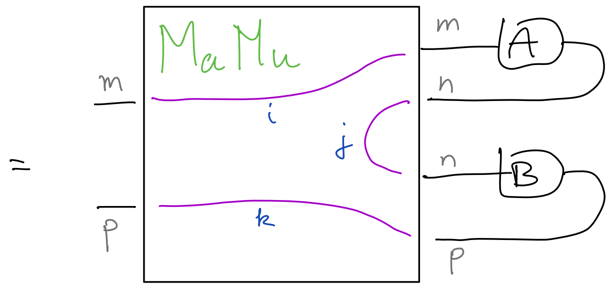
is a bilinear operation

$$C_{ik} = \sum_{j=1}^{m} A_{ij} B_{jk} \qquad i\in[m],\; k\in[p]$$

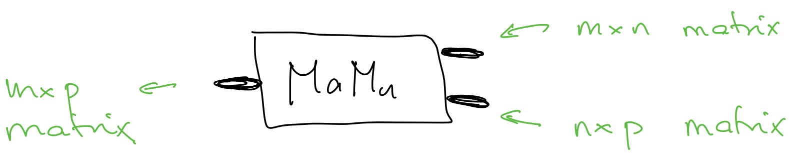


In vectorized form:

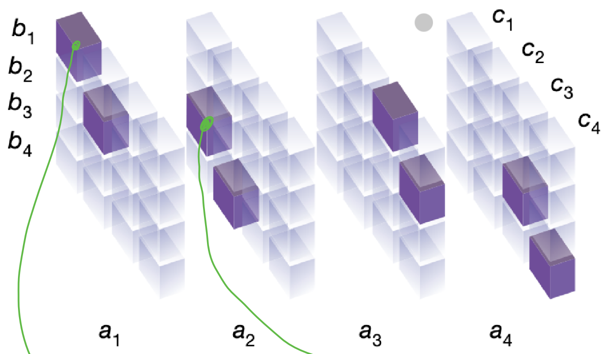




$$\mathrm{MaMu}_{m,n,p} = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} \underbrace{|j\rangle\langle i|}_{\text{"}A\text{"}} \otimes \underbrace{|k\rangle\langle j|}_{\text{"}B\text{"}} \otimes \underbrace{|i\rangle\langle k|}_{\text{"}C\text{"}}$$



$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \cdot \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$



$\mathrm{MaMu}(i,j,k) = 1 \to$ (purple box)

$= 0 \to$ (grey box)

$$= \mathrm{MaMu}_{2,2,2}$$

$$\cap$$

$$\underset{\text{"A"}}{M_2} \otimes \underset{\text{"B"}}{M_2} \otimes \underset{\text{"C"}}{M_2}$$

$$\cong$$

$$\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$$

$$c_1 = a_1 b_1 + a_2 b_3$$

Claim $\quad R(\mathrm{MaMu})$ = minimal number of multiplications needed to compute the matrix product.

$$\Rightarrow R(\mathrm{MaMu}_{2,2,2}) \leq 8$$

In general

$$\mathrm{MaMu}_{2,2,2} = \sum_{i=1}^{r} \underset{\mathbb{C}^4}{\underset{\cap}{\alpha_i}} \otimes \underset{\mathbb{C}^4}{\underset{\cap}{\beta_i}} \otimes \underset{\mathbb{C}^4}{\underset{\cap}{\gamma_i}}$$

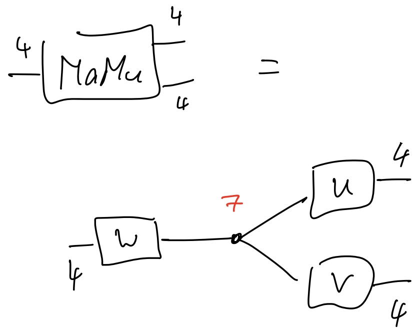
Claim : $R(\mathrm{MaMu}_{2,2,2}) = 7$ Strassen '69

$m_1 = (a_1 + a_4)(b_1 + b_4)$
$m_2 = (a_3 + a_4)\, b_1$
$m_3 = a_1\,(b_2 - b_4)$
$m_4 = a_4\,(b_3 - b_1)$
$m_5 = (a_1 + a_2)\, b_4$
$m_6 = (a_3 - a_1)(b_1 + b_2)$
$m_7 = (a_2 - a_4)(b_3 + b_4)$
$c_1 = m_1 + m_4 - m_5 + m_7$
$c_2 = m_3 + m_5$
$c_3 = m_2 + m_4$
$c_4 = m_1 - m_2 + m_3 + m_6$

$$\mathbf{U} = \overbrace{\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}}^{7}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\mathbf{W} = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$



Using this, the cost of multiplying matrices of size d goes from

\# of multiplications

$$d^3 \longrightarrow d^{\log_2 7} \simeq d^{2.808}$$

Open problem Can this be $d^{2+o(1)}$