Tensor norms

Norms $\quad X$ vector space over $\mathbb{C}$

$$\|\cdot\| : X \longrightarrow \mathbb{R}_+$$

- $\|x+y\| \le \|x\| + \|y\| \qquad x, y \in X$
- $\|\lambda x\| = |\lambda| \cdot \|x\| \qquad \lambda \in \mathbb{C},\ x \in X$
- $\|x\| = 0 \iff x = 0$

Examples (vectors)

ℓ_2^n · $(\mathbb{C}^n, \|\ \|_2) \qquad \|x\|_2 = \sqrt{\sum |x_i|^2}$

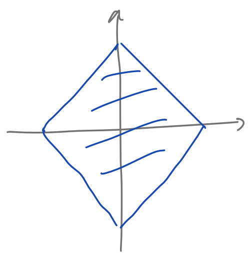
euclidean norm

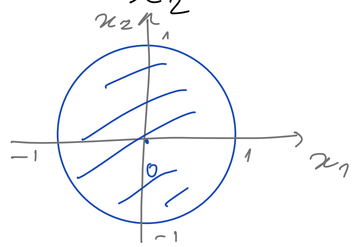
ℓ_1^n · $(\mathbb{C}^n, \|\ \|_1) \qquad \|x\|_1 = \sum |x_i|$

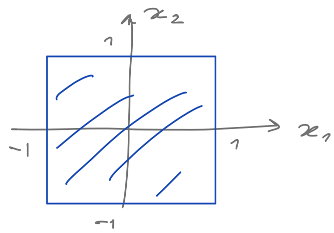
ℓ_∞^n · $(\mathbb{C}^n, \|\ \|_\infty) \qquad \|x\|_\infty = \max_i |x_i|$

Unit ball $\qquad B := \{ x \in X : \|x\| \le 1 \}$

$$B_{\ell_1} \subseteq B_{\ell_2}^{n=2} \subseteq B_{\ell_\infty}$$







$$\|\ \|_1 \ge \|\ \|_2 \ge \|\ \|_\infty$$

Examples (matrices) $A \in \mathcal{M}_{m\times n}(\mathbb{C})$

Frobenius norm $\|A\|_2 = \sqrt{\sum_{ij} |A_{ij}|^2} = \|sv(A)\|_2$

singular values of A

nuclear norm $\|A\|_1 = \|sv(A)\|_1 = \sum_i s_i(A)$

operator norm $\|A\|_\infty = \|sv(A)\|_\infty = \max_i s_i(A)$

Duality. Dual norms

$(X, \|\ \|)$ $\rightsquigarrow$ define a new norm on X^*

vector space, fin. dim

dual space

$X^* = \{ \varphi : X \longrightarrow \mathbb{C} \text{ linear} \}$

$$\|\varphi\|_* = \sup_{\substack{x\in X \\ \|x\|\le 1}} |\varphi(x)| = \sup_{x \in B_X} |\varphi(x)|$$

B_X unit ball

$(X^*, \|\ \|_*)$ is dual to $(X, \|\ \|)$

Fact : $(X^{**}, \|\ \|_{**}) = (X, \|\ \|)$

Examples $\ell_2^n = (\mathbb{C}^n, \|\ \|_2)$

$(\ell_2^n)^* = \ell_2^n$; $(\ell_1^n)^* = \ell_\infty^n$; $(\ell_\infty^n)^* = \ell_1^n$

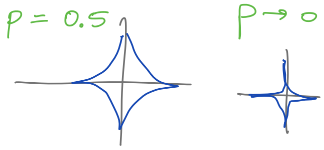
More generally: $1 < p < \infty$

$$\| x \|_p = \left(\sum |x_i|^p \right)^{1/p}$$

p-norm $\ell_p^n = (\mathbb{C}^n, \|\cdot\|_p)$

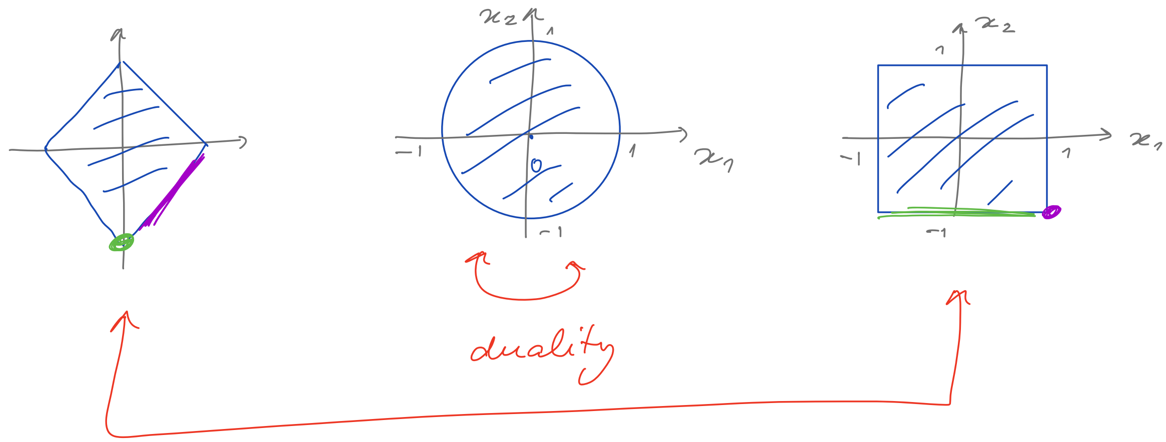
$$(\ell_p^n)^* = \ell_q^n \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = 1$$

$p = 0.5$ $\quad$ $p \to 0$



$\lim_{p \to 0} \|x\|_p = \#$ of non-zero elts of x

x is a prob. $\|x\|_1 = 1 = \sum x_i$ $\quad \lim_{p\to 1} \frac{\log \|x\|_p}{1-p} = H(x) = -\sum x_i \log x_i$



Tensor norms

$(X, \| \ \|_X)$ $\qquad$ $(Y, \| \ \|_Y)$

Goal: Define a norm on $X \otimes Y$, "compatible" with $\| \ \|_X$, $\| \ \|_Y$.

Def A norm $\|\cdot\|$ on $X \otimes Y$ is called a ==tensor norm== if

- $\|x \otimes y\| = \|x\|_X \cdot \|y\|_Y \quad \forall\, x \in X,\ y \in Y$
- $\|\alpha \otimes \beta\|_* = \|\alpha\|_{X^*} \cdot \|\beta\|_{Y^*} \quad \forall\, \alpha \in X^*,\ \beta \in Y^*$

dual norm of $\|\cdot\|$

<u>Example</u> $(\mathbb{R}^m, \| \ \|_2)$ $(\mathbb{R}^n, \| \ \|_2)$

On $\mathbb{R}^m \otimes \mathbb{R}^n \simeq \mathbb{R}^{m \cdot n}$, the euclidean norm is a tensor norm

- $\| x \otimes y \|_2 = \sqrt{\sum_k (x \otimes y)_k^2}$

$$= \sqrt{\sum_{ij} (x_i y_j)^2} = \sqrt{\sum_i x_i^2} \sqrt{\sum_j y_j^2}$$

$$= \| x \|_2 \cdot \| y \|_2$$

- dual property follows from $\ell_2^* = \ell_2$

The injective and projective tensor norms

$(X, \| \ \|_X)$ $(Y, \| \ \|_Y)$

On $X \otimes Y$, we define $z \in X \otimes Y$

- $\| z \|_\varepsilon := \sup_{\substack{\| \alpha \|_{X^*} \leq 1 \\ \| \beta \|_{Y^*} \leq 1}} | (\alpha \otimes \beta)(z) |$

injective tensor norm

- $\| z \|_\pi := \inf \left\{ \sum_{i=1}^n \| x_i \|_X \| y_i \|_Y : z = \sum_{i=1}^n x_i \otimes y_i \right\}$

projective tensor norm

Examples $\quad z \in (\mathbb{R}^m, \|\ \|_2) \otimes (\mathbb{R}^n, \|\ \|_2)$

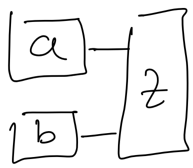
$$\|z\|_\varepsilon = \sup_{\substack{\|\alpha\|_{\ell_2^*} \le 1 \\ \|\beta\|_{\ell_2^*} \le 1}} \left| (\alpha \otimes \beta)(z) \right|$$

self dual

$$\left[\begin{array}{ll} \alpha(x) = \langle a, x \rangle & \|\alpha\|_2 = \|a\|_2 \\ \beta(x) = \langle b, x \rangle & \end{array} \right]$$

$$\|z\|_\varepsilon = \sup_{\substack{\|a\|_2 \le 1 \\ \|b\|_2 \le 1}} \langle a \otimes b, z \rangle$$

$$= \sup_{\|a\|, \|b\| \le 1}$$



$\leftarrow z \in \mathbb{R}^m \otimes \mathbb{R}^n$

$$= \sup_{\|a\|, \|b\| \le 1}$$

$$Z \in \underbrace{\mathcal{M}_{m \times n}(\mathbb{R})}_{= \mathbb{R}^m \otimes (\mathbb{R}^n)^*}$$

$$= \sup_{\substack{\|a\| \le 1 \\ \|b\| \le 1}} \langle a, Zb \rangle$$

$$= \sup_{\|b\| \le 1} \|Zb\| = \|Z\|_\infty$$

operator norm of Z

$$= \max_i s_i(Z)$$

$\sup_{\|a\| \le 1} \langle a, y \rangle = \|y\|$

$\le$ C-S

attained at $a = \frac{y}{\|y\|}$

<u>Conclusion</u> $(\mathcal{M}_{m\times n}, \|\ \|_\infty) = (\mathbb{R}^m, \|\ \|_2) \otimes_\varepsilon (\mathbb{R}^n, \|\ \|_2)$

<u>Important fact 1</u> ε and π norms are "dual":

$$[(X, \|\ \|_X) \otimes_\varepsilon (Y, \|\ \|_Y)]^* = (X, \|\ \|_X)^* \otimes_\pi (Y, \|\ \|_Y)^*$$

$$[(X, \|\ \|_X) \otimes_\pi (Y, \|\ \|_Y)]^* = (X, \|\ \|_X)^* \otimes_\varepsilon (Y, \|\ \|_Y)^*$$

<u>Example</u> $(\mathbb{R}^m, \|\ \|_2) \otimes_\pi (\mathbb{R}^n, \|\ \|_2) =$

$$\Big[\underbrace{(\mathbb{R}^m, \|\ \|_2)^*}_{\text{self-dual}} \otimes_\varepsilon \underbrace{(\mathbb{R}^n, \|\ \|_2)^*}_{\text{self-dual}}\Big]^*$$

$$= \Big[(\mathbb{R}^m, \|\ \|_2) \otimes_\varepsilon (\mathbb{R}^n, \|\ \|_2)\Big]^*$$

$$= (\mathcal{M}_{m\times n}, \|\ \|_\infty)^*$$

$$= (\mathcal{M}_{m\times n}, \|\ \|_1)$$

$\|A\|_1 = \sum_i s_i(A)$

For example: $A \in \mathcal{M}_{m\times n}$

$$\|A\|_\pi = \inf\{\sum \|x_i\|\,\|y_i\| : A = \sum x_i \otimes y_i\}$$

$$\underset{\substack{\uparrow \\ \text{use } A = \sum \underbrace{s_i |a_i\rangle}_{x_i} \underbrace{\langle b_i|}_{y_i}}}{\leq} \sum_i \|s_i a_i\| \cdot \|b_i\| = \sum_i s_i$$

Important fact 2 $(X, \|\ \|_X)$, $(Y, \|\ \|_Y)$ and $\|\ \|$ a tensor norm. Then

$$\|z\|_\varepsilon \leq \|z\| \leq \|z\|_\pi \quad \forall z$$

Proof of

Let $z \in X \otimes Y$ and let $z = \sum x_i \otimes y_i$ be the optimal decomp for π:

$$\|z\|_\pi = \sum \|x_i\|\,\|y_i\|$$

$$\|z\| = \|\sum x_i \otimes y_i\| \underset{\substack{\uparrow \\ \triangle \text{ ineq for } \|\ \|}}{\leq} \sum_i \|x_i \otimes y_i\|$$

1st point in the def of tensor norm

$$\leq \sum_i \|x_i\|_X \|y_i\|_Y = \|z\|_\pi$$

Injective norm for tensors

$$(\mathbb{C}^m, \|\ \|_2) \otimes_\varepsilon (\mathbb{C}^n, \|\ \|_2) \cong (\mathcal{M}_{m\times n}, \|\ \|_\infty)$$

How about

$$(\mathbb{C}^{d_1}, \|\ \|_2) \otimes_\varepsilon (\mathbb{C}^{d_2}, \|\ \|_2) \otimes_\varepsilon \cdots \otimes_\varepsilon (\mathbb{C}^{d_k}, \|\ \|_2) \ ?$$

$z \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_k}$ a tensor

$$\|z\|_\varepsilon = \sup_{\substack{\|a_1\| \le 1 \\ \vdots \\ \|a_k\| \le 1}} |\langle \overbrace{a_1 \otimes a_2 \otimes \cdots \otimes a_k}^{\text{separable pure q. state}} | z \rangle |$$

can be taken "="

$a_i \in \mathbb{C}^{d_i}$

So $\|z\|_\varepsilon$ = max overlap with separable states

$-\log_2 \|z\|_\varepsilon^2$ is called the geometric measure of entanglement.

<u>Remarks</u>

- ψ q. state (multipartite, pure)

  $\|\psi\|_\varepsilon \le \|\psi\|_2 = 1$

- ψ is separable $\Longleftrightarrow$ $\|\psi\|_\varepsilon = 1$

- if $\psi = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Omega\rangle$

$$\|\Omega\|_\varepsilon = \max_{\|a\|, \|b\| \le 1} \langle a \otimes b | \Omega \rangle$$

$$= \left\| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\|_\infty = \frac{1}{\sqrt{2}} < 1$$

but $-\log \|\Omega\|_\varepsilon^2 = 1$

Conclusion $\|z\|_{\ell_2 \otimes_\varepsilon \ell_2 \otimes_\varepsilon \cdots \otimes_\varepsilon \ell_2}$ is a measure of entanglement for multipartite pure q. states ($z \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_k}$)

Important open question

$$\min_{\|z\|_2 = 1} \|z\|_\varepsilon$$

pure q. state

- z is sep $\iff \|z\|_\varepsilon = 1$

corresponds to the state z which is the farthest from separable states i.e. the most entangled

$k = 2$, $d_1 \le d_2$: $\quad \min \|z\|_\varepsilon = \frac{1}{\sqrt{d_1}}$

achieved by $\frac{1}{\sqrt{d_1}} \sum_{i=1}^{d_1} |ii\rangle$

Fact It is NP-hard to compute $\|z\|_\varepsilon$ ($k \geq 3$)

Projective norm and mixed state entanglement

ρ density matrix $\quad \rho \in \mathcal{M}_d(\mathbb{C})^{sa}$

$\rho \geq 0$ positive semidefinite

$\operatorname{Tr} \rho = 1$ trace 1.

Important observation $\rho = \rho^*$ s.a.

$$\rho \geq 0 \iff \operatorname{Tr} \rho = \|\rho\|_1$$

nuclear norm

$$\|\rho\|_1 = \sum_i s_i(\rho) = \sum_i |\lambda_i(\rho)|$$

ρ self-adjoint $s_i = |\lambda_i|$

$$\|\rho\|_1 = \sum |\lambda_i(\rho)| \geq \sum \lambda_i(\rho) = \operatorname{Tr} \rho$$

eq iff $\lambda_i(\rho) \geq 0$

Separability and entanglement for mixed states

ρ is separable : $\rho = \sum t_i \, \alpha_i \otimes \beta_i$

probabilities density matrices

Theorem $\rho \in \mathcal{M}_{d_1}(\mathbb{C}) \otimes \cdots \otimes \mathcal{M}_{d_k}(\mathbb{C})$

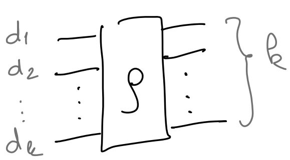
k-partite mixed q. state.

(A) ρ is a density matrix $\iff$

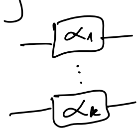
$$\|\rho\|_{\mathcal{M}_{\underbrace{d_1 \cdots d_k}_{D}}, \, \| \; \|_1} = \operatorname{Tr} \rho = 1$$

(B) ρ is a separable density matrix. $\iff$

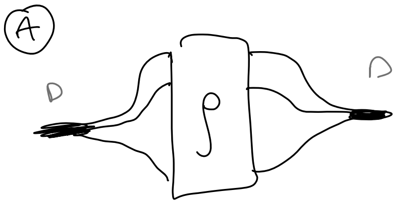
$$\|\rho\|_{(\mathcal{M}_{d_1}, \| \; \|_1) \otimes_\pi \cdots \otimes_\pi (\mathcal{M}_{d_k}, \| \; \|_1)} = \operatorname{Tr} \rho = 1$$



ρ sep: ρ as a convex comb of



density matrices



Remark computing

$(\mathcal{M}_{d_1}, \| \, \|_1) \otimes_{\pi} (\mathcal{M}_{d_2}, \| \, \|_1)$

is NP-hard