Tensors and graphical notation 19/03/25

① Tensor product

V, W two vector spaces fin. dim. over $\mathbb{C}$ ($\mathbb{R}, ..$)

$V \otimes W = ?$

Def 1: a basis of $V \otimes W$ is

$$\{ e_i \otimes f_j \}_{i,j} \quad \text{where}$$

$\{e_i\}$ basis of V

$\{f_j\}$ —— W

→ $\dim V \otimes W = \dim V \cdot \dim W$

→ contrast this w/ direct sum:

basis of $V \oplus W$ is $\{e_i\} \cup \{f_j\}$

$\dim V \oplus W = \dim V + \dim W$.

→ $\dim \mathcal{H}_{n \underbrace{\text{qubits}}_{\mathbb{C}^2}} = 2^n$ → exponential in n !

Def 2 V, W vector spaces.

Any $F : V \times W \longrightarrow A$ vector space

$(v, w) \longmapsto F(v, w)$

F bilinear. factorizes through $V \otimes W$

$$\begin{tikzcd}
V \times W \arrow[r, "F"] \arrow[dr, "i"'] & A \\
& V \otimes W \arrow[u, "\tilde{F}"']
\end{tikzcd}$$

This means: $\exists$ vector space "$V \otimes W$" s.t.

$$F(v,w) = \tilde{F}(\, i(v,w)\,)$$

$\tilde{F}$ linear map!

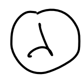
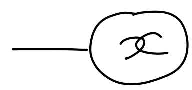
$$i(v,w) = v \otimes w$$

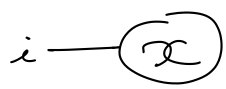
$$V \otimes W = \overline{\operatorname{span}\left\{ x \otimes y : \begin{matrix} x \in V \\ y \in W \end{matrix} \right\}}$$

Any $z \in V \otimes W$ can be decomposed as

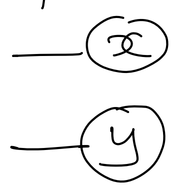
$$z = \sum \underbrace{x_i \otimes y_i}_{\text{simple tensors}} \qquad x_i \in V \quad y_i \in W$$

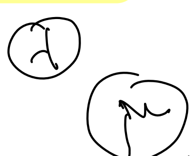
② Graphical notation

- scalars $\lambda \in \mathbb{C} \to$ 
- vectors $x \in V \to$ 

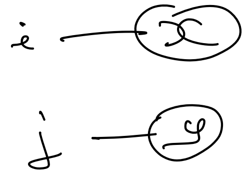
  $x_i \rightsquigarrow$ 
- tensor products of vectors

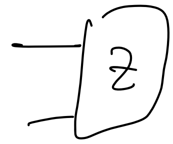
  $\to$ juxtaposing pictures

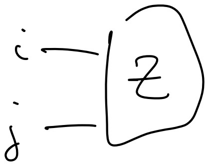
  $x \otimes y \to$ 
- product of scalars:

  $\lambda \cdot \mu \to$ 

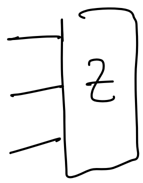
Rk: diagram with no edges sticking out $\to$ scalar

- $(x \otimes y)_{ij}$ =  $= x_i \cdot y_j$

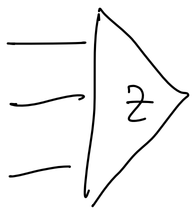
- for $z \in V \otimes W$ $\rightsquigarrow$ 

z_{ij} $\longrightarrow$ 

- more generally: $z \in V_1 \otimes V_2 \otimes V_3$



etc.

= kets 

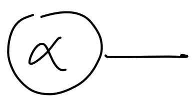
③ Dual spaces (i.e. bras)

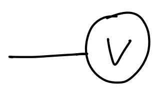
V vector space $\rightsquigarrow$ dual V^*

$$V^* = \{ \alpha : V \to \mathbb{C} \text{ linear} \}.$$

is again a vector space.

- graphically: $\alpha \in V^*$ i.e. $\alpha : V \to \mathbb{C}$

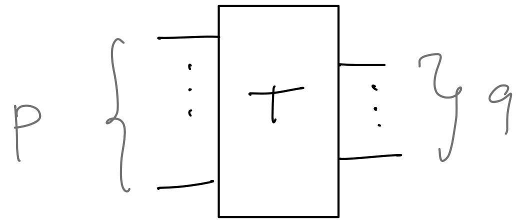




read this way

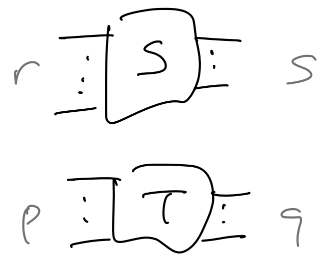
- more generally

$$T : \underbrace{W_1 \otimes \cdots \otimes W_p}_{p \text{ primal spaces}} \otimes \underbrace{V_1^* \otimes \cdots \otimes V_q^*}_{q \text{ dual spaces.}}$$



- $S : A_1 \otimes \cdots \otimes A_r \otimes B_1^* \otimes \cdots \otimes B_s^*$

$S \otimes T \rightsquigarrow$



$\boxed{④ \text{ Contraction}}$ (connecting boxes)

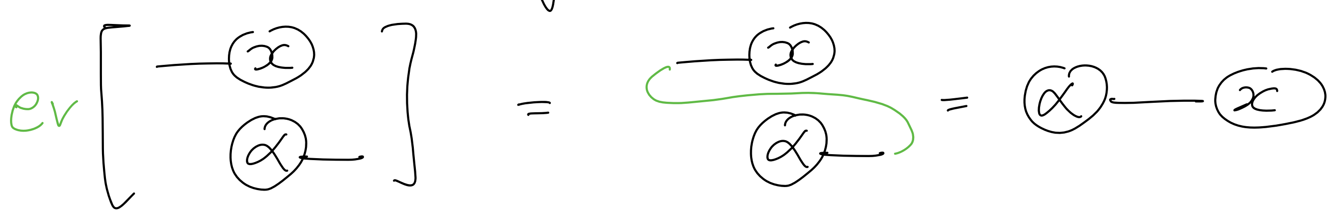
$$V \times V^* \longrightarrow V \otimes V^* \xrightarrow{ev} \mathbb{C}$$

$$(x, \alpha) \longmapsto (x \otimes \alpha) \longmapsto \alpha(x)$$

evaluation map.

⇣ graphical picture

"contracting" the corresponding half-edges.

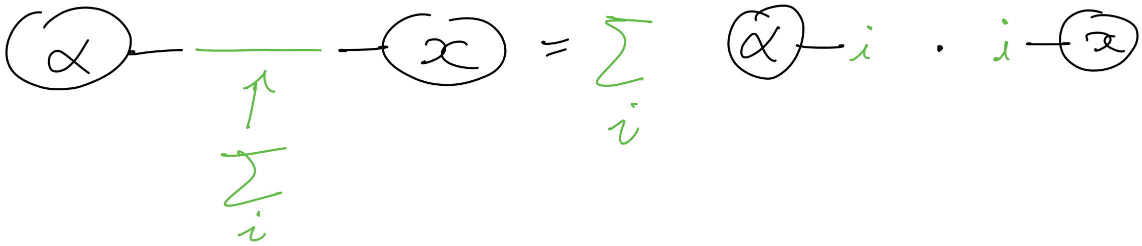


Example. $x \in \mathbb{C}^d$ $\quad \alpha \in (\mathbb{C}^d)^*$

$$\alpha(v) = \sum \overline{\alpha_i}\, v_i \qquad \alpha_i \in \mathbb{C}$$

"$\alpha = \langle \alpha |$"

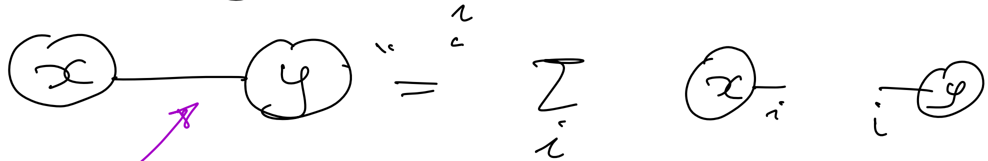
$$\alpha(x) = \sum \overline{\alpha_i}\, x_i$$



Examples.

- dot product in $\mathbb{R}^n$ $\quad x, y \in \mathbb{R}^n$

$$\langle x, y \rangle = \sum_i x_i y_i$$



! this is not canonical.

non-canonical identification

$$\tilde{x} \in V^* \qquad \tilde{x}(v) = \langle x, v \rangle$$

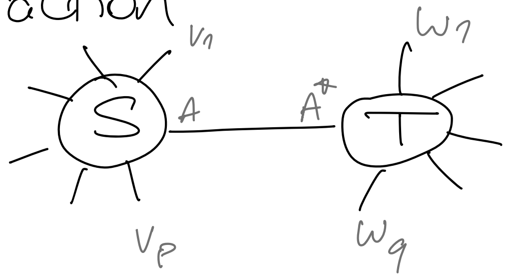
scalar product on V

$$\langle\langle a, b \rangle\rangle = a_1 b_1 + 2 a_2 b_2$$

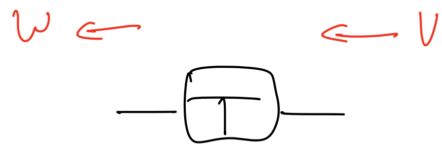
- more generally $S \in V_1 \otimes \cdots \otimes V_p \otimes A$

$T \in W_1 \otimes \cdots \otimes W_q \otimes A^*$

contraction



5 Matrices / Linear maps

$T : V \longrightarrow W \quad \rightsquigarrow$ 

$|||$

$T \in W \otimes V^*$

canonical.

$\{\text{linear maps } V \to W\} \equiv \{\text{tensors in } W \otimes V^*\}$

<u>Example</u> $\alpha \in V^*$ and $w \in W$

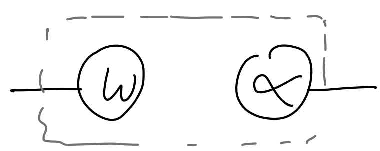
$$\left[\begin{array}{l} T : V \longrightarrow W \\ T(x) = \alpha(x) \cdot w \end{array}\right] \longleftarrow \underset{\substack{\cap \\ W}}{w} \otimes \underset{\substack{\cap \\ V^*}}{\alpha}$$

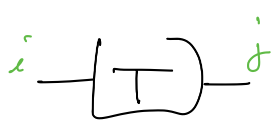
$x \in V$

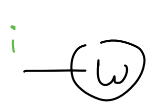
$|w\rangle\langle\alpha| \longleftarrow |w\rangle \otimes \langle\alpha|$

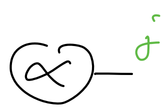
$$\Big(|w\rangle\langle\alpha|\Big)\Big(|x\rangle\Big) = |w\rangle\langle\alpha|x\rangle = \alpha(x) \cdot |w\rangle$$

anti-lin linear



In coordinates: 

$=$ 

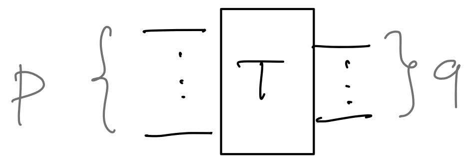


$(|w\rangle\langle\alpha|)_{ij} = T_{ij} = w_i \, \overline{\alpha_j}$ ← it's a bra

- **More generally:**

$$T : V_1 \otimes \cdots \otimes V_q \longrightarrow W_1 \otimes \cdots \otimes W_p$$

$$\equiv \quad T \in W_1 \otimes \cdots \otimes W_p \otimes V_1^* \otimes \cdots \otimes V_q^*$$

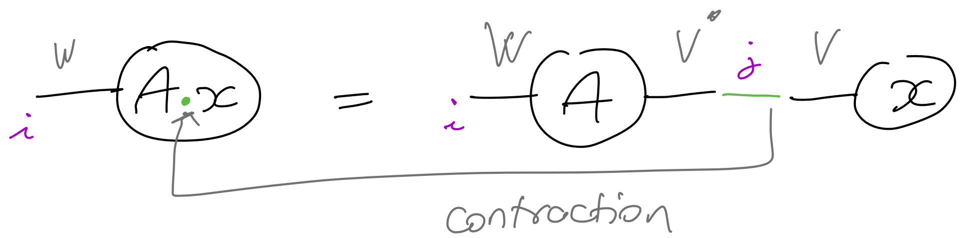


⑥ Applications

- **Matrix-vector product.**

$A : V \longrightarrow W$

$x \in V$. $\qquad Ax \in W$



In coordinates : $(Ax)_i = \sum_j A_{ij} \, x_j$

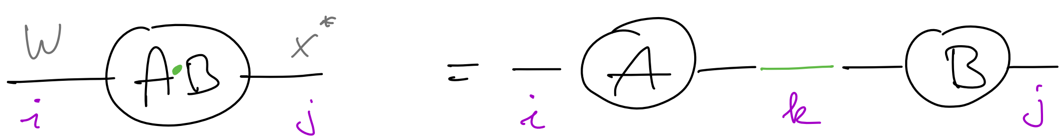
- **Matrix-matrix product**

$A : V \to W$

$B : X \to V$

$AB : X \longrightarrow W$

W V X



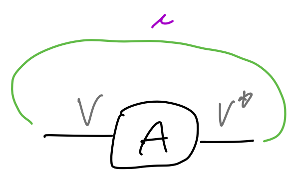
In coordinates: $(AB)_{ij} = \sum_k A_{ik} B_{kj}$

- Trace

$A : V \to V$

$\mathrm{Tr}\, A \in \mathbb{C}$

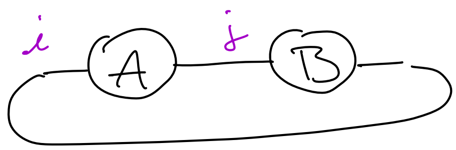
$\mathrm{Tr}\, A$ =



! no half-edges left ⇒ scalar

In coordinates: $\mathrm{Tr}\, A = \sum A_{ii}$

- A first graphical proof: $A, B : V \to V$



=

↑ by moving the boxes around the wires

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

i.e. trace is cyclic.

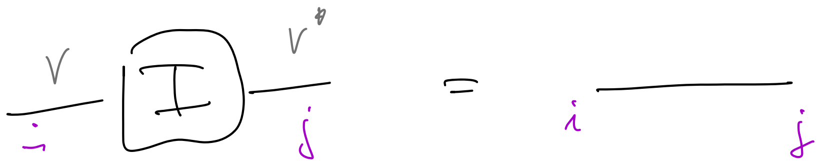
In coordinates

$$\mathrm{Tr}(AB) = \sum_i (AB)_{ii} = \sum_{ij} A_{ij} B_{ji}$$

$$\mathrm{Tr}(BA) = \sum_{ij} B_{ij} A_{ji}$$

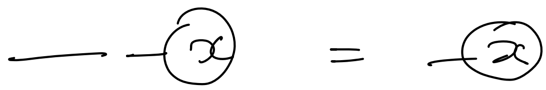
- Identity matrix (linear map).

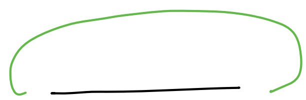
$$I : V \longrightarrow V$$
$$x \longmapsto x$$



In coordinates $I_{ij} = \delta_{ij}$

- $I \cdot x = x$

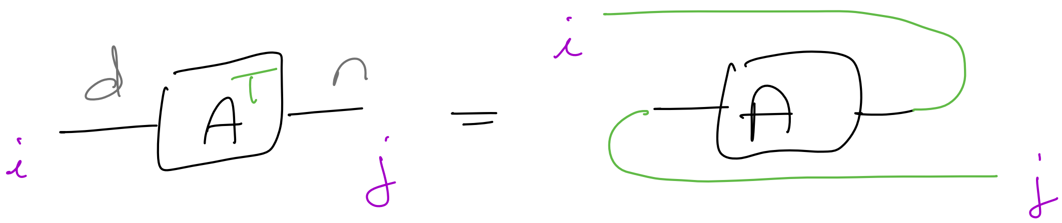


- $Tr(I)$ =  = loop = dim V

Loops associated to V = dim V.

- Transpose $A : \mathbb{C}^d \longrightarrow \mathbb{C}^n$

$$A^T : \mathbb{C}^n \longrightarrow \mathbb{C}^d.$$



In coordinates: $(A^T)_{ij} = A_{ji}$

! not canonical.

in real case: A^T is defined by $\langle A^T v, w \rangle = \langle v, Aw \rangle$

scalar product

in the complex case

$\shortparallel$
A dagger

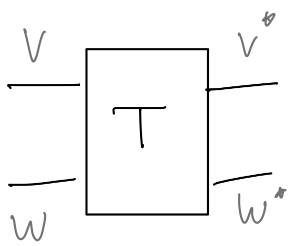
for example : if $x \in V$

$\langle x|$ = 

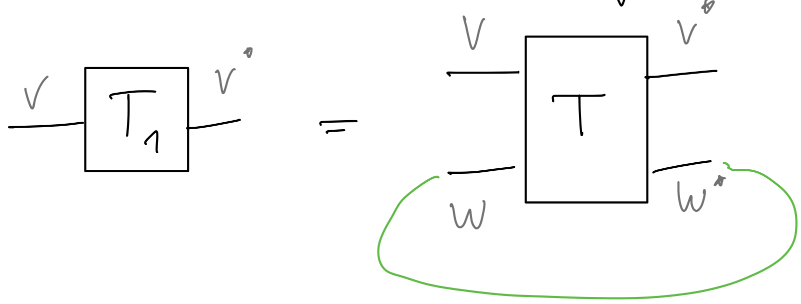
because $\langle x | v \rangle$ = 

$$= \sum_i \bar{x}_i v_i$$

- Partial trace. $T : V \otimes W \to V \otimes W$



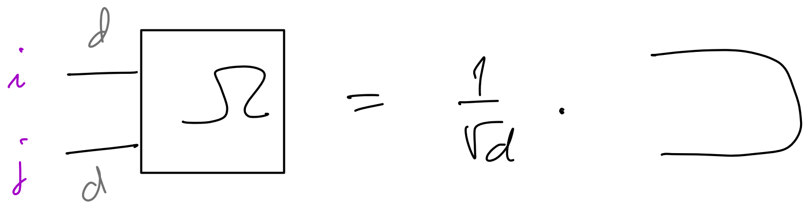
partial trace : $T_1 = [\mathrm{id}_V \otimes \mathrm{Tr}_W](T) : V \to V$



- Maximally entangled state.

$$\mathbb{C}^d \otimes \mathbb{C}^d \ni |\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$$

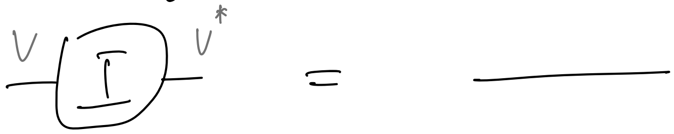
$d=2 \qquad \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$ Bell state



$|\Omega\rangle_{ij} = \frac{1}{\sqrt{d}}\,\delta_{ij}$

/// up to $\cdot \frac{1}{\sqrt{d}}$

$\circledast$ $\circ\ (\mathbb{C}^d)^* \equiv \mathbb{C}^d$



$I \in \mathbb{C}^d \otimes (\mathbb{C}^d)^*$

"The max. ent state is just$^{\circledast}$ the identity matrix"

- Graphical computation:

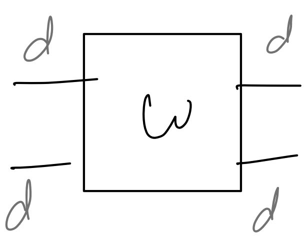
compute $\mathrm{Tr}_2 \underbrace{|\Omega\rangle\langle\Omega|}_{\omega}$

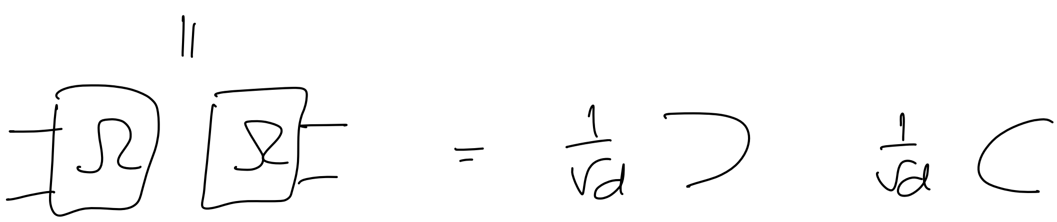
$\omega = \underbrace{|\Omega\rangle}_{\mathbb{C}^d\otimes\mathbb{C}^d}\underbrace{\langle\Omega|}_{(\mathbb{C}^d\otimes\mathbb{C}^d)^*} \in \mathbb{C}^d\otimes\mathbb{C}^d\otimes(\mathbb{C}^d)^*\otimes(\mathbb{C}^d)^*$

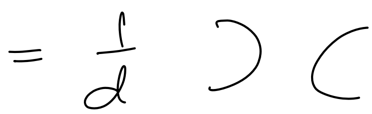
$\cong \mathrm{Lin}(\mathbb{C}^d\otimes\mathbb{C}^d \to \mathbb{C}^d\otimes\mathbb{C}^d)$

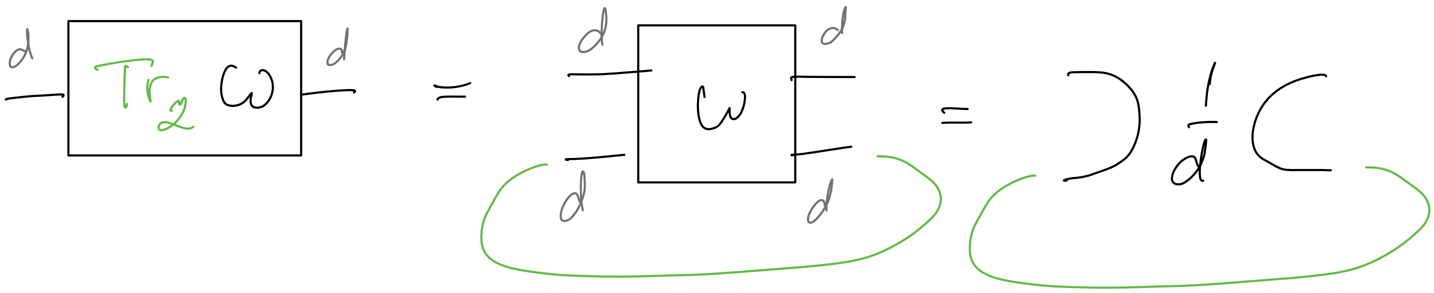
$\cong \mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$

$\left[\ \cong \mathcal{M}_{d^2}(\mathbb{C})\ \right]$









$= \frac{1}{d}$

So : $\mathrm{Tr}_2 |\Omega\rangle\langle\Omega| = \frac{1}{d} I$

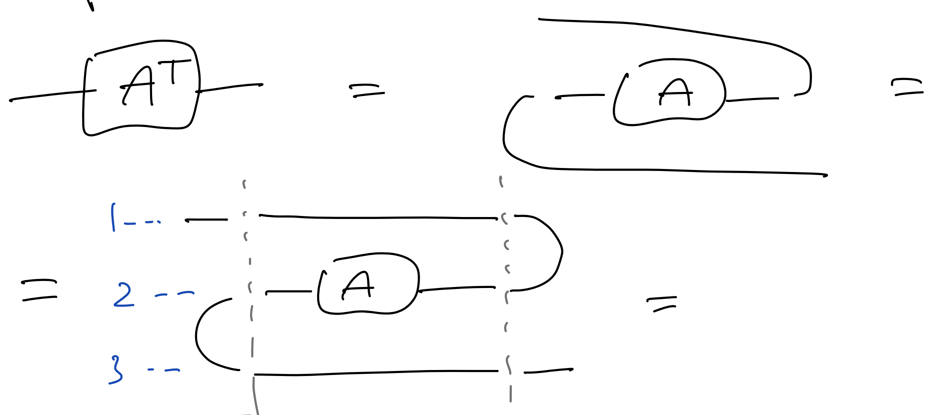
In coordinates :

$$\mathrm{Tr}_2 |\Omega\rangle\langle\Omega| = \mathrm{Tr}_2\left(\left(\frac{1}{\sqrt{d}}\sum_i |ii\rangle\right)\left(\frac{1}{\sqrt{d}}\sum_j \langle jj|\right)\right)$$

$$= \frac{1}{d}\sum_{ij} \mathrm{Tr}_2 |ii\rangle\langle jj| = \frac{1}{d}\sum_{ij} (\mathrm{id}\otimes\mathrm{Tr})\left(|i\rangle\langle j| \otimes |i\rangle\langle j|\right)$$

$$= \frac{1}{d}\sum_{ij} |i\rangle\langle j| \cdot \underbrace{\mathrm{Tr}\,|i\rangle\langle j|}_{=\delta_{ij}} = \frac{1}{d}\sum_i |i\rangle\langle i| = \frac{1}{d}\cdot I$$

- Transpose vs ω



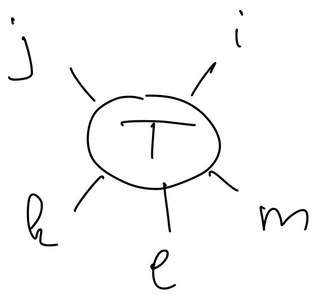
$$\underbrace{\left[\underset{1}{I}\otimes\sqrt{d}\,\underset{23}{\langle\Omega|}\right]}_{(\mathbb{C}^d)^{\otimes 3}\longrightarrow\mathbb{C}^d}\cdot\underbrace{\left[\underset{1}{I}\otimes\underset{2}{A}\otimes\underset{3}{I}\right]}_{(\mathbb{C}^d)^{\otimes 3}\longrightarrow(\mathbb{C}^d)^{\otimes 3}}\cdot\underbrace{\left[\sqrt{d}\,\underset{12}{|\Omega\rangle}\otimes\underset{3}{I}\right]}_{\mathbb{C}^d\longrightarrow\mathbb{C}^d\otimes\mathbb{C}^d\otimes\mathbb{C}^d}$$

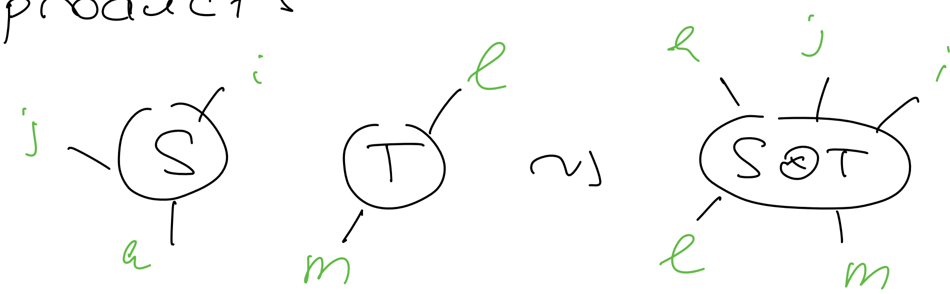
We have shown:

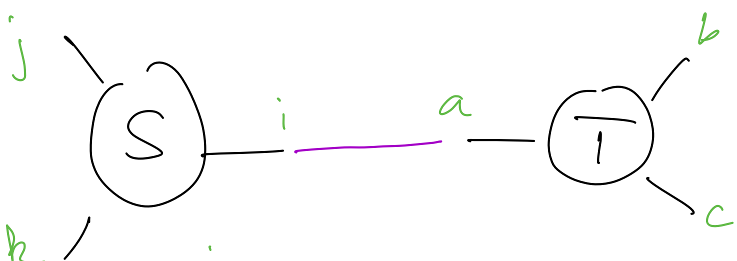
$$A^T = d\cdot\left(I\otimes\langle\Omega|\right)\cdot\left(I\otimes A\otimes I\right)\cdot\left(|\Omega\rangle\otimes I\right)$$

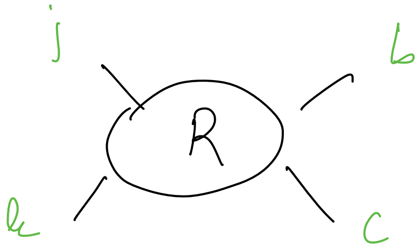
24/03

- Tensors  $T\in(\mathbb{R}^d)^{\otimes 5}$

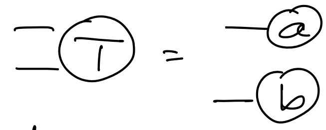
- Coordinates  $T_{ijklm}\in\mathbb{R}$

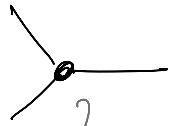
- Tensor products 

- Tensor contraction 



$$R_{jkbc}=\sum_{i=a} S_{ijk}\,T_{abc}$$

- $T \in V \otimes W$
  - $\to$ T simple $\quad T = a \otimes b$ 
  - $\to$ not simple $\quad T = \sum a_i \otimes b_i$

$\to$ Copy tensor $\quad C =$  $\in (\mathbb{R}^d)^{\otimes 3}$

$$C = \sum_{i=1}^{d} |ii\rangle\langle i| \qquad C : \mathbb{R}^d \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^d$$

$C_{ijk} = \delta_{ijk}$ in coord. $\qquad \sum |iii\rangle$

$$C|i\rangle = |ii\rangle$$

↳ C copies the canonical basis $\{|i\rangle\}$

However it is not true that
no-cloning theorem

$$C|x\rangle = |x\rangle \otimes |x\rangle \quad \text{for } \underline{\text{all}}\ |x\rangle$$

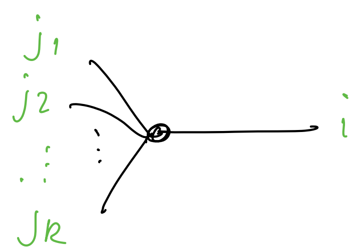
e.g. $\underbrace{C|+\rangle}_{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)} = \frac{1}{\sqrt{2}}\left(C|0\rangle + C|1\rangle\right)$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$\neq |+\rangle|+\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

- more generally

$$C : \mathbb{R}^d \longrightarrow (\mathbb{R}^d)^{\otimes k}$$



$$C|i\rangle = |i\rangle^{\otimes k}$$

$$C_{i j_1 \cdots j_k} = \delta_{i j_1 j_2 \cdots j_k}$$

- An arbitrary tensor $T \in V_1 \otimes \cdots \otimes V_k$

$$T = \sum_{i=1}^{n} a_i^{(1)} \otimes a_i^{(2)} \otimes \cdots \otimes a_i^{(k)}$$

$$a_i^{(x)} \in V_x \qquad \forall \begin{matrix} i \in [n] \\ x \in [k] \end{matrix}$$

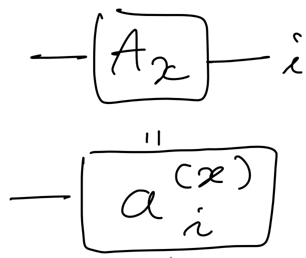
→ first, pack all the vectors $a_i^{(x)} \in V_x$ into a matrix A_x

$$A_x := \sum |a_i^{(x)}\rangle\langle i| : \mathbb{R}^n \longrightarrow V_x$$

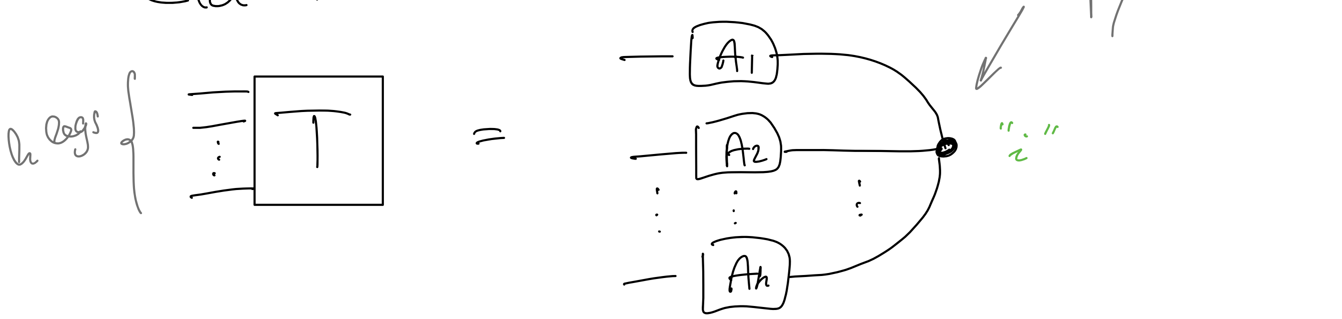
dim V_x rows

$$\boxed{A_x} = \begin{pmatrix} | & & | \\ a_1^{(x)} & \cdots & a_n^{(x)} \\ | & & | \end{pmatrix}$$

n columns



Claim



$$T = \sum_{i=1}^{n} a_i^{(1)} \otimes a_i^{(2)} \otimes \cdots \otimes a_i^{(k)}$$

→ unitary matrices. $U \cdot \underbrace{U^*}_{\bar{U}^T} = I_d$ ← dagger

$$-\boxed{U}-\boxed{U^*}- \;=\; \text{———} \;=\; -\boxed{U^*}-\boxed{U}-$$

→ more generally: isometries

$$V : \mathbb{C}^d \longrightarrow \mathbb{C}^n \qquad d \leq n$$

$$_d V^*_{\;n} V_d = I_d$$

$$_d-\boxed{V^*}-_n-\boxed{V}-_d \;=\; {}_d\text{———}_d$$

However: $V V^* =: P$ is the projection on the range of V

$$\text{ran } V = \underbrace{V(\mathbb{C}^d)}_{\text{subspace of } \mathbb{C}^n \text{ of dim } d} \subseteq \mathbb{C}^n$$

Check that P is a projection:

$$-\boxed{P^2}- = -\boxed{P}-\boxed{P}- = -\boxed{V}-\underbrace{\boxed{V^*}-\boxed{V}}_{=\,\text{———}}-\boxed{V^*}-$$

$$= -\boxed{V}-\boxed{V^*}- = -\boxed{P}-$$

def of isometry

$$\langle x,y\rangle_{\mathbb{C}^d} = \langle Vx, Vy\rangle_{\mathbb{C}^n} = \langle x, V^* V y\rangle \qquad \forall x,y \in \mathbb{C}^d$$

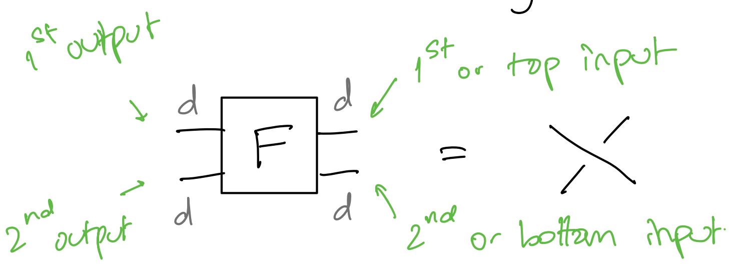
$$\Longrightarrow V^* V = I_d$$

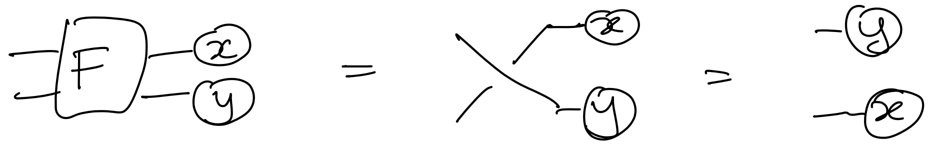
→ Flip (or SWAP) operator $F \in \mathcal{U}(d^2)$

$$F : \mathbb{C}^d \otimes \mathbb{C}^d \longrightarrow \mathbb{C}^d \otimes \mathbb{C}^d$$

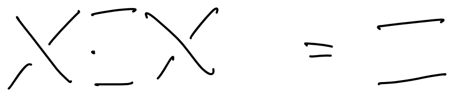
$$|x\rangle \otimes |y\rangle \longrightarrow |y\rangle \otimes |x\rangle$$

⊕ extend by linearity



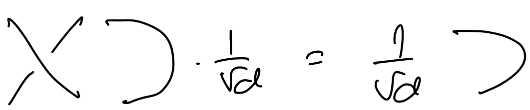


- Computations with the flip
  - $F^2 = I_{d^2}$

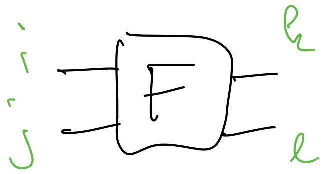
 $= I_{d^2}$

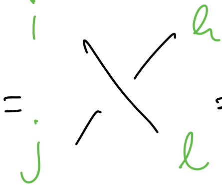
  - $\mathrm{Tr}_2(F) = I_d$

  - $F \cdot |\Omega_d\rangle = |\Omega_d\rangle$ $\frac{1}{\sqrt{d}} \sum |ii\rangle$

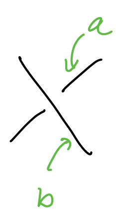


- In coordinates   It's canonical!



$F_{ij,kl} =$  $= \delta_{il}\,\delta_{jk}$

$$F = \sum_{ab=1}^{d} |ba\rangle\langle ab| =$$



- $F = F^*$: $\langle F x \otimes y, z \otimes t \rangle =$

$= \langle y \otimes x, z \otimes t \rangle = \langle y, z \rangle \langle x, t \rangle$

$\langle x \otimes y, F z \otimes t \rangle = \langle x, t \rangle \langle y, z \rangle$

In particular, $F \cdot F^* = F^2 = I_{d^2}$

$\Rightarrow$ F is unitary. ✓

+ F is a Hermitian unitary op. $\Rightarrow$ eigenvalues $= \pm 1$

$F = 1 \cdot P_s + (-1) \cdot P_a$

P_s = projection on the symmetric subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$

P_a = projection on the anti-symmetric subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$

S = range of P_s = span $\{ |x\rangle \otimes |x\rangle \}$ $\quad |00\rangle, |+\rangle|+\rangle$

$|v\rangle \in S \Rightarrow F \cdot |v\rangle = |v\rangle$

A = range of P_a = span $\{ |x\rangle \otimes |y\rangle - |y\rangle \otimes |x\rangle$, $x \perp y \}$

$|v\rangle \in A \Rightarrow F|v\rangle = -|v\rangle$

$|01\rangle - |10\rangle$

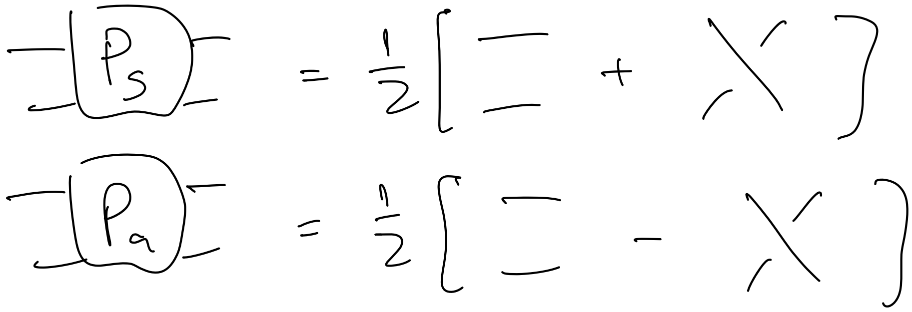
For qubits $d = 2$

S = span $\{ |00\rangle, |11\rangle, |++\rangle \}$ dim 3

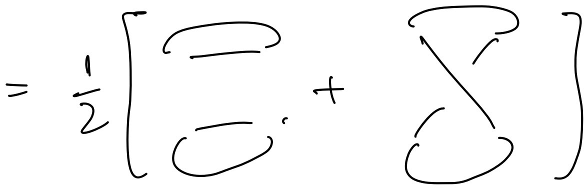
A = Span $\{ |01\rangle - |10\rangle \}$ dim 1

$S \oplus A = \mathbb{C}^2 \otimes \mathbb{C}^2$

Recall : $F = P_s - P_a$, $I = P_s + P_a$ $\Rightarrow$ $P_s = \dfrac{I+F}{2}$, $P_a = \dfrac{I-F}{2}$

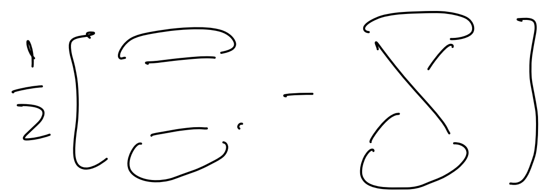


• $\dim S \subseteq \mathbb{C}^d \otimes \mathbb{C}^d = \operatorname{Tr} P_s$



$= \dfrac{d^2+d}{2} = \dfrac{d(d+1)}{2}$

• $\dim A =$



$= \dfrac{d^2-d}{2} = \binom{d}{2}$

$\hookrightarrow = \operatorname{span}\{ |i\rangle|j\rangle - |j\rangle|i\rangle \}_{1 \le i < j \le d}$

→ Same procedure works for more than 2 particles $\underbrace{\mathbb{C}^d \otimes \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d}_{n \text{ particles}}$

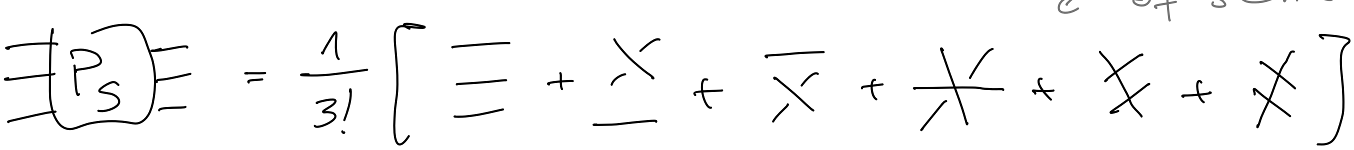
$S := \operatorname{ran} P_s \subseteq (\mathbb{C}^d)^{\otimes n}$

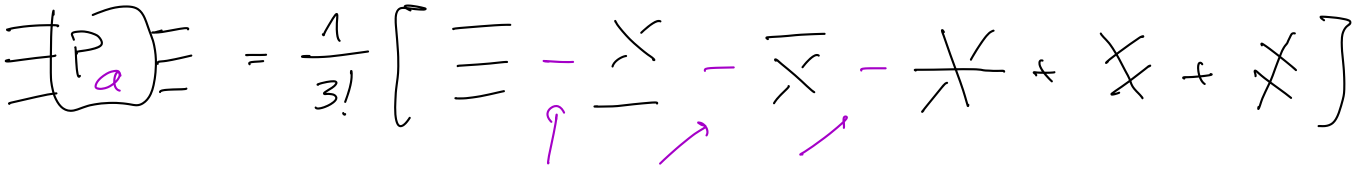
$A := \operatorname{ran} P_a \subseteq (\mathbb{C}^d)^{\otimes n}$

$n \geq 3$ $(\mathbb{C}^d)^{\otimes n} = S \oplus A \oplus \cdots$ other repr.

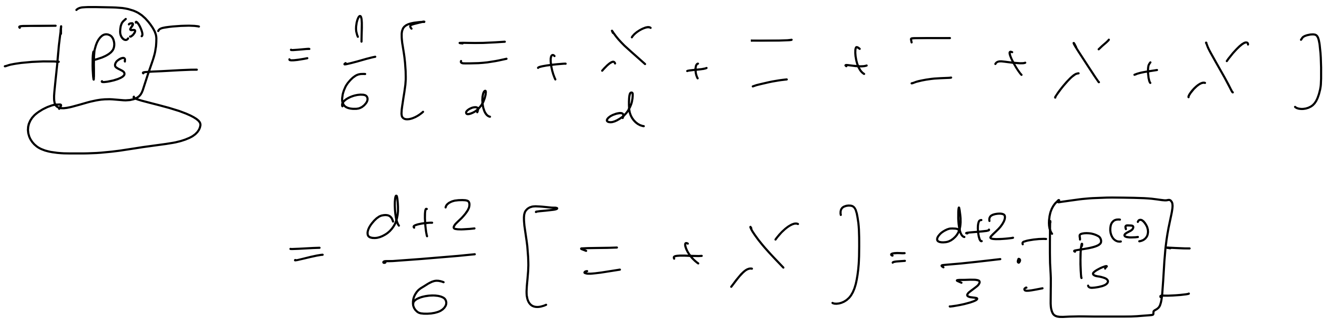
6 perm. of 3 elts.

$n=3$





signs of the perm.



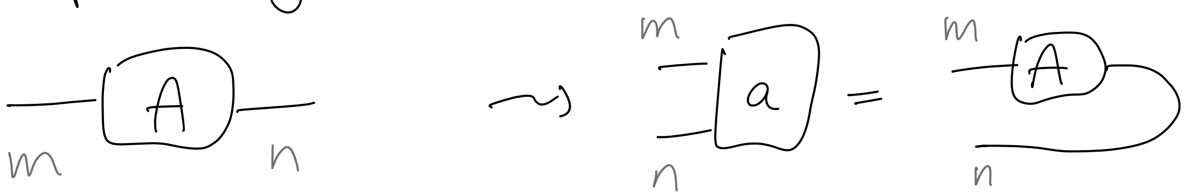
→ Vectorization (flattening)

Idea: $A \in \mathcal{M}_{m\times n}(\mathbb{R}) \longrightarrow a \in \mathbb{R}^m \otimes \mathbb{R}^n$

non-canonical

$$A \in \mathcal{M}_{m\times n} \simeq \{\mathbb{R}^n \to \mathbb{R}^m\} \simeq \mathbb{R}^m \otimes (\mathbb{R}^n)^*$$

graphically:

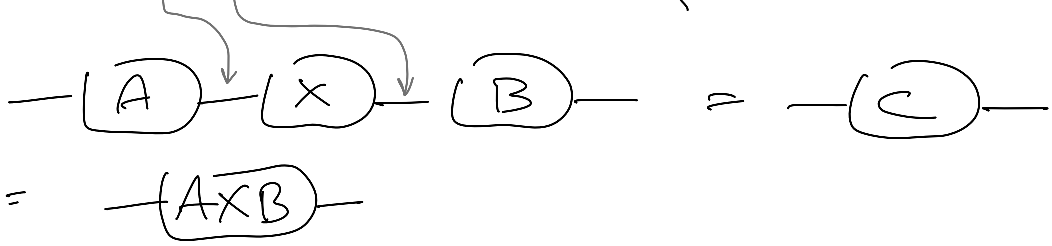


$$\underbrace{a_{ij}}_{\text{outputs}} = A_{i,j}$$

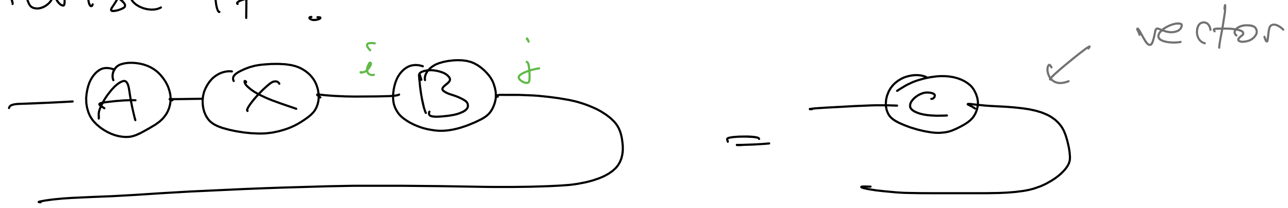
output input

Why? eg. solving matrix equations

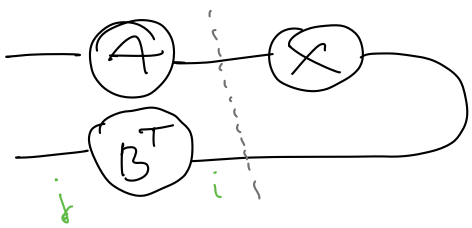
Solve $A \cdot X \cdot B = C$ for X



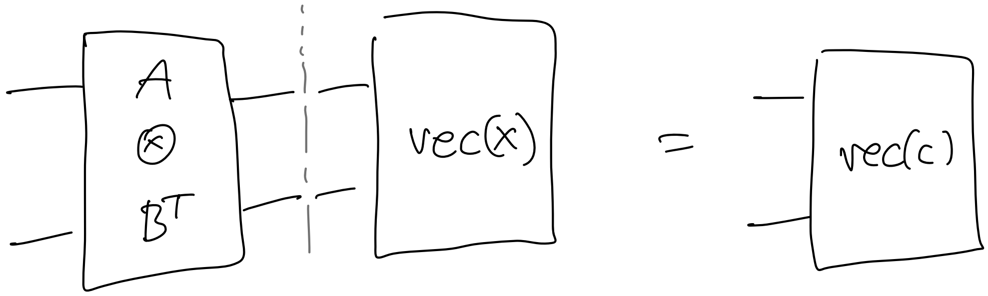
vectorize it !



$=$



$=$

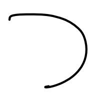


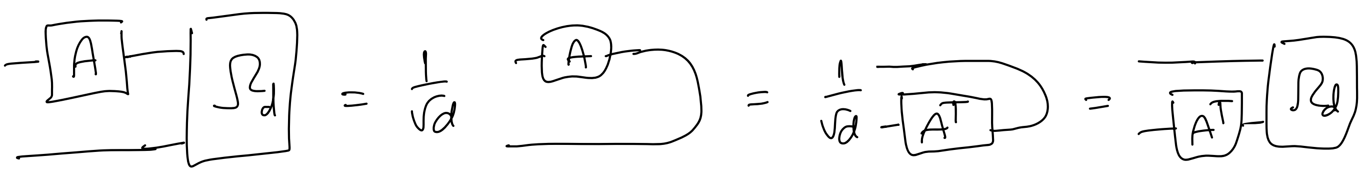
$$(A \otimes B^T) \cdot \text{vec}(X) = \text{vec}(C)$$

$$\mathcal{A} \cdot x = \mathcal{C} \quad \equiv \text{linear system}$$

matrix   vector   vector

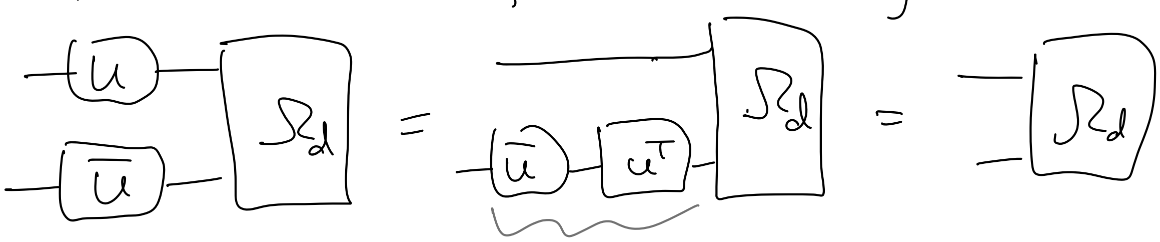
→ Important property of the maximally entangled state

$|\Omega\rangle = \frac{1}{\sqrt{d}} |ii\rangle = \frac{1}{\sqrt{d}}$  $= \frac{1}{\sqrt{d}} \text{vec}(I_d)$



with algebra : $(A \otimes I)|\Omega_d\rangle = (I \otimes A^T)|\Omega_d\rangle$

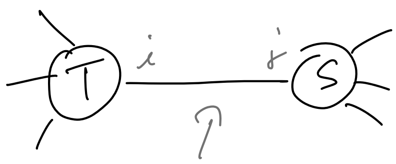
In particular: if u unitary



$\bar{u} \cdot u^T = I$

$$\forall\ U \in \mathcal{U}(d) :\quad (U \otimes \bar{U})|\Omega_d\rangle = |\Omega_d\rangle$$

7 More about contracting tensors.



$\to$ compute $|\{i=j\}|$ products

$$\sum_{i=j} T_{i\ldots} \, S_{j\ldots}$$

<u>Example</u>: Compute the matrix product ABC where

A	10×100
B	100×5
C	5×50

$ABC \to (A \cdot B)\cdot C$ I

$ABC \to A \cdot (B \cdot C)$ II

In general $\underset{p}{(}X \underset{r}{\cdot} Y\underset{q}{)}_{ij} = \sum_k X_{ik} Y_{kj}$

compute $p \cdot q$ entries; each entry r products $\Rightarrow pqr$ "$\times$"

I $\quad A \cdot B \to$ need $10 \cdot 100 \cdot 5 = 5000$ products

$AB \cdot C \to 2500$ products.

10 5 50

7500 products

II $\quad A \cdot (B \cdot C)$

$B \cdot C \to 25000$ products

100 5 50

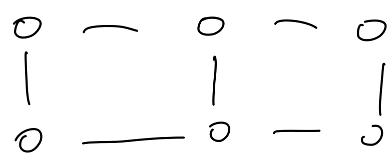
$A \cdot BC \to 50000$ products

10 100 50

75000 products

$(AB) \cdot C$ requires $10\times$ less products than $A(BC)$

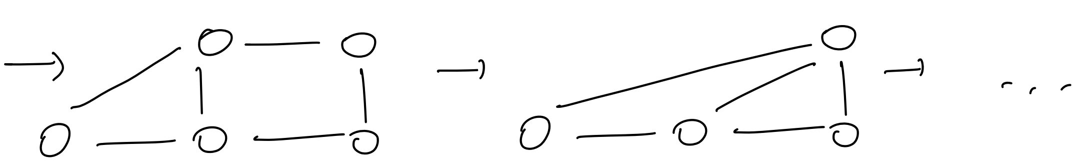
Example



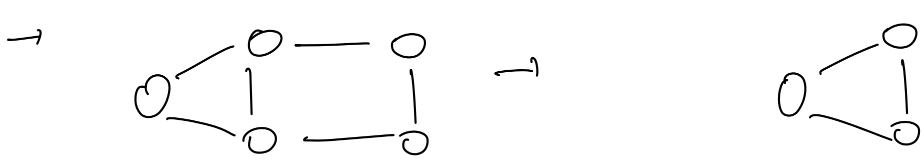
"bubbling"

see [Tensor networks in a nutshell]

I $\to$



II $\to$



What is the optimal contraction order?

→ easy if the network is a tree.

→ easy-ish if network is "tree-like"
treewidth

→ NP-hard in general

In particular

$$\overset{d_0}{}\;A_1 \overset{d_1}{\cdot} A_2 \overset{d_2}{\cdot} \dots \overset{d_{k-2}}{\cdot} A_{k-1} \overset{d_{k-1}}{\cdot} A_k \;\overset{d_k}{}$$

dynamic programming

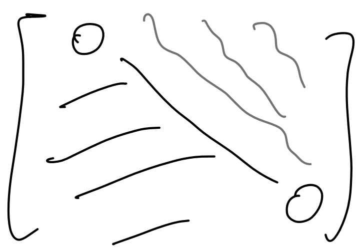
C_{ij} = optimal cost of mult. $A_i \cdot A_{i+1} \cdots \cdot A_j$

$i \le j$

$C_{ii} = 0$

$C_{i\,i+1} = d_{i-1} \cdot d_i \cdot d_{i+1}$

$= \text{cost of } A_i \cdot A_{i+1}$

$C =$ 

$$C_{ij} = \min_{i \le k < j} \text{cost}\left[(A_i \cdots A_k) \underset{\overset{\uparrow}{k}}{\cdot} (A_{k+1} \cdots A_j)\right]$$

$$= \min_{i \le k < j} d_{i-1} \cdot d_k \cdot d_j + C_{ik} + C_{k+1,j}$$

→ compute $C_{i,i+L}$ for $L = 1, 2, \dots$