TU BERLIN
Martin Henk

Convex Geometry II

WS 2015/16

Exercise sheet 11

Discussion: Thursday, 11.02.2016.

Exercise 11.1 *Let P be a polytope with (symmtric) facet data $\{(\boldsymbol{u}_i, \phi_i), (-\boldsymbol{u}_i, \phi_i) : 1 \leq i \leq m\}$. Show that P is centrally symmetric (w.r.t. some point).*

Exercise 11.2 *Let $K, L \in \mathcal{K}^n$. The set*

$$K \sim L = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} + L \subseteq K\}$$

is called Minkwoski difference *of K and L. Show that*

$$K \sim L = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}, \boldsymbol{x} \rangle \leq h(K, \boldsymbol{u}) - h(L, \boldsymbol{u}) \text{ for all } \boldsymbol{u} \in S^{n-1}\}.$$

Is (in general) $h(K, \boldsymbol{u}) - h(L, \boldsymbol{u})$ the support function of $K \sim L$?

Exercise 11.3 *For $K, B \in \mathcal{K}^n$, let $r(K, B) = \max\{r > 0 : K \sim r\,B \neq \emptyset\}$ be the inradius of K w.r.t. B and we set*

$$K_\rho := \begin{cases} K + \rho B, & \rho \geq 0, \\ K \sim (-\rho)B, & -r(K, B) \leq \rho \leq 0. \end{cases}$$

Show that $(1-\lambda)K_\rho + \lambda K_\sigma \subseteq K_{(1-\lambda)\rho + \lambda\sigma}$.

Exercise 11.4 *Let $K, L \in \mathcal{K}^n$, $\mathbf{0} \in K \cap L$, $p \geq 1$, and let*

$$f(\boldsymbol{u}) := [h(K, \boldsymbol{u})^p + h(L, \boldsymbol{u})^p]^{1/p}.$$

Show that $h(K_f, \boldsymbol{u}) = f(\boldsymbol{u})$ for all $\boldsymbol{u} \in S^{n-1}$ where K_f is the Wulff-shape w.r.t. f.

Exercise sheet 10

Discussion: Thursday, 04.02.2016.

Exercise 10.1 *Let $K \in \mathcal{K}_o^n$, $t > 0$. Show that*

$$\mathrm{N}\,(K, t\,B_n) \leq \mathrm{N}\,(K, 4t\,B_n)\,\mathrm{N}\,(B_n, (t/16)K^\star),$$
$$\mathrm{N}\,(B_n, t\,K) \leq \mathrm{N}\,(B_n, 4t\,K)\,\mathrm{N}\,(K^\star, (t/16)B_n).$$

Hence, e.g., if $B_n \subseteq 4K$ then $\mathrm{N}\,(B_n, K) \leq \mathrm{N}\,(K^\star, \frac{1}{16}B_n)$.

There is a theorem by Artstein-Milman-Szarek saying that there exists absolut constants $\alpha, \beta > 0$ such that for all $K \in \mathcal{K}_o^n$

$$\mathrm{N}\,(B_n, \alpha^{-1}K^\star)^{\frac{1}{\beta}} \leq \mathrm{N}\,(K, B_n) \leq \mathrm{N}\,(B_n, \alpha K^\star)^\beta.$$

Exercise 10.2 *Let $K \in \mathcal{K}_o^n$, $r = (\mathrm{vol}\,(K)/\mathrm{vol}\,(B_n))^{1/n}$ and let $\mathrm{N}\,(K, rB_n) \leq \mathrm{e}^{c\,n}$ for an absolute constant c. Then K is in M-position with constant c.*

Exercise 10.3 *Let $K \in \mathcal{K}^n$ be centered, i.e., the centroid is at the origin and let $K - K$ be in M-position with constant C. Then $K \cap (-K)$ is in M-position with constant $c(C)$ depending only on C.*

Exercise 10.4 *There exists an universal constant $c_1 > 0$ such that for $K \in \mathcal{K}^n$ with $\mathbf{0} \in K$ we have*

$$\mathrm{vol}\,(K)\mathrm{vol}\,(K^\star) \geq c_1^n \mathrm{vol}\,(B_n)^2.$$

Exercise sheet 9

Discussion: Thursday, 21.01.2015.

Exercise 9.1 *Let* $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in \mathbb{R}^n$ *and* $|\cdot|$ *be a norm. Show that*

$$2^{-n} \sum_{\epsilon \in \{-1,1\}^n} \left| \sum_{i=1}^{n} \epsilon_i \boldsymbol{v}_i \right| \geq \max\{|\boldsymbol{v}_1|, \dots, |\boldsymbol{v}_n|\}.$$

Exercise 9.2 *Let* $K, L \in \mathcal{K}^n$. *For* $\boldsymbol{v} \in \operatorname{int}((K-L)/2)$ *show hat*

$$\psi(\boldsymbol{v}) = \operatorname{vol}((-\boldsymbol{v}+K) \cap (\boldsymbol{v}+L))$$

is log-concave.

Exercise 9.3 *Let* $K \in \mathcal{K}_o^n$ *and* $\boldsymbol{t} \in \mathbb{R}^n$. *Show that*

$$\gamma_n(\boldsymbol{t}+K) \geq \mathrm{e}^{-\|\boldsymbol{t}\|^2/2} \gamma_n(K).$$

Exercise 9.4 (Concentration of volume in convex bodies) *Let* $K \in \mathcal{K}^n$, $\operatorname{vol}(K) = 1$, *and* $L \in \mathcal{K}_o^n$ *such that* $\operatorname{vol}(K \cap L) = r > 1/2$. *Then for* $t \geq 1$

$$\operatorname{vol}(K \cap (t\,L)^c) \leq r \left(\frac{1-r}{r}\right)^{(t+1)/2},$$

where $A^c = \mathbb{R}^n \setminus A$ *for a set* $A \subset \mathbb{R}^n$.

First show that

$$\frac{2}{t+1}(t\,L)^c + \frac{t-1}{t+1} L \subseteq L^c.$$

and then ask Brunn or Minkowski or both.

Exercise sheet 8

Discussion: Thursday, 14.01.2015.

Exercise 8.1 *Let* $v \in S^{n-1}$, $t \in [0,1]$ *and let* $U = \{u \in S^{n-1} : |\langle v, u\rangle| \leq t\}$. *Then* $\mu(U) \geq 1 - 4\mathrm{e}^{-nt^2/4}$ *and, in particular, for* $t \geq 4/\sqrt{n}$ *it holds*

$$\mu(U) \geq 0.9.$$

Exercise 8.2 *There exists always a* δ*-net on* S^{n-1} *consisting of at most* $(4/\delta)^n$ *points.*

Exercise 8.3 *For* $K \in \mathcal{K}_o^n$ *let* $M(K) = \int_{S^{n-1}} |\boldsymbol{u}|_K \,\mathrm{d}\mu(\boldsymbol{v})$.

i) *Show that*
$$M(K)\, M(K^\star) \geq 1.$$

ii) *Show that* $M(B_n^p) \geq (\sqrt{2/\pi})\, n^{1/p-1/2}$ *for* $1 \leq p < \infty$.

Rem.: For ii) Hölder and the next inequality might/could help.

Exercise 8.4 *Let* $\kappa_n = \mathrm{vol}\,(B_n) = \pi^{n/2}/\Gamma(n/2+1)$. *Show that*

$$\sqrt{\frac{2\pi}{n}} \geq \frac{\kappa_n}{\kappa_{n-1}} \geq \sqrt{\frac{2\pi}{n+1}}.$$

Show first that $\gamma_n = \frac{\kappa_n}{\kappa_{n-1}}$ is a decresing function in n and consider $\gamma_n\,\gamma_{n-1}$.

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Exercise sheet 7

Discussion: Thursday, 07.01.2016.

Exercise 7.1 *Show that* $\mathcal{K}^n$ *equipped with* $\log \mathrm{d}_{\mathrm{BM}}(\cdot,\cdot)$ *is a compact metric space, and that for* $K, L \in \mathcal{K}^n_o$

$$\mathrm{d}_{\mathrm{BM}}(K, L) = \mathrm{d}_{\mathrm{BM}}(K^\star, L^\star).$$

Exercise 7.2 *Let* $B^p_n = \{\boldsymbol{x} \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \leq 1\}$ *with* $B^\infty_m = [-1, 1]^n$. *Show that for* $1 \leq p \leq q \leq 2$ *or* $2 \leq p \leq q \leq \infty$

$$\mathrm{d}_{\mathrm{BM}}(B^p_n, B^q_n) = n^{1/p-1/q}.$$

The Minkowski-sum of finitely many line segments is called a *zonotope* Z, i.e.,

$$Z = \sum_{i=1}^m \mathrm{conv}\,\{\boldsymbol{v}_i, \boldsymbol{w}_i\}.$$

Exercise 7.3 *Let* Z *be a zonotope. Then there exists a* $\boldsymbol{c} \in \mathbb{R}^n$ *and* $\boldsymbol{u}_i \in \mathbb{R}^n$, $1 \leq i \leq m$, *such that*

$$Z = \boldsymbol{c} + \sum_{i=1}^m \mathrm{conv}\,\{-\boldsymbol{u}_i, \boldsymbol{u}_i\}.$$

For Z *as above show that*

$$\mathrm{vol}\,(Z) = 2^n \sum_{J \subseteq [m], |J|=n} |\det(\boldsymbol{u}_j : j \in J)|.$$

Exercise 7.4

i) *Show that the projection body (see Exercise sheet 4) of a polytope is a zonotope.*

ii) *What is the projection body of the cube* $[-1/2, 1/2]^n$*?*

iii) *What is the projection body of an ellipsoid?*

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Exercise sheet 6

Discussion: Thursday, 17.12.2015.

Exercise 6.1 *Let K, L be 2-dimensional o-symmetric convex bodies. Show that the inequality* $\mathrm{vol}\,(K) \leq \mathrm{vol}\,(L)$ *is a consequence of either of the next two properties*

i) $\mathrm{vol}_1(K \cap \mathrm{lin}\,\{\boldsymbol{u}\}) \leq \mathrm{vol}_1(L \cap \mathrm{lin}\,\{\boldsymbol{u}\})$ *for all* $\boldsymbol{u} \in S^1$,

ii) $\mathrm{vol}_1(K|\mathrm{lin}\,\{\boldsymbol{u}\}^\perp) \leq \mathrm{vol}_1(L|\mathrm{lin}\,\{\boldsymbol{u}\}^\perp)$ *for all* $\boldsymbol{u} \in S^1$.

Is symmetry needed?

Exercise 6.2 *Let $K \in \mathcal{K}^n_c$ and let $c_1, c_2 \in \mathbb{R}_{>0}$ such that for all $\boldsymbol{u} \in S^{n-1}$*

$$c_1 \leq \int_K \langle \boldsymbol{x}, \boldsymbol{u} \rangle^2 \, \mathrm{d}\boldsymbol{x} \leq c_2.$$

Show that $\sqrt{c_1} \leq \mathrm{L}_K \leq \sqrt{c_2}$.

Exercise 6.3 *Let $K \subset \mathbb{R}^n$, $L \subset \mathbb{R}^m$ be two convex bodies in isotropic position, and let*

$$M = \left(\frac{\mathrm{L}_L}{\mathrm{L}_K}\right)^{\frac{m}{n+m}} K \times \left(\frac{\mathrm{L}_K}{\mathrm{L}_L}\right)^{\frac{n}{n+m}} L \subset \mathbb{R}^{n+m}.$$

Show that M is in isotropic position and $\mathrm{L}_{K\times L} = \mathrm{L}_K^{\frac{n}{n+m}} \mathrm{L}_L^{\frac{m}{n+m}}$.

Exercise 6.4 *Let $\boldsymbol{a} \in \mathbb{Z}^n$, $\boldsymbol{a} \neq \boldsymbol{0}$, $\gcd(\boldsymbol{a}) = 1$ and let $S(\boldsymbol{a}) = \{\langle \boldsymbol{a}, \boldsymbol{z} \rangle : \boldsymbol{z} \in \mathbb{N}^n_{\geq 0}\}$. For $\boldsymbol{x} \in \mathbb{R}^n$ let $|\boldsymbol{x}|_0 = \#\{x_i \neq 0 : 1 \leq i \leq n\}$.*

i) *Let $\|\boldsymbol{a}\| < 2^{n-1}$ and $\alpha \in S(\boldsymbol{a})$. Then there exists a $\boldsymbol{z} \in \mathbb{N}^n_{\geq 0}$ with $\langle \boldsymbol{a}, \boldsymbol{z} \rangle = \alpha$ and $|\boldsymbol{z}|_0 < n$.*

ii) *Let $\alpha \in S(\boldsymbol{a})$. Then there exists a $\boldsymbol{z} \in \mathbb{N}^n_{\geq 0}$ with $\langle \boldsymbol{a}, \boldsymbol{z} \rangle = \alpha$ and $|\boldsymbol{z}|_0 \leq c \log_2 |\boldsymbol{a}|_\infty$, where c is an absolute constant and $|\boldsymbol{a}|_\infty$ is the maximum norm.*

Exercise 4.4 could be helpful.

Exercise sheet 5

Discussion: Thursday, 03.12.2015.

For $K, L \in \mathcal{K}^n$ let

$$\mathrm{N}\,(K,L) = \min\{|S| : S \subset \mathbb{R}^n \text{ with } K \subseteq S + L\}$$
$$\overline{\mathrm{N}}\,(K,L) = \min\{|S| : S \subset K \text{ with } K \subseteq S + L\}$$

$\mathrm{N}\,(K,L)$ is called the *covering number* of K by L.
For $K, L \in \mathcal{K}^n$, $L = -L$ let

$$\mathrm{M}\,(K,L) = \max\{|S| : S \subset K \text{ with } |\boldsymbol{x}_i - \boldsymbol{x}_j|_L > 1 \text{ for all } \boldsymbol{x}_i \neq \boldsymbol{x}_j \in S\}.$$

$\mathrm{M}\,(K,L)$ is called the *separation number* of K by L.

Exercise 5.1 *Show that*

i) *for* $K, L, M \in \mathcal{K}^n$, $K \subseteq L$*:* $\mathrm{N}\,(K,M) \leq \mathrm{N}\,(L,M)$, $\mathrm{N}\,(M,L) \leq \mathrm{N}\,(M,K)$, *and* $\overline{\mathrm{N}}\,(M,L) \leq \overline{\mathrm{N}}\,(M,K)$

ii) *for* $K, L \in \mathcal{K}^n$*:* $\overline{\mathrm{N}}\,(K, L-L) \leq \mathrm{N}\,(K,L) \leq \overline{\mathrm{N}}\,(K,L)$.

iii) *for* $K \in \mathcal{K}^n$, $\lambda > 0$*:* $\mathrm{N}\,(K, \lambda\, B_n) = \overline{\mathrm{N}}\,(K, \lambda\, B_n)$.

iv) *for* $K, L, M \in \mathcal{K}^n$*:* $\mathrm{N}\,(K,L) \leq \mathrm{N}\,(K,M)\,\mathrm{N}\,(M,L)$.

Exercise 5.2 *Let* $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{K}^n$ *ellipsoids centered at the origin. Show that*

$$\mathrm{N}\,(\mathcal{E}_1, \mathcal{E}_2) = \mathrm{N}\,({\mathcal{E}_2}^\star, {\mathcal{E}_1}^\star).$$

Exercise 5.3 *Let* $K, L \in \mathcal{K}^n$*. Show that*

$$\overline{\mathrm{N}}\,(K, (K-K) \cap L) = \overline{\mathrm{N}}\,(K,L).$$

Exercise 5.4 *Show that*

$$\mathrm{M}\,(K, 2\,L) \leq \mathrm{N}\,(K,L) \leq \overline{\mathrm{N}}\,(K,L) \leq \mathrm{M}\,(K,L)$$

Exercise 5.5 *Let* $K, L \in \mathcal{K}^n$, $\dim K, \dim L = n$. *Show that*

$$\mathrm{vol}\,(K)/\mathrm{vol}\,(L) \le \mathrm{N}\,(K, L).$$

If $L = -L$ *then*

$$\mathrm{N}\,(K, L) \le 2^n \mathrm{vol}\,(K + L/2)/\mathrm{vol}\,(L).$$

Exercise 5.6 *Let* $K \in \mathcal{K}^n_c$ *be in isotropic position. Then*

$$c_1\,\mathrm{L}_K \le \mathrm{r}(K) \le \mathrm{R}(K) \le c_2\, n\, \mathrm{L}_K,$$

where $c_1, c_2 > 0$ *are absolute constants.*

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Exercise sheet 4

Discussion: Thursday, 26.11.2015.

Exercise 4.1 *Show that among all n-simplices of inradius 1 the regular simplex has minimal volume.*
The inradius is the maximal radius of an n-dimensional ball contained in a body.

Exercise 4.2 *Show that*

$$\prod_{i=1}^{n} x_i^{x_i}$$

attains its minimum on the set $\{\boldsymbol{x} \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^{n} x_i = \alpha\}$, $\alpha > 0$, *if* $x_1 = x_2 = \cdots = x_n$.

Exercise 4.3 *Let $K \in \mathcal{K}^n$. The set*

$$\Pi(K) = \left\{x \in \mathbb{R}^n : |\langle \boldsymbol{u}, \boldsymbol{x} \rangle| \leq \mathrm{vol}_{n-1}(K|\boldsymbol{u}^{\perp}),\ \text{for all } \boldsymbol{u} \in S^{n-1}\right\}$$

is called the projection body *of K. Show that*

i) $h\big(\Pi(K), \boldsymbol{u}\big) = \mathrm{vol}_{n-1}(K|\boldsymbol{u}^{\perp})$.

ii) $\Pi(A\,K) = |\det A|\, A^{-\intercal}\Pi(K)$ *for* $A \in \mathrm{GL}(n, \mathbb{R})$, *and* $\Pi(\boldsymbol{t} + K) = \Pi(K)$ *for* $\boldsymbol{t} \in \mathbb{R}^n$.

Exercise 4.4 (Bombieri&Vaaler) *Let* $\boldsymbol{a} \in \mathbb{Z}^n$, $\boldsymbol{a} \neq \boldsymbol{0}$, $\gcd(\boldsymbol{a}) = 1$. *Show that there exists a* $\boldsymbol{z} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$ *with* $\langle \boldsymbol{a}, \boldsymbol{z} \rangle = 0$ *and* $\max_{1 \leq j \leq n} |z_j| \leq \|\boldsymbol{a}\|^{1/(n-1)}$.

What could help is i) Minkowski's theorem, saying that every o-symmetric convex body with volume not less than $2^n \det \Lambda$ contains a non-trivial lattice point of a lattice Λ, and ii) the set $\{\boldsymbol{z} \in \mathbb{Z}^n : \langle \boldsymbol{a}, \boldsymbol{z} \rangle = 0\}$ is an $(n-1)$-dimensional lattice of determinant $\|\boldsymbol{a}\|$.

Exercise 4.5 (McMullen) *Let L be a k-dimensional linear subspace with orthogonal complement $L^\perp$. Show that*

$$\operatorname{vol}_k(C_n|L) = \operatorname{vol}_{n-k}(C_n|L^\perp).$$

For a zonotope $Z = \sum_{i=1}^m \operatorname{conv}\{\mathbf{0}, \boldsymbol{v}_i\}$ the volume is given by

$$\operatorname{vol}(Z) = \sum_{J\subseteq[m],|J|=n} |\det(\boldsymbol{v}_j : j \in J)|.$$

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Exercise sheet 3

Discussion: Thursday, 12.11.2015!!

A function $f : M \subseteq \mathbb{R}^n \to \mathbb{R}_{>0}$ is called *log-concave* if $\ln f$ is a concave function. It is called *centered* if $\int_M \boldsymbol{x}\, f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \mathbf{0}$.

Exercise 3.1 *Let $K \in \mathcal{K}^n$ be a* centered convex bdoy*, i.e., $\int_K \boldsymbol{x}\,\mathrm{d}\boldsymbol{x} = \mathbf{0}$, and let $L \subseteq \mathbb{R}^n$ be a k-dimensional linear subspace with orthogonal subspace $L^\perp$. For $\boldsymbol{y} \in \operatorname{relint}(K|L)$ let $f(\boldsymbol{y}) = \operatorname{vol}_{n-k}(K \cap (\boldsymbol{y} + L^\perp))$. Show that f is a centered log-concave function.*

Exercise 3.2 *Let $F, G : \mathbb{R}^n \to \mathbb{R}_{>0}$ be integrable log-concave functions. Then, their convolution*

$$(F \star G)(\boldsymbol{x}) = \int_{\mathbb{R}^n} F(\boldsymbol{x} - \boldsymbol{y})\, G(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$

is also a log-concave function.
The Prékopa-Leindler inequality might help.

Exercise 3.3 *Let $K, L \in \mathcal{K}^n$ with $\mathbf{0} \in \operatorname{int} K, \operatorname{int} L$, and let $\lambda \in (0,1)$. Show that*

i)

$$\operatorname{vol}(K^\star) = \frac{1}{n!}\int_{\mathbb{R}^n} \mathrm{e}^{-h(K,\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x}.$$

ii)

$$\ln \operatorname{vol}[\lambda K + (1-\lambda)L]^\star \leq \lambda \ln \operatorname{vol} K^\star + (1-\lambda)\ln \operatorname{vol} L^\star.$$

Here the Hölder inequality might help

Exercise 3.4 *Let $Q \subset \mathbb{R}^{n-1} \times \{0\}$ be a polytope, $\boldsymbol{y} \in \mathbb{R}^n$, $y_n \neq 0$, and $P = \operatorname{conv}\{Q, \boldsymbol{y}\}$ be the pyramid over Q with apex $\boldsymbol{y}$. Let $c(P)$ be the centroid of P, and let $\boldsymbol{q} \in Q$ be the intersection of Q with the ray $\boldsymbol{y} + \lambda(c(P) - \boldsymbol{y})$, $\lambda \geq 0$. Then*

$$|\boldsymbol{y} - c(P)| = n|c(P) - \boldsymbol{q}| = \frac{n}{n+1}|\boldsymbol{y} - \boldsymbol{q}|.$$

Exercise 3.5

i) *Let $P \in \mathcal{K}^n$ be an n-dimensional polytope, and let $Q_i \subset P$, $i \in I$, be polytopes forming a subdivision of P, i.e., $\cup_{i\in I} Q_i = P$ and $\dim(Q_i \cap Q_j) \leq n-1$, $i \neq j$. Find a relation between the centroids $c(Q_i)$ and $c(P)$.*

ii) *Let $S = \operatorname{conv}\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n\}$ be a n-dimensional simplex. Show that $c(P) = \frac{1}{n+1}\sum_{i=0}^{n} \boldsymbol{v}_i$.*

Exercise 3.6 *Let $K \in \mathcal{K}^n$ mit $c(K) = \mathbf{0}$.*

i) *For $\boldsymbol{u} \in \mathbb{R}^n$ let $w(K, \boldsymbol{u}) = h(K, \boldsymbol{u}) + h(K, -\boldsymbol{u})$. Show that*

$$\frac{1}{n+1} w(K, \boldsymbol{u}) \leq h(K, \boldsymbol{u}) \leq \frac{n}{n+1} w(K, \boldsymbol{u}).$$

ii) *Show that $-K \subset n\, K$.*

Exercise sheet 2

Discussion: Thursday, 29.10.2015.

Exercise 2.1 *Let $H(\boldsymbol{a},0)$ be a $\mathbf{0}$ containing hyperplane and let $K \in \mathcal{K}^n$ be symmetric with respect to $H(\boldsymbol{a},0)$. Let $\boldsymbol{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\} \in H(\boldsymbol{a},0)$. Show that $\mathrm{st}_{H(\boldsymbol{v},0)}(K)$ is still symmetric to $H(\boldsymbol{a},0)$.*

Exercise 2.2 *Similar to the first part of the proof of the Rogers-Shephard inequality show that for $K, L \in \mathcal{K}^n$ holds*

$$\int_{\mathbb{R}^n} \mathrm{vol}\,(K \cap (\boldsymbol{x} - L))\,\mathrm{d}\boldsymbol{x} = \mathrm{vol}\,(K)\mathrm{vol}\,(L),$$

and conclude that that there exists a $\boldsymbol{x} \in 2\,K$ such that

$$\mathrm{vol}\,(K \cap (\boldsymbol{x} - K)) \geq 2^{-n}\mathrm{vol}\,(K).$$

Hence every convex body K of $\mathrm{vol}\,(K) = 2^n$ contains a centrally symmetric subset of volume at least 1.

Exercise 2.3 *Let $K \in \mathcal{K}^n$ containing $\mathbf{0}$ in the interior and let $h(K,\cdot)$ be its support function. The function $\rho(K,\cdot) : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}_{\geq 0}$ given by*

$$\rho(K,\boldsymbol{u}) = \max\{\rho \geq 0 : \rho\boldsymbol{u} \in K\}$$

is called radial function *of K. Show that*

$$h(K,\boldsymbol{u})\rho(K^\star,\boldsymbol{u}) = 1.$$

Exercise 2.4 *Let $K \in \mathcal{K}^n$, $K = -K$, $\dim K = n$, and let $L \subset \mathbb{R}^n$ be a k-dimensional linear subspace with orthogonal complement $L^\perp$. Show that*

$$\binom{n}{k}^{-1} \leq \frac{\mathrm{vol}\,(K)}{\mathrm{vol}_{\,k}(K \cap L) \cdot \mathrm{vol}_{\,n-k}(K|L^\perp)} \leq 1.$$

For the lower bound it might be useful to observe that for $\boldsymbol{x} \in (K|L^\perp)$ a suitable translation of $(1 - \rho(K|L^\perp,\boldsymbol{x})^{-1})(K \cap L) + \boldsymbol{x}$ is contained in $K \cap (\boldsymbol{x} + L)$.

Exercise 2.5 *Cauchy's surface are formula states that for $K \in \mathcal{K}^n$*

$$\mathrm{F}(K) = \frac{1}{\mathrm{vol}_{n-1}(B_{n-1})} \int_{S^{n-1}} \mathrm{vol}_{n-1}(K|\boldsymbol{u}^{\perp})\,\mathrm{d}\boldsymbol{u},$$

$K|\boldsymbol{u}^{\perp}$ *denotes he orthogonal projection of* K *onto the hyperplane with normal* $\boldsymbol{u}$. *Use it in order to show*

$$F\left(\frac{1}{2}(K-K)\right) \geq F(K).$$

Exercise sheet 1

Discussion: Thursday, 22.05.2015.

Exercise 1.1 *Let $K, K_i \in \mathcal{K}^n$, $i \in \mathbb{N}$, with $\mathbf{0} \in \operatorname{int} K$. Show that $K_i \to K$ if and only if for all $\epsilon > 0$ there exists an $i_\epsilon \in \mathbb{N}$ such that for all $i \geq i_\epsilon$*

$$(1-\epsilon)\, K_i \subseteq K \subseteq (1+\epsilon) K_i.$$

Exercise 1.2 *Let $K \in \mathcal{K}^n$. Show that the Steiner-symmetral $\mathrm{st}_H(K)$ of K with respect to a hyperplane H is a compact set.*

Exercise 1.3 *Show that the minimal width of a convex boy may descrease or increase under Steiner-symemmetrizations.*

Exercise 1.4 *Let $K \in \mathcal{K}^n$. Show that (without using Exercise 1.5)*

$$\mathrm{vol}\,(K) \leq \left(\frac{\mathrm{D}(K)}{2}\right)^n \mathrm{vol}\,(B_n).$$

Exercise 1.5 *Let $K \in \mathcal{K}^n$. The functional*

$$\mathrm{w}(K) = \frac{1}{n\mathrm{vol}\,(B_n)} \int_{S^{n-1}} [h(K,\boldsymbol{u}) + h(K,-\boldsymbol{u})]\ \mathrm{d}\boldsymbol{u}$$

is called the mean width of K. Here the integration "$\mathrm{d}\boldsymbol{u}$" is meant with respect to the $(n-1)$-dimensional Hausdorff-measure. Let H be a hyperplane. Show that

$$\mathrm{w}(\mathrm{st}_H(K)) \leq \mathrm{w}(K),$$

and conclude "Urysohn's inequality"

$$\mathrm{vol}\,(K) \leq \left(\frac{\mathrm{w}(K)}{2}\right)^n \mathrm{vol}\,(B_n).$$

Rem: The mean width is a continuous functional on $\mathcal{K}^n$.

`one more on the next page...`

Exercise 1.6 *Let T be an n-dimensional simplex, i.e.,* $T = \mathrm{conv}\,\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n\}$, $\boldsymbol{v}_i \in \mathbb{R}^n$, *and* $\dim T = n$.

i) *Let $H_{i,j}$, $i \neq j$, be the hyperplane through $\frac{1}{2}(\boldsymbol{v}_i + \boldsymbol{v}_j)$ and with normal vector $\boldsymbol{v}_i - \boldsymbol{v}_j$ (orthogonal to the edge* $\mathrm{conv}\,\{\boldsymbol{v}_i, \boldsymbol{v}_j\}$*). Show that* $\mathrm{st}_{H_{i,j}}(T)$ *is again an n-dimensional simplex.*

ii) *Show that among all n-dimensional simplices T the ratio* $\mathrm{R}(T)/\mathrm{r}(T)$ *becomes minimal for a regular simplex.* Here it might be helpful to use the fact the surface area of the Steiner symmetral $\mathrm{st}_H(K)$ is strictly decreasing if K is not symmetric to the hypeprlane H.