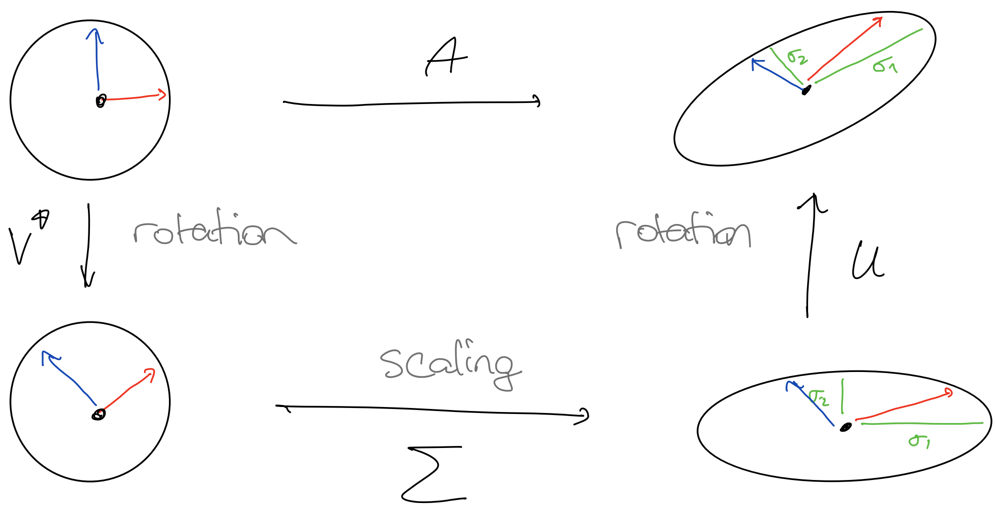
The singular value decomposition (SVD)

→ cornerstone of linear algebra

$$A = \underset{\text{unitary}}{U} \cdot \underset{\substack{\text{diagonal} \\ \text{with non-negative entries}}}{\Sigma} \cdot \underset{\text{unitary}}{V^*}$$

→ nice geometrical interpretation.



The SVD

$A \in \mathcal{M}_{m\times n}(\mathbb{C})$ $\quad r := \operatorname{rank}(A)$

There exist

- $U \in \mathcal{U}(m)$ — columns of U: left sing. vectors
- $V \in \mathcal{U}(n)$ — columns of V: right singular vectors
- $\Sigma \in \mathcal{M}_{m\times n}$ "diagonal"

$$\Sigma = \begin{array}{c} \\ m \end{array} \overset{n}{\boxed{\begin{matrix} \sigma_1 & & & & 0 \\ & \sigma_2 & & & \\ & & \ddots & & \\ 0 & & & \sigma_r & \\ & & & & 0 \ddots 0 \end{matrix}}}$$

singular values

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0.$

$\forall\ i \leq \min(m,n) \qquad \Sigma_{ii} = \sigma_i$

$\sigma_{r+1} = \dots = \sigma_{\min(m,n)} = 0$

$i \neq j \qquad \Sigma_{ij} = 0$

Such that $A = U \Sigma V^\dagger$

- In bra-ket notation

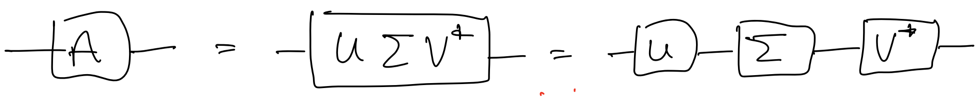
$$A = \sum_{i=1}^{\min(m,n) \text{ or } r} \sigma_i \, |u_i\rangle\langle v_i|$$

$$U = \sum_{i=1}^{m} |u_i\rangle\langle i| \qquad V = \sum_{j=1}^{n} |v_j\rangle\langle j|$$

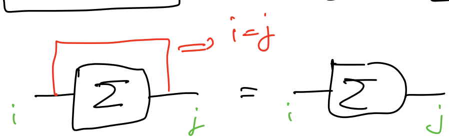
$$U \Sigma V^\dagger = \sum_{ij} |u_i\rangle\langle v_j| \cdot \Sigma_{ij}$$

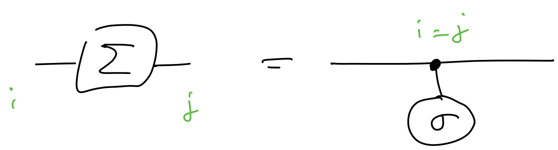
$$= \sum_i \sigma_i \, |u_i\rangle\langle v_i|$$

- In graphical notation:



Σ diagonal





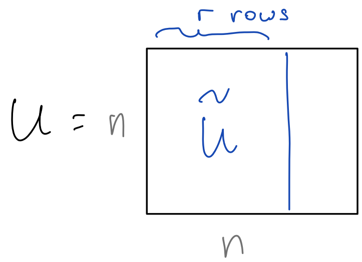
$= \delta_{ij} \cdot \sigma_i$

- "reduced" SVD : only keep the s.v. > 0

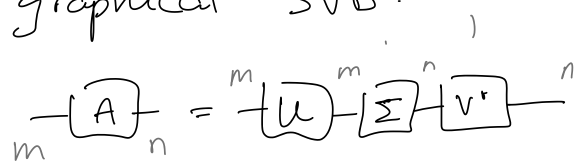
$$A = {}_m\tilde{U}_r \underbrace{\mathrm{diag}(\sigma_1, \dots, \sigma_r)}_{r \times r \text{ matrix}} {}_r\tilde{V}^\dagger_n$$

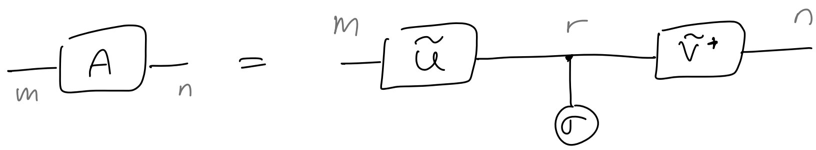
$\tilde{U} : \mathbb{C}^r \longrightarrow \mathbb{C}^m$
$\tilde{V} : \mathbb{C}^r \longrightarrow \mathbb{C}^n$ $\Big\}$ isometries

$$U = \sum_{i=1}^{n} |u_i\rangle\langle i| \qquad \tilde{U} = \sum_{i=1}^{r} |u_i\rangle\langle i|$$

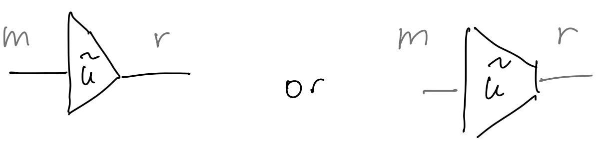


graphical SVD:





Sometimes $\tilde{U} : \mathbb{C}^r \longrightarrow \mathbb{C}^m$ isometry



$\tilde{U}^\dagger \tilde{U} = I_r$

Uniqueness of the SVD

- the numbers $\sigma_1 \geq \cdots \geq \sigma_r > 0$ are unique
- if $\sigma_1 > \sigma_2 > \cdots > \sigma_r$ then $|u_1\rangle, \ldots, |u_r\rangle$, $|v_1\rangle, \ldots, |v_r\rangle$ are unique

up to phases

$$A = \sum \sigma_i \, |u_i \rangle \langle v_i|$$

phases

<u>Example</u> $\quad I = \sum 1 \cdot |u_i \rangle \langle u_i|$

for any onb $\{u_i\}$

"Uniqueness" $\Rightarrow$ # parameters of A =

parameters of $\{\sigma, U, V\}$

real parameters of $A \in M_n(\mathbb{C}) = 2n^2$.

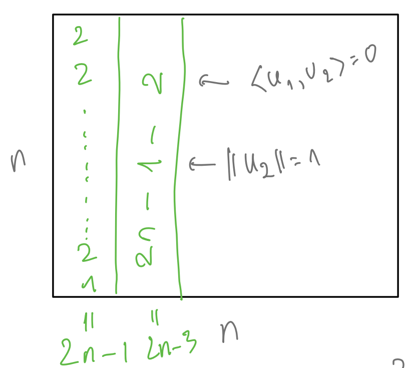
($\mathbb{R}$ = real)

parameters (σ, U, V)

$n : (\sigma_1, \dots, \sigma_n)$

parameters in $U \in \mathcal{U}(n)$

$U =$



$n = 2: \quad 3 + 1 = 4.$

$n = 3: \quad 5 + 3 + 1 = 9$

$u_3: \quad 2n - \underbrace{1}_{\text{norm}} - \underbrace{2 \cdot 2}_{\perp \text{ to } u_1 \text{ and } u_2} = 2n - 5$

Re and Im

In general : $\dim_{\mathbb{R}} \mathcal{U}(n) = \sum_{i=1}^{n} 2n - (2i-1) = n^2$

$$= 2n^2 - 2 \, \frac{n(n+1)}{2} + n$$

$$\dim_{\mathbb{R}} \mathcal{U}(n) = n^2$$

OR : $\dim_{\mathbb{R}} \mathcal{U}(n) = \dim_{\mathbb{R}}$ its Lie algebra

Lie group

$= \dim_{\mathbb{R}}$ skew-sym. mat

$= n^2$

params of (σ, U, V)

phases $|u_i\rangle \leftrightarrow |v_i\rangle$

$$n + n^2 + n^2 - n = 2n^2$$

$$(U,V) \in \underbrace{\mathcal{U}(d)/\mathcal{U}(1)^N}_{\text{no phase in columns}} \times \mathcal{U}(d)/\mathcal{U}(1)^N \times \mathcal{U}(1)^N$$

i.e. $U = \begin{bmatrix} \cdot & \cdot & \cdot \\ & * & \end{bmatrix}$ ← real numbers

How about tensor SVD ?

$\Big\{$ $T \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ general tensors

\# params $= 2n^3$

of order n^3 $\Big\}$

$\Big\{$ SVD-able tensors

$$T = \sum \sigma_i \, |u_i\rangle \otimes |v_i\rangle \otimes |w_i\rangle$$

n, n^2, n^2, n^2

$\sigma_i \geq 0$ $(-2n)$

$\{u_i\}, \{v_i\}, \{w_i\}$ o.n.b. $\}$

\# params $= 3n^2 - n$

of order n^2 $\Big\}$

SVD-able tensors in $(\mathbb{C}^n)^{\otimes k}$ form a manifold of $\dim_{\mathbb{R}} \sim k \cdot n^2$

vs (at largen $\hat{}$)

all tensors in $(\mathbb{C}^n)^{\otimes k}$ $\dim_{\mathbb{R}} = 2n^k$

Back to matrices

SVD: $A = U \Sigma V^\dagger$ — general matrices, even rectangular

eigenvalue decomposition $A = A^\dagger$ (hence square)

$$A = W \cdot \Lambda \cdot W^\dagger$$

eigenvectors eigenvalues

$$A = \sum \lambda_i |w_i\rangle\langle w_i|$$

$$\Lambda = \mathrm{diag}(\lambda) = \sum \lambda_i |i\rangle\langle i|$$

$$W = \sum |w_i\rangle\langle i|$$

Relation between SVD and eig. decomp

$$A = U \Sigma V^\dagger \Rightarrow AA^\dagger = U \Sigma V^\dagger V \Sigma U^\dagger$$

$$= U \underbrace{\Sigma^2}_{\mathrm{diag}(\sigma_i^2)} U^\dagger$$

$\Rightarrow$ $\begin{cases} \sigma_i\text{'s are } \sqrt{\text{eigs of } AA^\dagger} \\ U_i\text{'s are the eig of } AA^\dagger \\ V_i\text{'s } \text{———} \quad A^\dagger A \end{cases}$

This is how you compute the SVD !

• polar decomposition $\underset{\text{unitary}}{A = W} \cdot \underset{\text{positive semidefinite}}{P}$

(matrix version of $z = e^{i\theta} \cdot |z|$)

→ unique, up to degeneracies

$$A = U \Sigma V^{\dagger}$$

$$= \underbrace{U V^{\dagger}}_{W} \underbrace{V \Sigma V^{\dagger}}_{P}$$

P ← is positive semidef. since it has eigs σ_i's

Schmidt decomposition for q. states

Any $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ can be decomposed as

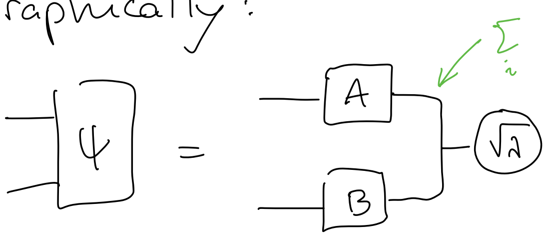
$$|\psi\rangle = \sum_i \sqrt{\lambda_i}\, |a_i\rangle \otimes |b_i\rangle$$

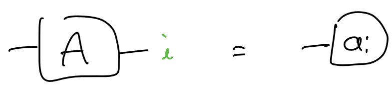
Schmidt coefficients

where • $\lambda_i \geq 0$, $\sum \lambda_i = 1$

• $\{a_i\}$, $\{b_i\}$ o.n.b. of $\mathbb{C}^d$.

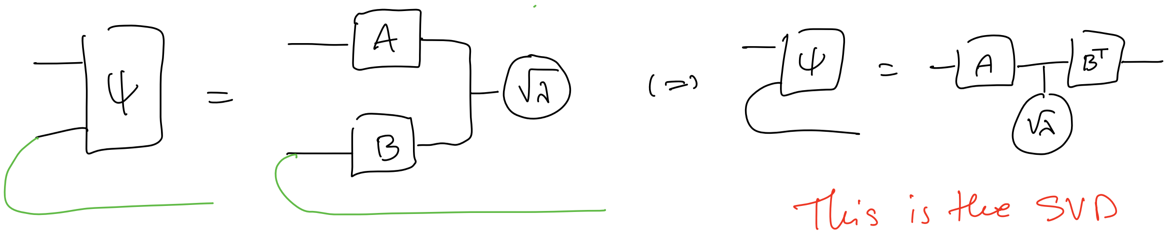
graphically:



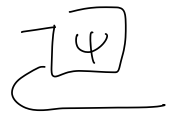


Remark $|\psi\rangle = \sum_{ij} \underset{\in\mathbb{C}}{\psi_{ij}}\, |i\rangle \otimes |j\rangle$

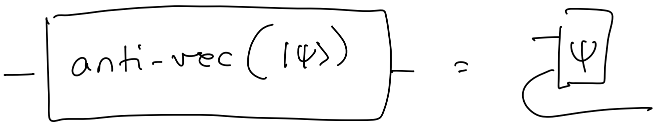
2 indices → $= \sum_{ij} |\psi_{ij}|\, |\arg(\psi_{ij})\, i\rangle \otimes |j\rangle$



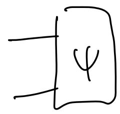
This is the SVD of



= anti-vectorization of $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$

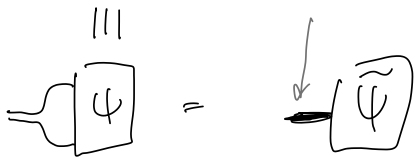


Remark



$\rightsquigarrow \quad |\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$

thick wire $\sim d^2$



$|\tilde{\psi}\rangle \in \mathbb{C}^{d^2}$

$\Rightarrow$ the schmitd coeffs λ_i are the squared s.v.

$$\lambda_i = \sigma_i^2$$

The partial traces of $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$

$$\rho_1 = \mathrm{Tr}_2 |\psi\rangle\langle\psi|$$

$$= \mathrm{Tr}_2 \Big(\sum_{ij} \sqrt{\lambda_i}\sqrt{\lambda_j} \, |a_i\rangle|b_i\rangle\langle a_j|\langle b_j| \Big)$$

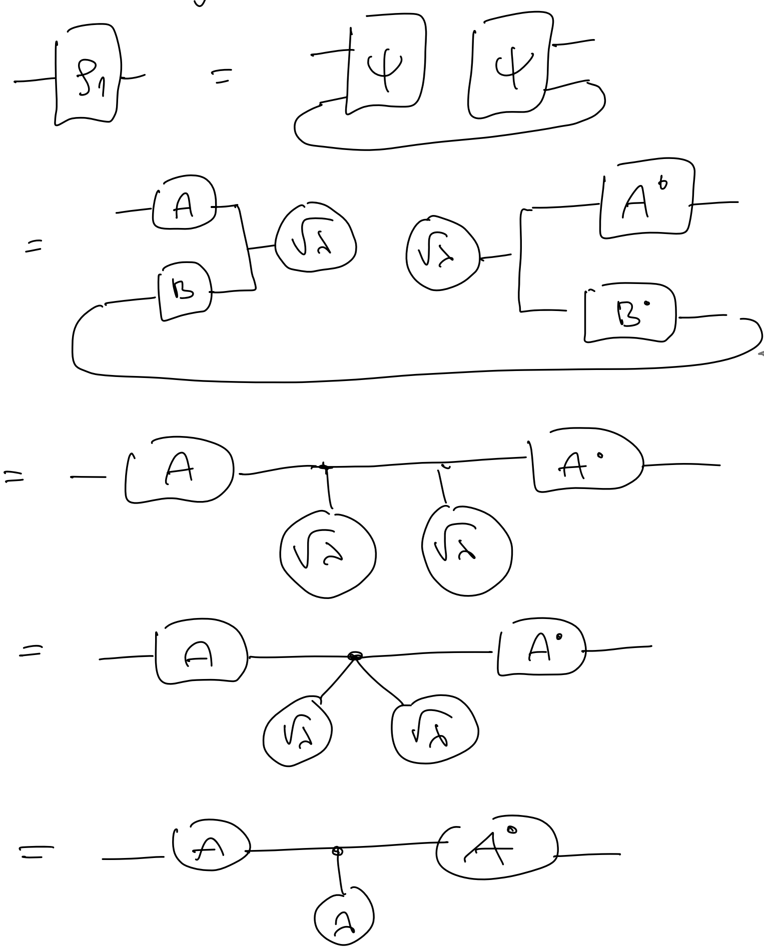
$$= \sum_{ij} \sqrt{\lambda_i \lambda_j} \, |a_i\rangle\langle a_j| \, \underbrace{\mathrm{Tr}(|b_i\rangle\langle b_j|)}_{\delta_{ij}}$$

$$= \sum \lambda_i \, |a_i\rangle\langle a_i|$$ eigenvalue decomp of ρ_1

$$\left(\mathrm{Tr}\, \rho_1 = \sum \lambda_i = 1 \right)$$

Similarly $\rho_2 = \sum_i \lambda_i \, |b_i\rangle\langle b_i|$

Graphically

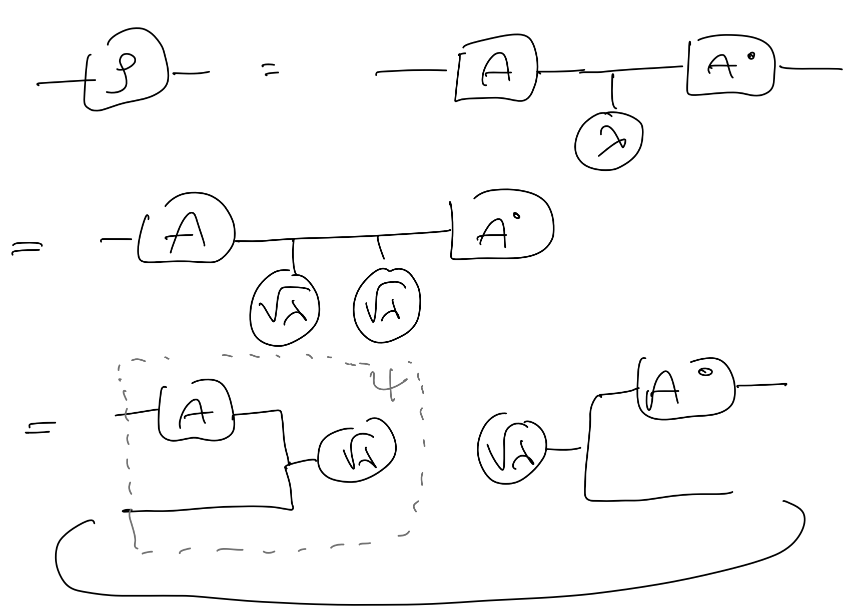


slide B's and use unitarity

<u>Purification</u> for any density matrix ρ

$\exists\ \psi_{1,2}$ pure s.t $\mathrm{Tr}_2 |\psi\rangle\langle\psi| = \rho$

ρ density matrix $\Rightarrow \rho = \sum \lambda_i \, |a_i\rangle\langle a_i|$



A unitary
$\lambda_i \geq 0$
$\sum \lambda_i = 1$

The Eckart - Young - Mirsky theorem

$A \in \mathcal{M}_n(\mathbb{C})$ a matrix

Question: Given $r < n$, what is the best rank - r approximation of A?

Best?

$$\min_{B:\ \mathrm{rank}\, B \leq r} \| A - B \|$$

What norm / distance?

- operator norm / spectral norm

$$\|A\|_\infty = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

- euclidean norm / Frobenius norm

$$\|A\|_2 = \sqrt{\sum_{i,j} |A_{ij}|^2} = \|\operatorname{vec}(A)\|$$

$$= \operatorname{Tr}(A^* A)$$

$$\|A\|_\infty = \max_i \sigma_i = \sigma_1$$

$$\|A\|_2 = \|\sigma\| = \sqrt{\sum_{i=1}^{n} \sigma_i^2}$$

EYM theorem

$$\min_{\operatorname{rank} B \le r} \|A - B\|_\infty = \sigma_{r+1}$$

$$\min_{\operatorname{rank} B \le r} \|A - B\|_2 = \sqrt{\sum_{i=r+1}^{n} \sigma_i^2}$$

In both cases, $B = \overbrace{\sum_{i=1}^{r} \sigma_i |a_i\rangle\langle b_i|}^{\text{SVD truncated to its first } r \text{ terms}}$ achieves the minimum.