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Martin Henk

Convex Geometry II

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## Exercise sheet 11

Discussion: Thursday, 11.02.2016.

**Exercise 11.1** Let  $P$  be a polytope with (symmetric) facet data  $\{(\mathbf{u}_i, \phi_i), (-\mathbf{u}_i, \phi_i) : 1 \leq i \leq m\}$ . Show that  $P$  is centrally symmetric (w.r.t. some point).

**Exercise 11.2** Let  $K, L \in \mathcal{K}^n$ . The set

$$K \sim L = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} + L \subseteq K\}$$

is called Minkowski difference of  $K$  and  $L$ . Show that

$$K \sim L = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{u}, \mathbf{x} \rangle \leq h(K, \mathbf{u}) - h(L, \mathbf{u}) \text{ for all } \mathbf{u} \in S^{n-1}\}.$$

Is (in general)  $h(K, \mathbf{u}) - h(L, \mathbf{u})$  the support function of  $K \sim L$ ?

**Exercise 11.3** For  $K, B \in \mathcal{K}^n$ , let  $r(K, B) = \max\{r > 0 : K \sim rB \neq \emptyset\}$  be the inradius of  $K$  w.r.t.  $B$  and we set

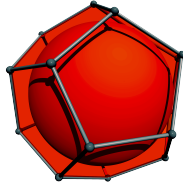
$$K_\rho := \begin{cases} K + \rho B, & \rho \geq 0, \\ K \sim (-\rho)B, & -r(K, B) \leq \rho \leq 0. \end{cases}$$

Show that  $(1 - \lambda)K_\rho + \lambda K_\sigma \subseteq K_{(1-\lambda)\rho + \lambda\sigma}$ .

**Exercise 11.4** Let  $K, L \in \mathcal{K}^n$ ,  $\mathbf{0} \in K \cap L$ ,  $p \geq 1$ , and let

$$f(\mathbf{u}) := [h(K, \mathbf{u})^p + h(L, \mathbf{u})^p]^{1/p}.$$

Show that  $h(K_f, \mathbf{u}) = f(\mathbf{u})$  for all  $\mathbf{u} \in S^{n-1}$  where  $K_f$  is the Wulff-shape w.r.t.  $f$ .



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## Exercise sheet 10

Discussion: Thursday, 04.02.2016.

**Exercise 10.1** Let  $K \in \mathcal{K}_o^n$ ,  $t > 0$ . Show that

$$\begin{aligned} N(K, tB_n) &\leq N(K, 4tB_n) N(B_n, (t/16)K^\star), \\ N(B_n, tK) &\leq N(B_n, 4tK) N(K^\star, (t/16)B_n). \end{aligned}$$

Hence, e.g., if  $B_n \subseteq 4K$  then  $N(B_n, K) \leq N(K^\star, \frac{1}{16}B_n)$ .

There is a theorem by Artstein-Milman-Szarek saying that there exists absolute constants  $\alpha, \beta > 0$  such that for all  $K \in \mathcal{K}_o^n$

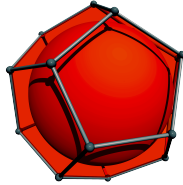
$$N(B_n, \alpha^{-1}K^\star)^{\frac{1}{\beta}} \leq N(K, B_n) \leq N(B_n, \alpha K^\star)^\beta.$$

**Exercise 10.2** Let  $K \in \mathcal{K}_o^n$ ,  $r = (\text{vol}(K)/\text{vol}(B_n))^{1/n}$  and let  $N(K, rB_n) \leq e^{cn}$  for an absolute constant  $c$ . Then  $K$  is in  $M$ -position with constant  $c$ .

**Exercise 10.3** Let  $K \in \mathcal{K}^n$  be centered, i.e., the centroid is at the origin and let  $K - K$  be in  $M$ -position with constant  $C$ . Then  $K \cap (-K)$  is in  $M$ -position with constant  $c(C)$  depending only on  $C$ .

**Exercise 10.4** There exists an universal constant  $c_1 > 0$  such that for  $K \in \mathcal{K}^n$  with  $\mathbf{0} \in K$  we have

$$\text{vol}(K)\text{vol}(K^\star) \geq c_1^n \text{vol}(B_n)^2.$$



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## Exercise sheet 9

Discussion: Thursday, 21.01.2015.

**Exercise 9.1** Let  $v_1, \dots, v_n \in \mathbb{R}^n$  and  $|\cdot|$  be a norm. Show that

$$2^{-n} \sum_{\epsilon \in \{-1,1\}^n} \left| \sum_{i=1}^n \epsilon_i v_i \right| \geq \max\{|v_1|, \dots, |v_n|\}.$$

**Exercise 9.2** Let  $K, L \in \mathcal{K}^n$ . For  $v \in \text{int}((K - L)/2)$  show that

$$\psi(v) = \text{vol}((-v + K) \cap (v + L))$$

is log-concave.

**Exercise 9.3** Let  $K \in \mathcal{K}_o^n$  and  $t \in \mathbb{R}^n$ . Show that

$$\gamma_n(t + K) \geq e^{-\|t\|^2/2} \gamma_n(K).$$

**Exercise 9.4 (Concentration of volume in convex bodies)** Let  $K \in \mathcal{K}^n$ ,  $\text{vol}(K) = 1$ , and  $L \in \mathcal{K}_o^n$  such that  $\text{vol}(K \cap L) = r > 1/2$ . Then for  $t \geq 1$

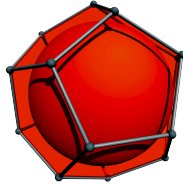
$$\text{vol}(K \cap (tL)^c) \leq r \left( \frac{1-r}{r} \right)^{(t+1)/2},$$

where  $A^c = \mathbb{R}^n \setminus A$  for a set  $A \subset \mathbb{R}^n$ .

First show that

$$\frac{2}{t+1}(tL)^c + \frac{t-1}{t+1}L \subseteq L^c.$$

and then ask Brunn or Minkowski or both.



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## Exercise sheet 8

Discussion: Thursday, 14.01.2015.

**Exercise 8.1** Let  $v \in S^{n-1}$ ,  $t \in [0, 1]$  and let  $U = \{u \in S^{n-1} : |\langle v, u \rangle| \leq t\}$ . Then  $\mu(U) \geq 1 - 4e^{-nt^2/4}$  and, in particular, for  $t \geq 4/\sqrt{n}$  it holds

$$\mu(U) \geq 0.9.$$

**Exercise 8.2** There exists always a  $\delta$ -net on  $S^{n-1}$  consisting of at most  $(4/\delta)^n$  points.

**Exercise 8.3** For  $K \in \mathcal{K}_o^n$  let  $M(K) = \int_{S^{n-1}} |u|_K \, d\mu(v)$ .

i) Show that

$$M(K) M(K^\star) \geq 1.$$

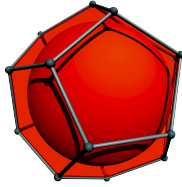
ii) Show that  $M(B_n^p) \geq (\sqrt{2/\pi}) n^{1/p-1/2}$  for  $1 \leq p < \infty$ .

Rem.: For ii) Hölder and the next inequality might/could help.

**Exercise 8.4** Let  $\kappa_n = \text{vol}(B_n) = \pi^{n/2}/\Gamma(n/2 + 1)$ . Show that

$$\sqrt{\frac{2\pi}{n}} \geq \frac{\kappa_n}{\kappa_{n-1}} \geq \sqrt{\frac{2\pi}{n+1}}.$$

Show first that  $\gamma_n = \frac{\kappa_n}{\kappa_{n-1}}$  is a decreasing function in  $n$  and consider  $\gamma_n \gamma_{n-1}$ .



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## Exercise sheet 7

Discussion: Thursday, 07.01.2016.

**Exercise 7.1** Show that  $\mathcal{K}^n$  equipped with  $\log d_{\text{BM}}(\cdot, \cdot)$  is a compact metric space, and that for  $K, L \in \mathcal{K}_o^n$

$$d_{\text{BM}}(K, L) = d_{\text{BM}}(K^*, L^*).$$

**Exercise 7.2** Let  $B_n^p = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \leq 1\}$  with  $B_m^\infty = [-1, 1]^n$ . Show that for  $1 \leq p \leq q \leq 2$  or  $2 \leq p \leq q \leq \infty$

$$d_{\text{BM}}(B_n^p, B_n^q) = n^{1/p-1/q}.$$

The Minkowski-sum of finitely many line segments is called a *zonotope*  $Z$ , i.e.,

$$Z = \sum_{i=1}^m \text{conv} \{\mathbf{v}_i, \mathbf{w}_i\}.$$

**Exercise 7.3** Let  $Z$  be a zonotope. Then there exists a  $\mathbf{c} \in \mathbb{R}^n$  and  $\mathbf{u}_i \in \mathbb{R}^n$ ,  $1 \leq i \leq m$ , such that

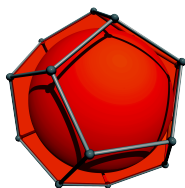
$$Z = \mathbf{c} + \sum_{i=1}^m \text{conv} \{-\mathbf{u}_i, \mathbf{u}_i\}.$$

For  $Z$  as above show that

$$\text{vol}(Z) = 2^n \sum_{J \subseteq [m], |J|=n} |\det(\mathbf{u}_j : j \in J)|.$$

## Exercise 7.4

- i) Show that the projection body (see Exercise sheet 4) of a polytope is a zonotope.
- ii) What is the projection body of the cube  $[-1/2, 1/2]^n$ ?
- iii) What is the projection body of an ellipsoid?



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## Exercise sheet 6

Discussion: Thursday, 17.12.2015.

**Exercise 6.1** Let  $K, L$  be 2-dimensional  $o$ -symmetric convex bodies. Show that the inequality  $\text{vol}(K) \leq \text{vol}(L)$  is a consequence of either of the next two properties

- i)  $\text{vol}_1(K \cap \text{lin}\{\mathbf{u}\}) \leq \text{vol}_1(L \cap \text{lin}\{\mathbf{u}\})$  for all  $\mathbf{u} \in S^1$ ,
- ii)  $\text{vol}_1(K|\text{lin}\{\mathbf{u}\}^\perp) \leq \text{vol}_1(L|\text{lin}\{\mathbf{u}\}^\perp)$  for all  $\mathbf{u} \in S^1$ .

Is symmetry needed?

**Exercise 6.2** Let  $K \in \mathcal{K}_c^n$  and let  $c_1, c_2 \in \mathbb{R}_{>0}$  such that for all  $\mathbf{u} \in S^{n-1}$

$$c_1 \leq \int_K \langle \mathbf{x}, \mathbf{u} \rangle^2 d\mathbf{x} \leq c_2.$$

Show that  $\sqrt{c_1} \leq L_K \leq \sqrt{c_2}$ .

**Exercise 6.3** Let  $K \subset \mathbb{R}^n$ ,  $L \subset \mathbb{R}^m$  be two convex bodies in isotropic position, and let

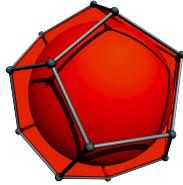
$$M = \left( \frac{L_L}{L_K} \right)^{\frac{m}{n+m}} K \times \left( \frac{L_K}{L_L} \right)^{\frac{n}{n+m}} L \subset \mathbb{R}^{n+m}.$$

Show that  $M$  is in isotropic position and  $L_{K \times L} = L_K^{\frac{n}{n+m}} L_L^{\frac{m}{n+m}}$ .

**Exercise 6.4** Let  $\mathbf{a} \in \mathbb{Z}^n$ ,  $\mathbf{a} \neq \mathbf{0}$ ,  $\gcd(\mathbf{a}) = 1$  and let  $S(\mathbf{a}) = \{\langle \mathbf{a}, \mathbf{z} \rangle : \mathbf{z} \in \mathbb{N}_{\geq 0}^n\}$ . For  $\mathbf{x} \in \mathbb{R}^n$  let  $|\mathbf{x}|_0 = \#\{x_i \neq 0 : 1 \leq i \leq n\}$ .

- i) Let  $\|\mathbf{a}\| < 2^{n-1}$  and  $\alpha \in S(\mathbf{a})$ . Then there exists a  $\mathbf{z} \in \mathbb{N}_{\geq 0}^n$  with  $\langle \mathbf{a}, \mathbf{z} \rangle = \alpha$  and  $|\mathbf{z}|_0 < n$ .
- ii) Let  $\alpha \in S(\mathbf{a})$ . Then there exists a  $\mathbf{z} \in \mathbb{N}_{\geq 0}^n$  with  $\langle \mathbf{a}, \mathbf{z} \rangle = \alpha$  and  $|\mathbf{z}|_0 \leq c \log_2 \|\mathbf{a}\|_\infty$ , where  $c$  is an absolute constant and  $\|\mathbf{a}\|_\infty$  is the maximum norm.

Exercise 4.4 could be helpful.



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## Exercise sheet 5

**Discussion: Thursday, 03.12.2015.**

For  $K, L \in \mathcal{K}^n$  let

$$N(K, L) = \min\{|S| : S \subset \mathbb{R}^n \text{ with } K \subseteq S + L\}$$

$$\overline{N}(K, L) = \min\{|S| : S \subset K \text{ with } K \subseteq S + L\}$$

$N(K, L)$  is called the *covering number* of  $K$  by  $L$ .

For  $K, L \in \mathcal{K}^n$ ,  $L = -L$  let

$$M(K, L) = \max\{|S| : S \subset K \text{ with } |x_i - x_j|_L > 1 \text{ for all } x_i \neq x_j \in S\}.$$

$M(K, L)$  is called the *separation number* of  $K$  by  $L$ .

**Exercise 5.1** Show that

i) for  $K, L, M \in \mathcal{K}^n$ ,  $K \subseteq L$ :  $N(K, M) \leq N(L, M)$ ,  $N(M, L) \leq N(M, K)$ , and  $\overline{N}(M, L) \leq \overline{N}(M, K)$

ii) for  $K, L \in \mathcal{K}^n$ :  $\overline{N}(K, L - L) \leq N(K, L) \leq \overline{N}(K, L)$ .

iii) for  $K \in \mathcal{K}^n$ ,  $\lambda > 0$ :  $N(K, \lambda B_n) = \overline{N}(K, \lambda B_n)$ .

iv) for  $K, L, M \in \mathcal{K}^n$ :  $N(K, L) \leq N(K, M) N(M, L)$ .

**Exercise 5.2** Let  $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{K}^n$  ellipsoids centered at the origin. Show that

$$N(\mathcal{E}_1, \mathcal{E}_2) = N(\mathcal{E}_2^*, \mathcal{E}_1^*).$$

**Exercise 5.3** Let  $K, L \in \mathcal{K}^n$ . Show that

$$\overline{N}(K, (K - K) \cap L) = \overline{N}(K, L).$$

**Exercise 5.4** Show that

$$M(K, 2L) \leq N(K, L) \leq \overline{N}(K, L) \leq M(K, L)$$

**Exercise 5.5** Let  $K, L \in \mathcal{K}^n$ ,  $\dim K, \dim L = n$ . Show that

$$\text{vol}(K)/\text{vol}(L) \leq N(K, L).$$

If  $L = -L$  then

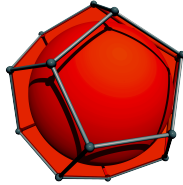
$$N(K, L) \leq 2^n \text{vol}(K + L/2)/\text{vol}(L).$$

**Exercise 5.6** Let  $K \in \mathcal{K}_c^n$  be in isotropic position. Then

$$c_1 L_K \leq r(K) \leq R(K) \leq c_2 n L_K,$$

where  $c_1, c_2 > 0$  are absolute constants.





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### Exercise sheet 4

Discussion: Thursday, 26.11.2015.

**Exercise 4.1** *Show that among all  $n$ -simplices of inradius 1 the regular simplex has minimal volume.*

The inradius is the maximal radius of an  $n$ -dimensional ball contained in a body.

**Exercise 4.2** *Show that*

$$\prod_{i=1}^n x_i^{x_i}$$

*attains its minimum on the set  $\{\mathbf{x} \in \mathbb{R}_{\geq 0}^n : \sum_{i=1}^n x_i = \alpha\}$ ,  $\alpha > 0$ , if  $x_1 = x_2 = \dots = x_n$ .*

**Exercise 4.3** *Let  $K \in \mathcal{K}^n$ . The set*

$$\Pi(K) = \{x \in \mathbb{R}^n : |\langle \mathbf{u}, \mathbf{x} \rangle| \leq \text{vol}_{n-1}(K|\mathbf{u}^\perp), \text{ for all } \mathbf{u} \in S^{n-1}\}$$

*is called the projection body of  $K$ . Show that*

$$\text{i) } h(\Pi(K), \mathbf{u}) = \text{vol}_{n-1}(K|\mathbf{u}^\perp).$$

$$\text{ii) } \Pi(AK) = |\det A| A^{-\top} \Pi(K) \text{ for } A \in \text{GL}(n, \mathbb{R}), \text{ and } \Pi(\mathbf{t} + K) = \Pi(K) \text{ for } \mathbf{t} \in \mathbb{R}^n.$$

**Exercise 4.4 (Bombieri&Vaaler)** *Let  $\mathbf{a} \in \mathbb{Z}^n$ ,  $\mathbf{a} \neq \mathbf{0}$ ,  $\gcd(\mathbf{a}) = 1$ . Show that there exists a  $\mathbf{z} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$  with  $\langle \mathbf{a}, \mathbf{z} \rangle = 0$  and  $\max_{1 \leq j \leq n} |z_j| \leq \|\mathbf{a}\|^{1/(n-1)}$ .*

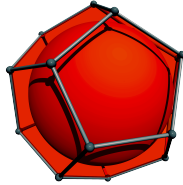
What could help is i) Minkowski's theorem, saying that every  $o$ -symmetric convex body with volume not less than  $2^n \det \Lambda$  contains a non-trivial lattice point of a lattice  $\Lambda$ , and ii) the set  $\{\mathbf{z} \in \mathbb{Z}^n : \langle \mathbf{a}, \mathbf{z} \rangle = 0\}$  is an  $(n-1)$ -dimensional lattice of determinant  $\|\mathbf{a}\|$ .

**Exercise 4.5 (McMullen)** *Let  $L$  be a  $k$ -dimensional linear subspace with orthogonal complement  $L^\perp$ . Show that*

$$\text{vol}_k(C_n|L) = \text{vol}_{n-k}(C_n|L^\perp).$$

For a zonotope  $Z = \sum_{i=1}^m \text{conv}\{\mathbf{0}, \mathbf{v}_i\}$  the volume is given by

$$\text{vol}(Z) = \sum_{J \subseteq [m], |J|=n} |\det(\mathbf{v}_j : j \in J)|.$$



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### Exercise sheet 3

**Discussion: Thursday, 12.11.2015!!**

A function  $f : M \subseteq \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  is called *log-concave* if  $\ln f$  is a concave function. It is called *centered* if  $\int_M \mathbf{x} f(\mathbf{x}) d\mathbf{x} = \mathbf{0}$ .

**Exercise 3.1** Let  $K \in \mathcal{K}^n$  be a centered convex body, i.e.,  $\int_K \mathbf{x} d\mathbf{x} = \mathbf{0}$ , and let  $L \subseteq \mathbb{R}^n$  be a  $k$ -dimensional linear subspace with orthogonal subspace  $L^\perp$ . For  $\mathbf{y} \in \text{relint}(K|L)$  let  $f(\mathbf{y}) = \text{vol}_{n-k}(K \cap (\mathbf{y} + L^\perp))$ . Show that  $f$  is a centered log-concave function.

**Exercise 3.2** Let  $F, G : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  be integrable log-concave functions. Then, their convolution

$$(F \star G)(\mathbf{x}) = \int_{\mathbb{R}^n} F(\mathbf{x} - \mathbf{y}) G(\mathbf{y}) d\mathbf{y}$$

is also a log-concave function.

The Prékopa-Leindler inequality might help.

**Exercise 3.3** Let  $K, L \in \mathcal{K}^n$  with  $\mathbf{0} \in \text{int } K, \text{int } L$ , and let  $\lambda \in (0, 1)$ . Show that

i)

$$\text{vol}(K^\star) = \frac{1}{n!} \int_{\mathbb{R}^n} e^{-h(K, \mathbf{x})} d\mathbf{x}.$$

ii)

$$\ln \text{vol}[\lambda K + (1 - \lambda)L]^\star \leq \lambda \ln \text{vol } K^\star + (1 - \lambda) \ln \text{vol } L^\star.$$

Here the Hölder inequality might help

**Exercise 3.4** Let  $Q \subset \mathbb{R}^{n-1} \times \{0\}$  be a polytope,  $\mathbf{y} \in \mathbb{R}^n$ ,  $y_n \neq 0$ , and  $P = \text{conv}\{Q, \mathbf{y}\}$  be the pyramid over  $Q$  with apex  $\mathbf{y}$ . Let  $c(P)$  be the centroid of  $P$ , and let  $\mathbf{q} \in Q$  be the intersection of  $Q$  with the ray  $\mathbf{y} + \lambda(c(P) - \mathbf{y})$ ,  $\lambda \geq 0$ . Then

$$|\mathbf{y} - c(P)| = n|c(P) - \mathbf{q}| = \frac{n}{n+1}|\mathbf{y} - \mathbf{q}|.$$

**Exercise 3.5**

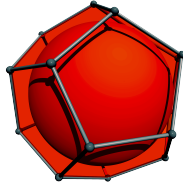
- i) Let  $P \in \mathcal{K}^n$  be an  $n$ -dimensional polytope, and let  $Q_i \subset P$ ,  $i \in I$ , be polytopes forming a subdivision of  $P$ , i.e.,  $\cup_{i \in I} Q_i = P$  and  $\dim(Q_i \cap Q_j) \leq n-1$ ,  $i \neq j$ . Find a relation between the centroids  $c(Q_i)$  and  $c(P)$ .
- ii) Let  $S = \text{conv}\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$  be a  $n$ -dimensional simplex. Show that  $c(P) = \frac{1}{n+1} \sum_{i=0}^n \mathbf{v}_i$ .

**Exercise 3.6** Let  $K \in \mathcal{K}^n$  mit  $c(K) = \mathbf{0}$ .

- i) For  $\mathbf{u} \in \mathbb{R}^n$  let  $w(K, \mathbf{u}) = h(K, \mathbf{u}) + h(K, -\mathbf{u})$ . Show that

$$\frac{1}{n+1} w(K, \mathbf{u}) \leq h(K, \mathbf{u}) \leq \frac{n}{n+1} w(K, \mathbf{u}).$$

- ii) Show that  $-K \subset nK$ .



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## Exercise sheet 2

Discussion: Thursday, 29.10.2015.

**Exercise 2.1** Let  $H(\mathbf{a}, 0)$  be a  $\mathbf{0}$  containing hyperplane and let  $K \in \mathcal{K}^n$  be symmetric with respect to  $H(\mathbf{a}, 0)$ . Let  $\mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\} \in H(\mathbf{a}, 0)$ . Show that  $\text{st}_{H(\mathbf{v}, 0)}(K)$  is still symmetric to  $H(\mathbf{a}, 0)$ .

**Exercise 2.2** Similar to the first part of the proof of the Rogers-Shephard inequality show that for  $K, L \in \mathcal{K}^n$  holds

$$\int_{\mathbb{R}^n} \text{vol}(K \cap (\mathbf{x} - L)) \, d\mathbf{x} = \text{vol}(K) \text{vol}(L),$$

and conclude that there exists a  $\mathbf{x} \in 2K$  such that

$$\text{vol}(K \cap (\mathbf{x} - K)) \geq 2^{-n} \text{vol}(K).$$

Hence every convex body  $K$  of  $\text{vol}(K) = 2^n$  contains a centrally symmetric subset of volume at least 1.

**Exercise 2.3** Let  $K \in \mathcal{K}^n$  containing  $\mathbf{0}$  in the interior and let  $h(K, \cdot)$  be its support function. The function  $\rho(K, \cdot) : \mathbb{R}^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}_{\geq 0}$  given by

$$\rho(K, \mathbf{u}) = \max\{\rho \geq 0 : \rho \mathbf{u} \in K\}$$

is called radial function of  $K$ . Show that

$$h(K, \mathbf{u}) \rho(K^*, \mathbf{u}) = 1.$$

**Exercise 2.4** Let  $K \in \mathcal{K}^n$ ,  $K = -K$ ,  $\dim K = n$ , and let  $L \subset \mathbb{R}^n$  be a  $k$ -dimensional linear subspace with orthogonal complement  $L^\perp$ . Show that

$$\binom{n}{k}^{-1} \leq \frac{\text{vol}(K)}{\text{vol}_k(K \cap L) \cdot \text{vol}_{n-k}(K|L^\perp)} \leq 1.$$

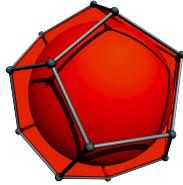
For the lower bound it might be useful to observe that for  $\mathbf{x} \in (K|L^\perp)$  a suitable translation of  $(1 - \rho(K|L^\perp, \mathbf{x})^{-1})(K \cap L) + \mathbf{x}$  is contained in  $K \cap (\mathbf{x} + L)$ .

**Exercise 2.5** *Cauchy's surface area formula states that for  $K \in \mathcal{K}^n$*

$$F(K) = \frac{1}{\text{vol}_{n-1}(B_{n-1})} \int_{S^{n-1}} \text{vol}_{n-1}(K|u^\perp) \, du,$$

*$K|u^\perp$  denotes the orthogonal projection of  $K$  onto the hyperplane with normal  $u$ . Use it in order to show*

$$F\left(\frac{1}{2}(K + K)\right) \geq F(K).$$



## Exercise sheet 1

Discussion: Thursday, 22.05.2015.

**Exercise 1.1** Let  $K, K_i \in \mathcal{K}^n$ ,  $i \in \mathbb{N}$ , with  $\mathbf{0} \in \text{int } K$ . Show that  $K_i \rightarrow K$  if and only if for all  $\epsilon > 0$  there exists an  $i_\epsilon \in \mathbb{N}$  such that for all  $i \geq i_\epsilon$

$$(1 - \epsilon) K_i \subseteq K \subseteq (1 + \epsilon) K_i.$$

**Exercise 1.2** Let  $K \in \mathcal{K}^n$ . Show that the Steiner-symmetral  $\text{st}_H(K)$  of  $K$  with respect to a hyperplane  $H$  is a compact set.

**Exercise 1.3** Show that the minimal width of a convex body may decrease or increase under Steiner-symmetrizations.

**Exercise 1.4** Let  $K \in \mathcal{K}^n$ . Show that (without using Exercise 1.5)

$$\text{vol}(K) \leq \left( \frac{D(K)}{2} \right)^n \text{vol}(B_n).$$

**Exercise 1.5** Let  $K \in \mathcal{K}^n$ . The functional

$$w(K) = \frac{1}{n \text{vol}(B_n)} \int_{S^{n-1}} [h(K, \mathbf{u}) + h(K, -\mathbf{u})] \, d\mathbf{u}$$

is called the mean width of  $K$ . Here the integration “ $d\mathbf{u}$ ” is meant with respect to the  $(n-1)$ -dimensional Hausdorff-measure. Let  $H$  be a hyperplane. Show that

$$w(\text{st}_H(K)) \leq w(K),$$

and conclude “Urysohn’s inequality”

$$\text{vol}(K) \leq \left( \frac{w(K)}{2} \right)^n \text{vol}(B_n).$$

Rem: The mean width is a continuous functional on  $\mathcal{K}^n$ .

one more on the next page...

**Exercise 1.6** Let  $T$  be an  $n$ -dimensional simplex, i.e.,  $T = \text{conv}\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ ,  $\mathbf{v}_i \in \mathbb{R}^n$ , and  $\dim T = n$ .

- i) Let  $H_{i,j}$ ,  $i \neq j$ , be the hyperplane through  $\frac{1}{2}(\mathbf{v}_i + \mathbf{v}_j)$  and with normal vector  $\mathbf{v}_i - \mathbf{v}_j$  (orthogonal to the edge  $\text{conv}\{\mathbf{v}_i, \mathbf{v}_j\}$ ). Show that  $\text{st}_{H_{i,j}}(T)$  is again an  $n$ -dimensional simplex.
- ii) Show that among all  $n$ -dimensional simplices  $T$  the ratio  $R(T)/r(T)$  becomes minimal for a regular simplex. Here it might be helpful to use the fact the surface area of the Steiner symmetral  $\text{st}_H(K)$  is strictly decreasing if  $K$  is not symmetric to the hyperplane  $H$ .