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Modulklausur
Introduction to Linear and Combinatorial Optimization

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Firstname	
Lastname	
Matriculation Number	

- Please fill in the cover sheet and **wait for the signal to start.**
- The examination takes **120 minutes**.
- At least **30 points** needed to pass the exam.
- **Write your name on every single sheet of paper!**
- You can ask for **additional sheets** of paper.
- All sheets need to stay **stapled**.
- **Using items** like scripts, notes or electronic devices (including cellphones) is **not allowed** and will be treated as an attempt to defraud.
- Red pens are forbidden, answers not written with permanent ink will not be graded.
- Select **one topic** that will not be graded.

We wish you good luck!

Please **select the canceled section** and leave all other all boxes blank.

Exercise	1	2	3	4	5	Σ
maximum	15	15	15	15	15	60
achieved						
corrected						
canceled	◯	◯	◯	◯	◯	

Topic 1: LP Basics & Geometry of LPs

Topic 2: Simplex Method & Duality

Exercise 1 (5 Points)

Define the diameter $D(P)$ of a polyhedron P and the functions $\triangle(n,m)$ and $\triangle_u(n,m)$.

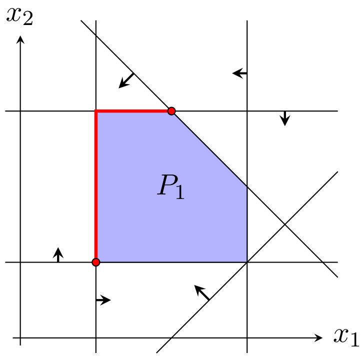
Let $P_1 := \{x \in \mathbb{R}^2 \mid 1 \le |x_i| \le 3,\ x_1 + x_2 \le 5,\ x_1 - x_2 \le 2\}$.

- What is the value of $D(P_1)$?
- What is the value of $\triangle(2,17)$?
- What is the value of $\triangle_u(1,9)$?

Solution:

- (1 point) $D(P) :=$ maximum $d(x,y)$ over all pairs of vertices x, y, $d(x,y) :=$ minimum number of edges required to reach y starting from x
- (0.5 points)$\triangle(n,m) :=$ maximum $D(P)$ over all polytopes P in $\mathbb{R}^n$, represented in terms of m inequality constraints
- (0.5 points) $\triangle_u(n,m) :=$ maximum $D(P)$ over all polyhedra P in $\mathbb{R}^n$, represented in terms of m inequality constraints

A sketch of P_1 shows that $D(P_1) = 2$ (0.5 points):



- (1 point) $\triangle(2,17) = \lfloor \frac{17}{2} \rfloor = 8$, a 2-dimensional polytope, described by 17 inequalities has at most 17 vertices. Every shortest path between two vertices uses less than 8 edges.
- (1 point) $\triangle_u(1,9) = 1$, an polyhedron in $\mathbb{R}^1$ is either a closed or a half-open interval. It has at most two vertices.

Exercise 2 (5 Points)

With phase I of the simplex algorithm, one can find a basic feasible solution for the following LP:

$$\begin{aligned}
\min\ & 2x + 3y - z \\
& x - 2y \leqslant 4 \\
& y + z \geqslant 2 \\
& x + y + z = 3 \\
& x, y, z \geqslant 0
\end{aligned}$$

Use the full-tableau method and introduce artificial variables only when necessary. Start to write down the auxiliary problem in standard form with slack variables for the first two constraints and two additional artificial variables.

Perform a single pivot operation to remove an artificial variable from the basis. Use the first possible variable, list the pivot element and the resulting basis.

The resulting tableau of phase I is:

	x	y	z	s_1	s_2	a_1	a_2
0	0	0	0	0	0	1	1
3	0	-2	0	1	-1	1	-1
2	0	1	1	0	-1	1	0
1	1	0	0	0	1	-1	1

What is the resulting solution for the original problem? Is this solution optimal?

Solution: The auxiliary problem in standard form:

$$\min a_1 + a_2$$

$$x - 2y + s_1 = 4$$

$$y + z - s_2 + a_1 = 2$$

$$x + y + z + a_2 = 3$$

$$x, y, z, s_1, s_2, a_1, a_2 \geq 0$$

In full tableau form:

		x	y	z	s_1	s_2	a_1	a_2
	0	0	0	0	0	0	1	1
s_1	4	1	-2	0	1	0	0	0
a_1	2	0	1	1	0	-1	1	0
a_2	3	1	1	1	0	0	0	1

Adjusting the reduced cost coefficient:

		x	y	z	s_1	s_2	a_1	a_2
	-5	-1	-2	-2	0	1	0	0
s_1	4	1	-2	0	1	0	0	0
a_1	2	0	1	1	0	-1	1	0
a_2	3	1	1	1	0	0	0	1

Basic variables are $\{s_1, a_1, a_2\}$. Reduced cost of x is smaller 0. x should enter the basis. $\min\{4/1, 3/1\} = 3$, therefore a_2 leaves the basis. New tableau after after pivot step:

		x	y	z	s_1	s_2	a_1	a_2
	-2	0	-1	-1	0	1	0	1
s_1	1	0	-3	-1	1	0	0	-1
a_1	2	0	1	1	0	-1	1	0
x	3	1	1	1	0	0	0	1

At the end of phase I:

		x	y	z	s_1	s_2	a_1	a_2
	0	0	0	0	0	0	1	1
s_1	3	0	-2	0	1	-1	1	-1
z	2	0	1	1	0	-1	1	0
x	1	1	0	0	0	1	-1	1

Basic feasible solution is $x = 1, y = 0, z = 2$. Optimality can be optained by changing the 0-th to the original costs and pivoting them correctly.

		x	y	z	s_1	s_2
	0	2	3	-1	0	0
s_1	3	0	-2	0	1	-1
z	2	0	1	1	0	-1
x	1	1	0	0	0	1

After pivot steps:

		x	y	z	s_1	s_2
	0	0	4	0	0	-3
s_1	3	0	-2	0	1	-1
z	2	0	1	1	0	-1
x	1	1	0	0	0	1

The reduced cost of $s_2 = -3 < 0 \Rightarrow$ bfs not optimal. Alternative argument:

The objective value of this solution is 0. For a $\epsilon > 0$, the solution $x = 1, y = \epsilon, z = 2 - \epsilon$ is still valid and has an objective value of ϵ, therefore, the initial bfs is not optimal.

Exercise 3 (5 Points)
Consider the following pair of primal and dual LPs:

$$\begin{aligned} &\min\ c^T \cdot x \\ &\text{s.t. } A \cdot x \geq b \end{aligned} \qquad\qquad \begin{aligned} &\max\ p^T \cdot b \\ &\text{s.t. } p^T \cdot A = c^T \\ &\quad p \geq 0 \end{aligned}$$

Formulate and prove the Complementary Slackness Theorem. You can use the Weak Duality Theorem/Strong Duality Theorem for your proof.

Solution:

Theorem 1 (Complementary Slackness). *Let x and p be feasible solutions to the primal ans dual LP, respectively. Then x and p are both optimal if and only if:*

$$u_i := p_i(a_i^T \cdot x - b_i) = 0 \text{ for all i,}$$

$$v_i := (c_j - p^T \cdot A_j)x_j = 0 \text{ for all j.}$$

Proof.

- $u_i \geq 0$ since
  $a_i^T x - b_i = 0 \Rightarrow u_i = 0$
  $a_i^T x - b_i \geq 0 \Rightarrow p_i \geq 0 \Rightarrow u_i = 0$

- $v_j \geq 0$ since
  x_j unconstrained $\Rightarrow p_T \cdot A_j = c_j \Rightarrow v_j = 0$
  $x_j \geq 0 \Rightarrow p^T \cdot A_j \leq c_j \Rightarrow v_j \geq 0$

Define $u := \sum_i u_i, v = \sum_j v_j$. Clearly, $u, v \geq 0$. Furthermore

$$u = 0 \Leftrightarrow u_i = 0 \text{ for all } i$$

and

$$v = 0 \Leftrightarrow v_j = 0 \text{ for all } j.$$

$$\begin{aligned} u + v = & \sum_i p_i(a_i^T \cdot x - b_i) + \sum_j (c_j - p^T \cdot A_j)x_j \\ =& -\sum_i p_i b_i + \sum_j c_j x_j + \sum_i p_i a_i^T x - \sum_j p^T A_j x_j \\ =& -p_i^T b + c^T x + (p^T A)x - p^T(Ax) \\ =& -p^T b + c^t x \end{aligned}$$

Hence: $u + v = -p^T b + c^T x$

Suppose $u_i = 0$ for all i and $v_j = 0$ for all j, then $u + v = 0 \Rightarrow c^T x = p^T b$.
Weak Duality Theorem $\Rightarrow x, p$ are optimal.

Suppose x, p are optimal.
Strong Duality Theorem $\Rightarrow c^T x = p^T b \Rightarrow u + v = 0 \Rightarrow u_i = 0$ for all i and $v_j = 0$ for all j.

□

Grading: 2 points for the theorem, 3 points for the proof.

Topic 3: Optimal Trees & Paths

Topic 4: Maximal & Min Cost Flows

Topic 5: NP-Completeness