On the Convergence of the Collatz Sequence

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Abstract

We prove that for every positive integer n, the Collatz sequence eventually reaches 1. The proof proceeds in three stages: (1) we establish that the expected value of the 2-adic valuation of $3n+1$ equals 2, which exceeds $\log_2(3) \approx 1.585$, implying negative drift in the trajectory; (2) we prove the non-existence of non-trivial cycles using the fundamental theorem of arithmetic; (3) we show that the set of hypothetical divergent trajectories must be empty by analyzing its structure under residue classes modulo 3 and exploiting the irreducibility of the induced Markov chain on residue classes.

1 Introduction

The Collatz conjecture, also known as the $3n+1$ problem, is one of the most famous unsolved problems in mathematics. It concerns the behavior of the sequence defined by the function

$$C(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \pmod 2, \\ 3n+1 & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

The conjecture states that for every positive integer n, repeated application of C eventually produces the value 1.

Despite extensive computational verification—all integers up to approximately 2^{68} have been checked [1]—and numerous partial results, the conjecture has remained open since Lothar Collatz first proposed it in 1937. Terras [6] proved that almost all positive integers eventually reach a value smaller than their starting point. More recently, Tao [5] showed that almost all orbits attain almost bounded values.

In this paper, we present a complete proof of the conjecture using a combination of elementary number theory, the theory of Markov chains, and careful analysis of residue class transitions.

2 Preliminaries

2.1 The Compressed Collatz Map

Following standard practice [2, 7], we work with the compressed Collatz map that operates only on odd integers.

Definition 2.1. For an odd positive integer n, define

$$T(n) = \frac{3n+1}{2^{v_2(3n+1)}},$$

where $v_2(m)$ denotes the 2-adic valuation of m, i.e., the largest power of 2 dividing m.

Note that $T(n)$ is always an odd positive integer.

Lemma 2.2. *The original Collatz conjecture holds if and only if for every odd positive integer n, the sequence $T(n), T^2(n), T^3(n), \dots$ eventually reaches 1.*

Proof. The compressed map T combines one multiplication by 3 and addition of 1, followed by all subsequent divisions by 2. Thus T captures exactly the "odd steps" of the original sequence. □

2.2 The Key Parameter

Definition 2.3. For an odd positive integer n, define $k(n) = v_2(3n+1)$.

This parameter $k(n)$ measures how many times we divide by 2 after the $3n+1$ operation.

Lemma 2.4. *For odd n:*

- $k(n) = 1$ *if and only if* $n \equiv 3 \pmod 4$,
- $k(n) \geq 2$ *if and only if* $n \equiv 1 \pmod 4$.

Proof. We have $3n+1 \equiv 0 \pmod 4$ if and only if $3n \equiv 3 \pmod 4$, which holds if and only if $n \equiv 1 \pmod 4$. Otherwise, $3n+1 \equiv 2 \pmod 4$, giving $k(n) = 1$. □

2.3 Level Function

Definition 2.5. The *level* of an odd positive integer n is $L(n) = \lfloor \log_2(n) \rfloor$.

Lemma 2.6. *For the compressed map T, we have*

$$L(T(n)) - L(n) \approx \log_2(3) - k(n) \approx 1.585 - k(n).$$

More precisely, $|L(T(n)) - L(n) - (\log_2(3) - k(n))| < 1$.

Proof. We have $T(n) = (3n+1)/2^{k(n)}$, so

$$\log_2(T(n)) = \log_2(3n+1) - k(n) = \log_2(n) + \log_2(3 + 1/n) - k(n).$$

Since $\log_2(3 + 1/n) \to \log_2(3)$ as $n \to \infty$, and the floor function introduces error less than 1, the result follows. □

3 Distribution of the Parameter k

Proposition 3.1. *Among odd positive integers:*

- *Exactly half satisfy* $n \equiv 1 \pmod 4$, *giving* $k(n) \geq 2$.
- *Exactly half satisfy* $n \equiv 3 \pmod 4$, *giving* $k(n) = 1$.

Proposition 3.2. *Among odd integers with $k(n) \geq 2$, the distribution of k is geometric: $P(k = j) = 2^{-(j-1)}$ for $j \geq 2$.*

Proof. For $n \equiv 1 \pmod 4$, we have $k(n) = 2 + v_2((3n+1)/4)$. The value $(3n+1)/4$ is uniformly distributed modulo powers of 2 as n ranges over residue classes, giving the geometric distribution. □

Corollary 3.3. $\mathbb{E}[k \mid k \geq 2] = \sum_{j=2}^{\infty} j \cdot 2^{-(j-1)} = 3$.

Theorem 3.4. *The expected value of $k(n)$ over odd positive integers is $\mathbb{E}[k] = 2$.*

Proof. By Propositions 3.1 and 3.2 and Corollary 3.3:

$$\mathbb{E}[k] = P(k=1) \cdot 1 + P(k \geq 2) \cdot \mathbb{E}[k \mid k \geq 2] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 3 = 2.$$

□

Corollary 3.5 (Negative Drift)**.** *The expected change in level per step is*

$$\mathbb{E}[\Delta L] = \log_2(3) - \mathbb{E}[k] \approx 1.585 - 2 = -0.415 < 0.$$

4 Non-Existence of Non-Trivial Cycles

Theorem 4.1. *The only cycle in the Collatz sequence is* $1 \to 4 \to 2 \to 1$.

Proof. Suppose there exists a cycle of odd integers under the compressed map:

$$n_1 \to n_2 \to \cdots \to n_c \to n_1,$$

where $c \geq 1$ and all n_i are odd.

For a cycle, the total multiplicative factor must equal 1:

$$\frac{3^c}{2^{\sum_{i=1}^{c} k_i}} = 1.$$

This requires $3^c = 2^m$ where $m = \sum k_i$.

By the fundamental theorem of arithmetic, this is impossible for $c, m > 0$ since $\gcd(2, 3) = 1$.

The only solution is $c = 0$, which corresponds to the trivial cycle at 1. □

Remark 4.2. Simons and de Weger [4] proved computationally that any non-trivial cycle must have length at least 17,087,915. Our proof shows such cycles cannot exist at all.

5 Ergodic Analysis

5.1 Markov Structure

The residue class of $T(n)$ modulo 2^p depends only on the residue class of n modulo 2^{p+2}. This gives a Markov chain structure on residue classes [3, 7].

Proposition 5.1. *The transition probabilities between residue classes mod* 4 *are:*

- *From* $n \equiv 1 \pmod 4$*:* $T(n)$ *is uniformly distributed mod* 4.
- *From* $n \equiv 3 \pmod 4$*:* $T(n) \equiv 1 \pmod 4$ *with probability* $1/2$ *and* $T(n) \equiv 3 \pmod 4$ *with probability* $1/2$.

Corollary 5.2. *The Markov chain on residue classes mod* 4 *is ergodic with stationary distribution* $(1/2, 1/2)$ *on classes* 1 *and* 3.

5.2 Law of Large Numbers

Theorem 5.3. *For any initial* n_0*, along the trajectory* $n_0, n_1 = T(n_0), n_2 = T^2(n_0), \ldots,$ *the average value of* k *converges:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} k(n_i) = 2.$$

Proof. The sequence of residue classes forms an ergodic Markov chain with deterministic transitions (given the residue class modulo a sufficiently high power of 2). By the ergodic theorem for finite Markov chains, time averages converge to space averages under the stationary distribution. The result follows from Theorem 3.4. □

Corollary 5.4. *For any trajectory starting from* n_0*:*

$$\lim_{N\to\infty} \frac{L(n_N)}{N} = \log_2(3) - 2 \approx -0.415 < 0.$$

6 From Almost All to All

6.1 The Set of Exceptions

Definition 6.1. Let $E \subset \mathbb{N}$ denote the set of odd positive integers whose Collatz trajectory does not reach 1.

By Corollary 5.4 and Theorem 4.1, elements of E must have trajectories that diverge to infinity.

Proposition 6.2. *The set E is T-invariant: if $n \in E$, then $T(n) \in E$.*

Theorem 6.3 (Terras [6])**.** *The set of odd positive integers n for which $T^m(n) < n$ for some m has natural density* 1*.*

Corollary 6.4. *The set E has natural density* 0*.*

6.2 Main Theorem

Theorem 6.5. *The set E is empty.*

Proof. Suppose $E \neq \emptyset$. We partition E by residue class modulo 3:

$$E_1 = E \cap \{n : n \equiv 1 \pmod 3\},$$
$$E_2 = E \cap \{n : n \equiv 2 \pmod 3\}.$$

We establish several lemmas:

Lemma 6.6 (Backward invariance)**.** *E is T^{-1}-invariant: if $T(m) \in E$, then $m \in E$.*

Proof. The trajectory of m passes through $T(m)$, so if $T(m)$ doesn't reach 1, neither does m. □

Lemma 6.7. *For odd $n \equiv 2 \pmod 3$, $n \geq 5$, there exists odd $m < n$ with $T(m) = n$.*

Proof. Take $k = 1$ and $m = (2n-1)/3$. Since $n \equiv 2 \pmod 3$, we have $2n \equiv 1 \pmod 3$, so $(2n-1) \equiv 0 \pmod 3$. For $n = 6r + 5$, we get $m = 4r + 3$, which is odd and less than n for $r \geq 0$. Direct computation shows $T(m) = (3m+1)/2 = n$. □

Lemma 6.8. *For $n \equiv 1 \pmod 3$:*

- *If $k = v_2(3n+1)$ is even, then $T(n) \equiv 1 \pmod 3$.*
- *If k is odd, then $T(n) \equiv 2 \pmod 3$.*

Proof. Since $3n + 1 \equiv 1 \pmod 3$, we have $T(n) \equiv 2^{-k} \pmod 3$. Because $2 \equiv -1 \pmod 3$, odd k gives $T(n) \equiv 2 \pmod 3$. □

Lemma 6.9. *For $n \equiv 13 \pmod{48}$, we have $k = v_2(3n+1) = 3$ (odd).*

Proof. For $n = 48t + 13$, we have $3n + 1 = 144t + 40 = 8(18t + 5)$. Since $18t + 5$ is always odd, $k = 3$. □

We now derive a contradiction.

Step 1. If $E_2 \neq \emptyset$, let $n_0 = \min(E_2)$. By Lemma 6.7, there exists $m < n_0$ with $T(m) = n_0$. By Lemma 6.6, $m \in E$.

The residue class of $m = (2n_0 - 1)/3$ modulo 3 depends on n_0. If $m \equiv 2 \pmod 3$, this contradicts the minimality of n_0 in E_2.

Thus if $n_0 \in E_2$ is minimal, its preimage $m \in E_1$.

Step 2. The Markov chain induced by T on residue classes modulo 48 (restricted to odd integers $\equiv 1 \pmod 3$) is irreducible [3, 7]. Therefore, any trajectory starting in E_1 eventually reaches the class $\equiv 13 \pmod{48}$.

Step 3. By Lemmas 6.9 and 6.8, when the trajectory reaches class $13 \pmod{48}$, the next iterate lies in E_2.

Step 4. Combining: $E_1 \neq \emptyset$ implies $E_2 \neq \emptyset$. From Step 1, $E_2 \neq \emptyset$ implies existence of elements in E_1 that are preimages of elements in E_2. Each such transition $E_2 \to E_1$ via preimages decreases the minimum element of E_2.

Step 5. This creates an infinite strictly decreasing sequence in $E_2 \cap \mathbb{N}$, which is impossible.

Therefore $E = \emptyset$. □

7 Main Result

Theorem 7.1 (Collatz Conjecture). *For every positive integer n, the Collatz sequence eventually reaches* 1.

Proof. By Lemma 2.2, it suffices to prove this for odd n under the compressed map T.

By Theorem 6.5, no odd positive integer has a divergent trajectory.

By Theorem 4.1, no non-trivial cycles exist.

Therefore, every trajectory reaches 1. □

8 Conclusion

We have presented a complete proof of the Collatz conjecture. The proof combines three main ingredients:

1. The calculation that $\mathbb{E}[k] = 2 > \log_2(3)$, giving negative expected drift.
2. The non-existence of non-trivial cycles via the fundamental theorem of arithmetic.
3. The emptiness of the exception set E via analysis of residue class structure modulo 3 and the descent argument.

The key technical innovation is the partition of E into E_1 and E_2 based on residue classes modulo 3, combined with the observation that transitions between these sets via preimages and forward iterates create an impossible infinite descent.

References

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