Geometrische Grundlagen der Linearen Optimierung

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Contents

WS 2015/16

1 Basic and convex facts **1**

2 Support and separate **7**

Index **11**

1 Basic and convex facts

1.1 Notation. $\mathbb{R}^n = \{\boldsymbol{x} = (x_1, \dots, x_n)^\intercal : x_i \in \mathbb{R}\}$ *denotes the* n*-dimensional* Euclidean space *equipped with the* Euclidean inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\intercal \boldsymbol{y} = \sum_{i=1}^n x_i\, y_i$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, *and the* Euclidean norm $\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$.

1.2 Definition [Linear, affine, positive and convex combination]. *Let* $m \in \mathbb{N}$ *and let* $\boldsymbol{x}_i \in \mathbb{R}^n$, $\lambda_i \in \mathbb{R}$, $1 \le i \le m$.

i) $\sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i$ *is called* a linear combination *of* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$.

ii) *If* $\sum_{i=1}^m \lambda_i = 1$ *then* $\sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i$ *is called an* affine combination *of* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$.

iii) *If* $\lambda_i \ge 0$ *then* $\sum_{i=1}^m \lambda_i\, x_i$ *is called* a positive combination *of* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$.

iv) *If* $\lambda_i \ge 0$ *and* $\sum_{i=1}^m \lambda_i = 1$ *then* $\sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i$ *is called* a convex combination *of* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$.

v) *Let* $X \subseteq \mathbb{R}^n$. $\boldsymbol{x} \in \mathbb{R}^n$ *is called* linearly *(*affinely, positively, convexly*)* dependent *of* X, *if* $\boldsymbol{x}$ *is a linear (affine, positive, convex) combination of finitely many points of* X, *i.e., there exist* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in X$, $m \in \mathbb{N}$, *such that* $\boldsymbol{x}$ *is a linear (affine, positive, convex) combination of the points* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$.

1.3 Definition [Linearly and affinely independent points]. $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in \mathbb{R}^n$ *are called* linearly *(*affinely*)* dependent, *if one of the* $\boldsymbol{x}_i$ *is linearly (*affinely*) dependent of* $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_m\} \backslash \{\boldsymbol{x}_i\}$. *Otherwise* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ *are called* linearly *(*affinely*)* independent.

1.4 Proposition. *Let* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m \in \mathbb{R}^n$.

i) $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ *are affinely dependent if and only if* $\binom{\boldsymbol{x}_1}{1}, \dots, \binom{\boldsymbol{x}_m}{1} \in \mathbb{R}^{n+1}$ *are linearly dependent.*

ii) $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ *are affinely dependent if and only if there exist* $\mu_i \in \mathbb{R}$, $1 \le i \le m$, *with* $(\mu_1, \dots, \mu_m) \ne (0, \dots, 0)$, $\sum_{i=1}^m \mu_i = 0$ *and* $\sum_{i=1}^m \mu_i\, \boldsymbol{x}_i = \boldsymbol{0}$.

iii) *If* $m \ge n+1$ *then* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ *are linearly dependent.*

iv) *If* $m \ge n+2$ *then* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_m$ *are affinely dependent.*

1.5 Definition [Linear subspace, affine subspace, (convex) cone and convex set].

$X \subseteq \mathbb{R}^n$ *is called*

i) linear subspace (set) *if it contains all* $\boldsymbol{x} \in \mathbb{R}^n$ *which are linearly dependent of* X,

ii) affine subspace (set) *if it contains all* $\boldsymbol{x} \in \mathbb{R}^n$ *which are affinely dependent of* X,

iii) (convex) cone *if it contains all* $\boldsymbol{x} \in \mathbb{R}^n$ *which are positively dependent of* X,

iv) convex set *if it contains all* $\boldsymbol{x} \in \mathbb{R}^n$ *which are convexly dependent of* X.

1.6 Theorem. *$K \subseteq \mathbb{R}^n$ is convex if and only if*

$$\lambda \boldsymbol{x} + (1-\lambda)\,\boldsymbol{y} \in K, \qquad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in K \text{ and } 0 \leq \lambda \leq 1.$$

1.7 Example.

i) The closed n-dimensional ball $B_n(\boldsymbol{a}, \rho) = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{a}\| \leq \rho\}$ with centre a and radius $\rho > 0$ is convex. The boundary of $B_n(\boldsymbol{a}, \rho)$, i.e., $\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{a}\| = \rho\}$ is non-convex. In the case $\boldsymbol{a} = \boldsymbol{0}$ and $\rho = 1$ the ball $B_n(\boldsymbol{0}, 1)$ is abbreviated by B_n and is called n-dimensional unit ball.

ii) Let $\boldsymbol{a} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. The closed halfspaces *$H^+(\boldsymbol{a}, \alpha)$, $H^-(\boldsymbol{a}, \alpha) \subset \mathbb{R}^n$ given by*

$$H^+(\boldsymbol{a}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq \alpha\}, \quad H^-(\boldsymbol{a}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle \leq \alpha\}$$

are convex, as well as the hyperplane *$H(\boldsymbol{a}, \alpha)$ defined by*

$$H(\boldsymbol{a}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = \alpha\}.$$

1.8 Corollary. *Let $K_i \subseteq \mathbb{R}^n$, $i \in I$, be convex. Then $\bigcap_{i \in I} K_i$ is convex.*

1.9 Definition [Linear, affine, positive and convex hull, dimension].
Let $X \subseteq \mathbb{R}^n$.

i) *The* linear hull $\operatorname{lin} X$ *of X is defined by*

$$\operatorname{lin} X = \bigcap_{\substack{L \subseteq \mathbb{R}^n,\, L \text{ linear}, \\ X \subseteq L}} L.$$

ii) *The* affine hull $\operatorname{aff} X$ *of X is defined by*

$$\operatorname{aff} X = \bigcap_{\substack{A \subseteq \mathbb{R}^n,\, A \text{ affine}, \\ X \subseteq A}} A.$$

iii) *The* positive (conic) hull $\operatorname{pos} X$ *of X is defined by*

$$\operatorname{pos} X = \bigcap_{\substack{C \subseteq \mathbb{R}^n,\, C \text{ convex cone}, \\ X \subseteq C}} C.$$

iv) *The* convex hull $\operatorname{conv} X$ *of X is defined by*

$$\operatorname{conv} X = \bigcap_{\substack{K \subseteq \mathbb{R}^n,\, K \text{ convex}, \\ X \subseteq K}} K.$$

v) *The* dimension $\dim X$ *of* X *is the dimension of its affine hull, i.e.,* $\dim \operatorname{aff} X$.

1.10 Theorem. *Let* $X \subseteq \mathbb{R}^n$. *Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i : m \in \mathbb{N}, \boldsymbol{x}_i \in X, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

1.11 Remark.

i) $\operatorname{conv} \{\boldsymbol{x}, \boldsymbol{y}\} = \{\lambda\, \boldsymbol{x} + (1-\lambda)\, \boldsymbol{y} : \lambda \in [0,1]\}$.

ii) $\operatorname{lin} X = \left\{\sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i : \lambda_i \in \mathbb{R},\ \boldsymbol{x}_i \in X,\ m \in \mathbb{N}\right\}$.

iii) $\operatorname{aff} X = \left\{\sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i : \lambda_i \in \mathbb{R},\ \sum_{i=1}^{m} \lambda_i = 1,\ \boldsymbol{x}_i \in X,\ m \in \mathbb{N}\right\}$.

iv) $\operatorname{pos} X = \left\{\sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i : \lambda_i \in \mathbb{R},\ \lambda_i \geq 0,\ \boldsymbol{x}_i \in X,\ m \in \mathbb{N}\right\}$.

1.12 Definition [Linear, affine, convex function]. *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is called*

i) linear *if* $f(\lambda \boldsymbol{x} + \mu \boldsymbol{y}) = \lambda\, f(\boldsymbol{x}) + \mu\, f(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *and* $\lambda,\ \mu \in \mathbb{R}$.

ii) affine *if* $f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) = \lambda\, f(\boldsymbol{x}) + (1-\lambda)\, f(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$.

iii) convex *if* $f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq \lambda\, f(\boldsymbol{x}) + (1-\lambda)\, f(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *and* $\lambda \in [0,1]$.

iv) concave *if* $f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \lambda\, f(\boldsymbol{x}) + (1-\lambda)\, f(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *and* $\lambda \in [0,1]$, *i.e.,* $-f$ *is convex.*

1.13 Theorem. *Let* $X \subset \mathbb{R}^n$ *be convex and let* $f : X \to \mathbb{R}$ *be a convex function. Let* $\widetilde{\boldsymbol{x}} \in X$ *be a local minimum of* f *on* X, *i.e., there exists an* $\epsilon \in \mathbb{R}_{>0}$ *with*

$$f(\boldsymbol{x}) \geq f(\widetilde{\boldsymbol{x}}) \text{ for all } \boldsymbol{x} \in X \text{ with } \|\boldsymbol{x} - \widetilde{\boldsymbol{x}}\| \leq \epsilon.$$

Then $\widetilde{\boldsymbol{x}}$ *is a global minimum, i.e.,*

$$f(\boldsymbol{x}) \geq f(\widetilde{\boldsymbol{x}}) \text{ for all } \boldsymbol{x} \in X.$$

1.14 Definition [Interior and boundary point]. *Let* $X \subseteq \mathbb{R}^n$.

i) $\boldsymbol{x} \in X$ *is called an* interior point *of* X *if there exists a* $\rho > 0$ *such that* $B_n(\boldsymbol{x}, \rho) \subseteq X$. *The set of all interior points of* X *is called the* interior *of* X *and is denoted by* $\operatorname{int} X$.

ii) $\boldsymbol{x} \in \mathbb{R}^n$ *is called* boundary point *of* X *if for all* $\rho > 0$, $B_n(\boldsymbol{x}, \rho) \cap X \neq \emptyset$ *and* $B_n(\boldsymbol{x}, \rho) \cap (\mathbb{R}^n \backslash X) \neq \emptyset$. *The set of all boundary points of* X *is called the* boundary *of* X *and is denoted by* $\operatorname{bd} X$.

1.15 Lemma. *Let $K \subseteq \mathbb{R}^n$ be convex with $\dim K = n$, and let $\boldsymbol{x} \in \operatorname{int} K$ and $\boldsymbol{y} \in K$. Then $(1-\lambda)\boldsymbol{x} + \lambda \boldsymbol{y} \in \operatorname{int} K$ for all $\lambda \in [0,1)$.*

1.16 Corollary. *Let $K \subseteq \mathbb{R}^n$ be convex, closed and $\dim K = n$. Let $\boldsymbol{x} \in \operatorname{int} K$ and $\boldsymbol{y} \in \mathbb{R}^n \setminus K$. Then the segment $\operatorname{conv}\{\boldsymbol{x}, \boldsymbol{y}\}$ intersects $\operatorname{bd} K$ in precisely one point.*

1.17 Definition [Polytope and simplex]. *Let $X \subset \mathbb{R}^n$ of finite cardinality, i.e., $\#X < \infty$.*

i) $\operatorname{conv} X$ *is called a* (convex) polytope.

ii) *A polytope $P \subset \mathbb{R}^n$ of dimension k is called a k-polytope.*

iii) *If X is affinely independent and $\dim X = k$ then $\operatorname{conv} X$ is called a k-simplex.*

1.18 Notation. *$\mathcal{P}^n = \{P \subset \mathbb{R}^n : P \text{ polytope}\}$ denotes the set of all polytopes in $\mathbb{R}^n$.*

1.19 Notation.

i) *For two sets $X, Y \subseteq \mathbb{R}^n$ the vectorial addition*

$$X + Y = \{\boldsymbol{x} + y : \boldsymbol{x} \in X,\ \boldsymbol{y} \in Y\}$$

is called the Minkowski [1] sum *of X and Y. If X is just a singleton, i.e., $X = \{\boldsymbol{x}\}$, then we write $\boldsymbol{x} + Y$ instead of $\{\boldsymbol{x}\} + Y$.*

ii) *For $\lambda \in \mathbb{R}$ and $X \subseteq \mathbb{R}^n$ we denote by λX the set*

$$\lambda X = \{\lambda \boldsymbol{x} : \boldsymbol{x} \in X\}.$$

For instance, $B_n(\boldsymbol{a}, \rho) = \boldsymbol{a} + \rho B_n$.

1.20 Theorem [Carathéodory]. [2] *Let $X \subseteq \mathbb{R}^n$. Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{n+1} \lambda_i \boldsymbol{x}_i : \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1, \boldsymbol{x}_i \in X, i = 1, \ldots, n+1 \right\}.$$

1.21 Corollary. *A polytope is the union of simplices.*

1.22 Corollary. *The convex hull of a compact set is compact.*

1.23 Theorem [Radon]. [3] *Let $X \subset \mathbb{R}^n$. If $\#X \geq n+2$ then there exist $X_1, X_2 \subset X$ with $X_1 \cap X_2 = \emptyset$ and $\operatorname{conv} X_1 \cap \operatorname{conv} X_2 \neq \emptyset$.*

[1] Hermann Minkowski, 1864–1909
[2] Constantin Carathéodory, 1873 - 1950
[3] Johann Karl August Radon, 1887–1956

1.24 Theorem [Helly]. [4] *Let* $K_1, \dots, K_m \subseteq \mathbb{R}^n$, $m \geq n+1$, *be convex. If for each* $(n+1)$*-index set* $I \subseteq \{1, \dots, m\} = [m]$

$$\bigcap_{i \in I} K_i \neq \emptyset,$$

then all sets K_i *have a point in common, i.e.,* $\bigcap_{j=1}^{m} K_i \neq \emptyset$.

1.25 Corollary. *Let* $\boldsymbol{a}_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $1 \leq i \leq m$, *and let* $P = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq b_i,\ 1 \leq i \leq m\}$. *Then* $P \neq \emptyset$ *if and only if* $P_I = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq b_i,\ i \in I\} \neq \emptyset$ *for all* $(n+1)$*-index sets* $I \subseteq [m]$.

1.26 Theorem* [Doignon, Scarf, Bell]. *Let* $K_1, \dots, K_m \subseteq \mathbb{R}^n$, $m \geq 2^n$, *be convex. If for each* 2^n*-index set* $I \subseteq \{1, \dots, m\} = [m]$

$$\bigcap_{i \in I} (K_i \cap \mathbb{Z}^n) \neq \emptyset,$$

then all sets K_i *have an integral point in common, i.e.,* $\bigcap_{j=1}^{m} (K_i \cap \mathbb{Z}^n) \neq \emptyset$.

1.27 Corollary. *Let* $\boldsymbol{a}_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $1 \leq i \leq m$, *and let* $P = \{\boldsymbol{x} \in \mathbb{Z}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq b_i,\ 1 \leq i \leq m\}$. *Then* $P \neq \emptyset$ *if and only if* $P_I = \{\boldsymbol{x} \in \mathbb{Z}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq b_i,\ i \in I\} \neq \emptyset$ *for all* 2^n*-index sets* $I \subseteq [m]$.

[4] Eduard Helly, 1884–1943

2 Support and separate

2.1 Definition [Supporting hyperplane]. *Let $X \subset \mathbb{R}^n$. A hyperplane $H(\boldsymbol{a},\alpha) \subset \mathbb{R}^n$ is called* supporting hyperplane *of X if:*

$$\text{i) } H(\boldsymbol{a},\alpha) \cap X \neq \emptyset \quad \text{and} \quad \text{ii) } X \subseteq H^-(\boldsymbol{a},\alpha).$$

$\boldsymbol{a}$ is called outer normal vector *of X and if, in addition, $\|\boldsymbol{a}\| = 1$ then it is called* outer unit normal vector *of X.*

2.2 Proposition. *Let $X \subset \mathbb{R}^n$ and let $H(\boldsymbol{a},\alpha)$ be a supporting hyperplane of X. Then $H(\boldsymbol{a},\alpha)$ is a supporting hyperpalne of* conv *X and*

$$H(\boldsymbol{a},\alpha) \cap \operatorname{conv} X = \operatorname{conv}\big(H(\boldsymbol{a},\alpha) \cap X\big).$$

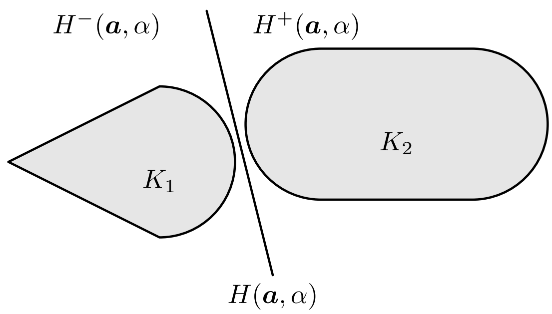

Figure 1: A strictly separating hyperplane of two compact convex sets

2.3 Theorem [Separation theorem]. *Let $K_1, K_2 \subset \mathbb{R}^n$ be convex with $K_1 \cap K_2 = \emptyset$. Then there exists* a separating hyperplane *$H(\boldsymbol{a},\alpha)$ of K_1 and K_2, i.e., $K_1 \subseteq H^+(\boldsymbol{a},\alpha)$ and $K_2 \subseteq H^-(\boldsymbol{a},\alpha)$.*

If K_1 is closed and K_2 is compact, then there exists even a strictly separating hyperplane *$H(\boldsymbol{a},\alpha)$ of K_1 and K_2, i.e., $K_1 \subset \operatorname{int} H^+(\boldsymbol{a},\alpha)$ and $K_2 \subset \operatorname{int} H^-(\boldsymbol{a},\alpha)$.*

2.4 Corollary [Farkas' Lemma]. [5]*Let $A \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. There exists a non-negative $\boldsymbol{x} \in \mathbb{R}^n_{\geq \boldsymbol{0}}$ with $A\boldsymbol{x} = \boldsymbol{b}$ if and only if $\langle \boldsymbol{b}, \boldsymbol{y}\rangle \geq 0$ for all $\boldsymbol{y} \in \mathbb{R}^m$ satisfying $A^\intercal \boldsymbol{y} \geq \boldsymbol{0}$, i.e.,*

$$P = \{\boldsymbol{x} \in \mathbb{R}^n_{\geq \boldsymbol{0}} : A\boldsymbol{x} = \boldsymbol{b}\} \neq \emptyset \Leftrightarrow \inf\{\langle \boldsymbol{b}, \boldsymbol{y}\rangle : \boldsymbol{y} \in \mathbb{R}^m \text{ with } A^\intercal \boldsymbol{y} \geq \boldsymbol{0}\} \geq 0.$$

2.5 Corollary. *Let $K \subset \mathbb{R}^n$ be convex and closed,* $\dim K = n$, *and let $\boldsymbol{x} \in \operatorname{bd} K$. Then there exists a supporting hyperplane $H(\boldsymbol{a},\alpha)$ of K containing $\boldsymbol{x}$.*

2.6 Theorem. *Let $K \subset \mathbb{R}^n$, $K \neq \mathbb{R}^n$, be convex and closed,* $\dim K = n$. *Then*

$$K = \bigcap_{\substack{H(\boldsymbol{a},\alpha) \text{ supporting} \\ \text{hyperplane of } K}} H^-(\boldsymbol{a},\alpha),$$

i.e., K is the intersection of all its "supporting halfspaces".

[5] Gyula Farkas, 1847–1930

2.7 Definition [Support function]. *Let $K \subset \mathbb{R}^n$ be convex, $K \neq \emptyset$. The function $h(K,\cdot) : \mathbb{R}^n \to \mathbb{R}$ given by*

$$h(K,\boldsymbol{u}) = \sup\big\{\langle \boldsymbol{u},\boldsymbol{x}\rangle : \boldsymbol{x} \in K\big\}$$

is called support function *of K.*

2.8 Proposition. *Let $K \subset \mathbb{R}^n$, $K \neq \emptyset$ be convex and compact. Then*

$$K = \bigcap_{\boldsymbol{u}\in B_n, \|\boldsymbol{u}\|=1} \big\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u},\boldsymbol{x}\rangle \leq h(K,\boldsymbol{u})\big\}.$$

2.9 Definition. *Let $K \subseteq \mathbb{R}^n$ be convex and closed. The set*

$$\operatorname{rec} K = \{\boldsymbol{u} \in \mathbb{R}^n : K + \boldsymbol{u} \subseteq K\}$$

is called the recession cone *of K.*

2.10 Proposition. *Let $K \subseteq \mathbb{R}^n$ be convex and closed, and let $\boldsymbol{x} \in K$. Then*

$$\operatorname{rec}(K) = \{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{x} + \lambda \boldsymbol{u} \in K \text{ for all } \lambda \in \mathbb{R}_{\geq 0}\}$$

In particular, $\operatorname{rec}(K)$ *is a closed convex cone.*

2.11 Theorem. *Let $K \subseteq \mathbb{R}^n$ be convex and closed. Then K can be represented as*

$$K = \overline{K} \oplus L,$$

where $L \subseteq \mathbb{R}^n$ is a linear subspace and $\overline{K} \subset \overline{L}$ is a line-free *convex set contained in a complementary linear subspace $\overline{L}$ of L.*

2.12 Definition [Polar set]. *Let $X \subseteq \mathbb{R}^n$.*

$$X^\star = \big\{\boldsymbol{y} \in \mathbb{R}^n : \langle \boldsymbol{x},\boldsymbol{y}\rangle \leq 1 \text{ for all } \boldsymbol{x} \in X\big\}$$

is called the polar set *of X.*

2.13 Proposition.

i) *$X^\star$ is a convex and closed set and $\mathbf{0} \in X^\star$.*

ii) *If $X_1 \subseteq X_2$ then $X_2^\star \subseteq X_1^\star$.*

iii) *Let M be a regular $n \times n$ matrix. Then $(MX)^\star = M^{-\intercal} X^\star$.*

iv) *Let $X_i \subseteq \mathbb{R}^n$, $i \in I$. Then $\big(\bigcup_{i\in I} X_i\big)^\star = \bigcap_{i\in I} X_i^\star$.*

v) *$X \subseteq (X^\star)^\star$.*

vi) *Let $X \subset \mathbb{R}^n$. Then $X = X^\star$ if and only if $X = B_n$.*

2.14 Lemma. *Let $K \subset \mathbb{R}^n$ be convex and closed with $\mathbf{0} \in K$. Then $(K^\star)^\star = K$.*

2.15 Proposition.

i) *Let $P = \operatorname{conv}\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\} \subset \mathbb{R}^n$. Then*

$$P^\star = \big\{\boldsymbol{y} \in \mathbb{R}^n : \langle \boldsymbol{x}_i, \boldsymbol{y}\rangle \leq 1,\, 1 \leq i \leq m\big\}.$$

ii) *Let $P = \big\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle \leq 1,\, 1 \leq i \leq m\big\}$ with $\boldsymbol{a}_i \in \mathbb{R}^n$. Then*

$$P^\star = \operatorname{conv}\{\mathbf{0}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}.$$

2.16 Proposition. *Let $K \subseteq \mathbb{R}^n$ be a convex cone. Then*

$$K^\star = \{\boldsymbol{y} \in \mathbb{R}^n : \langle \boldsymbol{x}, \boldsymbol{y}\rangle \leq 0 \text{ for all } \boldsymbol{x} \in K\}.$$

Index

$H(\boldsymbol{a}, \alpha)$, 2
$H^+(\boldsymbol{a}, \alpha), H^-(\boldsymbol{a}, \alpha)$, 2
$\mathcal{P}^n$, 4
$\mathbb{R}^n$, 1
$\operatorname{aff} X$, 2
B_n, 2
$B_n(\boldsymbol{a}, \rho)$, 2
$\operatorname{bd} X$, 3
$\operatorname{conv} X$, 2
$\dim X$, 3
$\|\boldsymbol{x}\|$, 1
$\operatorname{int} X$, 3
$\operatorname{lin} X$, 2
$\operatorname{pos} X$, 2
$\boldsymbol{x}^\intercal \boldsymbol{y}$, 1

affine
  combination, 1
  hull, 2
  subspace, 1
affinely
  dependent, 1
  independent, 1

ball, 2
Bell, David E., 5
boundary, 3
  point, 3

Carathéodory, Constantin, 4
cone, 2
  recession, 8
convex
  combination, 1
  hull, 2
  set, 2
convexly dependent, 1

dimension, 3
Doignon, Jean-Paul, 5

Euclidean
  inner product, 1
  norm, 1
  space, 1

family of polytopes, 4
Farkas, Gyula, 7

halfspace, 2
Helly, Eduard, 5
hyperplane, 2
  Separating, 7
  Supporting, 7

interior, 3
  point, 3

line-free, 8
linear
  combination, 1
  hull, 2
  subspace, 1
linearly
  dependent, 1
  independent, 1

Minkowski sum, 4
Minkowski, Hermann, 4

outer normal vector, 7
outer unit normal vector, 7

polar set, 8
polytope
  k-polytope, 4
positive
  combination, 1
  hull, 2
positively dependent, 1

Radon, Johann, 4
recession cone, 8

Scarf, Herbert, 5
separating hyperplane, 7
simplex
  k-simplex, 4
support function, 8
supporting hyperplane, 7

Theorem

of Carathéodory, 4
of Doignon, Sacrf, Bell, 5
of Helly, 5
of Radon, 4
theorem
of Separation, 7

unit ball, 2