Geometrische Grundlagen der Linearen Optimierung

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# Chapter 5: Introduction to Linear Programming

# Optimization Problems

### Generic optimization problem

Given: set $X$, function $f : X \to \mathbb{R}$
Task: find $x^* \in X$ maximizing (minimizing) $f(x^*)$, i. e.,

$$f(x^*) \geq f(x) \quad (f(x^*) \leq f(x)) \qquad \text{for all } x \in X.$$

- An $x^*$ with these properties is called optimal solution (optimum).
- Here, $X$ is the set of feasible solutions, $f$ is the objective function.

Short form:

$$\begin{array}{rl} \text{maximize} & f(x) \\ \text{subject to} & x \in X \end{array}$$

or simply: $\max\{f(x) \mid x \in X\}$.

Problem: Too general to say anything meaningful!



# Reminder: Convex Optimization Problems

### Definition. (cp. Def. 1.5, Def. 1.12)

Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}$.

a $X$ is convex if for all $x, y \in X$ and $0 \leq \lambda \leq 1$ it holds that

$$\lambda \cdot x + (1 - \lambda) \cdot y \in X \ .$$

b $f$ is convex if for all $x, y \in X$ and $0 \leq \lambda \leq 1$ with $\lambda \cdot x + (1 - \lambda) \cdot y \in X$ it holds that

$$\lambda \cdot f(x) + (1 - \lambda) \cdot f(y) \geq f(\lambda \cdot x + (1 - \lambda) \cdot y) \ .$$

c If $X$ and $f$ are both convex, then $\min\{f(x) \mid x \in X\}$ is a convex optimization problem.

Note: $f : X \to \mathbb{R}$ is called concave if $-f$ is convex.

## Reminder: Local and Global Optimality

**Definition. (cp. Theorem 1.13)**

Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}$.
$x' \in X$ is a local optimum of the optimization problem $\min\{f(x) \mid x \in X\}$ if there is an $\varepsilon > 0$ such that

$$f(x') \leq f(x) \qquad \text{for all } x \in X \text{ with } \|x' - x\|_2 \leq \varepsilon.$$

**Theorem 1.13.**

For a convex optimization problem, every local optimum is a (global) optimum.



## Optimization Problems Considered in this Course:

$$\begin{array}{ll} \text{maximize} & f(x) \\ \text{subject to} & x \in X \end{array}$$

- $X \subseteq \mathbb{R}^n$ polyhedron, $f$ linear function
  $\longrightarrow$ linear optimization problem (in particular convex)
- $X \subseteq \mathbb{Z}^n$ integer points of a polyhedron, $f$ linear function
  $\longrightarrow$ integer linear optimization problem
- $X$ related to some combinatorial structure (e. g., graph)
  $\longrightarrow$ combinatorial optimization problem
- $X$ finite (but usually huge)
  $\longrightarrow$ discrete optimization problem

# Example: Shortest Path Problem

Given: directed graph $D = (V, A)$, weight function $w : A \to \mathbb{R}_{\geq 0}$,
start node $s \in V$, destination node $t \in V$.

Task: find $s$-$t$-path of minimum weight in $D$.

That is, $X = \{P \subseteq A \mid P \text{ is } s\text{-}t\text{-path in } D\}$ and $f : X \to \mathbb{R}$ is given by

$$f(P) = \sum_{a \in P} w(a) \ .$$

**Remark.**
Note that the finite set of feasible solutions $X$ is only implicitly given by $D$.
This holds for all interesting problems in combinatorial optimization!



# Example: Minimum Spanning Tree (MST) Problem

Given: undirected graph $G = (V, E)$, weight function $w : E \to \mathbb{R}_{\geq 0}$.

Task: find connected subgraph of $G$ containing all nodes in $V$ with minimum total weight.

That is, $X = \{E' \subseteq E \mid E' \text{ connects all nodes in } V\}$ and $f : X \to \mathbb{R}$ is given by

$$f(E') = \sum_{e \in E'} w(e) \ .$$

Remarks.

- Notice that there always exists an optimal solution without cycles.
- A connected graph without cycles is called a tree.
- A subgraph of $G$ containing all nodes in $V$ is called spanning.

# Example: Minimum Cost Flow Problem

Given: directed graph $D = (V, A)$, with arc *capacities* $u : A \to \mathbb{R}_{\geq 0}$, arc costs $c : A \to \mathbb{R}$, and node *balances* $b : V \to \mathbb{R}$.

Interpretation:

- nodes $v \in V$ with $b(v) > 0$ ($b(v) < 0$) have *supply* (*demand*) and are called *sources* (*sinks*)
- the capacity $u(a)$ of arc $a \in A$ limits the amount of flow that can be sent through arc $a$.

Task: find a *flow* $x : A \to \mathbb{R}_{\geq 0}$ obeying capacities and satisfying all supplies and demands, that is,

$$
\begin{aligned}
0 \leq x(a) \leq u(a) & \qquad \text{for all } a \in A, \\
\sum_{a \in \delta^+(v)} x(a) - \sum_{a \in \delta^-(v)} x(a) = b(v) & \qquad \text{for all } v \in V,
\end{aligned}
$$

such that $x$ has minimum cost $c(x) := \sum_{a \in A} c(a) \cdot x(a)$.



# Example: Minimum Cost Flow Problem (cont.)

Formulation as a linear program (LP):

$$
\begin{aligned}
\text{minimize} \quad & \sum_{a \in A} c(a) \cdot x(a) & & \qquad (5.1) \\
\text{subject to} \quad & \sum_{a \in \delta^+(v)} x(a) - \sum_{a \in \delta^-(v)} x(a) = b(v) & \text{for all } v \in V, & \qquad (5.2) \\
& x(a) \leq u(a) & \text{for all } a \in A, & \qquad (5.3) \\
& x(a) \geq 0 & \text{for all } a \in A. & \qquad (5.4)
\end{aligned}
$$

- Objective function given by (5.1). Set of feasible solutions:

$$X = \{x \in \mathbb{R}^A \mid x \text{ satisfies (5.2), (5.3), and (5.4)}\} \ .$$

- Notice that (5.1) is a linear function of $x$ and (5.2) – (5.4) are linear equations and linear inequalities, respectively. $\longrightarrow$ linear program

# Example (cont.): Adding Fixed Cost

Fixed costs $w : A \to \mathbb{R}_{\geq 0}$.

If arc $a \in A$ shall be used (i. e., $x(a) > 0$), it must be bought at cost $w(a)$.

Add variables $y(a) \in \{0,1\}$ with $y(a) = 1$ if arc $a$ is used, 0 otherwise.

This leads to the following mixed-integer linear program (MIP):

$$
\begin{array}{rlr}
\text{minimize} & \displaystyle\sum_{a \in A} c(a) \cdot x(a) + \sum_{a \in A} w(a) \cdot y(a) & \\
\text{subject to} & \displaystyle\sum_{a \in \delta^+(v)} x(a) - \sum_{a \in \delta^-(v)} x(a) = b(v) & \text{for all } v \in V, \\
& x(a) \leq u(a) \cdot y(a) & \text{for all } a \in A, \\
& x(a) \geq 0 & \text{for all } a \in A. \\
& y(a) \in \{0,1\} & \text{for all } a \in A.
\end{array}
$$

MIP: Linear program where some variables may only take integer values.



# Example: Maximum Weighted Matching Problem

Given: undirected graph $G = (V, E)$, weight function $w : E \to \mathbb{R}$.

Task: find matching $M \subseteq E$ with maximum total weight.

($M \subseteq E$ is a matching if every node is incident to at most one edge in $M$.)

Formulation as an integer linear program (IP):

Variables: $x_e \in \{0,1\}$ for $e \in E$ with $x_e = 1$ if and only if $e \in M$.

$$
\begin{array}{rlr}
\text{maximize} & \displaystyle\sum_{e \in E} w(e) \cdot x_e & \\
\text{subject to} & \displaystyle\sum_{e \in \delta(v)} x_e \leq 1 & \text{for all } v \in V, \\
& x_e \in \{0,1\} & \text{for all } e \in E.
\end{array}
$$

IP: Linear program where all variables may only take integer values.

# Example: Traveling Salesperson Problem (TSP)

Given: complete graph $K_n$ on $n$ nodes, weight function $w : E(K_n) \to \mathbb{R}$.

Task: find a Hamiltonian circuit with minimum total weight.

(A Hamiltonian circuit visits every node exactly once.)

Formulation as an integer linear program? (later!)



# Typical Questions

For a given optimization problem:

- How to find an optimal solution?
- How to find a feasible solution?
- Does there exist an optimal/feasible solution?
- How to prove that a computed solution is optimal?
- How difficult is the problem?
- Does there exist an *efficient algorithm* with "small" worst-case running time?
- How to formulate the problem as a (mixed integer) linear program?
- Is there a useful special structure of the problem?

## Outline of the Remainder of this Course

- linear programming and the simplex algorithm
- geometric interpretation of the simplex algorithm
- LP duality, complementary slackness
- maybe: efficient algorithms for maximum flows and minimum cost flows
- ...



## Literature on Linear Optimization (not complete)


- D. Bertsimas, J. N. Tsitsiklis, *Introduction to Linear Optimization*, Athena, 1997.
- V. Chvatal, *Linear Programming*, Freeman, 1983.
- G. B. Dantzig, *Linear Programming and Extensions*, Princeton University Press, 1998 (1963).
- M. Grötschel, L. Lovàsz, A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*. Springer, 1988.
- J. Matousek, B. Gärtner, *Using and Understanding Linear Programming*, Springer, 2006.
- M. Padberg, *Linear Optimization and Extensions*, Springer, 1995.
- A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, 1986.
- R. J. Vanderbei, *Linear Programming*, Springer, 2001.

# Chapter 6:
# Linear Programming Basics

(cp. Bertsimas & Tsitsiklis, Chapter 1)

## Example of a Linear Program

$$
\begin{array}{rrcrcrcrl}
\text{minimize} & 2x_1 & - & x_2 & + & 4x_3 & & & \\
\text{subject to} & x_1 & + & x_2 & & & + & x_4 & \leq 2 \\
& & & 3x_2 & - & x_3 & & & = 5 \\
& & & & & x_3 & + & x_4 & \geq 3 \\
& x_1 & & & & & & & \geq 0 \\
& & & & & x_3 & & & \leq 0
\end{array}
$$

Remarks.

- objective function is linear in vector of variables $x = (x_1, x_2, x_3, x_4)^T$
- constraints are linear inequalities and linear equations
- last two constraints are special
  (non-negativity and non-positivity constraint, respectively)

# General Linear Program

$$
\begin{aligned}
\text{minimize} \quad & c^T \cdot x & & \\
\text{subject to} \quad & {a_i}^T \cdot x \geq b_i & \text{for } i \in M_1, & \quad (6.1)\\
& {a_i}^T \cdot x = b_i & \text{for } i \in M_2, & \quad (6.2)\\
& {a_i}^T \cdot x \leq b_i & \text{for } i \in M_3, & \quad (6.3)\\
& x_j \geq 0 & \text{for } j \in N_1, & \quad (6.4)\\
& x_j \leq 0 & \text{for } j \in N_2, & \quad (6.5)
\end{aligned}
$$

with $c \in \mathbb{R}^n$, $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for $i \in M_1 \dot\cup M_2 \dot\cup M_3$ (finite index sets), and $N_1, N_2 \subseteq \{1, \dots, n\}$ given.

- $x \in \mathbb{R}^n$ satisfying constraints (6.1) – (6.5) is a feasible solution;
- set of feasible solutions $\{x \in \mathbb{R}^n \mid (6.1) - (6.5)\}$ is polyhedron in $\mathbb{R}^n$;
- feasible solution $x^*$ is optimal solution if

$$c^T \cdot x^* \leq c^T \cdot x \qquad \text{for all feasible solutions } x;$$

- linear program is unbounded if, for all $k \in \mathbb{R}$, there is a feasible solution $x \in \mathbb{R}^n$ with $c^T \cdot x \leq k$.



# Special Forms of Linear Programs

- maximizing $c^T \cdot x$ is equivalent to minimizing $(-c)^T \cdot x$.
- any linear program can be written in the form

$$
\begin{aligned}
\text{minimize} \quad & c^T \cdot x\\
\text{subject to} \quad & A \cdot x \geq b
\end{aligned}
$$

  for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$:

  - rewrite ${a_i}^T \cdot x = b_i$ as: ${a_i}^T \cdot x \geq b_i \;\wedge\; {a_i}^T \cdot x \leq b_i$,
  - rewrite ${a_i}^T \cdot x \leq b_i$ as: $(-a_i)^T \cdot x \geq -b_i$.

- Linear program in standard form:

$$
\begin{aligned}
\min \quad & c^T \cdot x\\
\text{s.t.} \quad & A \cdot x = b\\
& x \geq 0
\end{aligned}
$$

  with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

# Example: Diet Problem

Given:

- $n$ different foods, $m$ different nutrients
- $a_{ij}$ := amount of nutrient $i$ in one unit of food $j$
- $b_i$ := requirement of nutrient $i$ in some ideal diet
- $c_j$ := cost of one unit of food $j$

Task: find a cheapest ideal diet consisting of foods $1, \ldots, n$.

LP formulation: Let $x_j$ := number of units of food $j$ in the diet:

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x \geq b \\ & x \geq 0 \end{aligned}$$

with $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, $b = (b_i) \in \mathbb{R}^m$, $c = (c_j) \in \mathbb{R}^n$.



# Reduction to Standard Form

Any linear program can be brought into standard form:

- elimination of free (unbounded) variables $x_j$:

  replace $x_j$ with $x_j^+, x_j^- \geq 0$: $\quad x_j = x_j^+ - x_j^-$

- elimination of non-positive variables $x_j$:

  replace $x_j \leq 0$ with $(-x_j) \geq 0$.

- elimination of inequality constraint $a_i{}^T \cdot x \leq b_i$:

  introduce slack variable $s \geq 0$ and rewrite: $a_i{}^T \cdot x + s = b_i$

- elimination of inequality constraint $a_i{}^T \cdot x \geq b_i$:

  introduce slack variable $s \geq 0$ and rewrite: $a_i{}^T \cdot x - s = b_i$

## Example

The linear program

$$\begin{array}{lrcrl} \min & 2\,x_1 & + & 4\,x_2 & \\ \text{s.t.} & x_1 & + & x_2 & \geq 3 \\ & 3\,x_1 & + & 2\,x_2 & = 14 \\ & x_1 & & & \geq 0 \end{array}$$

is equivalent to the standard form problem

$$\begin{array}{lrcrcrcrl} \min & 2\,x_1 & + & 4\,x_2^+ & - & 4\,x_2^- & & & \\ \text{s.t.} & x_1 & + & x_2^+ & - & x_2^- & - & x_3 & = 3 \\ & 3\,x_1 & + & 2\,x_2^+ & - & 2\,x_2^- & & & = 14 \\ & & & & & & & x_1,\, x_2^+,\, x_2^-,\, x_3 & \geq 0 \end{array}$$



## Affine Linear and Convex Functions

**Lemma 6.1.**

a. An affine linear function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = c^T \cdot x + d$ with $c \in \mathbb{R}^n$, $d \in \mathbb{R}$, is both convex and concave.
b. If $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ are convex functions, then $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) := \max_{i=1,\ldots,k} f_i(x)$ is also convex.

Proof: . . . $\square$

## Piecewise Linear Convex Objective Functions

Let $c_1, \ldots, c_k \in \mathbb{R}^n$ and $d_1, \ldots, d_k \in \mathbb{R}$.

Consider piecewise linear convex function: $x \mapsto \max_{i=1,\ldots,k} {c_i}^T \cdot x + d_i$:

$$\begin{array}{ll} \min & \max\limits_{i=1,\ldots,k} {c_i}^T \cdot x + d_i \\ \text{s.t.} & A \cdot x \geq b \end{array} \quad \longleftrightarrow \quad \begin{array}{lll} \min & z & \\ \text{s.t.} & z \geq {c_i}^T \cdot x + d_i & \text{for all } i \\ & A \cdot x \geq b & \end{array}$$

Example: let $c_1, \ldots, c_n \geq 0$

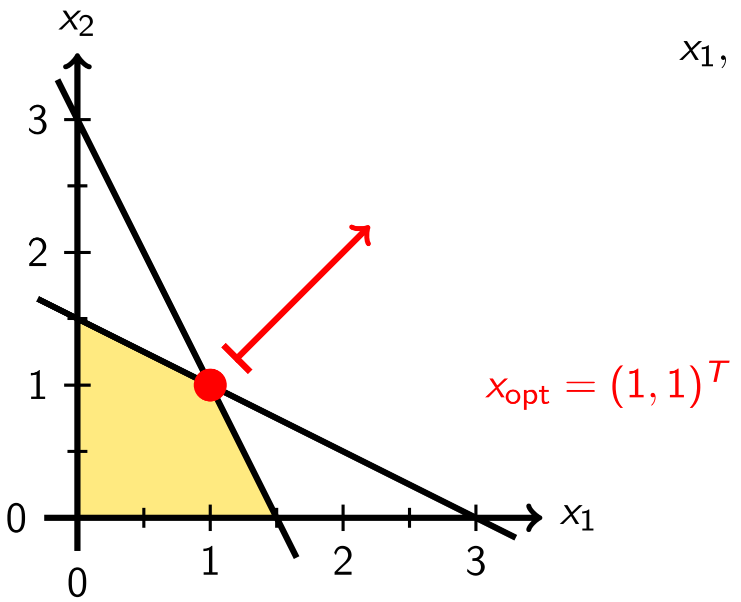
$$\begin{array}{ll} \min & \sum\limits_{i=1}^{n} c_i \cdot |x_i| \\ \text{s.t.} & A \cdot x \geq b \end{array} \quad \leftrightarrow \quad \begin{array}{ll} \min & \sum\limits_{i=1}^{n} c_i \cdot z_i \\ \text{s.t.} & z_i \geq x_i \\ & z_i \geq -x_i \\ & A \cdot x \geq b \end{array} \quad \leftrightarrow \quad \begin{array}{ll} \min & \sum\limits_{i=1}^{n} c_i \cdot (x_i^+ + x_i^-) \\ \text{s.t.} & A \cdot (x^+ - x^-) \geq b \\ & x^+, x^- \geq 0 \end{array}$$



## Graphical Representation and Solution

2D example:

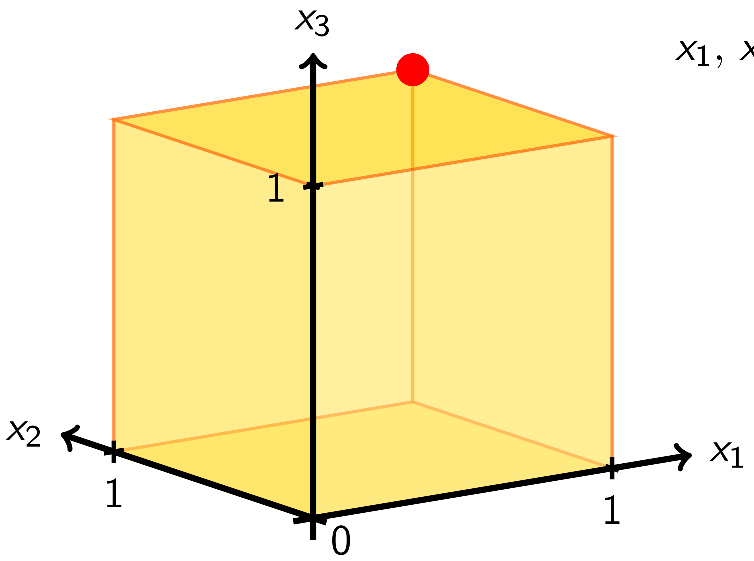
$$\begin{array}{lrcrl} \min & -x_1 & - & x_2 & \\ \text{s.t.} & x_1 & + & 2\,x_2 & \leq 3 \\ & 2\,x_1 & + & x_2 & \leq 3 \\ & & & x_1,\, x_2 & \geq 0 \end{array}$$



## Graphical Representation and Solution (cont.)

3D example:

$$\begin{array}{rrcrcrl} \text{min} & -x_1 & - & x_2 & - & x_3 & \\ \text{s.t.} & x_1 & & & & & \leq 1 \\ & & & x_2 & & & \leq 1 \\ & & & & & x_3 & \leq 1 \\ & & & & & x_1,\, x_2,\, x_3 & \geq 0 \end{array}$$




$x_{\text{opt}} = (1, 1, 1)^T$



## Graphical Representation and Solution (cont.)

another 2D example:

$$\begin{array}{rrcrl} \text{min} & c_1\, x_1 & + & c_2\, x_2 & \\ \text{s.t.} & -x_1 & + & x_2 & \leq 1 \\ & & & x_1,\, x_2 & \geq 0 \end{array}$$

- for $c = (1, 1)^T$, the unique optimal solution is $x = (0, 0)^T$
- for $c = (1, 0)^T$, the optimal solutions are exactly the points
  $$x = (0, x_2)^T \qquad \text{with } 0 \leq x_2 \leq 1$$
- for $c = (0, 1)^T$, the optimal solutions are exactly the points
  $$x = (x_1, 0)^T \qquad \text{with } x_1 \geq 0$$
- for $c = (-1, -1)^T$, the problem is unbounded, optimal cost is $-\infty$
- if we add the constraint $x_1 + x_2 \leq -1$, the problem is infeasible

## Properties of the Set of Optimal Solutions

In the last example, the following 5 cases occurred:

i there is a unique optimal solution

ii there exist infinitely many optimal solutions, but the set of optimal solutions is bounded

iii there exist infinitely many optimal solutions and the set of optimal solutions is unbounded

iv the problem is unbounded, i. e., the optimal cost is $-\infty$ and no feasible solution is optimal

v the problem is infeasible, i. e., the set of feasible solutions is empty

These are indeed all cases that can occur in general (see also later).

(Notice that the set of optimal solutions is a face of the polyhedron given by the set of feasible solutions.)



## Visualizing LPs in Standard Form

Example:
Let $A = (1, 1, 1) \in \mathbb{R}^{1\times 3}$, $b = (1) \in \mathbb{R}^1$ and consider the set of feasible solutions

$$P = \{x \in \mathbb{R}^3 \mid A \cdot x = b,\ x \geq 0\} \ .$$

More general:

- if $A \in \mathbb{R}^{m\times n}$ with $m \leq n$ and the rows of $A$ are linearly independent, then

$$\{x \in \mathbb{R}^n \mid A \cdot x = b\}$$

  is an $(n - m)$-dimensional affine subspace of $\mathbb{R}^n$.
- set of feasible solutions lies in this affine subspace and is only constrained by non-negativity constraints $x \geq 0$.

## Reminder: Extreme Points and Vertices of Polyhedra

**Definition. (cp. Notation 3.6, Remark 3.7)**

Let $P \subseteq \mathbb{R}^n$ be a polyhedron.

a. $x \in P$ is an extreme point of $P$ if

$$x \neq \lambda \cdot y + (1 - \lambda) \cdot z \qquad \text{for all } y, z \in P \setminus \{x\},\ 0 \leq \lambda \leq 1,$$

i. e., $x$ is not a convex combination of two other points in $P$.

b. $x \in P$ is a vertex of $P$ if there is some $c \in \mathbb{R}^n$ such that

$$c^T \cdot x < c^T \cdot y \qquad \text{for all } y \in P \setminus \{x\},$$

i. e., $x$ is the unique optimal solution to the LP $\min\{c^T \cdot z \mid z \in P\}$.



## Active and Binding Constraints

In the following, let $P \subseteq \mathbb{R}^n$ be a polyhedron defined by

$$\begin{aligned} {a_i}^T \cdot x &\geq b_i &\qquad \text{for } i \in M_1,\\ {a_i}^T \cdot x &= b_i &\qquad \text{for } i \in M_2, \end{aligned}$$

with $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, for all $i$.

**Definition 6.2.**

If $x^* \in \mathbb{R}^n$ satisfies ${a_i}^T \cdot x^* = b_i$ for some $i$, then the corresponding constraint is active (or binding) at $x^*$.

# Basic Facts from Linear Algebra

**Theorem 6.3.**

Let $x^* \in \mathbb{R}^n$ and $I = \{i \mid {a_i}^T \cdot x^* = b_i\}$. The following are equivalent:

i. there are $n$ vectors in $\{a_i \mid i \in I\}$ which are linearly independent;
ii. the vectors in $\{a_i \mid i \in I\}$ span $\mathbb{R}^n$;
iii. $x^*$ is the unique solution to the system of equations ${a_i}^T \cdot x = b_i$, $i \in I$.



# Vertices, Extreme Points, and Basic Feasible Solutions

**Definition 6.4.**

a. $x^* \in \mathbb{R}^n$ is a basic solution of $P$ if
   - all equality constraints are active and
   - there are $n$ linearly independent constraints that are active.
b. A basic solution satisfying all constraints is a basic feasible solution.

**Theorem 6.5 (cp. Remark 3.7).**

For $x^* \in P$, the following are equivalent:

i. $x^*$ is a vertex of $P$;
ii. $x^*$ is an extreme point of $P$;
iii. $x^*$ is a basic feasible solution of $P$.

Proof: . . . $\square$

# Reminder: Number of Vertices

**Corollary.**

a) A polyhedron has a finite number of vertices and basic solutions.

b) For a polyhedron in $\mathbb{R}^n$ given by linear equations and $m$ linear inequalities, this number is at most $\binom{m}{n}$.

Example:

$P := \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1,\ i = 1, \ldots, n\}$ ($n$-dimensional unit cube)

- number of constraints: $m = 2n$
- number of vertices: $2^n$



# Adjacent Basic Solutions and Edges

**Definition 6.6 (cp. Notation 3.6).**

Let $P \subseteq \mathbb{R}^n$ be a polyhedron.

a) Two distinct basic solutions are adjacent if there are $n-1$ linearly independent constraints that are active at both of them.

b) If both solutions are feasible, the line segment that joins them is an edge of $P$.

# Polyhedra in Standard Form

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $P = \{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\}$ a polyhedron in standard form representation.

Observation.

One can assume without loss of generality that $\text{rank}(A) = m$.

**Theorem 6.7.**

$x \in \mathbb{R}^n$ is a basic solution of $P$ if and only if $A \cdot x = b$ and there are indices $B(1), \dots, B(m) \in \{1, \dots, n\}$ such that

- columns $A_{B(1)}, \dots, A_{B(m)}$ of matrix $A$ are linearly independent and
- $x_i = 0$ for all $i \notin \{B(1), \dots, B(m)\}$.

Proof: ... □

- $x_{B(1)}, \dots, x_{B(m)}$ are basic variables, the remaining variables non-basic.
- The vector of basic variables is denoted by $x_B := (x_{B(1)}, \dots, x_{B(m)})^T$.
- $A_{B(1)}, \dots, A_{B(m)}$ are basic columns of $A$ and form a basis of $\mathbb{R}^m$.
- The matrix $B := (A_{B(1)}, \dots, A_{B(m)}) \in \mathbb{R}^{m \times m}$ is called basis matrix.



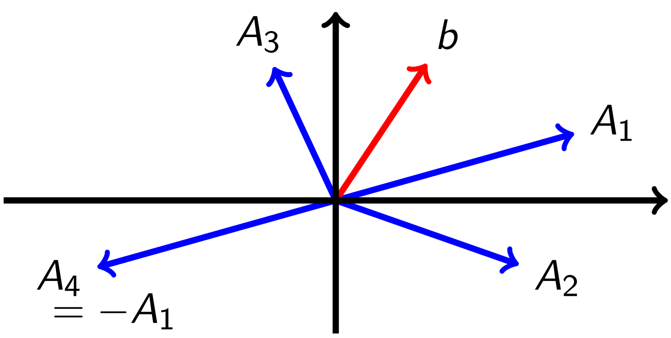
# Basic Columns and Basic Solutions

**Observation 6.8.**

Let $x \in \mathbb{R}^n$ be a basic solution, then:

- $B \cdot x_B = b$ and thus $x_B = B^{-1} \cdot b$;
- $x$ is a basic feasible solution if and only if $x_B = B^{-1} \cdot b \geq 0$.

Example: $m = 2$




- $A_1, A_3$ or $A_2, A_3$ form bases with corresp. basic feasible solutions.
- $A_1, A_4$ do not form a basis.
- $A_1, A_2$ and $A_2, A_4$ and $A_3, A_4$ form bases with infeasible basic solution.

## Bases and Basic Solutions

**Corollary 6.9.**

- Every basis $A_{B(1)}, \ldots, A_{B(m)}$ determines a unique basic solution.
- Thus, different basic solutions correspond to different bases.
- But: two different bases might yield the same basic solution.

Example: If $b = 0$, then $x = 0$ is the only basic solution.



## Adjacent Bases

**Definition 6.10.**

Two bases $A_{B(1)}, \ldots, A_{B(m)}$ and $A_{B'(1)}, \ldots, A_{B'(m)}$ are adjacent if they share all but one column.

**Observation 6.11.**

a. Two adjacent basic solutions can always be obtained from two adjacent bases.
b. If two adjacent bases lead to distinct basic solutions, then the latter are adjacent.

## Degeneracy

**Definition 6.12 (cp. Def. 3.21 of simple polytope).**
A basic solution $x$ of a polyhedron $P$ is degenerate if more than $n$ constraints are active at $x$.

**Observation 6.13.**
Let $P = \{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\}$ be a polyhedron in standard form with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

a. A basic solution $x \in P$ is degenerate if and only if more than $n - m$ components of $x$ are zero.
b. For a non-degenerate basic solution $x \in P$, there is a unique basis.



## Three Different Reasons for Degeneracy

i. redundant variables
Example:
$$\begin{array}{rcrcr} x_1 & + & x_2 & & = 1 \\ & & & x_3 & = 0 \\ & & x_1,\ x_2,\ x_3 & & \geq 0 \end{array} \quad \longleftrightarrow \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ii. redundant constraints
Example:
$$\begin{array}{rcrl} x_1 & + & 2\,x_2 & \leq 3 \\ 2\,x_1 & + & x_2 & \leq 3 \\ x_1 & + & x_2 & \leq 2 \\ & & x_1,\ x_2 & \geq 0 \end{array}$$

iii. geometric reasons (non-simple polyhedra)
Example: Octahedron

**Observation 6.14 (cp. Proof of Lemma 3.26).**
Perturbing the right hand side vector $b$ may remove degeneracy.

## Existence of Extreme Points

### Definition 6.15 (cp. Proof of Theorem 2.11).

A polyhedron $P \subseteq \mathbb{R}^n$ contains a line if there is $x \in P$ and a direction $d \in \mathbb{R}^n \setminus \{0\}$ such that

$$x + \lambda \cdot d \in P \qquad \text{for all } \lambda \in \mathbb{R}.$$

### Theorem 6.16.

Let $P = \{x \in \mathbb{R}^n \mid A \cdot x \geq b\} \neq \emptyset$ with $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following are equivalent:

i. There exists an extreme point $x \in P$.
ii. $P$ does not contain a line.
iii. $A$ contains $n$ linearly independent rows.

Proof: ... □



## Existence of Extreme Points (cont.)

### Corollary 6.17.

a. A non-empty polytope contains an extreme point.
b. A non-empty polyhedron in standard form contains an extreme point.

Proof of b:

$$\begin{aligned} A \cdot x &= b \\ x &\geq 0 \end{aligned} \qquad \longleftrightarrow \qquad \begin{pmatrix} A \\ -A \\ I \end{pmatrix} \cdot x \geq \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}$$

□

Example:

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{rcrl} x_1 & + & x_2 & \geq 1 \\ x_1 & + & 2\,x_2 & \geq 0 \end{array} \right\}$$

contains a line since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in P \quad \text{for all } \lambda \in \mathbb{R}.$

# Optimality of Extreme Points

**Theorem 6.18.**

Let $P \subseteq \mathbb{R}^n$ a polyhedron and $c \in \mathbb{R}^n$. If $P$ has an extreme point and $\min\{c^T \cdot x \mid x \in P\}$ is bounded, there is an extreme point that is optimal.

Proof: . . . $\square$

**Corollary 6.19.**

Every linear programming problem is either infeasible or unbounded or there exists an optimal solution.

Proof: Every linear program is equivalent to an LP in standard form. The claim thus follows from Corollary 6.17 and Theorem 6.18. $\square$