Tensors and graphical notation 19/03/25

① Tensor product

V, W two vector spaces fin. dim. over $\mathbb{C}$ ($\mathbb{R}, ..$)

$V \otimes W = ?$

Def 1: a basis of $V \otimes W$ is

$\{ e_i \otimes f_j \}_{i,j}$ where

$\{ e_i \}$ basis of V

$\{ f_j \}$ —— W

→ $\dim V \otimes W = \dim V \cdot \dim W$

→ contrast this w/ direct sum:

basis of $V \oplus W$ is $\{e_i\} \cup \{f_j\}$

$\dim V \oplus W = \dim V + \dim W$.

→ $\dim \mathcal{H}_{n \underbrace{\text{qubits}}_{\mathbb{C}^2}} = 2^n$ → exponential in n !

Def 2 V, W vector spaces.

Any $F : V \times W \longrightarrow A$ ← vector space

$(v, w) \longmapsto F(v, w)$

F bilinear. factorizes through $V \otimes W$

$$\begin{tikzcd} V \times W \arrow[r, "F"] \arrow[d, "i"'] & A \\ V \otimes W \arrow[ur, "\tilde{F}"'] & \end{tikzcd}$$

This means: $\exists$ vector space "$V \otimes W$" s.t.

$$F(v,w) = \tilde{F}(\, i(v,w)\,)$$

linear map!

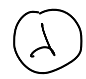
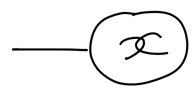
$$i(v,w) = v \otimes w$$

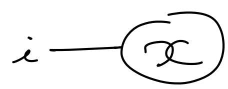
$$V \otimes W = \overline{\operatorname{span} \left\{ x \otimes y \ : \ \begin{matrix} x \in V \\ y \in W \end{matrix} \right\}}$$

Any $z \in V \otimes W$ can be decomposed as

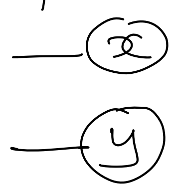
$$z = \sum \underbrace{x_i \otimes y_i}_{\text{simple tensors}} \qquad x_i \in V \quad y_i \in W$$

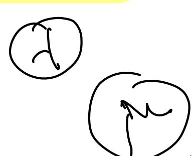
② Graphical notation

- scalars $\lambda \in \mathbb{C} \to$ 
- vectors $x \in V \to$ 

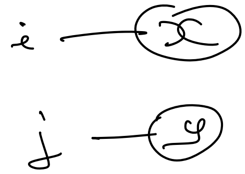
  $x_i \rightsquigarrow$ 
- tensor products of vectors

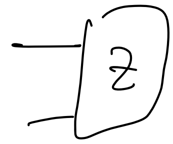
  $\to$ juxtaposing pictures

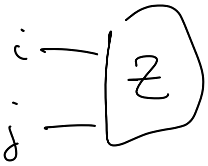
  $x \otimes y \to$ 
- product of scalars:

  $\lambda \cdot \mu \to$ 

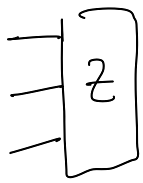
Rk diagram with no edges sticking out $\to$ scalar

- $(x \otimes y)_{ij} =$  $= x_i \cdot y_j$

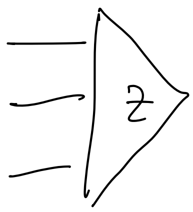
- for $z \in V \otimes W \rightsquigarrow$ 

$z_{ij} \longrightarrow$ 

- more generally: $z \in V_1 \otimes V_2 \otimes V_3$



etc.

= kets 

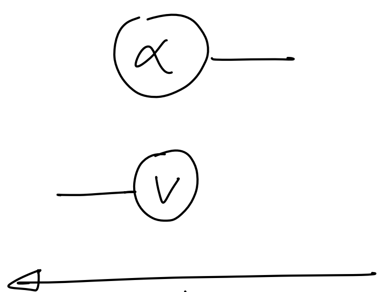
$\boxed{\text{(3) Dual spaces}}$ (i.e. bras)

V vector space $\rightsquigarrow$ dual V^*

$$V^* = \{ \alpha : V \to \mathbb{C} \text{ linear} \}.$$

is again a vector space.

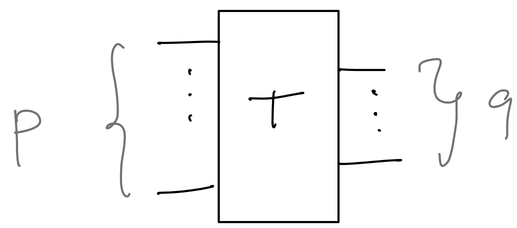
- graphically: $\alpha \in V^*$ i.e. $\alpha : V \to \mathbb{C}$



read this way

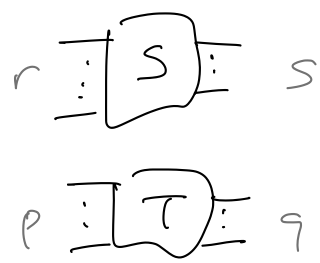
- more generally

$$T : \underbrace{W_1 \otimes \cdots \otimes W_p}_{p \text{ primal spaces}} \otimes \underbrace{V_1^* \otimes \cdots \otimes V_q^*}_{q \text{ dual spaces.}}$$



- $S : A_1 \otimes \cdots \otimes A_r \otimes B_1^* \otimes \cdots \otimes B_s^*$

$S \otimes T \rightsquigarrow$



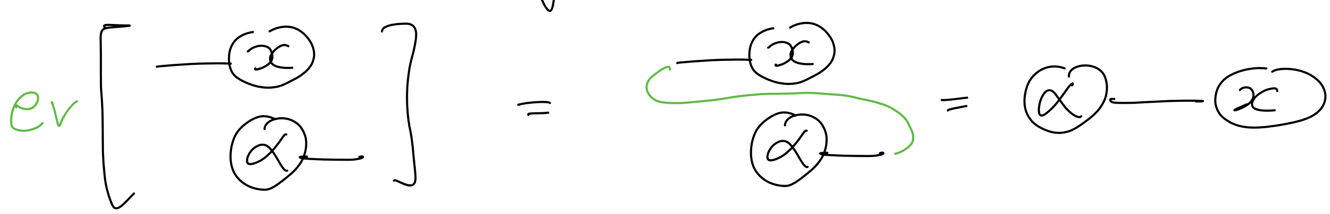
$\boxed{\text{④ Contraction}}$ (connecting boxes)

$$V \times V^* \longrightarrow V \otimes V^* \xrightarrow{ev} \mathbb{C}$$
$$(x, \alpha) \longmapsto (x \otimes \alpha) \longmapsto \alpha(x)$$

evaluation map.

$\Big\downarrow$ graphical picture

"contracting" the corresponding half-edges.

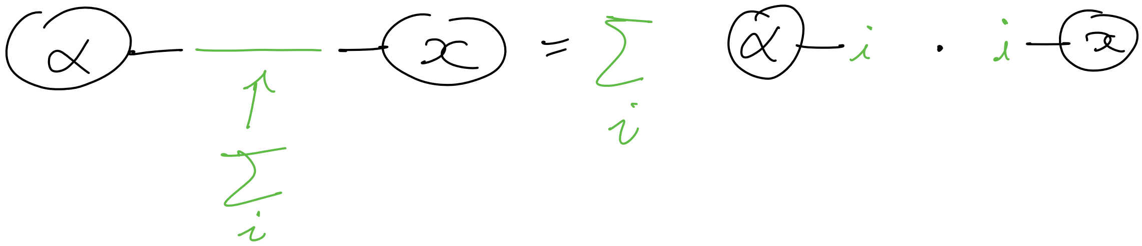


Example. $x \in \mathbb{C}^d$ $\quad \alpha \in (\mathbb{C}^d)^*$

$$\alpha(v) = \sum \overline{\alpha_i}\, v_i \qquad \alpha_i \in \mathbb{C}$$

$$\text{“}\ \alpha = \langle \alpha | \ \text{”}$$

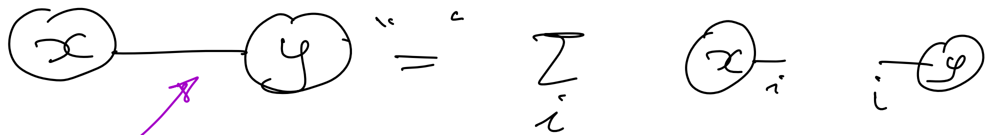
$$\alpha(x) = \sum \overline{\alpha_i}\, x_i$$



Examples.

- dot product in $\mathbb{R}^n$ $\quad x, y \in \mathbb{R}^n$

$$\langle x, y \rangle = \sum_i x_i y_i$$



! this is not canonical.

non-canonical identification

$$\tilde{x} \in V^* \qquad \tilde{x}(v) = \langle x, v \rangle$$

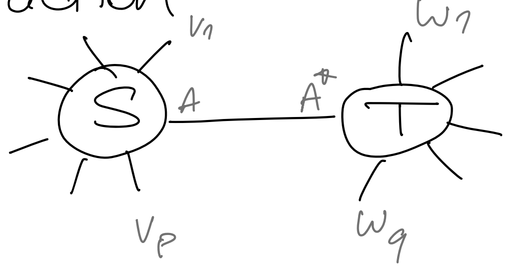
scalar product on V

$$\langle\!\langle a, b \rangle\!\rangle = a_1 b_1 + 2 a_2 b_2$$

· more generally $S \in V_1 \otimes \cdots \otimes V_p \otimes A$

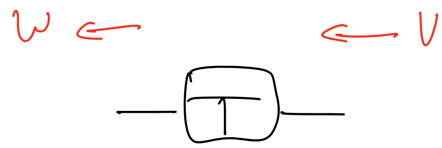
$T \in W_1 \otimes \cdots \otimes W_q \otimes A^*$

contraction



⑤ Matrices / Linear maps

$T : V \longrightarrow W \quad \rightsquigarrow$



$|||$

$T \in W \otimes V^*$

canonical.

$\{\text{linear maps } V \to W\} \equiv \{\text{tensors in } W \otimes V^*\}$

<u>Example</u> $\alpha \in V^*$ and $w \in W$

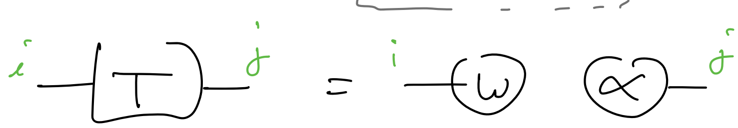
$$\begin{bmatrix} T : V \longrightarrow W \\ T(x) = \alpha(x) \cdot w \end{bmatrix} \longleftarrow \underset{W}{w} \otimes \underset{V^*}{\alpha}, \qquad x \in V$$

$$|w\rangle\langle\alpha| \longleftarrow |w\rangle \otimes \langle\alpha|$$

$$\Big(|w\rangle\langle\alpha|\Big)\Big(|x\rangle\Big) = |w\rangle\langle\alpha|x\rangle = \alpha(x)\cdot|w\rangle$$

anti-lin   linear

In coordinates:

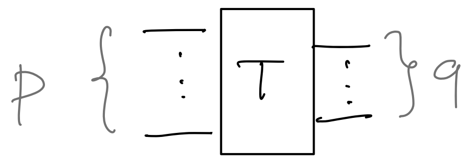


$$(|w\rangle\langle\alpha|)_{ij} = T_{ij} = w_i \, \overline{\alpha_j}$$ ← it's a bra

- **More generally:**

$$T : V_1 \otimes \cdots \otimes V_q \longrightarrow W_1 \otimes \cdots \otimes W_p$$

$$\equiv \quad T \in W_1 \otimes \cdots \otimes W_p \otimes V_1^* \otimes \cdots \otimes V_q^*$$

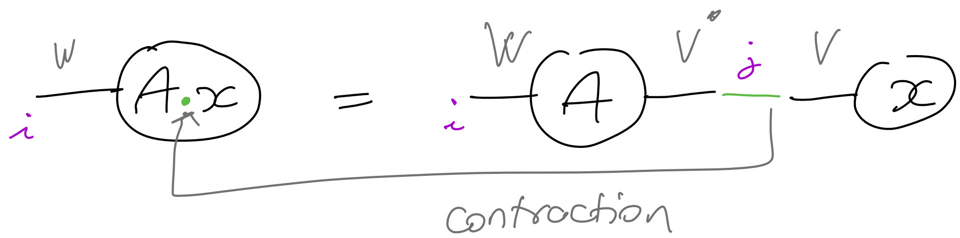


6 Applications

- **Matrix-vector product.**

$A : V \longrightarrow W$

$x \in V.$ $\qquad Ax \in W$



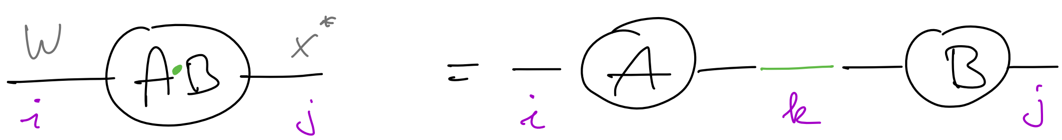
In coordinates : $(Ax)_i = \sum_j A_{ij} \, x_j$

- **Matrix - matrix product**

$A : V \to W$

$B : X \to V$ $\qquad AB : X \longrightarrow W$

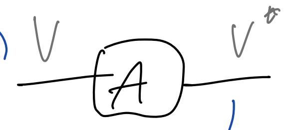
W V X ←



In coordinates: $(AB)_{ij} = \sum_{k} A_{ik} B_{kj}$

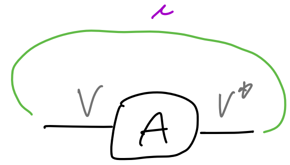
- Trace

$A : V \to V$



$\operatorname{Tr} A \in \mathbb{C}$

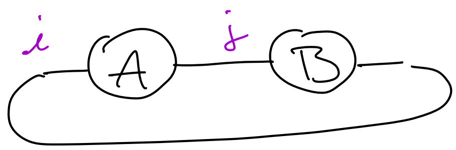
$\operatorname{Tr} A$ =



! no half-edges left ⇒ scalar

In coordinates: $\operatorname{Tr} A = \sum A_{ii}$

- A first graphical proof: $A, B : V \to V$



=

↑ by moving the boxes around the wires

$\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$

i.e. trace is cyclic.

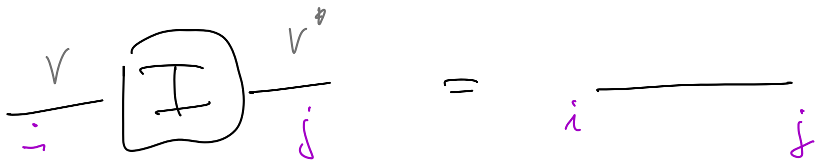
In coordinates

$$\operatorname{Tr}(AB) = \sum_{i} (AB)_{ii} = \sum_{ij} A_{ij} B_{ji}$$

$$\operatorname{Tr}(BA) = \sum_{ij} B_{ij} A_{ji}$$

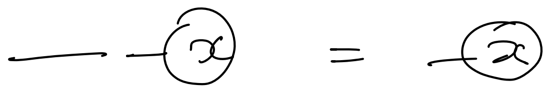
- **Identity matrix** (linear map).

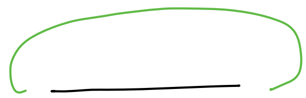
$$I : V \longrightarrow V$$
$$x \longmapsto x$$



In coordinates $I_{ij} = \delta_{ij}$

- $I \cdot x = x$

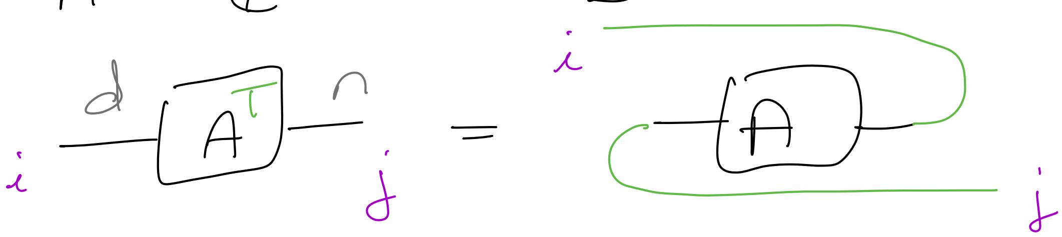


- $Tr(I) =$  $=$ loop $= \dim V$

Loops associated to $V = \dim V$.

- **Transpose** $A : \mathbb{C}^d \longrightarrow \mathbb{C}^n$

$$A^T : \mathbb{C}^n \longrightarrow \mathbb{C}^d.$$



In coordinates: $(A^T)_{ij} = A_{ji}$

! not canonical.

scalar product

in real case: A^T is defined by $\langle A^T v, w \rangle = \langle v, Aw \rangle$

in the complex case

=
A dagger

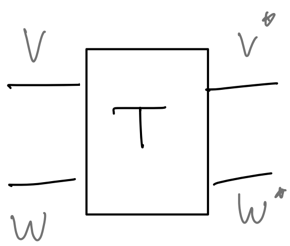
for example: if $x \in V$

$\langle x| =$ 

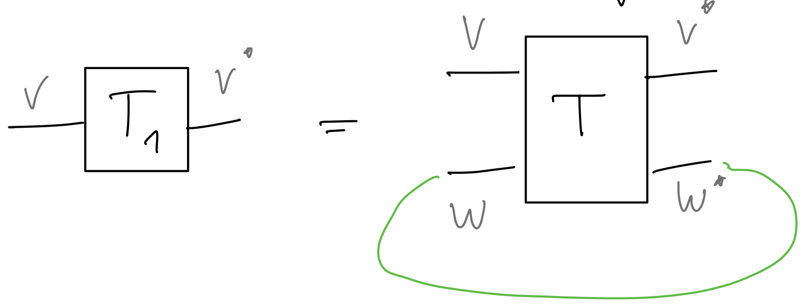
because $\langle x | v \rangle =$ 

$= \sum_i \overline{x}_i v_i$

- Partial trace. $T : V \otimes W \to V \otimes W$



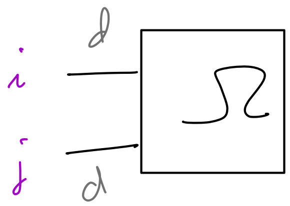
partial trace: $T_1 = [\mathrm{id}_V \otimes Tr_W](T) : V \to V$

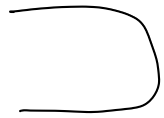


- Maximally entangled state.

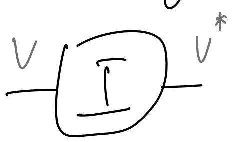
$\mathbb{C}^d \otimes \mathbb{C}^d \ni |\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$

$d=2 \qquad \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$ Bell state



$= \frac{1}{\sqrt{d}} \cdot$ 

$|\Omega\rangle_{ij} = \frac{1}{\sqrt{d}} \delta_{ij}$ $\qquad /// \text{ up to } \circledast \qquad \cdot \frac{1}{\sqrt{d}} \qquad \circ \left(\mathbb{C}^d\right)^* \equiv \mathbb{C}^d$



$=$ $\qquad I \in \mathbb{C}^d \otimes \left(\mathbb{C}^d\right)^*$

"The max. ent state is just the identity$^{\circledast}$ matrix"

- Graphical computation:

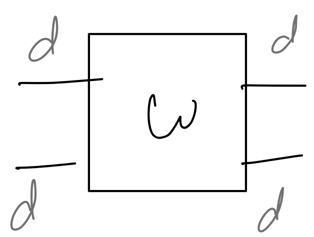
compute $\mathrm{Tr}_2 \underbrace{|\Omega\rangle\langle\Omega|}_{\omega}$

$$\omega = \underbrace{|\Omega\rangle}_{\mathbb{C}^d \otimes \mathbb{C}^d}\underbrace{\langle\Omega|}_{\left(\mathbb{C}^d\otimes\mathbb{C}^d\right)^*} \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \left(\mathbb{C}^d\right)^* \otimes \left(\mathbb{C}^d\right)^*$$

$$\text{|||} \quad \mathrm{Lin}\left(\mathbb{C}^d\otimes\mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^d\right)$$

$$\text{|||} \quad \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$$

$$\left[\ \text{|||} \quad \mathcal{M}_{d^2}(\mathbb{C})\ \right]$$



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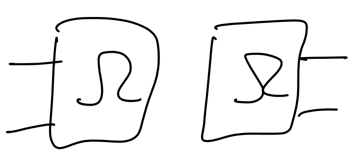
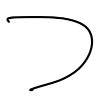
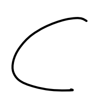
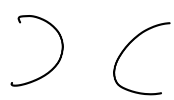
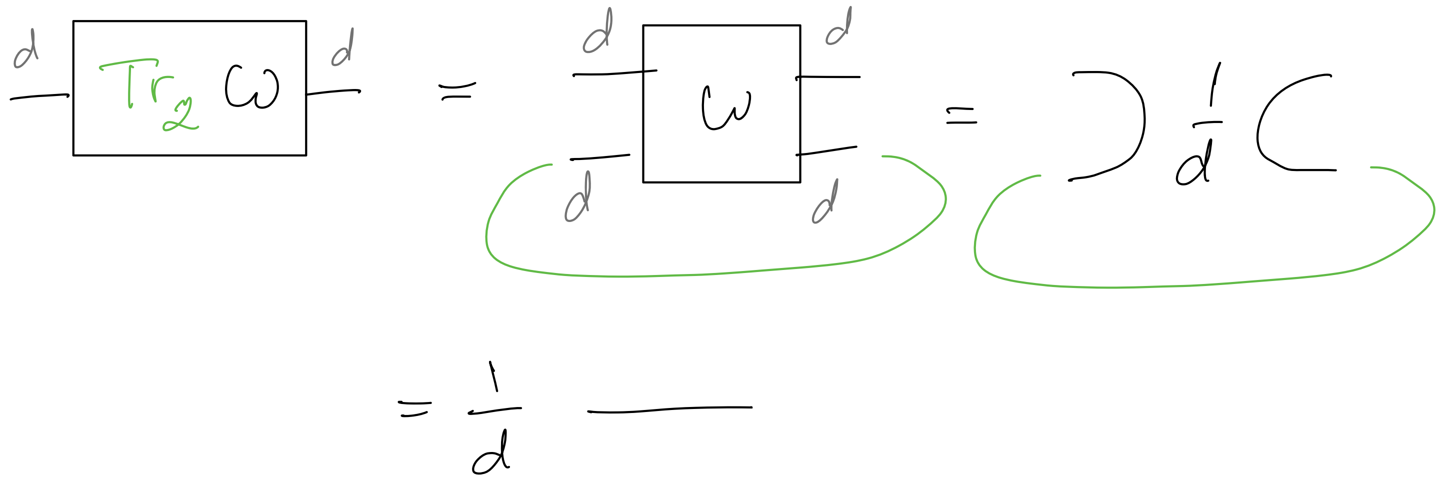


$= \frac{1}{\sqrt{d}}$  $\quad \frac{1}{\sqrt{d}}$ 

$= \frac{1}{d}$ 



So : $\mathrm{Tr}_2\, |\Omega\rangle\langle\Omega| = \frac{1}{d}\, I$

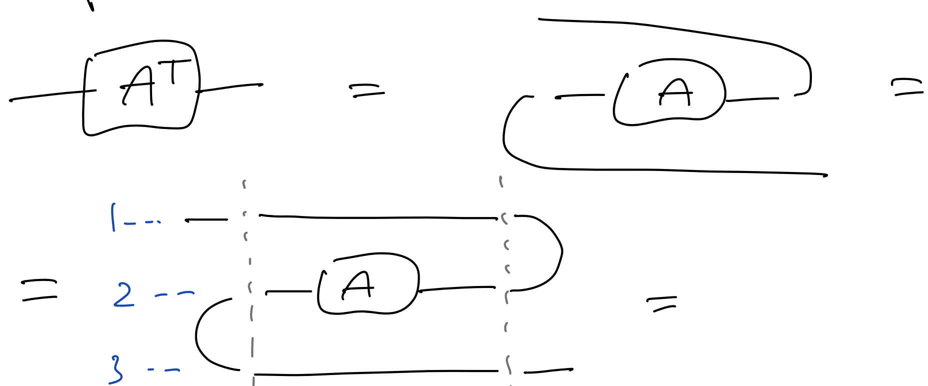
In coordinates :

$$\mathrm{Tr}_2\, |\Omega\rangle\langle\Omega| = \mathrm{Tr}_2\left(\left(\frac{1}{\sqrt{d}}\sum_i |ii\rangle\right)\left(\frac{1}{\sqrt{d}}\sum_j \langle jj|\right)\right)$$

$$= \frac{1}{d}\sum_{ij} \mathrm{Tr}_2\, |ii\rangle\langle jj| = \frac{1}{d}\sum_{ij} (id \otimes \mathrm{Tr})\left(|i\rangle\langle j| \otimes |i\rangle\langle j|\right)$$

$$= \frac{1}{d}\sum_{ij} |i\rangle\langle j| \cdot \underbrace{\mathrm{Tr}\, |i\rangle\langle j|}_{=\delta_{ij}} = \frac{1}{d}\sum_i |i\rangle\langle i| = \frac{1}{d}\cdot I$$

- Transpose vs ω



$$\underbrace{\left[\underset{1}{I} \otimes \sqrt{d}\, \underset{23}{\langle\Omega|}\right]}_{(\mathbb{C}^d)^{\otimes 3} \to \mathbb{C}^d} \cdot \underbrace{\left[\underset{1}{I} \otimes \underset{2}{A} \otimes \underset{3}{I}\right]}_{(\mathbb{C}^d)^{\otimes 3} \to (\mathbb{C}^d)^{\otimes 3}} \cdot \underbrace{\left[\sqrt{d}\, \underset{12}{|\Omega\rangle} \otimes \underset{3}{I}\right]}_{\mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d}$$

We have shown:

$$A^T = d \cdot \left(I \otimes \langle \Omega | \right) \cdot \left(I \otimes A \otimes I \right) \cdot \left(|\Omega\rangle \otimes I \right)$$