Universitat de les Illes Balears

MASTER'S THESIS

IMPROVING THE CONSENSUS MODELS FOR GROUP DECISION-MAKING PROBLEMS BASED ON DISCRETE FUZZY NUMBERS

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Master's Degree in Intelligent Systems (MUSI)

Specialisation: Artificial Intelligence

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***Resumen*– El modelo lingüístico computacional basado en números borrosos discretos ha ganado gran interés entre los investigadores por sus interesantes propiedades. Sin embargo, la investigación realizada hasta ahora en el tema de consenso en grupo con este modelo lingüístico no es suficiente y necesita un mayor desarrollo. Por esta razón, en este proyecto, se propone un modelo de consenso basado en este marco que soluciona algunas de las principales desventajas de las anteriores propuestas en la literatura. Para ello, se presenta una nueva función de agregación en el conjunto de números borrosos discretos y se propone un algoritmo semiautomático que permite a los expertos interactuar y modificar sus opiniones durante el proceso de consenso. Este nuevo modelo logra una tasa de convergencia significativamente mayor en los problemas de toma de decisiones grupales en comparación con los algoritmos existentes.**

***Abstract*—The linguistic computational model based on discrete fuzzy numbers has gained significant interest among scholars due to its attractive properties. However, the investigations on group consensus with this linguistic model are not enough and need further exploration. For this reason, in this research study, we propose a novel consensus model based on this framework that overcomes some of the main disadvantages of the previously proposed methods in the literature. Moreover, we provide a new aggregation function on the set of discrete fuzzy numbers. We propose a semi-automatic algorithm that allows the experts to interact and modify their opinions during the consensus process. This novel method achieves a significantly greater convergence rate in Group Decision Making problems than the existing algorithms.**

***Index Terms*—Consensus, Discrete Fuzzy Numbers, Aggregation Function, Linguistic Computing Models, Group Decision Making**

I. INTRODUCTION

A Group Decision Making problem (GDM) can be understood as a dynamic process where a group of experts, known as decision-makers, must decide the best alternative from a given set of other options. To pick out the best choice(s), two different processes are carried out [7, 15, 20]. The first one, called *the consensus process*, has as the primary goal that the group of experts achieve the maximum possible agreement (the best scenario would be to accomplish a unanimous decision) about which alternatives are going to be selected among all the possible ones. The second process, called *the selection process*, deals with the choice of the best alternative(s) according to the preferences expressed by the experts. Frequently, the consensus process [7] is defined as a group discussion dynamic process managed by a moderator whose role is to help the experts to bring their opinions closer together. Indeed, using a consensus measure, the moderator knows the consensus level of the experts in each stage of the process. While this level is unsatisfactory (usually lower than a predefined threshold), the moderator suggests that the experts change their opinions to increase the consensus level. Whenever a satisfactory consensus level is achieved, the selection process obtains the final consensus solution to the general decision-making problem.

Depending on the nature of the problem, the experts can provide their opinions by using different models. One of the most frequent models in the scientific literature is the one based on preference relations to provide their evaluations. In this way, depending on the nature and the complexity of the problem, these preference relations can be modelled using different frameworks. We can highlight the preference relations based on the concepts related to fuzzy logic and fuzzy sets: Interval-valued fuzzy preference relations, linguistic interval fuzzy preference relations or incomplete fuzzy linguistic preference relations are interesting examples among many others [5, 6, 11, 15, 20, 22]. In this direction, in [2] the authors propose to explore consensus under new preference structures. Thus, it is suggested to extend the existing consensus models to work with hesitant fuzzy sets (see [24]) or to investigate the consensus model based on the linguistic computational model based on discrete fuzzy numbers [13].

Discrete fuzzy numbers [23] and, specifically, discrete fuzzy numbers whose support is a closed interval of a finite chain $L_n = \{0, 1, \ldots, n\}$ have been investigated in detail in the literature (see for instance [4, 18]). The main reason is that this family of fuzzy subsets have provided the theoretical foundations of:

i) The multi-granular linguistic model based on discrete fuzzy numbers (see references[8, 13, 17]).

ii) The adaptation of the linguistic model based on Z-numbers called mixed-discrete Z-numbers, recently published in [12].

In the framework of the multi-granular linguistic model based on discrete fuzzy numbers, as far as we know, only two works on consensus have been proposed in [10] and [14]. Specifically, in [10] the concept of subjective linguistic preference relation is introduced to collect the subjective evaluations given by the experts as the preference value for a pair of alternatives. Meanwhile, in [14] the authors introduce a novel method to measure and reach the consensus using an automatic algorithm based on discrete fuzzy numbers. This second paper also presents the use of weights on the initial preferences of the experts for aggregation purposes, the idea of the uncertainty associated with a discrete fuzzy number, and an automatic algorithm to change the experts' opinions to reach the desired consensus level.

Nevertheless, on some occasions, these two methods do not converge, and therefore, the consensus is not reached. In particular, the proposal considered in [10] has a significant drawback related to the automation of the improvement of the consensus degree. Namely, the experts only take part at the beginning of the process. Their opinions are changed without their approval. Therefore, their final assessment may be very different from their actual view. For this reason, it is important that experts must be allowed to participate during the whole process. Some experts can defend their positions and convince others to modify their opinions to others closer to those supported by themselves throughout the discussion. Otherwise, it is not a valid consensus process but an exploitation model to obtain an automatic final decision representative of the initial opinions. A second drawback already mentioned above is that the method to improve the group consensus level in [10] does not always work since some inputs can lead the algorithm to a loop in which the consensus level does not improve.

The main goal of this master's final project is to propose a novel consensus method based on discrete fuzzy numbers which combine the assets of the two methods presented in [10, 14] but also solves the main drawbacks such as the low rate of convergence, the high average number of required iterations and the fact that the experts do not play any role during the consensus process in which their initial opinions are automatically changed without their validation.

The contributions of this work are the following ones:

1) Propose a new aggregation function on the set of discrete fuzzy numbers in line with [4, 9, 18].
2) Propose a new method to improve the consensus degree where it allows the experts to modify their opinions but ensures the improvement of the consensus degree.
3) Improve the convergence rate of the algorithm.

The rest of the document is organized as follows. Section II is devoted to the preliminaries. Section III introduces the reference algorithm. The proposed algorithm, as well as an application example, is introduced in Section IV. Then, in Section V, some experiments to validate the algorithm and results are carried out and the analysis of the results is presented followed by the conclusions in Section VI.

II. Preliminaries

This section is devoted to presenting the main concepts and results which we will use in this work. By a fuzzy subset of $\mathbb{R}$, we mean a function $A : \mathbb{R} \to [0, 1]$. For each fuzzy subset A, let $A^\alpha = \{x \in \mathbb{R} : A(x) \geq \alpha\}$ for any $\alpha \in (0, 1]$ be its α-level set (or α-cut). By supp(A) or A^0, we mean the support of A, i.e., the set $\{x \in \mathbb{R} : A(x) > 0\}$.

Definition 1 ([23]) *A fuzzy subset A of $\mathbb{R}$ with membership mapping $A : \mathbb{R} \to [0, 1]$ is called a* discrete fuzzy number*, or* dfn *for short, if its support is finite, i.e., there exist $x_1, ..., x_n \in \mathbb{R}$ with $x_1 < x_2 < ... < x_n$ such that supp$(A) = \{x_1, ..., x_n\}$, and there are natural numbers s, t with $1 \leq s \leq t \leq n$ such that:*

1) *$A(x_i) = 1$ for all i with $s \leq i \leq t$. (*core*)*
2) *$A(x_i) \leq A(x_j)$ for all i, j with $1 \leq i \leq j \leq s$.*
3) *$A(x_i) \geq A(x_j)$ for all i, j with $t \leq i \leq j \leq n$.*

We will denote for short the set of discrete fuzzy numbers as DFN and a discrete fuzzy number as dfn. Similarly, let $A_1^{L_n}$ denote the set of discrete fuzzy numbers whose support is a closed interval of the finite chain $L_n = \{0, 1, ..., n\}$.

A dfn A with supp$(A) = \{x_1, ..., x_n\}$ will be denoted for short as $A = \{A(x_1)/x_1, \ldots, A(x_n)/x_n\}$. The α-level cuts of A will be represented by $A^\alpha = \left[x_1^\alpha, x_p^\alpha\right]$ with $1 \leq p \leq n$.

Definition 2 *Let $A \in \mathcal{A}_1^{L_n}$ be a discrete fuzzy number. We will say that $\alpha \in (0, 1]$ is a relevant α-level if there exists a $x \in$ supp(A) such that $A(x) = \alpha$.*

The following result presents a method to generate aggregation functions on the set $\mathcal{A}_1^{L_n}$ by using discrete aggregation functions on L_n. First, let us recall the definition of a discrete aggregation function.

Definition 3 ([16]) *A mapping $F : L_n \times L_n \to L_n$ is said to be a discrete aggregation function if it is increasing in each argument and such that $F(0, 0) = 0$ and $F(n, n) = n$.*

Theorem 1 ([4, 18]) *Let F be a discrete aggregation function on the finite chain L_n. The binary operation on $A_1^{L_n}$ defined as follows*

$$\begin{aligned} F : A_1^{L_n} \times A_1^{L_n} &\to A_1^{l_n} \\ (A, B) &\to F(A, B) \end{aligned}$$

being $F(A, B)$ the dfn whose α-cuts are the sets:

$$\{z \in L_n | F(\min A^\alpha, \min B^\alpha) \leq z \leq F(\max A^\alpha, \max B^\alpha)\} \tag{1}$$

for each $\alpha \in [0, 1]$, is an aggregation function on $A_1^{L_n}$.

The following subsection is devoted to recalling the main ideas about the linguistic model based on discrete fuzzy numbers whose support is a closed interval of a finite chain L_n that was presented in [13].

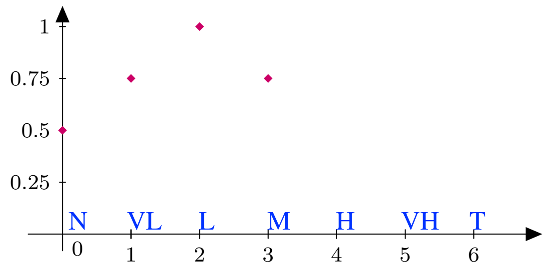

Figure 1: A possible flexibilization of the label $L =$ Low.

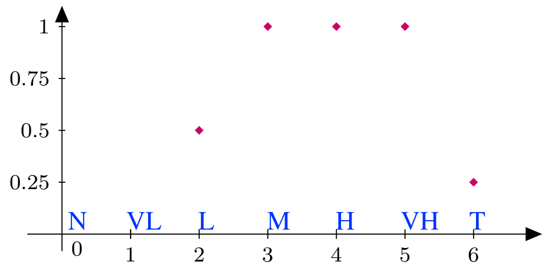

Figure 2: A possible flexibilization of "between Medium and Very High".

A. Linguistic model based on discrete fuzzy numbers

First, it is easy to check that we can consider a bijective mapping between the ordinal scale $\mathfrak{L} = \{s_0, \ldots, s_n\}$ and the finite chain L_n which keeps the original order. Moreover, each normal discrete convex fuzzy subset defined on the ordinal scale $\mathfrak{L}$ can be considered like a discrete fuzzy number belonging to $\mathcal{A}_1^{L_n}$, and vice-versa.

For example, consider the linguistic chain

$$\mathfrak{L} = \{N, VL, L, M, H, VH, T\} \quad (2)$$

where the letters refer to the linguistic terms None, Very Low, Low, Medium, High, Very High and Total and they are listed in an increasing order:

$$N \prec VL \prec L \prec M \prec H \prec VH \prec T$$

and the finite chain $L_6 = \{0, 1, 2, 3, 4, 5, 6\}$. Thus, for instance, the discrete fuzzy number $A = \{0.5/0, 0.75/1, 1/2, 0.75/3\} \in \mathcal{A}_1^{L_6}$ can be also expressed as $A = \{0.5/N, 0.75/VL, 1/L, 0.75/M\}$. Note that this discrete fuzzy number, A, can be interpreted as a possible flexibilization of the linguistic label L (Low). In this sense, the discrete fuzzy number A can be used by an expert that thinks the best valuation would be the linguistic label L. Still, he cannot discard other linguistic labels around it to some degree (see Figure 1). On the other hand, the discrete fuzzy number $A = \{1/2\}$, or equivalently $A = \{1/L\}$, would be considered by an expert who is completely sure of his opinion. This fact ensures that this model generalizes also any linguistic model where the experts' valuations are limited to choose a single linguistic label (for more details, see references [13, 17]). Furthermore, this linguistic model generalizes the concept of Interval-valued evaluations or Hesitant Fuzzy Linguistic Term Sets (HFLTS) (see [19] and [21] for details). In this way, discrete fuzzy numbers $A \in \mathcal{A}_1^{L_n}$ with $core(A) = [s_i, s_j]$, but with a different support, can be explained as flexibilizations of the subjective evaluations of the form "between s_i and s_j", "worse than s_i", etc. For instance, the discrete fuzzy number $B = \{0.5/2, 1/3, 1/4, 1/5, 0.25/6\}$, that can be also expressed as $B = \{0.5/L, 1/M, 1/H, 1/VH, 0.25/T\}$, is a possible flexibilization of the HFLTS "between Medium and Very High" (see Figure 2).

From the above discussion, we can define an evaluation based on discrete fuzzy numbers.

Definition 4 ([13]) *Let $L_n = \{0, \ldots, n\}$ be a finite chain. We call an evaluation based on discrete fuzzy numbers to each discrete fuzzy number belonging to the partially ordered set $\mathcal{A}_1^{L_n}$.*

This linguistic computational model presents some interesting properties (for more details, see [8, 13, 17]) concerning other linguistic approaches. We will highlight three interesting aspects:

- The model enables a higher flexibilization of the linguistic term sets, and hence, it allows the experts to select the evaluations that best suit their opinions.
- This model allows to use scales with different granularities (see [13]), enabling to increase the flexibility of the experts' opinions.
- This model does not need to make previous transformations when we wish to aggregate the information.

III. Reference Algorithm

In this section, we present the reference algorithm that was introduced in [10]. In this work, the authors propose a complete group consensus model for decision-making problems with the linguistic model based on discrete fuzzy numbers. In this way, this method includes the research of aggregation method, the method of measuring the deviation and the definition of group consensus index.

A. *Part 1: Aggregation of the Evaluations*

Recall that, according to [3], a Group Decision Making (GDM) problem is usually understood as a decision problem which consists in finding the best alternative(s) from a set of feasible alternatives, $X = \{x_1, \ldots, x_n\}$, according to the preferences provided by a group of experts, $E = \{e_1, \ldots, e_q\}$, characterized by their experience and knowledge. In this sense, suppose that these q experts express their evaluations using discrete fuzzy numbers and let $A^1, A^2, \ldots, A^q \in \mathcal{A}_1^{L_n}$ be such evaluations. Moreover, since the experts have different levels of expertise and authority, the so called subjective weights vector, denoted by $w_s = \{w_s^1, w_s^2, \cdots, w_s^q\}$ where $0 < w_s^i < 1$ for all $1 \leq i \leq q$ and $\sum_{i=1}^q w_s^i = 1$, is considered.

As it is proposed in [10], these initial subjective weights of the experts refer to the initial degree of authority of the experts. However, suppose that an expert provides a very uncertain evaluation. This is an indicator that the expert lacks a deep knowledge of the decision problem. Consequently, the weight of their evaluation in the decision problem must decrease. This idea is performed in [10] with the concept of *combination weights* (a combination of the subjective weights and the uncertainty of the evaluation). To calculate the combination weights, we need to compute first the uncertainty of a discrete fuzzy number.

Definition 5 ([10]) *Let $A \in A_1^{L_n}$ be a discrete fuzzy number with supp$(A) = \{x_1, \ldots, x_p\}$. Let $0 < \alpha < 1$. The* uncertainty level *of A is computed as:*

$$U(A) = \alpha \frac{p-1}{n} + (1-\alpha)(1 - D(A)) \tag{3}$$

where $D(A)$ is the standard deviation of the membership values of A defined as:

$$D(A) = \frac{\sqrt{\sum_{i=1}^{p} \left(A(x_i) - \sum_{i=1}^{p} \frac{A(x_i)}{p} \right)^2}}{p}. \tag{4}$$

Note that to calculate the uncertainty, we need to compute previously the standard deviation.

Example 6 *Let $A = \{0.5/N, 0.75/VL, 1/L, 0.75/M\} \in \mathcal{A}_1^{L_6}$ and $\alpha = 0.5$. Its uncertainty can be calculated by*

$$D(A) = \frac{\sqrt{\sum_{i=1}^{4} \left(A(x_i) - \sum_{i=1}^{4} \frac{A(x_i)}{4} \right)^2}}{4}$$

$$= \frac{\sqrt{\sum_{i=1}^{4} (A(x_i) - 0.75)^2}}{4} = 0.353.$$

The uncertainty is calculated as follows:

$$U(A) = \alpha \frac{p-1}{n} + (1-\alpha)(1 - D(A))$$

$$= 0.5 \cdot \frac{3}{6} + (1 - 0.5)(1 - 0.353) = 0.537.$$

At this point, we can already present the definition of the combination weights.

Definition 7 ([10]) *Let $A^1, A^2, \cdots, A^q \in A_1^{L_n}$ be discrete fuzzy numbers and $w_s = \left\{w_s^1, w_s^2, \cdots, w_s^q\right\}$ be the associated vector of subjective weights. Let $0 < \beta < 1$. The combination weights w_c^i for all $1 \leq i \leq q$ are computed as:*

$$w_c^i = \beta w_s^i + (1-\beta) \frac{1 - U(A^i)}{\sum_{j=1}^{q} (1 - U(A^j))}.$$

Example 8 *Consider $A^1 = \{0.5/0, 1/1, 0.6/2, 0.2/3\}, A^2 = \{0.3/0, 0.8/1, 0.9/2, 1/3\} \in A_1^{L_n}$, the evaluations of two experts with the subjective weights $w_s = \{0.7, 0.3\}$. We consider $\alpha = 0.5$ and $\beta = 0.6$, as the parameter values to compute the uncertainty and the combination weights.*

The uncertainty of experts A_1, A_2 are computed using Definition 4 obtaining $U(A_1) = 0.544$ and $U(A_2) = 0.553$. Now, using Definition 6, we get

$$w_c^1 = 0.6 \cdot 0.7 + 0.4 \cdot \frac{1-0.544}{0.456+0.477} = 0.6,$$

$$w_c^2 = 0.6 \cdot 0.3 + 0.4 \cdot \frac{1-0.553}{0.456+0.477} = 0.4.$$

We will now explain the different steps involved in the information aggregation process presented in [10].

Step 1: Join the linguistic terms that frequently comes in the evaluations presented by experts in the form of sets.

Let $U =$

$$\left\{ z \in L_n | \min_{1 \leq k \leq q} \left\{ \min_{1 \leq k \leq m^l} x_k^l \right\} \leq z \leq \max_{1 \leq k \leq q} \left\{ \max_{1 \leq k \leq m^l} x_k^l \right\} \right\}$$

with supp$(A^l) = \{x_1^l, \ldots, x_{m^l}^l\}$ for all $1 \leq l \leq q$. Consider now for each $p \in U$:

$$\bar{p} = \left| \left\{ l \mid p \in \text{supp}\left(A^l\right) \right\} \right|,$$

where $| \cdot |$ denotes the cardinal of a finite set. Finally, let us define:

$$\begin{aligned} Q &= \{z \in U | \bar{z} = \max_{p \in U} (\bar{p})\}, \\ \Delta &= \{z \in L_n | \min(Q) \leq z \leq \max(Q)\}. \end{aligned}$$

Step 2: Expand this set of linguistic terms according to the weights and evaluations made by each expert according to the next procedure. Namely, for each expert e_l, we will denote this expanded set of linguistic terms by Δ^l.

In the following operations, we will denote a decimal number as a and $a^\diamond$ will represent the result of rounding a.

Case 1: If supp$(A^l) \subseteq \Delta$, then $\Delta^l = \Delta$.

Case 2: If $\max(\text{supp}(A^l) \setminus \Delta) < \min(\Delta)$, then

$$\begin{aligned} \Delta^l = \{ & \min(\Delta) - (|\text{supp}(A^l) \setminus \Delta| \cdot w_c^l)^\diamond, \ldots, \\ & \min(\Delta) - 1, \Delta\} \end{aligned}$$

that is, add $(|\text{supp}(A^l) \setminus \Delta| \cdot w_c^l)^\diamond$ smaller elements to Δ.

Case 3: If $\min(\text{supp}(A^l) \setminus \Delta) > \max(\Delta)$, then

$$\begin{aligned} \Delta^l = \{ & \Delta, \max(\Delta) + 1, \ldots, \max(\Delta) + \\ & (\left|\text{supp}(A^l) \setminus \Delta\right| \cdot w_c^l)^\diamond\} \end{aligned}$$

that is, add $(\left|\text{supp}(A^l) \setminus \Delta\right| \cdot w_c^l)^\diamond$ greater elements to Δ.

Case 4: If $\Delta \subset \text{supp}(A^l), \min(\text{supp}(A^l)) \neq \min(\Delta)$ and $\max(\text{supp}(A^l)) \neq \max(\Delta)$, then $\Delta^l =$

$$\begin{aligned} & \{\min(\Delta) - \left(| \left\{ z < \min(\Delta) | z \in (\text{supp}(A^l) \setminus \Delta) \right\} | \cdot w_c^l \right)^\diamond, \\ & \ldots, \Delta, \ldots, \\ & \max(\Delta) + (| \left\{ z > \max(\Delta) | z \in (\text{supp}(A^l) \setminus \Delta) \right\} | \cdot w_c^l)^\diamond\} \end{aligned}$$

that is, add $(| \left\{ z < \min(\Delta) | z \in (\text{supp}(A^l) \setminus \Delta) \right\} | \cdot w_c^l)^\diamond$ smaller elements and $(| \left\{ z > \max(\Delta) | z \in (\text{supp}(A^l) \setminus \Delta) \right\} | \cdot w_c^l)^\diamond$ greater elements to Δ.

Step 3: We will denote by $\mathcal{F}_{w_c}$ the result of the aggregation, which will be a discrete fuzzy number. First of all, in this step, its support is determined by:

$$\text{supp}(\mathcal{F}_{w_c}) = \bigcup_{l=1}^{q} \Delta^l.$$

Step 4: Now, we compute the provisional membership values of the elements of the support of $\mathcal{F}_{w_c}$ using a weighted average method. Specifically, for all $x \in \text{supp}(\mathcal{F}_{w_c})$, we have that

$$\widetilde{\mathcal{F}_{w_c}}(x) = \sum_{\{l | x \in \text{supp}(A^l)\}} A^l(x) \omega_c^l.$$

Step 5: Normalization. For all $x \in \text{supp}(\mathcal{F}_{w_c})$, we compute the final values dividing by the maximum:

$$\mathcal{F}_{w_c}(x) = \widetilde{\mathcal{F}_{w_c}}(x) \cdot \frac{1}{\max_{y \in \text{supp}(\mathcal{F}_{w_c})} \widetilde{\mathcal{F}_{w_c}}(y)}.$$

Remark 9 *During this last step, the authors also round the membership values to one decimal place. Note moreover that* $\mathcal{F}_{w_c}(x) \in [0,1]$ *holds for all* $x \in supp(\mathcal{F}_{w_c})$*.*

The following example is presented for a better understanding of the aggregation process.

Example 10 *We consider 4 experts whose evaluations are given in* $\mathcal{A}_1^{L_4}$*:*

$$\begin{array}{l} A = \{1/3, 1/4\}, B = \{1/0, 1/1, 0.8/2\}, \\ C = \{0.1/0, 0.4/1, 1/2, 0.4/3, 0.3/4\}, \\ D = \{0.6/1, 1/2, 0.3/3\}. \end{array}$$

The subjective weights vector is $w_s = \{0.3, 0.1, 0.5, 0.1\}$*. We consider the parameter values* $\alpha = 0.5$*,* $\beta = 0.6$ *(for the computation of the combination weights vector).*

First of all, we obtain the combination weights vector using Definition 7:

$$w_c = (0.278, 0.155, 0.39, 0.177).$$

Then, we apply the aggregation algorithm defined above. The results obtained at each stage are:
In Step 1*,* $\Delta = \{1, 2, 3\}$*.*
In Step 2*, for each expert, we analyze the linguistic terms and compute the updated* Δ *value such as:*

$$\begin{array}{l} \Delta^1 = \{1, 2, 3\}, \\ \Delta^2 = \{1, 2, 3\}, \\ \Delta^3 = \{1, 2, 3\}, \\ \Delta^4 = \{1, 2, 3\}. \end{array}$$

In Step 3*, we compute the support of the aggregation result*

$$supp(\mathcal{F}_{w_c}) = \bigcup_{l=1}^{4} \Delta^l = \{1, 2, 3\}.$$

In Step 4*, the provisional membership values are obtained as shown below:*

$$\begin{array}{l} \widetilde{\mathcal{F}_{w_c}}(1) = 0.42, \\ \widetilde{\mathcal{F}_{w_c}}(2) = 0.69, \\ \widetilde{\mathcal{F}_{w_c}}(3) = 0.48. \end{array}$$

In Step 5*, normalization is applied on the membership values obtained from Step 4. In addition, the normalized results has been rounded to one decimal place:*

$$\begin{array}{l} \mathcal{F}_{w_c}(x_1) = 0.6, \\ \mathcal{F}_{w_c}(x_2) = 1.0, \\ \mathcal{F}_{w_c}(x_3) = 0.7. \end{array}$$

Finally the result of the aggregation is the following one:

$$\mathcal{F}_{w_c}(A, B, C, D) = \{0.6/1, 1/2, 0.7/3\}.$$

B. *Part 2: Computing the Group Consensus Index*

Once the aggregation function is defined, we can find the group consensus level obtained from the actual experts' evaluations.

Definition 11 ([10]) *Let* $A^1, A^2, \ldots, A^q \in \mathcal{A}_1^{L_n}$ *be the opinions of q experts, let* $w_c = \{w_c^1, w_c^2, \cdots, w_c^q\}$ *be the associated combination weights vector and let* $\mathcal{F}_{w_c}$ *be the result of the aggregation of* $A^1, A^2, \ldots, A^q$*. The* Group Consensus Index*, GCI for short, is computed as:*

$$GCI(A^1, A^2, \ldots, A^q) = \sum_{i=1}^{q} w_c^i \left(1 - D_{A^i, \mathcal{F}_{w_c}}\right) \tag{5}$$

where $D_{A,B}$ *is the deviation between two discrete fuzzy numbers* $A, B \in \mathcal{A}_1^{L_n}$ *given by*

$$\begin{aligned} D_{A,B} = \lambda &\left[\frac{1}{n(r+1)} \sum_{i=0}^{r} \left|x_{p,A}^{\alpha_i} - x_{p',B}^{\alpha_i}\right|\right] \\ + (1-\lambda) &\left[\frac{1}{n(r+1)} \sum_{i=0}^{r} \left|x_{1,A}^{\alpha_i} - x_{1,B}^{\alpha_i}\right|\right] \end{aligned} \tag{6}$$

where $\lambda \in [0,1]$*,* $r+1$ *is the number of* α*-levels considered to compute the deviation and* $A^{\alpha_i} = [x_{1,A}^{\alpha_i}, x_{p,A}^{\alpha_i}]$ *and* $B^{\alpha_i} = [x_{1,B}^{\alpha_i}, x_{p',B}^{\alpha_i}]$ *are the* α_i*-cuts of A and B, respectively, for all* $1 \leq i \leq r$*.*

Example 12 *Let us consider the following four evaluations*

$$\begin{array}{l} A = \{1/3, 1/4\}, B = \{1/0, 1/1, 0.8/2\}, \\ C = \{0.1/0, 0.4/1, 1/2, 0.4/3, 0.3/4\}, \\ D = \{0.6/1, 1/2, 0.3/3\}, \end{array}$$

defined in $\mathcal{A}_1^{L_4}$*, let* $w_c = \{0.278, 0.155, 0.39, 0.177\}$ *be the combination weights vector. Consider* $\lambda = 0.5$ *and* $r = 10$*, i.e., we consider* $0, 0.1, \ldots, 0.9, 1.0$ *as* α*-levels to compute the deviation. In this case, we have that* $\mathcal{F}_{w_c}(A, B, C, D) = \{0.6/1, 1/2, 0.7/3\}$*,*

$$\begin{array}{ll} D_{A,\mathcal{F}_{w_c}(A,B,C,D)} & = \frac{0.5}{44} \cdot 14 + \frac{0.5}{44} \cdot 18 = 0.3636, \\ D_{B,\mathcal{F}_{w_c}(A,B,C,D)} & = 0.2841, \\ D_{C,\mathcal{F}_{w_c}(A,B,C,D)} & = 0.125, \\ D_{D,\mathcal{F}_{w_c}(A,B,C,D)} & = 0.0455, \end{array}$$

obtaining that

$$\begin{array}{l} GCI(A, B, C, D) = 0.278(1 - 0.18) + 0.155(1 - 0.14) \\ +0.39(1 - 0.06) + 0.177(1 - 0.02) = 0.7981. \end{array}$$

C. *Part 3: Improving Group Consensus Index*

In decision-making problems it is usual to establish a minimum degree of consensus among experts called threshold. In this way, once the opinions of the experts have been established, the degree of consensus of the experts is measured by means of the so-called consensus indexes. If this index is equal to or greater than the threshold initially established, the selection process begins. Otherwise, it is necessary to establish a method to improve group consensus. We explain the method for improving the group consensus index as presented in the [10].

We will consider the same notations used for the aggregation algorithm. The discrete fuzzy numbers $A^1, A^2, \cdots, A^q$ are used for the evaluations and w_c represents the combination weights with the vector $w_c = (w_c^1, w_c^2, ..., w_c^q)$.

We will denote by θ the desired consensus threshold for the decision-making problem, i.e., the consensus algorithm will finish whenever $GCI(A^{1,p}, A^{2,p}, \ldots, A^{q,p}) \geq \theta$ where $A^{l,p}$ for all $1 \leq l \leq q$ denote the opinions of the experts at some iteration p of the algorithm to improve the consensus. With this notation, note that $A^{l,0} = A^l$, i.e., the initial experts' opinions.

Input: The experts' evaluations $A^1, \ldots, A^q \in \mathcal{A}_1^{L_n}$;
Combination weights w_c;
The desired group consensus index threshold θ;

Output: The modified experts' opinions $A^{1,p}, A^{2,p}, \ldots, A^{q,p}$ for some $p > 1$ which satisfy $GCI(A^{1,p}, A^{2,p}, \ldots, A^{q,p}) \geqslant \theta$;

Let $p = 0$.

Step 1: Using the uncertainty of the evaluation given by the expert and the deviation between the expert's evaluation and the aggregation result, calculate the so-called screening index value for each expert given by

$$S^{l,p} = \gamma D_{A^{l,p}, \mathcal{F}_{w_c}(A^{1,p},\ldots,A^{q,p})} + (1-\gamma) U(A^{l,p}),$$

where γ is the distribution importance of $U(A^l)$ and $D_{A^{l,p}, \mathcal{F}_{w_c}(A^{1,p},\ldots,A^{q,p})}$.

Step 2: Making adjustments on experts evaluations.

Select the expert with the highest screening index. Suppose that this is the expert whose opinion is represented by $A^{l_0,p}$. Consider the following cases depending on $\text{supp}(A^{l_0,p})$.

Case 1:
If $\min(\text{supp}(A^{l_0,p})) \leq \min(\text{supp}(\mathcal{F}_{w_c}(A^{1,p}, \ldots, A^{q,p})))$ and $\max(\text{supp}(A^{l_0,p})) < \max(\text{supp}(\mathcal{F}_{w_c}(A^{1,p}, \ldots, A^{q,p})))$ then:

$$\text{supp}(A^{l_0,p+1}) = \text{supp}(A^{l_0,p}) \cup \{\max(\text{supp}(A^{l_0,p})) + 1\},$$
$$A^{l_0,p+1}(\max(\text{supp}(A^{l_0,p})) + 1) = A^{l_0,p}(\max(\text{supp}(A^{l_0,p})),$$
$$A^{l_0,p+1}(x) = A^{l_0,p}(x), \text{ for all } x \in \text{supp}(A^{l_0,p}).$$

Case 2:
If $\min(\text{supp}(\mathcal{F}_{w_c}(A^{1,p}, \ldots, A^{q,p}))) < \min(\text{supp}(A^{l_0,p}))$ and $\max(\text{supp}(\mathcal{F}_{w_c}(A^{1,p}, \ldots, A^{q,p})) \leq \max(\text{supp}(A^{l_0,p}))$ then:

$$\text{supp}(A^{l_0,p+1}) = \{\min(\text{supp}(A^{l_0,p})) - 1\} \cup \text{supp}(A^{l_0,p}),$$
$$A^{l_0,p+1}(\min(\text{supp}(A^{l_0,p})) - 1) = A^{l_0,p}(\min(\text{supp}(A^{l_0,p})),$$
$$A^{l_0,p+1}(x) = A^{l_0,p}(x), \text{ for all } x \in \text{supp}(A^{l_0,p}).$$

Case 3:
If $\min(\text{supp}(\mathcal{F}_{w_c}(A^{1,p}, \ldots, A^{q,p})) < \min(\text{supp}(A^{l_0,p}))$ and $\max(\text{supp}(\mathcal{F}_{w_c}(A^{1,p}, \ldots, A^{q,p}))) > \max(\text{supp}(A^{l_0,p}))$ then:

$$\begin{aligned}\text{supp}(A^{l_0,p+1}) &= \{\min(\text{supp}(A^{l_0,p})) - 1\} \cup \text{supp}(A^{l_0,p}) \\ &\cup \{\max(\text{supp}(A^{l_0,p})) + 1\},\end{aligned}$$
$$A^{l_0,p+1}(\max(\text{supp}(A^{l_0,p})) + 1) = A^{l_0,p}(\max(\text{supp}(A^{l_0,p})),$$
$$A^{l_0,p+1}(\min(\text{supp}(A^{l_0,p})) - 1) = A^{l_0,p}(\min(\text{supp}(A^{l_0,p}))),$$
$$A^{l_0,p+1}(x) = A^{l_0,p}(x), \text{ for all } x \in \text{supp}(A^{l_0,p}).$$

Case 4:
If $\min(\text{supp}(\mathcal{F}_{w_c}(A^{1,p}, \ldots, A^{q,p}))) \geq \min(\text{supp}(A^{l_0,p}))$ and $\max(\text{supp}(\mathcal{F}_{w_c}(A^{1,p}, \ldots, A^{q,p}))) \leq \max(\text{supp}(A^{l_0,p}))$ then let us define first $A^{-1}(x) = \{z \in L_n \mid A(z) = x\}$ and now

$$\begin{aligned}V = \; & \{z \in L_n \mid \min((A^{l_0,p})^{-1}(1) \cup \text{supp}(\mathcal{F}_{w_c}(A^{1,p}, \ldots, A^{q,p}))) \\ & \leq z \leq \max((A^{l_0,p})^{-1}(1) \cup \text{supp}(\mathcal{F}_{w_c}(A^{1,p}, \ldots, A^{q,p})))\},\end{aligned}$$
$$T = \text{supp}(A^{l_0,p}) \setminus (V \cup (A^{l_0,p})^{-1}(0.1)\}.$$

When $T \neq \emptyset$ and $\min V > \min T$, set $A^{l_0,p+1}(\min T) = 0.1$ and $A^{l_0,p+1}(x) = A^{l_0,p}(x)$, for all $x \in \text{supp}(A^{l_0,p}) \setminus \{\min T\}$.
When $T \neq \emptyset$ and $\min V < \min T$, let $A^{l_0,p+1}(\max T) = 0.1$ and $A^{l_0,p+1}(x) = A^{l_0,p}(x)$, for all $x \in \text{supp}(A^{l_0,p}) \setminus \{\max T\}$.
When $T = \emptyset$, return to Step 2 but now do not consider this expert in any further execution of Step 2 until an evaluation is modified through one of the cases.
Set $A^{l,p+1} = A^{l,p}$ for all $l \neq l_0$. Set $p = p + 1$.

Step 3: Check whether $GCI(A^{1,p}, A^{2,p}, \ldots, A^{q,p}) \geq \theta$. If the answer is affirmative, stop. Otherwise, go to Step 1 and continue the algorithm until the group consensus value reaches θ.

D. A complete example

Let us apply the previous algorithms to the example presented in Table 5 in [10]. Consider 4 experts whose evaluations are given in $\mathcal{A}_1^{L_8}$:

$$\begin{aligned}A^1 &= \{0.6/2, 1/3, 0.7/4\}, \\ A^2 &= \{0.6/3, 1/4, 0.8/5, 0.7/6\}, \\ A^3 &= \{0.4/5, 1/6, 0.9/7\}, \\ A^4 &= \{0.6/5, 1/6, 0.8/7, 0.7/8\}.\end{aligned}$$

The subjective weights vector is $w_s = \{0.3, 0.1, 0.5, 0.1\}$. We consider $\theta = 0.9$ and the following parameter values $\alpha = 0.5$, $\beta = 0.6$ (for the computation of the combination weights vector), $\lambda = 0.5$ and $r = 10$ for the computation of GCI.

First of all, we obtain the combination weights vector using Definition 7: $w_c = (0.286, 0.149, 0.416, 0.149)$.

In the next step, we apply the aggregation algorithm to find the aggregated result $\mathcal{F}_{w_c}(A^1, A^2, A^3, A^4)$. The following results are found at each step:
In Step 1, we compute first the set Δ which is given by $\Delta = \{5, 6\}$.
In Step 2, for each expert e_l, we analyze the linguistic terms and we compute the extended Δ^l sets given by:

$$\begin{aligned}\Delta^1 &= \{4, 5, 6\}, \\ \Delta^2 &= \{5, 6\}, \\ \Delta^3 &= \{5, 6\}, \\ \Delta^4 &= \{5, 6\}.\end{aligned}$$

In Step 3, we compute the support of the aggregation result $\text{supp}(\mathcal{F}_{w_c}) = \bigcup_{l=1}^{4} \Delta^l = \{4, 5, 6\}$.
In Step 4, the provisional membership values are obtained as shown below:

$$\begin{aligned}\widetilde{\mathcal{F}_{w_c}}(4) &= 0.3492, \\ \widetilde{\mathcal{F}_{w_c}}(5) &= 0.375, \\ \widetilde{\mathcal{F}_{w_c}}(6) &= 0.6693.\end{aligned}$$

In Step 5, the normalization process is applied on the membership values obtained from Step 4. The normalized results are further rounded to one decimal place:

$$\mathcal{F}(4) = 0.5, \mathcal{F}(5) = 0.6, \mathcal{F}(6) = 1.$$

Finally the aggregated result is:

$$\mathcal{F}_{w_c}(A^1, A^2, A^3, A^4) = \{0.5/4, 0.6/5, 1.0/6\}.$$

Now, let us compute the group consensus index. First, we need the deviations:

$$\begin{aligned} D_{A^1,\mathcal{F}_{w_c}(A^1,A^2,A^3,A^4)} &= 0.296, \\ D_{A^2,\mathcal{F}_{w_c}(A^1,A^2,A^3,A^4)} &= 0.119, \\ D_{A^3,\mathcal{F}_{w_c}(A^1,A^2,A^3,A^4)} &= 0.102, \\ D_{A^4,\mathcal{F}_{w_c}(A^1,A^2,A^3,A^4)} &= 0.131. \end{aligned}$$

Thus, the GCI is computed as follows:

$$\begin{aligned} &GCI(A^1, A^2, A^3, A^4) = 0.286(1-0.296) + 0.149(1-0.119) \\ &+0.416(1-0.102) + 0.149(1-0.131) = 0.8357. \end{aligned}$$

As we considered $\theta = 0.9$ as the desired threshold and now we got 0.8357, we need to improve the group consensus index to meet the threshold. To do so, we have to apply the algorithm for improving this index changing the opinions of the experts.

Set $p = 0$. In Step 1, we compute the screening index ($S^{l,p}$) for each expert.

$$S^{1,0} = 0.345, S^{2,0} = 0.218, S^{3,0} = 0.180, S^{4,0} = 0.228.$$

Then, we select the first expert as the one with the highest screening index and update his/her opinion according to Step 2 of the algorithm. Note that Case 1 applies and we have that:

$$\begin{aligned} \text{supp}(A^{1,1}) &= \{2, 3, 4, 5\}, \\ A^{1,1}(5) &= A^{1,0}(4) = 0.7, \end{aligned}$$

and therefore, the opinion becomes now:

$$A^{1,1} = \{0.6/2, 1/3, 0.7/4, 0.7/5\}.$$

We set $p = 1$ and $A^{2,1} = A^{2,0}$, $A^{3,1} = A^{3,0}$ and $A^{4,1} = A^{4,0}$ and we return to Step 1. In this example, the desired consensus threshold is reached after 10 iterations (GCI$= 0.9163$) and the final modified opinions are the following ones:

$$\begin{aligned} A^{1,10} &= \{0.1/2, 1/3, 0.7/4, 0.7/5, 0.7/6\}, \\ A^{2,10} &= \{0.1/3, 1/4, 0.8/5, 0.7/6\}, \\ A^{3,10} &= \{0.4/3, 0.4/4, 0.4/5, 1/6, 0.1/7\}, \\ A^{4,10} &= \{0.6/3, 0.6/4, 0.6/5, 1/6, 0.8/7, 0.1/8\}. \end{aligned}$$

IV. Novel consensus algorithm based on discrete fuzzy numbers

This section will explain in detail the steps of the novel consensus algorithm based on discrete fuzzy numbers. The proposed algorithm relies on the strong points of the algorithm presented in [10]. Still, it incorporates a whole new aggregation method and strategy to improve the group consensus level, which involves the participation of the experts in the process.

We will consider the same framework and inputs as in previous section. Let us recall it. Consider a group decision making problem in which q experts must evaluate an alternative and reach a consensus on the group decision. Let us denote by $A^1, A^2, \cdots, A^q$ the experts' evaluations given as discrete fuzzy numbers. We denote by $w_s = \left\{w_s^1, w_s^2, \cdots, w_s^q\right\}$ the subjective weights vector where $0 < w_s^i < 1$ for all $1 \leq i \leq q$ and $\sum_{i=1}^q w_s^i = 1$.

This model will make use also of the concept of combination weights which are computed also using the Definition 7 as proposed in [10].

A. Aggregation Function

Our algorithm relies heavily on the proposal of an aggregation function to obtain a representative of the group opinion from the experts' evaluations. This representative will play a pivotal role in computing the group consensus level and knowing which expert must change their opinion when the desired consensus level is not attained.

We will make use of the following discrete aggregation function.

Definition 13 ([9]) *Let $w = (w_1, w_2, \ldots, w_q)$ be a weights vector. The so-called* discrete weighted arithmetic mean *is the operation $F : (L_n)^q \to L_n$ defined as*

$$F_w(x_1, \ldots, x_q) = \left\lceil \sum_{i=1}^q x_i w_i \right\rceil$$

where $\lceil x \rceil$ denotes the smaller element in L_n which is greater or equal to x.

By using Theorem 1, this discrete aggregation function can be extended to an aggregation function on the set $\mathcal{A}_1^{L_n}$.

Note that the direct use of aggregation functions on the set of discrete fuzzy numbers is an easier and clearer process than the algorithmic aggregation proposed in the previous section. Moreover, we can ensure that this function satisfies the properties of an aggregation function in $\mathcal{A}_1^{L_n}$ while we can not ensure that the algorithmic process proposed in the previous section satisfies the monotonicities and boundary conditions.

Example 14 *Let us consider $A = \{1/3, 1/4\}$ and $B = \{1/0, 1/1, 0.8/2\}$ the dfns in $\mathcal{A}_1^{L_4}$ and the weights vector $w = (0.5, 0.5)$. The union of the relevant α-levels of A and B are $\{0.8, 1\}$. Let us compute the cuts for these levels:*

- *A: $A^{0.8} = [3, 4]$ and $A^1 = [3, 4]$.*
- *B: $B^{0.8} = [0, 2]$ and $B^1 = [0, 1]$.*

Now, we can compute the α-levels of $\mathcal{F}_w(A, B)$ being $\mathcal{F}$ the extension of F_w by using Theorem 1:

- $\mathcal{F}_w(A, B)^{0.8} = \{z \in L_6 \mid F(3, 0) \leq z \leq F(4, 2)\} = \{z \in L_6 \mid \lceil 0.5 \times 3 + 0.5 \times 0 \rceil \leq z \leq \lceil 0.5 \times 4 + 0.5 \times 2 \rceil\} = [2, 3]$.
- $\mathcal{F}_w(A, B)^1 = \{z \in L_6 \mid F(3, 0) \leq z \leq F(4, 1)\} = \{z \in L_6 \mid \lceil 0.5 \times 3 + 0.5 \times 0 \rceil \leq z \leq \lceil 0.5 \times 4 + 0.5 \times 1 \rceil\} = [2, 3]$.

Therefore, the aggregated discrete fuzzy number is $\mathcal{F}_w(A, B) = \{\frac{1}{2}, \frac{1}{3}\}$.

B. Improving Group Consensus Index

Figure 3 presents the flowchart of our proposed algorithm. Whenever the desired group consensus index is not reached, the expert whose opinion is furthest from the aggregated value is identified. This step is based on the use of the distance between discrete fuzzy numbers $D_{A,B}$ introduced in Definition 11.

Once this expert is identified, the moderator asks him/her to modify his/her evaluation in order to improve the group consensus index. Formally, let us suppose that $GCI(A^1, \dots, A^q) < \theta$, where θ is the desired consensus level index. Let A^j be the evaluation such that $D_{A^j, \mathcal{F}_{w_c}(A^1,\dots,A^q)} \geq D_{A^i, \mathcal{F}_{w_c}(A^1,\dots,A^q)}$ for all $1 \leq i \leq q$, i.e., A^j is the evaluation which is furthest from the aggregated value $\mathcal{F}_{w_c}(A^1, \dots, A^q)$. Then the expert e_j must change his/her evaluation in order to improve the group consensus index. Thus, this expert must choose an evaluation $(A^j)'$ such that

$$GCI(A^1, \dots, (A^j)', \dots, A^q) > GCI(A^1, \dots, A^j, \dots, A^q).$$

It is clear that such evaluations always exist since $(A_j)' = \mathcal{F}_{w_c}(A^1, \dots, A^j, \dots, A^q)$ always improve the group consensus index.

Remark 15 *The moderator can provide a set of evaluations to expert e_j, which improves the group consensus index to help the expert update their evaluation. When none of these evaluations fit the expert, another set of feasible evaluations is presented. This process is repeated until the expert accepts one of the proposed evaluations or he/she provides an evaluation improving the consensus index. Note that the whole process relies on the collaboration of all the experts. However, in the case that one expert does not want to cooperate, some solutions are available such as reducing the combination weight of this expert to reduce his/her influence in the final result.*

Algorithm 1: Novel consensus algorithm

Input: The experts' evaluations $A^1, \dots, A^q \in \mathcal{A}_1^{L_n}$;
The subjective weights vector w_s;
The desired group consensus index θ;
Compute w_c using Def. 7;
while $GCI(A^1, \dots, A^q) < \theta$ **do**
  Let e_j be the expert such that his/her evaluation A^j satisfies $D_{A^j, \mathcal{F}_{w_c}(A^1,\dots,A^q)} \geq D_{A^i, \mathcal{F}_{w_c}(A^1,\dots,A^q)}$ for all $1 \leq i \leq q$;
  Expert e_j chooses $(A^j)'$ such that $GCI(A^1, \dots, (A^j)', \dots, A^q) > GCI(A^1, \dots, A^j, \dots, A^q)$;
  $A^j = (A^j)'$;
end

At this point, A^j is replaced by $(A^j)'$ and $GCI(A^1, \dots, (A^j)', \dots, A^q)$ is computed. While the desired group consensus index is not achieved, this stage is repeated, identifying in each iteration the expert whose opinion is farthest from the aggregated value (which is computed in each iteration). Then this expert must update their evaluation improving in each iteration the group consensus index.

C. The complete algorithm and an example

The pseudocode of the proposed algorithm is presented in Algorithm 1.

Let us apply the proposed algorithm to the same example presented in [10]. Let us consider 4 experts whose evaluations are given in $\mathcal{A}_1^{L_8}$:

$$\begin{aligned}
A^1 &= \{0.6/2, 1/3, 0.7/4\}, \\
A^2 &= \{0.6/3, 1/4, 0.8/5, 0.7/6\}, \\
A^3 &= \{0.4/5, 1/6, 0.9/7\}, \\
A^4 &= \{0.6/5, 1/6, 0.8/7, 0.7/8\}.
\end{aligned}$$

The subjective weights vector is $w_s = \{0.3, 0.1, 0.5, 0.1\}$. We consider $\theta = 0.9$ and the following parameter values $\alpha = 0.5$, $\beta = 0.6$ (for the computation of the combination weights vector), $\lambda = 0.5$ and $r = 10$ with the α-levels from 0 to 1, with step of 0.1 (for the computation of GCI).

First of all, we obtain the combination weights vector using Definition 7:

$$w_c = (0.286, 0.149, 0.416, 0.149).$$

Then, in order to compute GCI, we need first to compute

$$\mathcal{F}_{w_c}(A^1, A^2, A^3, A^4) = \{0.6/4, 1/5, 0.9/6, 0.7/7\}.$$

Now, we can compute

$$GCI(A^1, A^2, A^3, A^4) = 0.8372 < 0.9 = \theta.$$

Since the desired group consensus index is not achieved, we identify the expert whose opinion is farthest from $\mathcal{F}_{w_c}(A^1, A^2, A^3, A^4)$ which is expert 1 since:

$$\begin{aligned}
D_{A^1, \mathcal{F}_{w_c}(A^1,A^2,A^3,A^4)} &\approx 0.47, \\
D_{A^2, \mathcal{F}_{w_c}(A^1,A^2,A^3,A^4)} &\approx 0.22, \\
D_{A^3, \mathcal{F}_{w_c}(A^1,A^2,A^3,A^4)} &\approx 0.15, \\
D_{A^4, \mathcal{F}_{w_c}(A^1,A^2,A^3,A^4)} &\approx 0.18.
\end{aligned}$$

Now, this expert must change their opinion with the help of the moderator. Expert 1 chooses the following evaluation:

$$(A^1)' = \{0.1/2, 1/3, 0.6/4, 0.2/5, 0.3/6\},$$

which improves the GCI since:

$$GCI((A^1)', A^2, A^3, A^4) = 0.8464.$$

It is essential to note that it is necessary to update the aggregated value $\mathcal{F}_{w_c}((A^1)', A^2, A^3, A^4)$ to compute $GCI((A^1)', A^2, A^3, A^4)$. Thus, experts 2, 3 and 4 keep their initial evaluations, and since the new GCI is still less than $\theta = 0.9$, more iterations are needed. The algorithm ends when the group consensus index is greater or equal to $\theta = 0.9$.

Remark 16 *In this case, we do not provide the final result of this example (as we did in Section III-D) since the final result depends on the choices made by the experts during the consensus reaching process. Thus, it is not a fully automatic process, which is a good feature because now the experts control how their opinions are modified.*

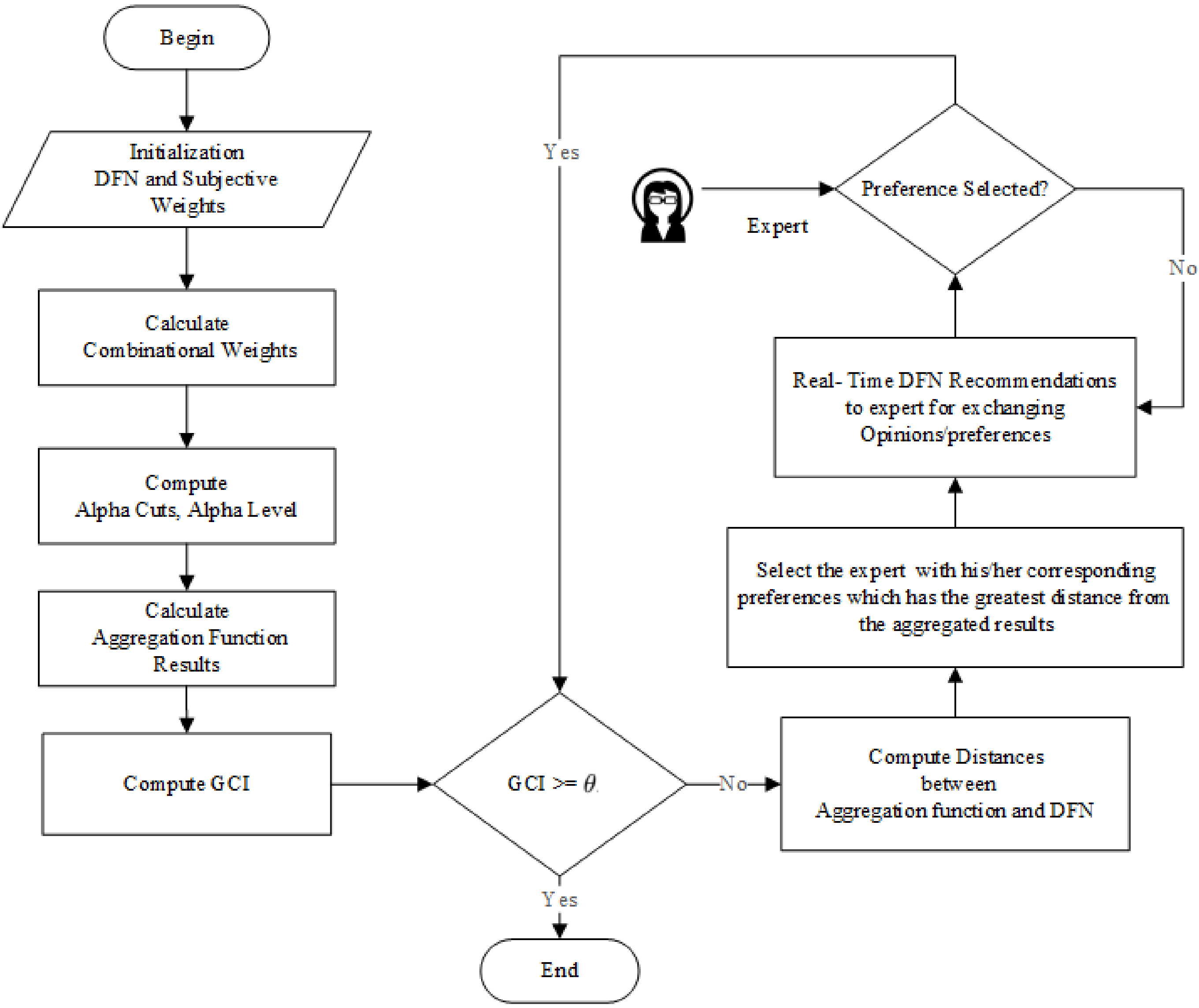

Figure 3: Our proposed approach.

V. Evaluation Comparison of the Algorithms through a Simulation Experiment

To further validate this novel algorithm, a comparative analysis has been performed to study the performance of the two considered algorithms.

Parameter	Value
α	0.5
β	0.6
λ	0.5
r	10

Table I: Simulation parameters.

A. Experimental Setup

In order to compare the algorithms, a simulation experiment has been designed in which a great number of executions of both algorithms are carried out from random evaluations given as discrete fuzzy numbers. We set the parameters shown in Table I.

These parameter values are the ones we have used throughout this work and previously considered in [10]. Moreover, we have considered different values of the following parameters in order to analyse their influence on the simulation results.

- We have considered different numbers of experts such as 4, 6 and 8.
- We have generated 1000 random groups of 4, 6 and 8 discrete random numbers (depending on the number of experts) in $\mathcal{A}_1^{L_n}$ with different values of n such as 4, 6 and 8, to model the evaluations of the experts.
- If one run did not end in 25 iterations of the algorithm, the run was stopped and we considered that the algorithm did not end for that input (although it could have ended with a greater number of iterations).
- We have considered two different values of the desired global consensus index θ. Namely, $\theta = 0.9$ and $\theta = 0.95$.

It is important to note that the algorithm presented in [10] is a fully automatic algorithm and therefore, from some fixed inputs, output is obtained without further external action from the experts. On the contrary, the algorithm presented in the previous section requires the intervention of the experts throughout the whole process since they must modify their evaluations to improve the global consensus index. In order to automatically model these choices, two models are proposed: the random and the probabilistic models. Both of them start similarly. Indeed, whenever an evaluation is selected as the one which is the furthest away from the result of the aggregation, the algorithm generates 50 random discrete fuzzy numbers in $\mathcal{A}_1^{L_n}$. However, they differ on how they choose one of them. Specifically,

- **Random model**: From those improving the GCI when that concrete discrete fuzzy number is considered, one of them is selected randomly (having all the choices the same probability) as the modified opinion of the expert.
- **Probalistic model**: Suppose that B^1, B^2, $\ldots$, B^k with $k \leq 50$ are the discrete fuzzy numbers that improve the GCI and A^l is the opinion that must be modified. Now, the algorithm computes for each B_j with $1 \leq j \leq k$ the following probability:

$$p_j = \frac{\frac{1}{D_{B^j,A^l}}}{\sum_{i=1}^{k} \frac{1}{D_{B^i,A^l}}}$$

These probabilities are considered in order to choose one of them randomly as the modified opinion of the expert. Note that as closer is B^j to A^l, higher is the probability. This tries to catch the idea that it is more probable that the expert modifies his/her evaluation for another evaluation which is not radically different from his/her former evaluation.

In the case that none of the 50 generated random discrete fuzzy numbers improves the GCI, the process is repeated until there is at least one that improves the global consensus index.

B. Analysis of the Results

First of all, it is important to note that some of the executions in the simulation caused some crashes in the algorithm described in [10]. After carefully checking the inputs that caused the problem, we can conclude that the step to improve the global consensus index in [10] do not work for some discrete fuzzy numbers. Let us show it. Let us consider the following discrete fuzzy numbers defined in $\mathcal{A}_1^{L_4}$:

$$\begin{aligned} A_1 &= \{0.6/1, 0.7/1, 1/3, 0.1/4\}, \\ A_2 &= \{1/1, 1/2, 1/3, 0.1/4\}, \\ A_3 &= \{1/0, 1/1, 1/2, 0.4/3\}, \\ A_4 &= \{1/0, 1/1, 0.6/2, 0.3/3\}, \end{aligned}$$

with the following combination vector $w_c = (0.292, 0.134, 0.405, 0.169)$. In this case, the aggregated result is

$$\mathcal{F}_{w_c}(A_1, A_2, A_3, A_4) = \{1/1, 1/2, 0.7/3\}.$$

Now, applying the algorithm to improve the GCI explained in Section III-C, each of these discrete fuzzy numbers go to Case 4 where $T = \emptyset$. Therefore, the algorithm does not modify any of the discrete fuzzy numbers and therefore, it enters an infinite loop. In order to bypass this problem, in these cases, we considered that the algorithm did not finish in 25 iterations.

At this point, let us analyze the results. For each algorithm, we will compute the following two measures:

- **Convergence rate**: Computed as the percentage of executions in which the algorithm achieved the desired GCI with 25 iterations.
- **Average**: Computed as the average number of iterations needed for the algorithm to achieve the desired GCI in those executions where the desired GCI is achieved with 25 iterations.

It is clear that a higher convergence rate and a smaller average indicate a better performance of the algorithm.

Table II presents the results of the case when $\theta = 0.9$ and Table III the ones when $\theta = 0.95$. Note that we compute both performance measures for each pair (n, q) where n is the length of the discrete chain L_n and q is the number of experts. Several conclusions emerge straightforwardly:

- Our proposed algorithm (either with the random or the probabilistic models) achieves greater convergence rate than the algorithm published in [10]. In some cases, it achieves an outstanding 100% of convergence rate (case $\theta = 0.9$) with $n = 8$ and 4 experts) while in this same case, the algorithm in [10] only obtains 53.3%. Comparing the two models of our algorithm, the probabilistic model obtains higher convergence rate than the random model in most of the cases. It should be further analyzed the underlying reason for this fact.
- There are clear differences in the convergence rates for the different values of θ. With $\theta = 0.9$, the GCI is easier to achieve in 25 iterations and therefore, more executions end with this number of steps, while with $\theta = 0.95$ the desired consensus is very demanding and 25 executions are not enough. Note that there are some cases where none of 1000 executions end. It has to be said that by construction, our algorithm would have eventually reached the desired GCI but many iterations would have been needed.
- There is no clear conclusion with respect to the average number of iterations. First of all, note that it is not a good measure to compare the algorithm published in [10] with respect to our algorithm since the average number of iterations is computed in each case with a very different number of executions (those in which the algorithm ends). When comparing the two proposed models for our algorithm, the probabilistic model reaches the consensus with a smaller average number of iterations, but on the contrary, when $\theta = 0.95$ now it is the random model which obtains better results. These differences could be caused by the different convergence rates for the two cases.
- With respect to the length of L_n, as higher is the value of n, higher is the convergence rate. This is unexpected since with a broader range of possibilities for the evaluations of the experts, it could be thought that more difficult is to achieve the consensus. However, note that in the

GCI formula, specifically when computing the deviation, n appears also in the denominator, so higher values of n make the factor $\frac{1}{n(r+1)}$ smaller. Moreover, when improving the consensus, by choosing random discrete fuzzy numbers in the simulation, a larger L_n provides more room for the improvement of GCI. Anyway, further investigations are also needed in this point.
- Finally, with respect to the number of experts, it is clear that when more experts are considered, the desired GCI is more difficult to achieve since the convergence rates for a fixed value of n decrease.

VI. Conclusions

This Master thesis introduces a novel algorithm to reach a consensus on group decision-making problems with a single alternative. The algorithm relies on the interesting features of the recent research work presented in [10]. Namely,

1) The concept of combination weights incorporating the uncertainty of the evaluations as a key aspect to evaluate the expert's knowledge.
2) The concept of Group Consensus Index is defined using the deviation between two discrete fuzzy numbers.

However, we have introduced, from our point of view, two important novelties:

1) A new weighted aggregation function on discrete fuzzy numbers with a solid background and theoretical properties.
2) A novel method to improve the group consensus index allows the experts to participate in the consensus process by updating their evaluations.

From these novelties, the most important one is the above second point. The algorithm presented in [10] works automatically but the experts are only engaged in the initial evaluations, not being able to modify their opinions throughout the consensus reaching process. This is a significant drawback which may lead to the rejection of experts in real life to participate in such procedure.

In order to assess the performance of our algorithm, a simulation experiment has been performed where for several parameter values 1000 executions with random inputs have been run and the results obtained by the algorithm [10] and our algorithm has been analyzed. To run the simulation, we have considered two different models in order to simulate the choices of the experts in order to modify their opinions: the random and the probabilistic models. The results presented in Tables II and III show that our algorithm obtains higher convergence rates than the algorithm in [10]. Among the two models considered for our algorithm, the probabilistic models seem to have a better performance. This fact is a strong asset because in real life the modifications of the experts' opinions are better simulated with the probabilistic model (experts are more keen to choose opinions that are closer to their previous opinions).

Finally, it is worthy to note that the simulation has illustrated some problems in the algorithm presented in [10]. This algorithm does not end for some inputs, entering to an infinite loop in which the GCI value does not improve.

To end this Master thesis, we want to compile the main personal contributions:

- The algorithm presented in [10] is described using a notation with some mathematical flaws which hinder the understanding of the steps by the reader. Therefore, all the steps and formulas have been corrected by using the standard notation in the literature of discrete fuzzy numbers.
- The design of the proposed algorithm and the set-up of the simulation experiment with the two models in order to simulate the modification of the evaluations by the experts.
- Both algorithms have been implemented from scratch. The codes are accessible in the following repository: https://github.com/ines1992/Master_Thesis_Consensus-Model-for-Group-Decision-Making-Problem.
- This Master thesis has derived in a scientific publication at the IEEE International Conference on Fuzzy Systems 2021 (see [1]).

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Length (L_n)	Algorithm [10]		Proposed Algorithm (Probabilistic)		Proposed Algorithm (Random)	
	Convergence (%)	Iterations (Average)	Convergence (%)	Iterations (Average)	Convergence (%)	Iterations (Average)
			No. of Experts = 4			
n=4	45.4	3.39	96.4	3.16	99.7	5.07
n=6	48.4	4.53	99.9	1.75	99.8	2.62
n=8	53.3	5.68	100	1.5	99.9	2.16
			No. of Experts = 6			
n=4	27.3	6.45	87.7	5.73	54.4	10.7
n=6	36.4	8.62	93.8	5.12	81.89	10.1
n=8	43.8	10.53	99.6	2.79	98.1	5.25
			No. of Experts = 8			
n=4	12.9	8.9	42.9	11.5	20.2	11.6
n=6	22.3	13.06	56.3	10.5	5.4	17.7
n=8	26.5	16.13	81.4	8.93	37.8	14.8

Table II: Results of the simulation with $\theta = 0.9$. The average number of iterations is computed only for those runs where the algorithm ends in 25 iterations.

Length (L_n)	Algorithm [10]		Proposed Algorithm (Probabilistic)		Proposed Algorithm (Random)	
	Convergence (%)	Iterations (Average)	Convergence (%)	Iterations (Average)	Convergence (%)	Iterations (Average)
			No. of Experts = 4			
n=4	5.2	4	30.4	12.28	30.5	8.25
n=6	5.4	5.24	24.6	13	53.8	8.66
n=8	5.8	6.48	24.8	13	69.2	9.33
			No. of Experts = 6			
n=4	0.2	8	7.4	15	2.4	6.59
n=6	0.4	10.25	1.9	17	1.2	6.38
n=8	1	12	1.6	14	4.6	5.65
			No. of Experts = 8			
n=4	NA	NA	3	16	NA	NA
n=6	NA	NA	2	14.5	NA	NA
n=8	NA	NA	NA	NA	NA	NA

Table III: Results of the simulation with $\theta = 0.9$. The average number of iterations is computed only for those runs where the algorithm ends in 25 iterations. NA means that no run ended in 25 iterations.

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