Coordinate Systems & Transformations

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Datum

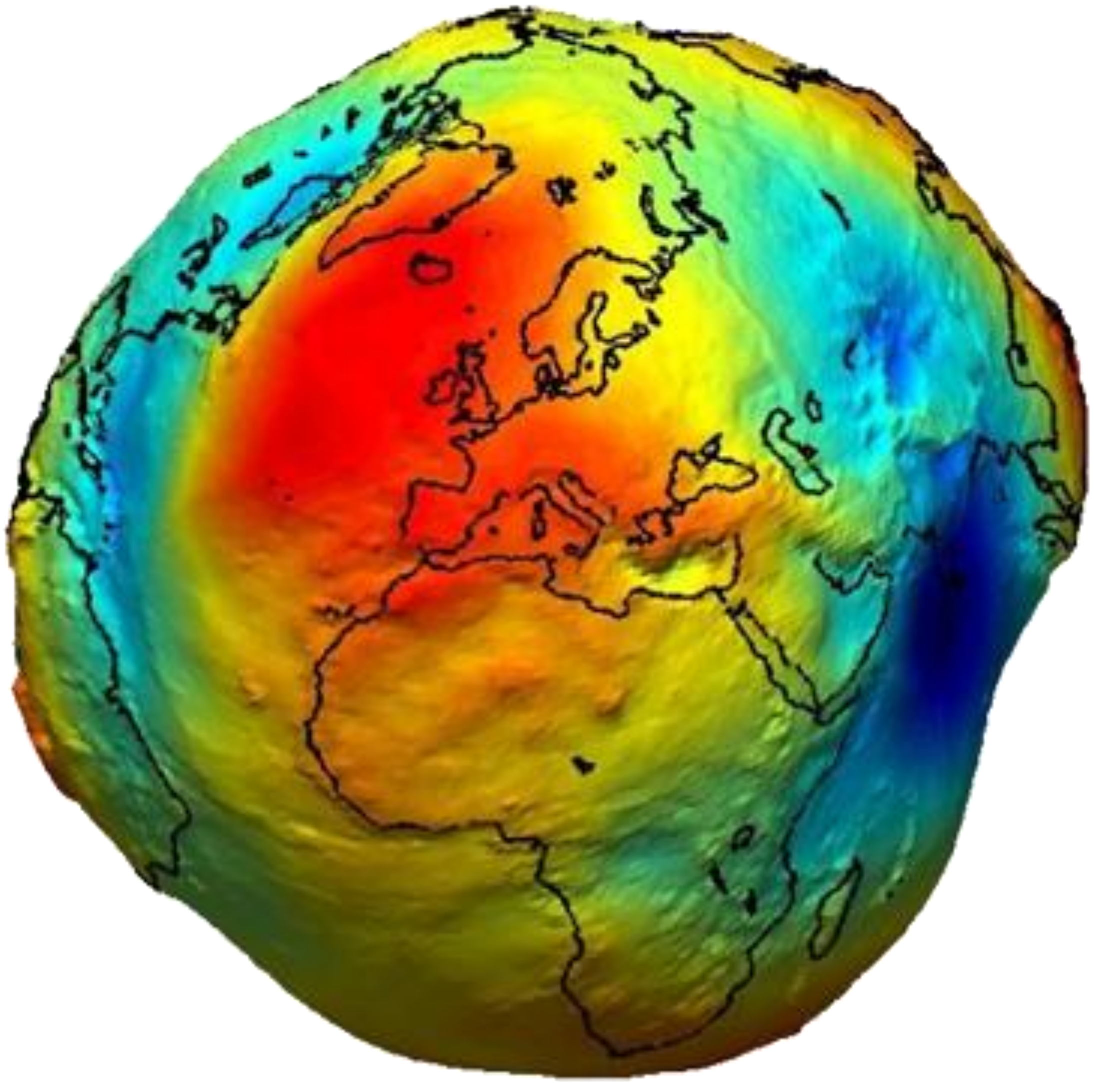
Shape of Earth

The shape of the earth is irregular due to

- topography and
- gravitational variations

Can we use irregular shape for calculations?

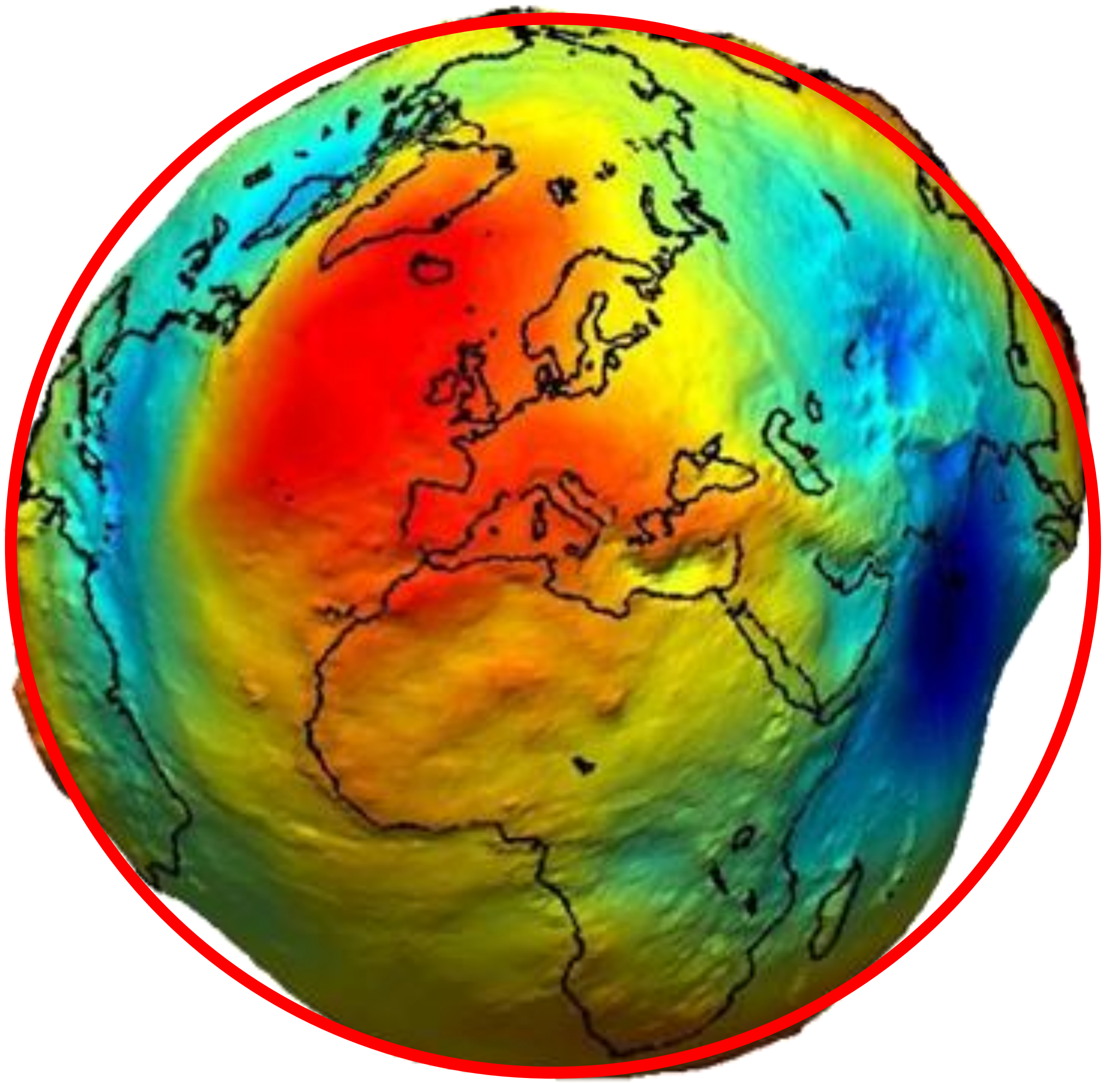
- It would be impractical for most of the applications because calculations of Coordinates become too complex.



Shape of Earth

How to make calculations efficient?

- A reference ellipsoid (simplified model of earth) can be used as Horizontal datum for measuring latitude and longitude.
- It is mathematically defined regular surface and approximates the overall shape of the earth.



What is datum?

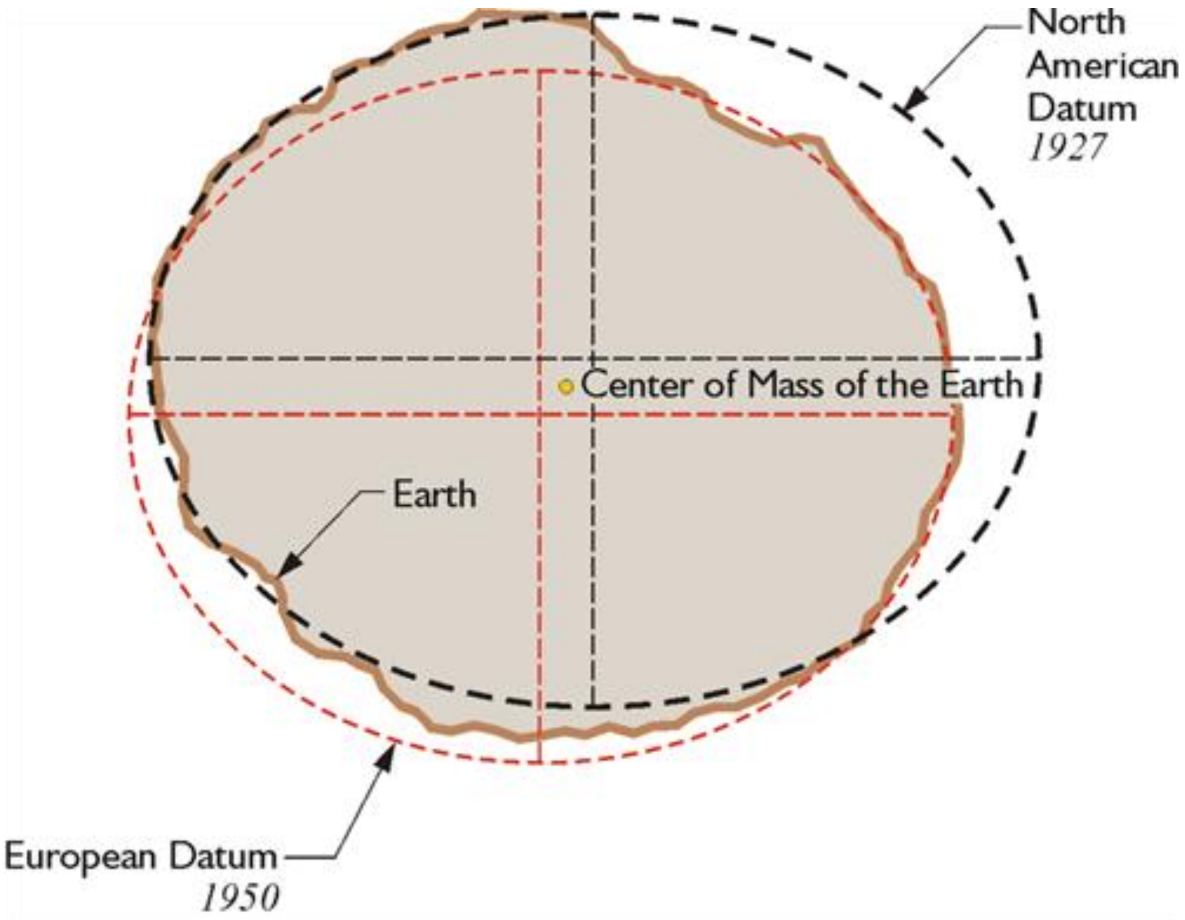
Datum is standardized reference framework for measuring positions, elevations.

It is the foundation for consistent and accurate geo-spatial measurement.

Types of datum

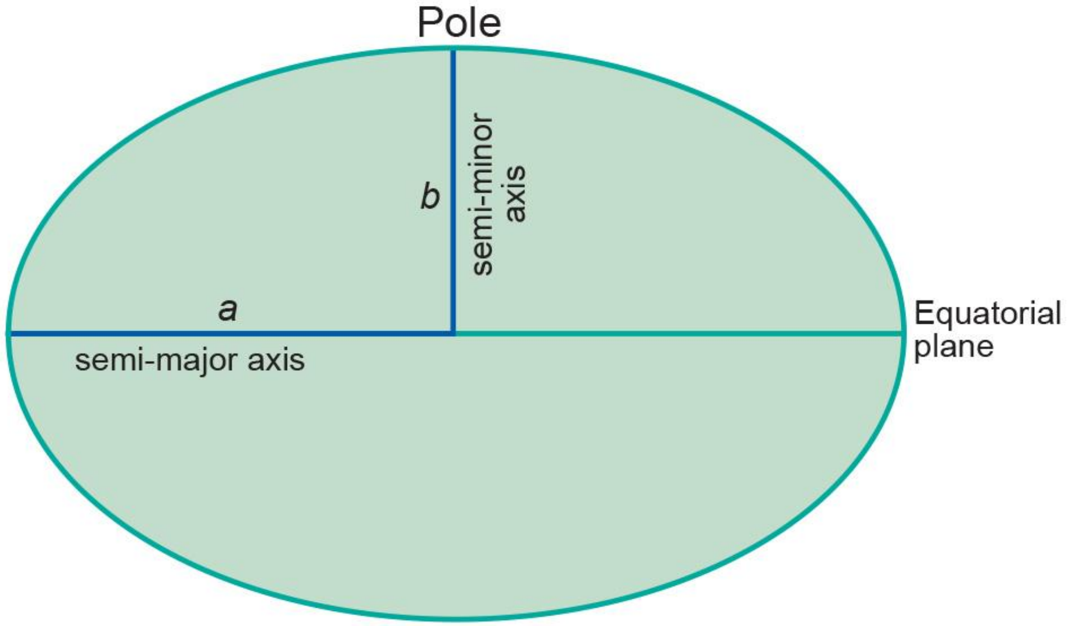
Horizontal datum (Longitude, Latitude)

Vertical datum (Height)

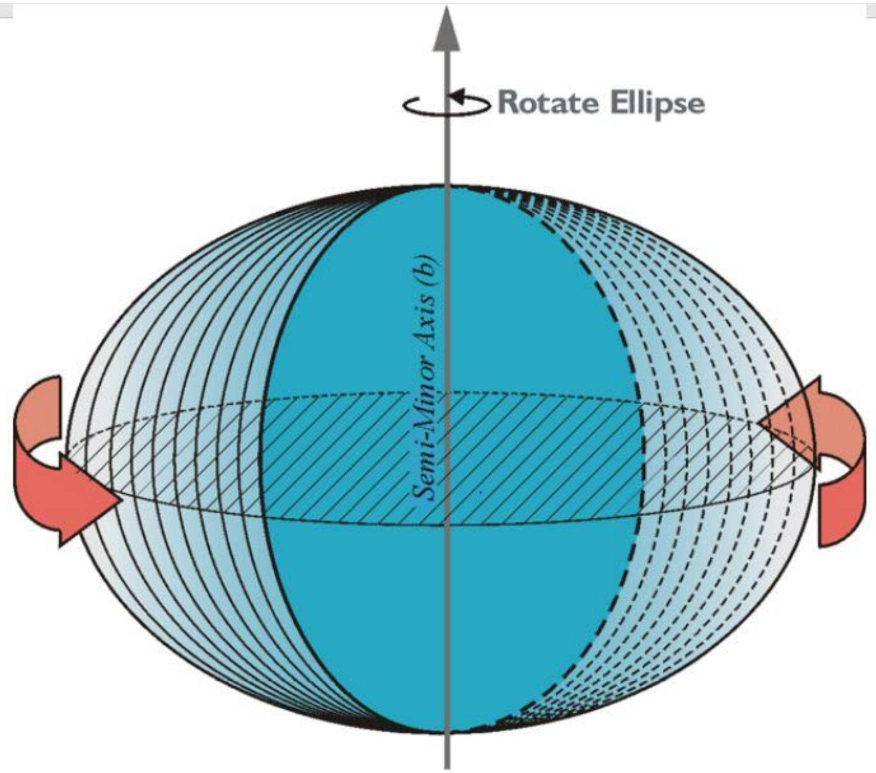


Pic: https://www.e-education.psu.edu/geog862/book/export/html/1669

Generation of a Reference Ellipsoid

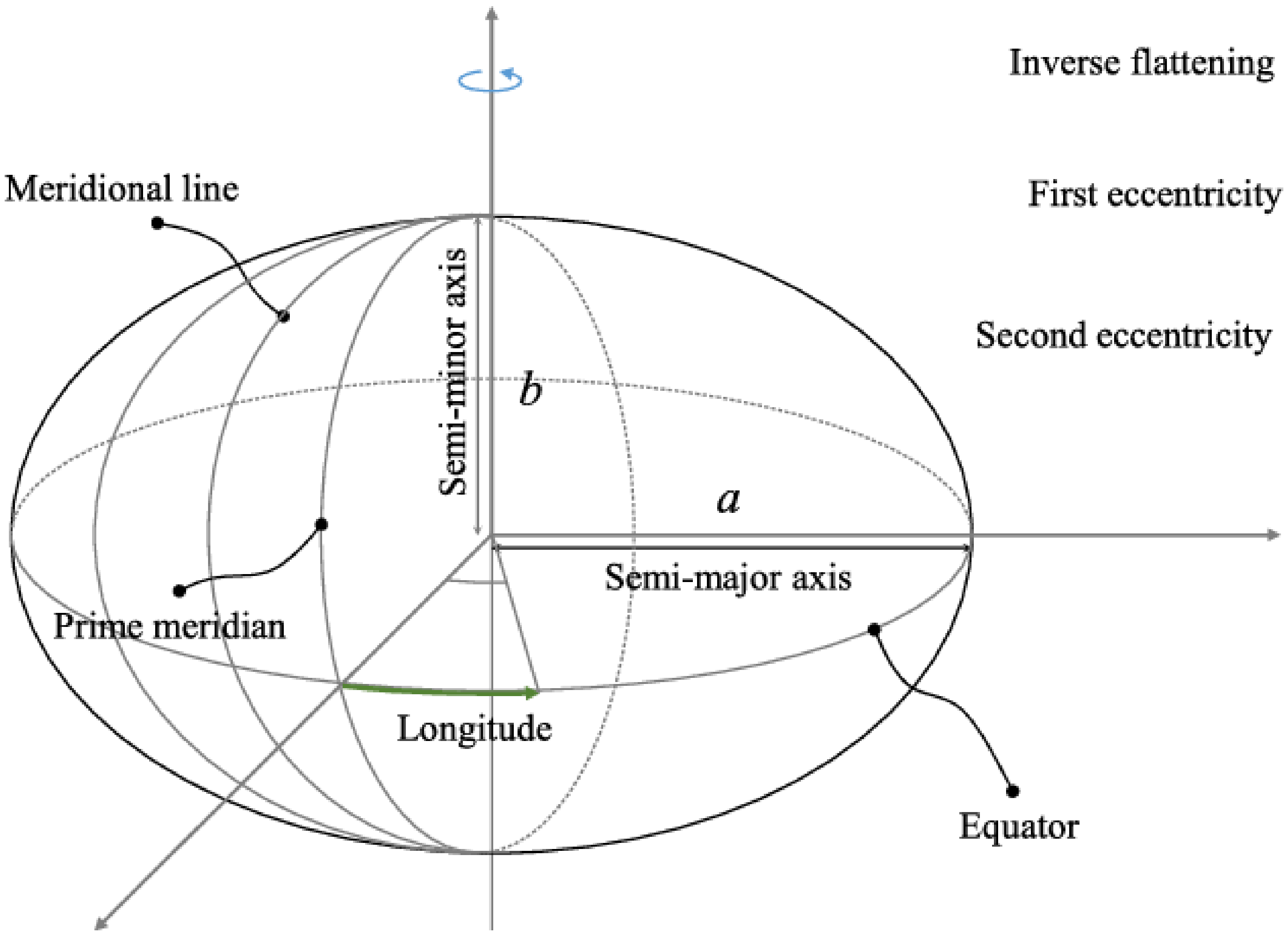


Ellipse



Ellipsoid

Ellipsoid as Mathematical surface of Earth

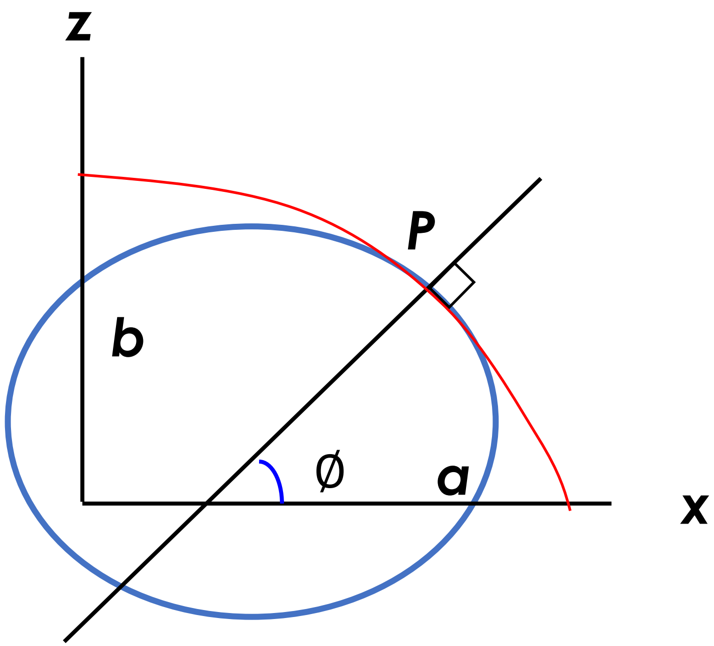


Inverse flattening $\frac{1}{f} = \frac{a}{a-b}$

First eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}}$

Second eccentricity $e' = \sqrt{\frac{a^2}{b^2} - 1}$

Radius of Meridional section



$$M = \frac{a(1 - e^2)}{(1 - e^2 sin\emptyset)^{3/2}}$$

a-Semi major axis
e-Eccentricity
Φ-Geodetic latitude

Meridian Radius of Curvature

M is the radius of curvature at a point on the curve is just the radius of osculating circle in the meridional plane.

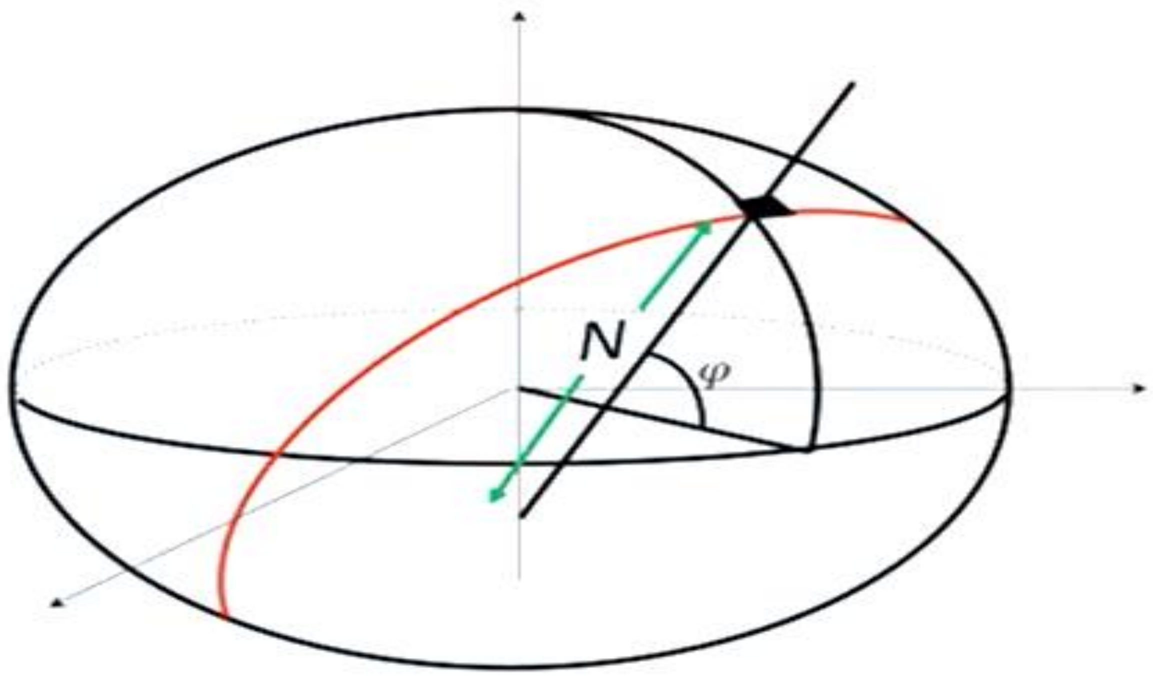
It is the measures how ‘curved’ the reference ellipsoid along a constant longitude(South-North direction).

At equator φ=0°,

M is the smallest.

At poles φ=90°,M is largest.

Radius of Prime Vertical



$$N = \frac{a}{\sqrt{(1 - e^2 sin\emptyset)}}$$

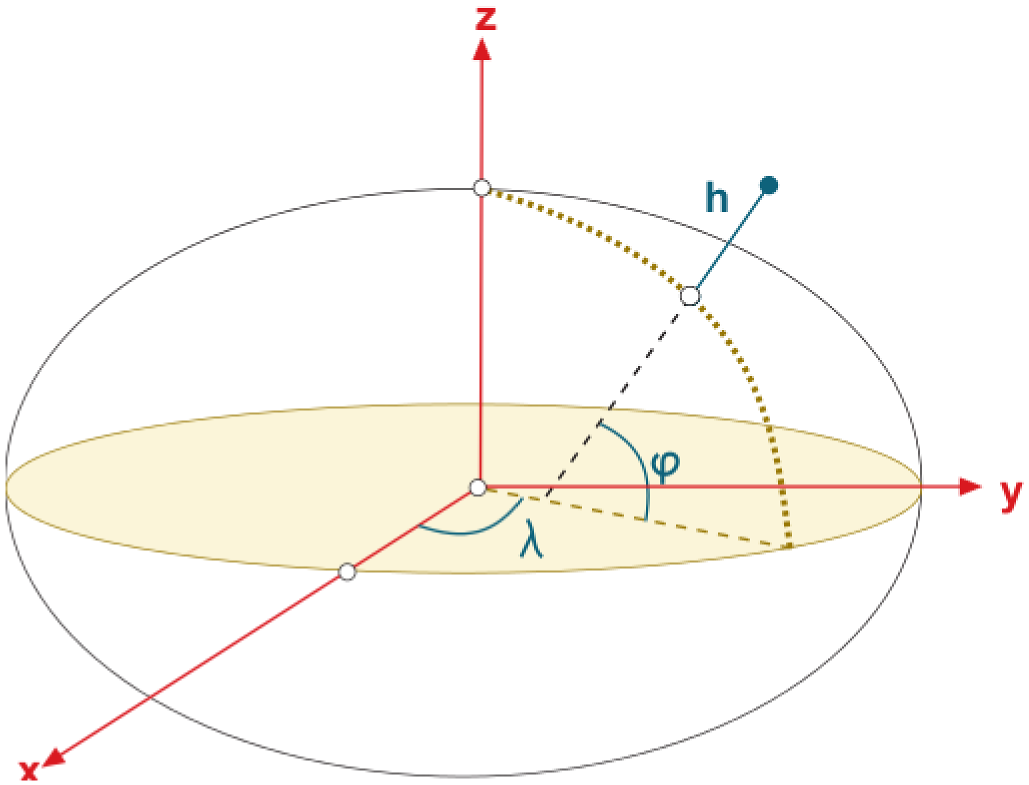
a-Semi major axis
e-Eccentricity
Φ-Geodetic latitude

Coordinate Systems

Geodetic Coordinate System (GCS)

The Geodetic Coordinate System represents positions on or near the Earth using angular measurements referenced to a mathematical **reference ellipsoid**.

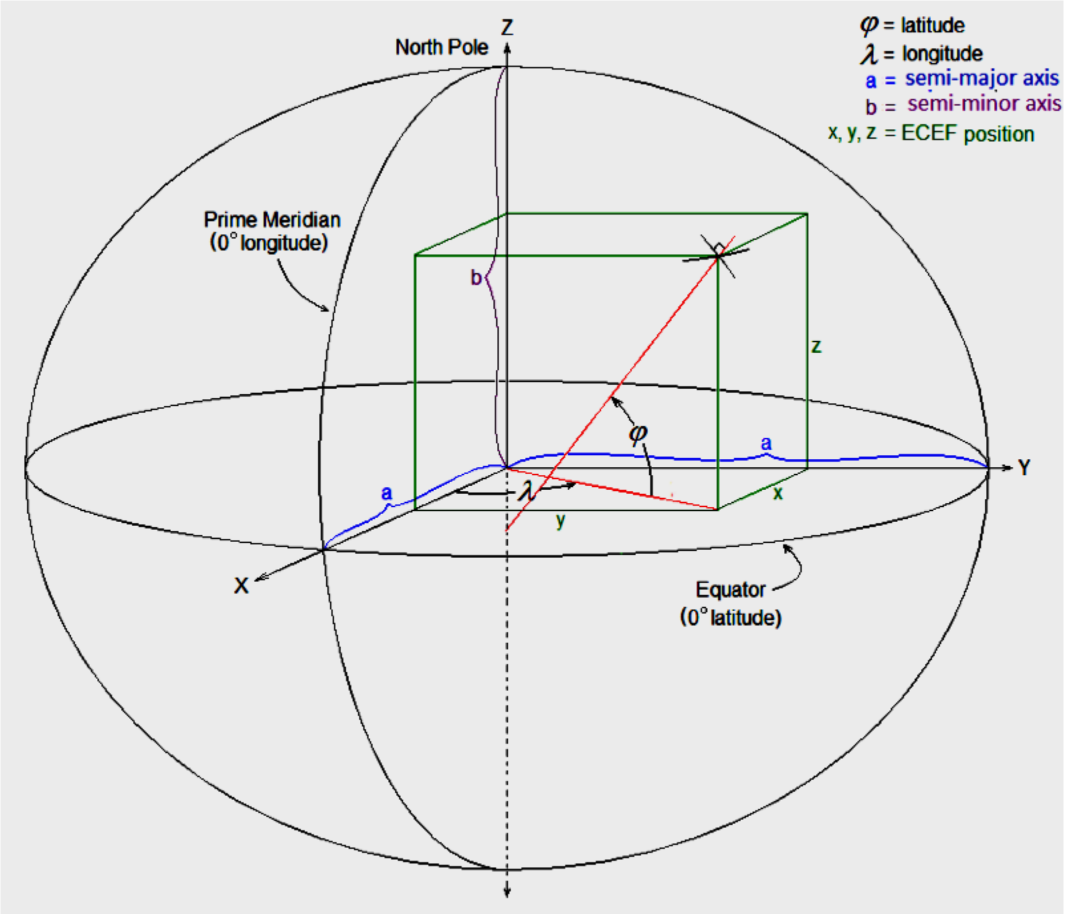
Locations are expressed by **Latitude (φ)**, **Longitude (λ)**, and **Ellipsoidal Height (h)**.



Earth-Centered Earth-Fixed (ECEF) Coordinates

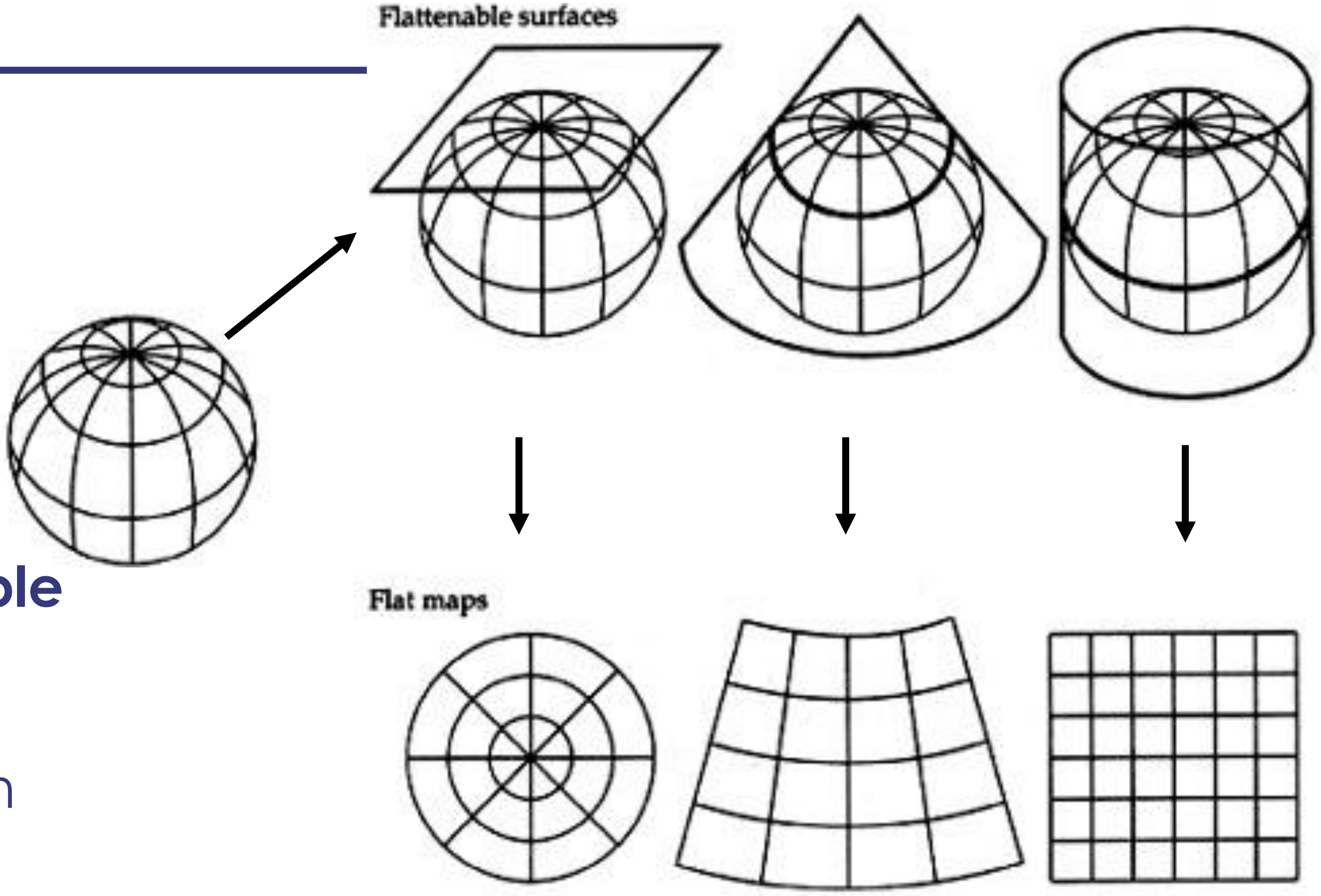
Earth-Centered Earth-Fixed (ECEF) is a three-dimensional Cartesian coordinate system with its origin at the Earth's center of mass.

The axes rotate with the Earth, making the system fixed relative to the Earth's surface.



Projected Coordinate System

A Projected Coordinate System represents locations on a **flat, two-dimensional surface** by mathematically projecting the Earth's curved surface from a Geographic Coordinate System (latitude–longitude).



Projection Surfaces (developable surface)

- Plane → Azimuthal Projection
- Cone → Conical Projection
- Cylinder → Cylindrical Projection

Coordinate Transformations

Introduction to coordinate transformations

Map and GIS users routinely perform coordinate transformations to integrate, analyze, and present spatial data accurately.

2D System Transformation
Conversion between different two-dimensional coordinate systems used in mapping and GIS.

Polar → Cartesian
Transformation of survey observations (distance, angle) into Cartesian map coordinates (X, Y).

Projection Change
Transformation between Cartesian (X, Y) systems of different map projections.

Datum Transformation
Alignment of datasets referenced to different horizontal datums, essential for large- and medium-scale mapping.

GNSS to Local Datum
Conversion of GNSS-collected field data to match published maps based on local or national datums.

Conversion of Geodetic (WGS-84) datum to ECEF Coordinates

- Geodetic (λ, ϕ) → ECEF (X, Y, Z)

$$X = (N + h)\cos\phi\cos\lambda$$
$$Y = (N + h)\cos\phi\sin\lambda$$
$$Z = [(1 - e^2)N + h]\sin\phi$$

Where,

Latitude (φ), Longitude (λ), and Ellipsoidal Height (h)

ECEF Coordinates are expressed in meters as (X, Y, Z)

Radius of Curvature in the Prime Vertical $N(\phi) = \frac{a}{\sqrt{1-e^2 sin^2\phi}}$

Conversion of ECEF to Geodetic(WGS-84) Coordinates

Direct Solution for Latitude, Longitude, and Height from X, Y, Z

This conversion is not exact and provides centimeter accuracy for heights < 1,000 km
(See Bowring, B. 1976. Transformation from spatial to geographical coordinates.
Survey Review, XXIII: pg. 323-327)

$$\phi = \operatorname{atan}\left(\frac{Z + e'^2 b \sin^3\theta}{p - e^2 a \cos^3\theta}\right)$$

$$\lambda = \operatorname{atan2}(Y, X)$$

$$h = \frac{p}{\cos(\phi)} - N(\phi)$$

where:

ϕ, λ, h = geodetic latitude, longitude, and height above ellipsoid

X, Y, Z = Earth Centered Earth Fixed Cartesian coordinates

and:

$$p = \sqrt{X^2 + Y^2} \qquad \theta = \operatorname{atan}\left(\frac{Za}{pb}\right) \qquad e'^2 = \frac{a^2 - b^2}{b^2}$$

$N(\phi) = a / \sqrt{1 - e^2 \sin^2\phi}$ = radius of curvature in prime vertical

a = semi - major earth axis (ellipsoid equatorial radius)

b = semi - minor earth axis (ellipsoid polar radius)

$f = \dfrac{a - b}{a}$ = flattening

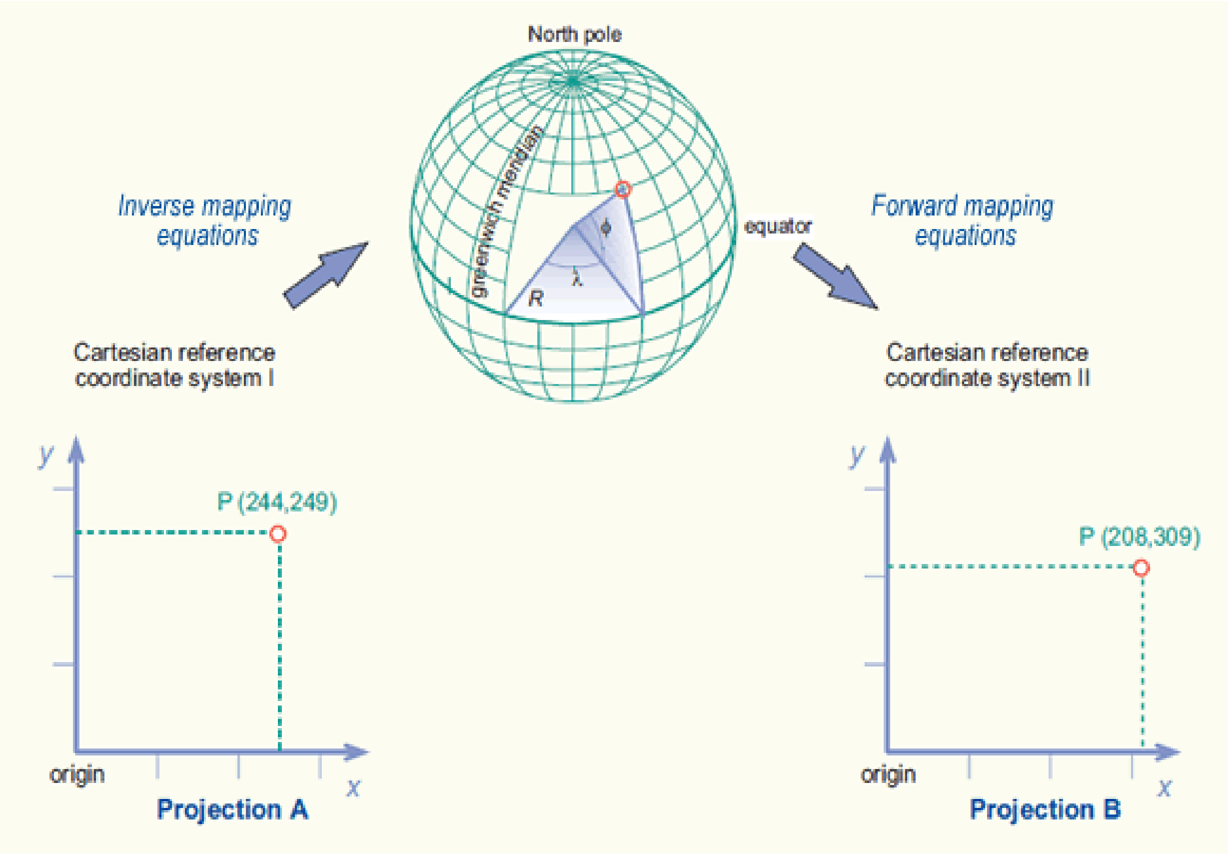
$e^2 = 2f - f^2$ = eccentricity squared

Changing Map Projection

Forward mapping converts geographic coordinates (φ, λ) to target projection coordinates (x′, y′).

Inverse mapping converts source projection coordinates (x, y) to geographic coordinates (φ, λ).

Workflow: Projection A (x, y) → Inverse equation → Geographic (φ, λ) → Forward equation → Projection B (x′, y′).



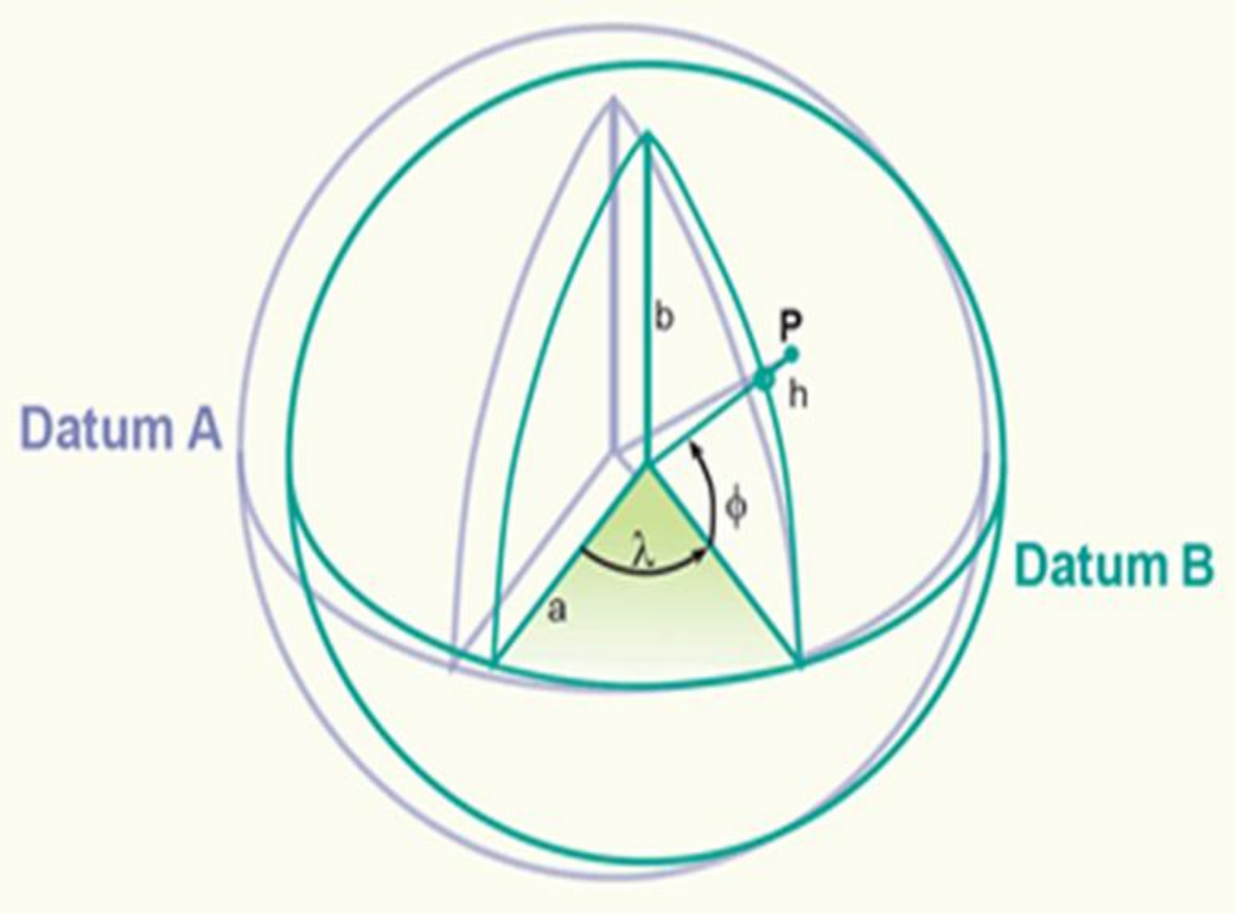
Datum Transformations

A change of map projection may also include a change of the horizontal datum (also called geodetic datum).

This is the case when the source projection is based upon a different horizontal datum than the target projection.

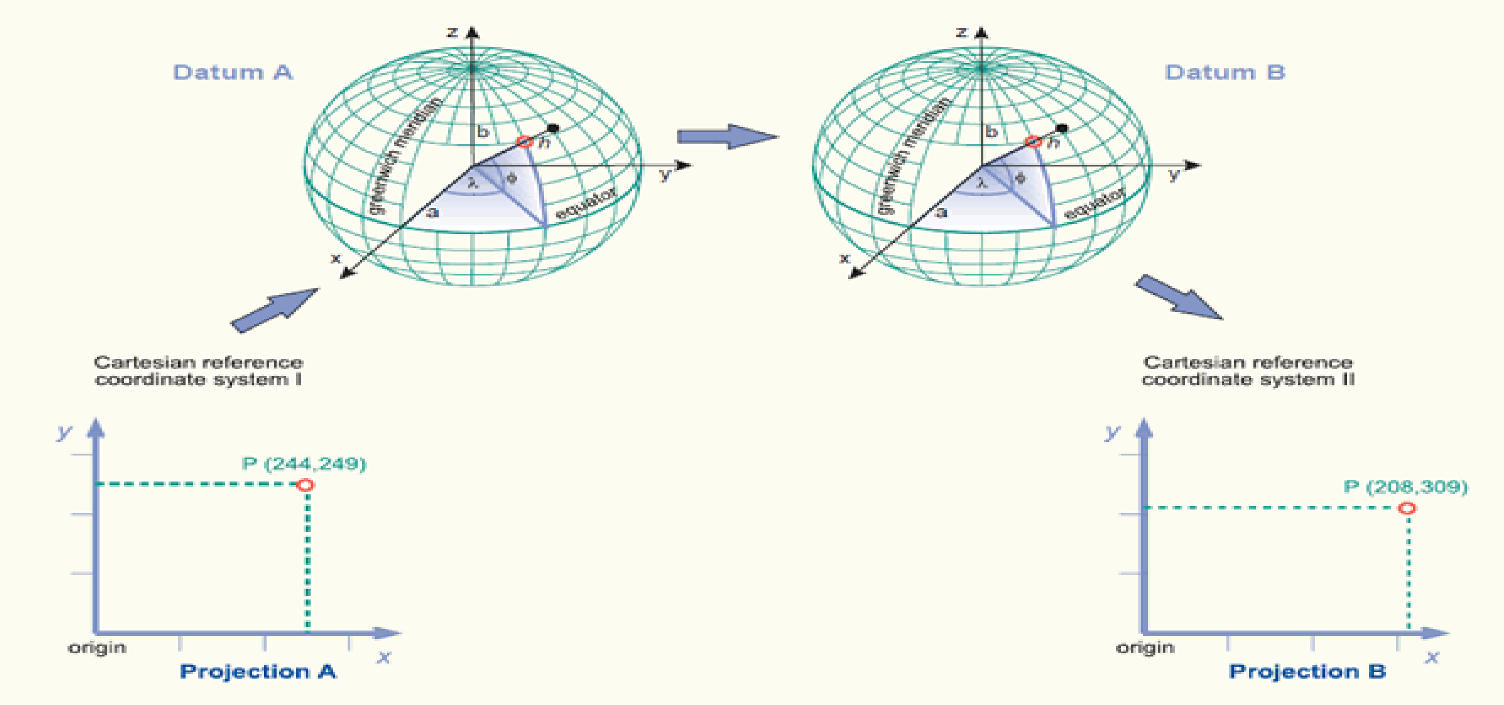
Datum transformations are transformations from a 3D coordinate system (i.e. horizontal datum) into another 3D coordinate system.

Example: Everest to WGS84



Datum Transformations

- Suppose we wish to transform spatial data from the Survey of India old maps (prior to 2005 edition) to the present Open Series Maps. In Old maps, the datum is EVEREST, and the projection is POLYCONIC while the new OSM maps are on WGS84 Datum and projection is UTM.



The principle of changing from one into another projection combined with a datum transformation from datum A to datum B.

Datum Transformations

The transformation parameters to take us from one datum system to another datum system are estimated on the basis of a set of selected points whose coordinates are known in both datum systems.

If the coordinates of these points are not correct, the estimated parameters may be inaccurate. Their estimate will depend on which particular common points are chosen, and they also will depend on the amount of transformation parameters estimated.

The transformation parameters are also not unique.

Mathematically a datum transformation can be realized by relating the Geographic Coordinates (φ, λ, h) of both datum systems directly, or indirectly by relating the Geocentric Coordinates (X, Y, Z) of the datums.

Datum transformation methods

Geocentric Translation (applies a shift between the centres of the two geocentric coordinate systems)

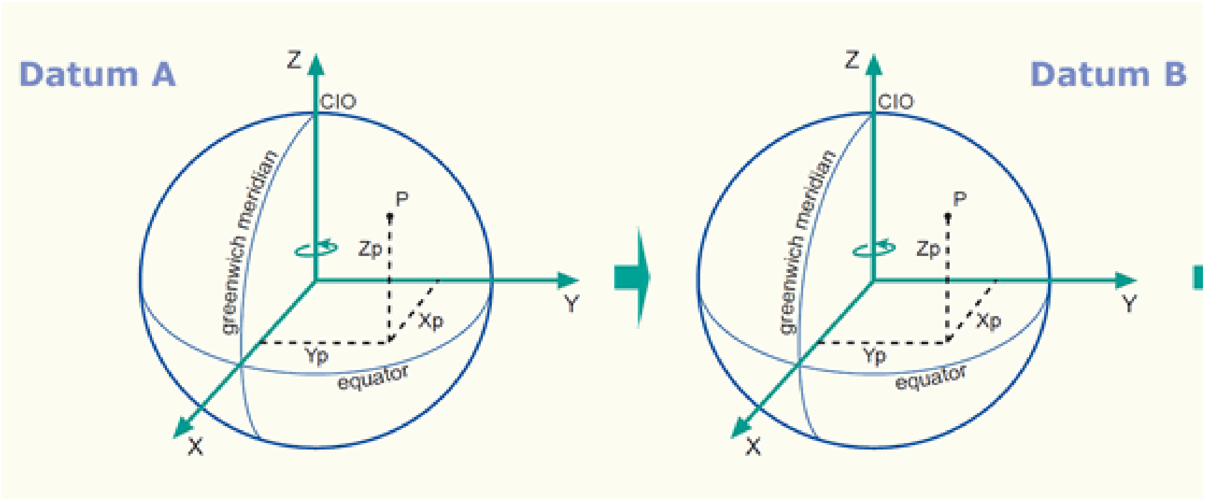
Helmert 7-parameter transformations (three rotation angles (α, β, γ), three origin shifts (ΔX, ΔY and ΔZ) and one scale factor (s). It is reversible.

Molodensky-Badekas 10-parameter transformation. (three rotation angles (α, β, γ), three origin shifts (ΔX, ΔY and ΔZ) three origin shifts (*ΔX, ΔY* and *ΔZ*), one scale factor (s) and the coordinates of the rotation point (*Xp*, *Yp*, *Zp*) given in the source geocentric coordinate system. Compared to the Helmert transformation, the Molodensky-badekas provides usually a better approximation, but the transformation is not reversible.

Datum Transformations via ECEF Coordinates

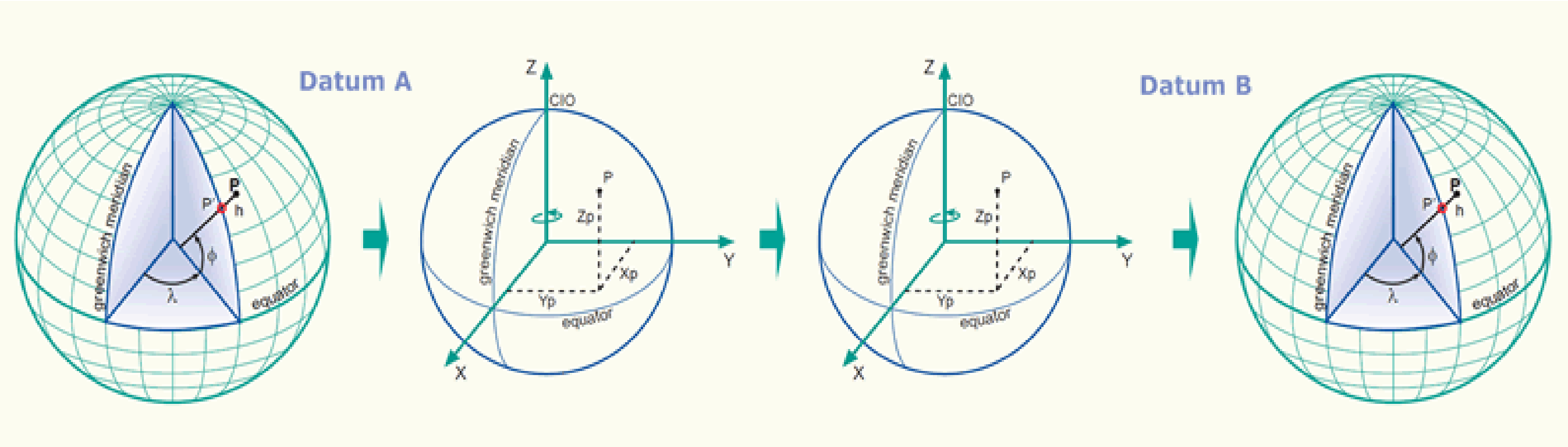
Datum transformations via the ECEF Coordinates (X,Y,Z) are ***3D similarity transformations***. Essentially, these are transformations between two orthogonal 3D Cartesian spatial reference frames together with some elementary tools from adjustment theory.

The principle of changing from one datum into another datum via the ECEF coordinates.



Datum Transformations via ECEF Coordinates

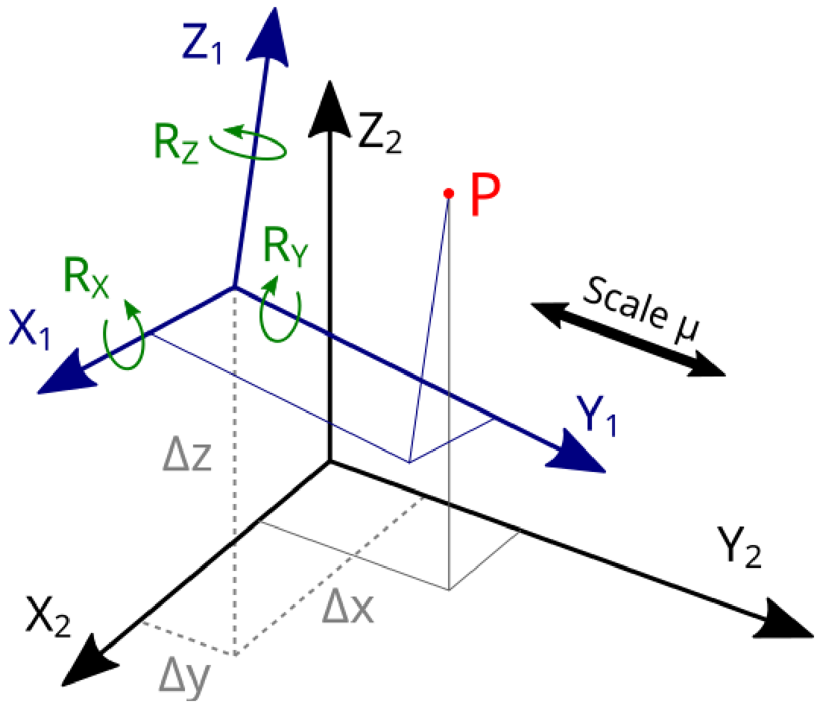
The three geocentric transformations described in Datum transformations via ECEF coordinates are usually combined with conversions between the ECEF coordinates (X,Y,Z) and the ellipsoidal latitude (ϕ) and longitude (λ) coordinates and height (h) in both datum systems.



The principle of a datum transformation via the ECEF coordinates.
The datum transformation is combined with conversions between the 3D geographic coordinates and ECEF coordinates in both datum systems.

Helmert 7-parameter Datum Transformation

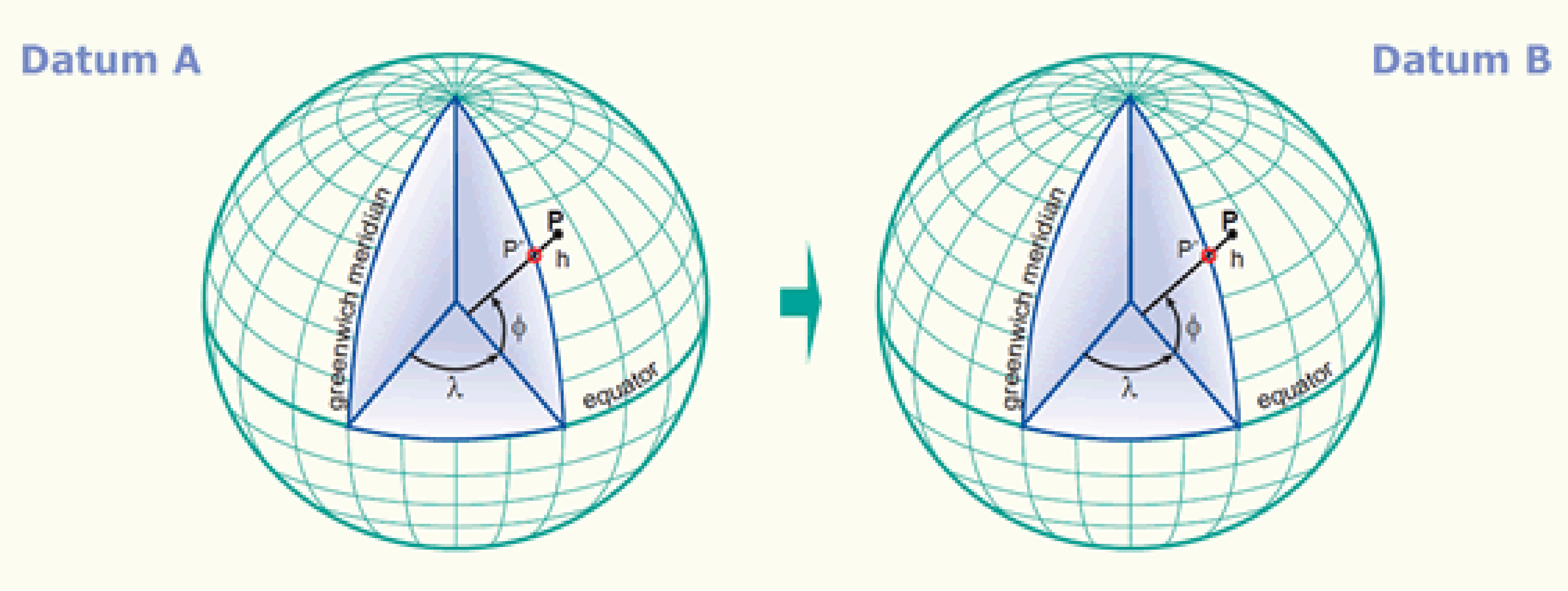
$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = (1+S)\begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} \triangle X \\ \triangle Y \\ \triangle Z \end{bmatrix}$$



https://www.ngs.noaa.gov/CORS/news/historical_helmert.shtml

Datum Transformations via Geographic Coordinates

Datum transformations via the geographic coordinates directly relate the ellipsoidal latitude (φ) and longitude (λ), and possibly also the ellipsoidal height (h), of both datum systems.



The principle of changing from one datum into another datum via the 3D geographic coordinates.

Methods of Datum Transformations via Geographic Coordinates

Geographic offsets: The method is only used for purposes where low accuracy can be tolerated.

$\phi_{datum\ B} = \phi_{datum\ A} + \Delta\phi$

$\lambda_{datum\ B} = \lambda_{datum\ A} + \Delta\lambda$

Molodensky and Abridged Molodensky transformation:

$\Delta\phi = f\ (\Delta x,\ \Delta y,\ \Delta z,\ \Delta a,\ \Delta f)$

$\Delta\lambda = f\ (\Delta x,\ \Delta y,\ \Delta z,\ \Delta a,\ \Delta f)$

$\Delta h = f\ (\Delta x,\ \Delta y,\ \Delta z,\ \Delta a,\ \Delta f)$

- This is quite satisfactory for small areas, but for larger areas such as large countries or continents significant errors occur .

Multiple regression transformation:

$\Delta\phi = f\ (\phi,\ \lambda,\ a_1,\ a_2,\ a_3,\ ....)$

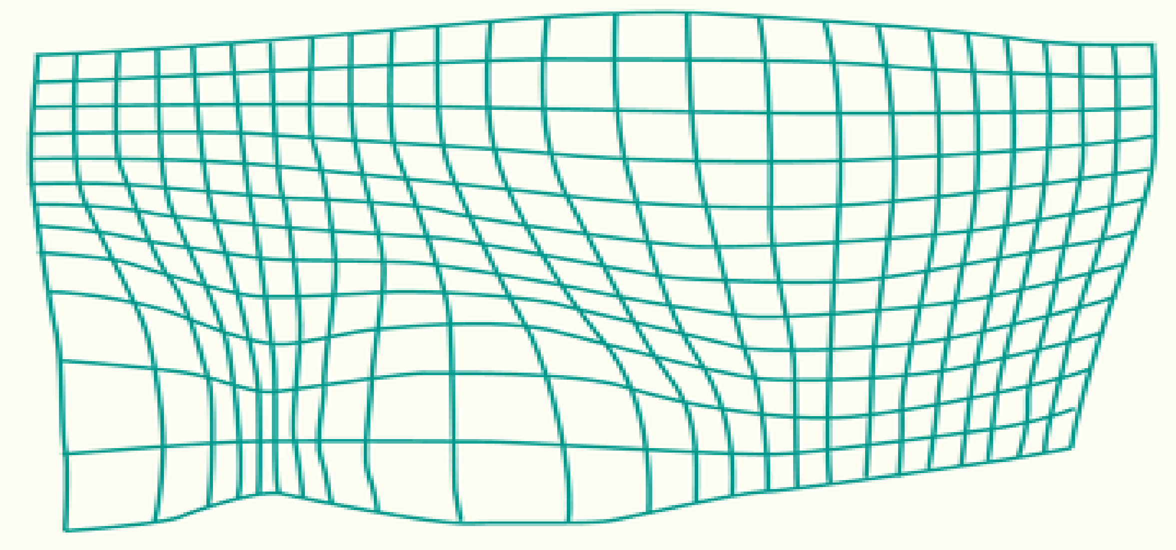
$\Delta\lambda = f\ (\phi,\ \lambda,\ b_1,\ b_2,\ b_3,\ ......\ )$

$\Delta h = f\ (\phi,\ \lambda,\ c_1,\ c_2,\ c_3,\ ....)$

- A series of best-fit equations provide the local shifts in latitude and longitude as a function of position. The main advantage of this method over Molodensky (often implemented in geo-software) is that a better fit over continental size land areas can be achieved

2D Cartesian coordinate transformations used in Georeferencing

An image or Grid with distortions



It may occur in an aerial photograph, caused by the tilting of the camera and the terrain relief (topography).

An approximate correction may be derived through a high-order polynomial transformation.

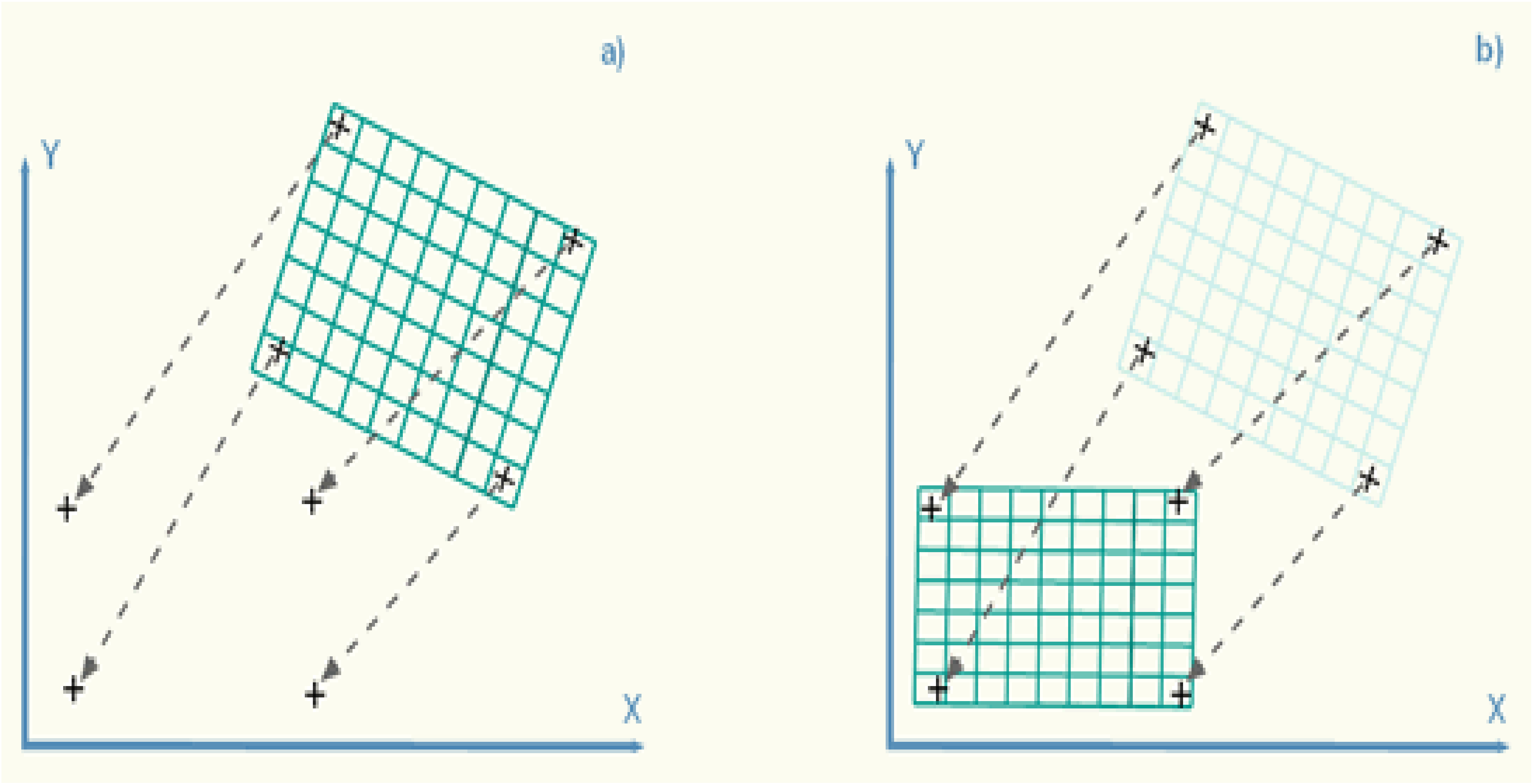
The displacements caused by relief differences can be corrected using a Digital Elevation Model (DEM).

2D Cartesian coordinate transformations

2D Cartesian coordinate transformations are generally used to assign map coordinates (x,y) to an uncorrected image or scanned map. The type of transformation (usually an affine transformation) depends on the geometric errors in the data set.

After Georeferencing, the image can be aligned (rectified) so that the pixels are exactly positioned within the map coordinate system.

For each image pixel in the new coordinate system, a new pixel value has to be determined by means of an interpolation from surrounding pixels in the old image. This is called ***image resampling***.



Coordinates are assigned to a raster image by means of a 2D transformation using a set of control points. The image is ***Georeferenced***

The Georeferenced image is ***rectified*** *to match it with the map coordinate system.*

2D Cartesian Coordinate Transformations

2D Cartesian coordinate transformations can be used to transform 2D Cartesian coordinates (x,y) from one 2D Cartesian coordinate system to another 2D Cartesian coordinate system.

The three primary transformation methods are:

- Conformal transformation
- Affine transformation
- Polynomial transformation.

Conformal transformation

It is a linear (or first-order) transformation and relates two 2D Cartesian coordinate systems through a *rotation*, a *uniform scale change*, followed by a *translation*. The rotation is defined by one rotation angle (α), and the scale change by one scale factor (s).

The translation is defined by two origin shift parameters (x_o,y_o). The equation is:

$$X' = s\,X\cos(\alpha) - s\,Y\sin(\alpha) + x_o$$

$$Y' = s\,X\sin(\alpha) + s\,Y\cos(\alpha) + y_o$$

The simplified equation is:

$$X' = aX - bY + x_o$$

$$Y' = bX + aY + y_o$$

where a=s cos(α) and b=s sin(α).
The transformation parameters (or coefficients) are a,b,x_o,y_o.

Affine transformation

It is a linear (or first-order) transformation and relates two 2D Cartesian coordinate systems through a *rotation*, a *scale change* in x- and y-direction, followed by a *translation*. The transformation function is expressed with 6 parameters: one rotation angle (α), two scale factors, a scale factor in the x-direction (s_x) and a scale factor in the y-direction (s_y), and two origin shifts (x_o, y_o).

The equation is:

$$X' = s_x X \cos(\alpha) - s_y Y \sin(\alpha) + x_o$$
$$Y' = s_x X \sin(\alpha) + s_y Y \cos(\alpha) + y_o$$

The simplified equation is

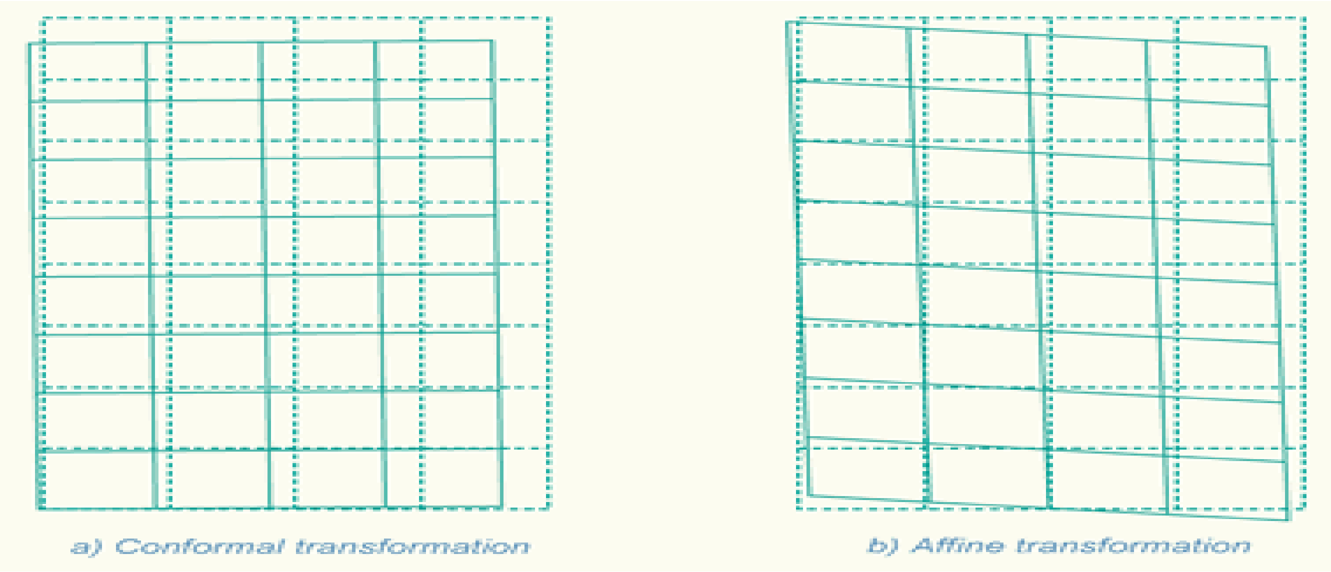
$$X' = aX - bY + x_o$$
$$Y' = cX + dY + y_o$$

where the transformation parameters (or coefficients) are a,b,c,d,x_o,y_o.

Conformal and Affine transformations

Both are linear transformations which means that the lines of the grid remain straight after the transformation.

The uniform scale change of the ***conformal*** *transformation retains the shape of the original rectangular grid.*



The different scale in x and y-direction of the ***affine transformation*** *changes the shape of the original rectangular grid, but the lines of the grid remain straight.*

Polynomial transformation

It is a non-linear transformation and relates two 2D Cartesian coordinate systems through a *translation*, a *rotation* and a *variable scale change*.
The transformation function can have an infinite number of terms.
The equation is:

$$X' = x_o + a_1X + a_2Y + a_3XY + a_4X^2 + a_5Y^2 + a_6X^2Y + a_7XY^2 + a_8X^3 + \ldots\ldots\ldots$$

$$Y' = y_o + b_1X + b_2Y + b_3XY + b_4X^2 + b_5Y^2 + b_6X^2Y + b_7XY^2 + b_8X^3 + \ldots\ldots\ldots$$

Polynomial transformations are sometimes used to Georeference uncorrected satellite imagery or aerial photographs or to match vector data layers that don't fit exactly by stretching or rubber sheeting them over the most accurate data layer.

Control Points in Coordinate Transformations

Transformation Parameters

- Parameters (coefficients/unknowns) of conformal, affine, and polynomial transformations are computed using **Ground Control Points (GCPs)** or **Common (Tie) Points** with known coordinates in both systems (e.g., building corners, road intersections).

Minimum Control Point Requirements

- **Conformal transformation:** Minimum **2 GCPs** → 4 parameters *(a, b, x_0, y_0)*
- **Affine transformation:** Minimum **3 GCPs** → 6 parameters *(a, b, c, d, x_0, y_0)*
- **2nd-order polynomial transformation:** Minimum **6 GCPs** → 12 parameters *(x_0, a_1–a_5, y_0, b_1–b_5)*

Accuracy in Coordinate Transformations

RMSE (Root Mean Square Error)

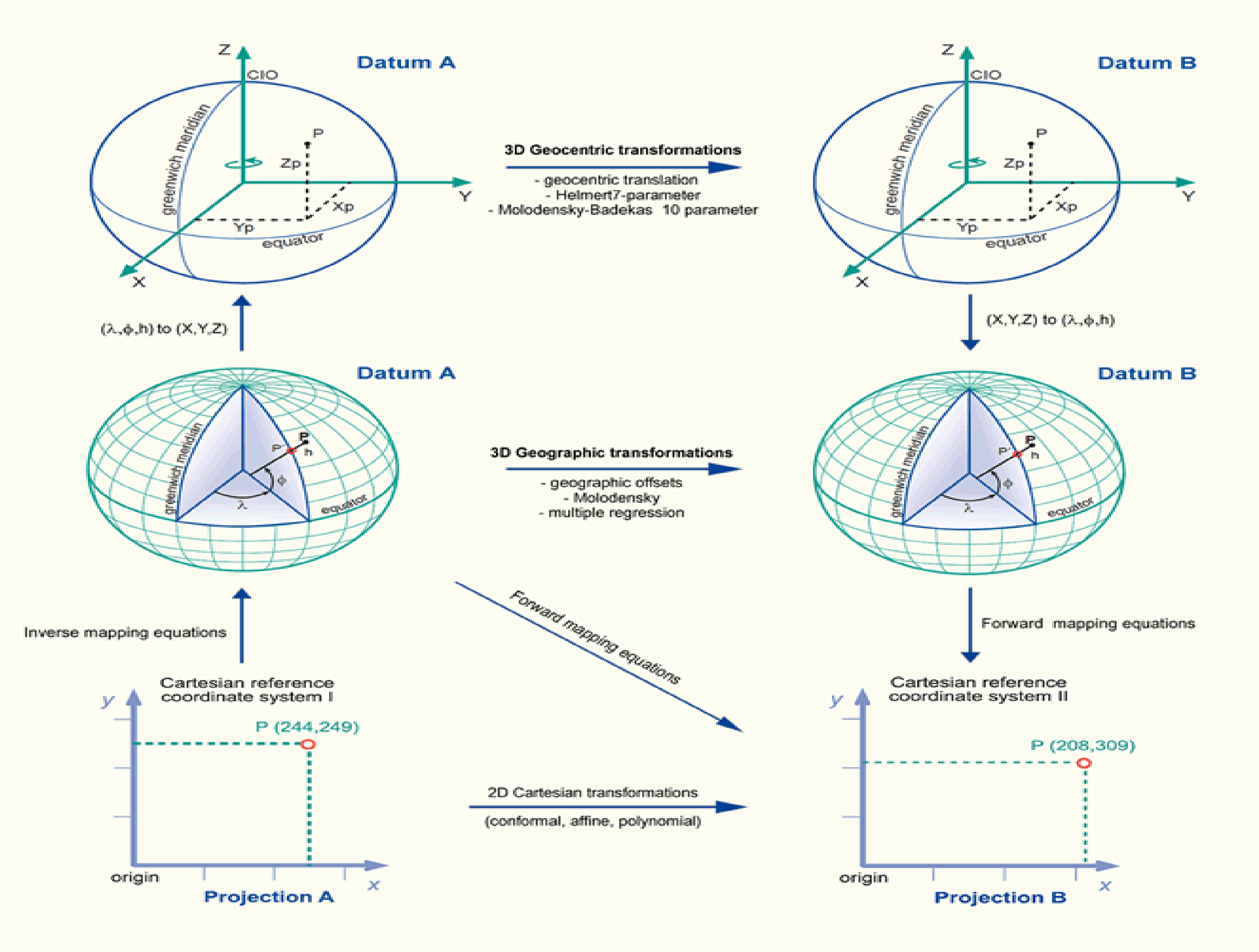
- Statistical measure of transformation accuracy
- Indicates average deviation of computed positions from true positions (residuals)
- Used as an independent quality check of the transformation

Quality Improvement

- If RMSE is unacceptable, improve results by selecting better-distributed GCPs, increasing the number of control points, or applying an alternative transformation model.

Summary

Overview of several types of coordinate transformations.



Thanks