Geometrische Grundlagen der Linearen Optimierung

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1 Basic and convex facts

1.1 Notation. $\mathbb{R}^n = \{\boldsymbol{x} = (x_1, \ldots, x_n)^\intercal : x_i \in \mathbb{R}\}$ *denotes the n-dimensional* Euclidean space *equipped with the* Euclidean inner product $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \boldsymbol{x}^\intercal \boldsymbol{y} = \sum_{i=1}^n x_i\, y_i$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, *and the* Euclidean norm $|\boldsymbol{x}| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}$.

1.2 Definition [Linear, affine, positive and convex combination]. *Let $m \in \mathbb{N}$ and let $\boldsymbol{x}_i \in \mathbb{R}^n$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq m$.*

i) $\sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i$ *is called* a linear combination *of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$.*

ii) *If $\sum_{i=1}^m \lambda_i = 1$ then $\sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i$ is called an* affine combination *of $\boldsymbol{x}_1$, $\ldots, \boldsymbol{x}_m$.*

iii) *If $\lambda_i \geq 0$ then $\sum_{i=1}^m \lambda_i\, x_i$ is called* a positive combination *of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$.*

iv) *If $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$ then $\sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i$ is called* a convex combination *of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$.*

v) *Let $X \subseteq \mathbb{R}^n$. $\boldsymbol{x} \in \mathbb{R}^n$ is called* linearly *(*affinely, positively, convexly*)* dependent *of X, if $\boldsymbol{x}$ is a linear (affine, positive, convex) combination of finitely many points of X, i.e., there exist $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in X$, $m \in \mathbb{N}$, such that $\boldsymbol{x}$ is a linear (affine, positive, convex) combination of the points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$.*

1.3 Definition [Linearly and affinely independent points]. *$\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in \mathbb{R}^n$ are called* linearly *(*affinely*)* dependent*, if one of the $\boldsymbol{x}_i$ is linearly (*affinely*) dependent of $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\}\backslash\{\boldsymbol{x}_i\}$. Otherwise $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are called* linearly *(*affinely*)* independent.

1.4 Proposition. *Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in \mathbb{R}^n$.*

i) *$\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are affinely dependent if and only if $\binom{\boldsymbol{x}_1}{1}, \ldots, \binom{\boldsymbol{x}_m}{1} \in \mathbb{R}^{n+1}$ are linearly dependent.*

ii) *$\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are affinely dependent if and only if there exist $\mu_i \in \mathbb{R}$, $1 \leq i \leq m$, with $(\mu_1, \ldots, \mu_m) \neq (0, \ldots, 0)$, $\sum_{i=1}^m \mu_i = 0$ and $\sum_{i=1}^m \mu_i\, \boldsymbol{x}_i = \boldsymbol{0}$.*

iii) *If $m \geq n+1$ then $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are linearly dependent.*

iv) *If $m \geq n+2$ then $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are affinely dependent.*

1.5 Definition [Linear subspace, affine subspace, (convex) cone and convex set].

$X \subseteq \mathbb{R}^n$ is called

i) linear subspace (set) *if it contains all $\boldsymbol{x} \in \mathbb{R}^n$ which are linearly dependent of X,*

ii) affine subspace (set) *if it contains all $\boldsymbol{x} \in \mathbb{R}^n$ which are affinely dependent of X,*

iii) (convex) cone *if it contains all* $\boldsymbol{x} \in \mathbb{R}^n$ *which are positively dependent of* X,

iv) convex set *if it contains all* $\boldsymbol{x} \in \mathbb{R}^n$ *which are convexly dependent of* X.

1.6 Theorem. $K \subseteq \mathbb{R}^n$ *is convex if and only if*

$$\lambda \boldsymbol{x} + (1-\lambda)\,\boldsymbol{y} \in K, \qquad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in K \text{ and } 0 \le \lambda \le 1.$$

1.7 Example.

i) The closed n-dimensional ball $B_n(\boldsymbol{a}, \rho) = \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x} - \boldsymbol{a}| \le \rho\}$ *with centre a and radius $\rho > 0$ is convex. The boundary of $B_n(\boldsymbol{a}, \rho)$, i.e.,* $\{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x} - \boldsymbol{a}| = \rho\}$ *is non-convex. In the case $\boldsymbol{a} = \mathbf{0}$ and $\rho = 1$ the ball $B_n(\mathbf{0}, 1)$ is abbreviated by B_n and is called n-*dimensional unit ball.

ii) Let $\boldsymbol{a} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. The closed halfspaces $H^+(\boldsymbol{a}, \alpha)$, $H^-(\boldsymbol{a}, \alpha) \subset \mathbb{R}^n$ *given by*

$$H^+(\boldsymbol{a}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle \ge \alpha\}, \quad H^-(\boldsymbol{a}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle \le \alpha\}$$

are convex, as well as the hyperplane $H(\boldsymbol{a}, \alpha)$ *defined by*

$$H(\boldsymbol{a}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = \alpha\}.$$

1.8 Corollary. *Let $K_i \subseteq \mathbb{R}^n$, $i \in I$, be convex. Then $\bigcap_{i \in I} K_i$ is convex.*

1.9 Definition [Linear, affine, positive and convex hull, dimension].
Let $X \subseteq \mathbb{R}^n$.

i) *The* linear hull $\operatorname{lin} X$ *of X is defined by*

$$\operatorname{lin} X = \bigcap_{\substack{L \subseteq \mathbb{R}^n,\ L \text{ linear}, \\ X \subseteq L}} L.$$

ii) *The* affine hull $\operatorname{aff} X$ *of X is defined by*

$$\operatorname{aff} X = \bigcap_{\substack{A \subseteq \mathbb{R}^n,\ A \text{ affine}, \\ X \subseteq A}} A.$$

iii) *The* positive (conic) hull $\operatorname{pos} X$ *of X is defined by*

$$\operatorname{pos} X = \bigcap_{\substack{C \subseteq \mathbb{R}^n,\ C \text{ convex cone}, \\ X \subseteq C}} C.$$

iv) *The* convex hull $\operatorname{conv} X$ *of X is defined by*

$$\operatorname{conv} X = \bigcap_{\substack{K \subseteq \mathbb{R}^n,\ K \text{ convex}, \\ X \subseteq K}} K.$$

v) *The* dimension $\dim X$ *of* X *is the dimension of its affine hull, i.e.,* $\dim \operatorname{aff} X$.

1.10 Theorem. *Let* $X \subseteq \mathbb{R}^n$. *Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{m} \lambda_i \, \boldsymbol{x}_i : m \in \mathbb{N}, \boldsymbol{x}_i \in X, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

1.11 Remark.

i) $\operatorname{conv}\{\boldsymbol{x}, \boldsymbol{y}\} = \{\lambda \, \boldsymbol{x} + (1-\lambda) \, \boldsymbol{y} : \lambda \in [0,1]\}$.

ii) $\operatorname{lin} X = \{\sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i : \lambda_i \in \mathbb{R}, \, \boldsymbol{x}_i \in X, \, m \in \mathbb{N}\}$.

iii) $\operatorname{aff} X = \{\sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i : \lambda_i \in \mathbb{R}, \, \sum_{i=1}^{m} \lambda_i = 1, \, \boldsymbol{x}_i \in X, \, m \in \mathbb{N}\}$.

iv) $\operatorname{pos} X = \{\sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i : \lambda_i \in \mathbb{R}, \, \lambda_i \geq 0, \, \boldsymbol{x}_i \in X, \, m \in \mathbb{N}\}$.

1.12 Definition [Linear, affine, convex function]. *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is called*

i) linear *if* $f(\lambda \boldsymbol{x} + \mu \boldsymbol{y}) = \lambda \, f(\boldsymbol{x}) + \mu \, f(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *and* λ, $\mu \in \mathbb{R}$.

ii) affine *if* $f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) = \lambda \, f(\boldsymbol{x}) + (1-\lambda) \, f(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$.

iii) convex *if* $f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq \lambda \, f(\boldsymbol{x}) + (1-\lambda) \, f(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *and* $\lambda \in [0,1]$.

iv) concave *if* $f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \geq \lambda \, f(\boldsymbol{x}) + (1-\lambda) \, f(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *and* $\lambda \in [0,1]$, *i.e.,* $-f$ *is convex.*

1.13 Theorem. *Let* $X \subset \mathbb{R}^n$ *be convex and let* $f : X \to \mathbb{R}$ *be a convex function. Let* $\widetilde{\boldsymbol{x}} \in X$ *be a local minimum of* f *on* X, *i.e., there exists an* $\epsilon \in \mathbb{R}_{>0}$ *with*

$$f(\boldsymbol{x}) \geq f(\widetilde{\boldsymbol{x}}) \text{ for all } \boldsymbol{x} \in X \text{ with } |\boldsymbol{x} - \widetilde{\boldsymbol{x}}| \leq \epsilon.$$

Then $\widetilde{\boldsymbol{x}}$ *is a global minimum, i.e.,*

$$f(\boldsymbol{x}) \geq f(\widetilde{\boldsymbol{x}}) \text{ for all } \boldsymbol{x} \in X.$$

1.14 Definition [Interior and boundary point]. *Let* $X \subseteq \mathbb{R}^n$.

i) $\boldsymbol{x} \in X$ *is called an* interior point *of* X *if there exists a* $\rho > 0$ *such that* $B_n(\boldsymbol{x}, \rho) \subseteq X$. *The set of all interior points of* X *is called the* interior *of* X *and is denoted by* $\operatorname{int} X$.

ii) $\boldsymbol{x} \in \mathbb{R}^n$ *is called* boundary point *of* X *if for all* $\rho > 0$, $B_n(\boldsymbol{x}, \rho) \cap X \neq \emptyset$ *and* $B_n(\boldsymbol{x}, \rho) \cap (\mathbb{R}^n \backslash X) \neq \emptyset$. *The set of all boundary points of* X *is called the* boundary *of* X *and is denoted by* $\operatorname{bd} X$.

1.15 Lemma. *Let $K \subseteq \mathbb{R}^n$ be convex with $\dim K = n$, and let $\boldsymbol{x} \in \operatorname{int} K$ and $\boldsymbol{y} \in K$. Then $(1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y} \in \operatorname{int} K$ for all $\lambda \in [0,1)$.*

1.16 Corollary. *Let $K \subseteq \mathbb{R}^n$ be convex, closed and $\dim K = n$. Let $\boldsymbol{x} \in \operatorname{int} K$ and $\boldsymbol{y} \in \mathbb{R}^n \setminus K$. Then the segment $\operatorname{conv}\{\boldsymbol{x}, \boldsymbol{y}\}$ intersects $\operatorname{bd} K$ in precisely one point.*

1.17 Definition [Polytope and simplex]. *Let $X \subset \mathbb{R}^n$ of finite cardinality, i.e., $\#X < \infty$.*

i) $\operatorname{conv} X$ *is called a* (convex) polytope.

ii) *A polytope $P \subset \mathbb{R}^n$ of dimension k is called a k-polytope.*

iii) *If X is affinely independent and $\dim X = k$ then $\operatorname{conv} X$ is called a k-simplex.*

1.18 Notation. *$\mathcal{P}^n = \{P \subset \mathbb{R}^n : P \text{ polytope}\}$ denotes the set of all polytopes in $\mathbb{R}^n$.*

1.19 Notation.

i) *For two sets $X, Y \subseteq \mathbb{R}^n$ the vectorial addition*

$$X + Y = \{\boldsymbol{x} + y : \boldsymbol{x} \in X,\ \boldsymbol{y} \in Y\}$$

is called the Minkowski [1] sum *of X and Y. If X is just a singleton, i.e., $X = \{\boldsymbol{x}\}$, then we write $\boldsymbol{x} + Y$ instead of $\{\boldsymbol{x}\} + Y$.*

ii) *For $\lambda \in \mathbb{R}$ and $X \subseteq \mathbb{R}^n$ we denote by λX the set*

$$\lambda X = \{\lambda \boldsymbol{x} : \boldsymbol{x} \in X\}.$$

For instance, $B_n(\boldsymbol{a}, \rho) = \boldsymbol{a} + \rho B_n$.

1.20 Theorem [Carathéodory]. [2] *Let $X \subset \mathbb{R}^n$. Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{n+1} \lambda_i \boldsymbol{x}_i : \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1, \boldsymbol{x}_i \in X, i = 1, \ldots, n+1 \right\}.$$

1.21 Corollary. *A polytope is the union of simplices.*

1.22 Corollary. *The convex hull of a compact set is compact.*

1.23 Theorem [Radon]. [3] *Let $X \subset \mathbb{R}^n$. If $\#X \geq n+2$ then there exist $X_1, X_2 \subset X$ with $X_1 \cap X_2 = \emptyset$ and $\operatorname{conv} X_1 \cap \operatorname{conv} X_2 \neq \emptyset$.*

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[1] Hermann Minkowski, 1864–1909
[2] Constantin Carathéodory, 1873 - 1950
[3] Johann Karl August Radon, 1887–1956

1.24 Theorem [Helly]. [4] *Let $K_1, \dots, K_m \subseteq \mathbb{R}^n$, $m \geq n+1$, be convex. If for each $(n+1)$-index set $I \subseteq \{1, \dots, m\} = [m]$*

$$\bigcap_{i \in I} K_i \neq \emptyset,$$

then all sets K_i have a point in common, i.e., $\bigcap_{j=1}^m K_i \neq \emptyset$.

1.25 Corollary. *Let $\boldsymbol{a}_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $1 \leq i \leq m$, and let $P = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq b_i, 1 \leq i \leq m\}$. Then $P \neq \emptyset$ if and only if $P_I = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq b_i, i \in I\} \neq \emptyset$ for all $(n+1)$-index sets $I \subseteq [m]$.*

1.26 Theorem* [Doignon, Scarf, Bell]. *Let $K_1, \dots, K_m \subseteq \mathbb{R}^n$, $m \geq 2^n$, be convex. If for each 2^n-index set $I \subseteq \{1, \dots, m\} = [m]$*

$$\bigcap_{i \in I} (K_i \cap \mathbb{Z}^n) \neq \emptyset,$$

then all sets K_i have an integral point in common, i.e., $\bigcap_{j=1}^m (K_i \cap \mathbb{Z}^n) \neq \emptyset$.

1.27 Corollary. *Let $\boldsymbol{a}_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $1 \leq i \leq m$, and let $P = \{\boldsymbol{x} \in \mathbb{Z}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq b_i, 1 \leq i \leq m\}$. Then $P \neq \emptyset$ if and only if $P_I = \{\boldsymbol{x} \in \mathbb{Z}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq b_i, i \in I\} \neq \emptyset$ for all 2^n-index sets $I \subseteq [m]$.*

[4] Eduard Helly, 1884–1943

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