Heat propagation along a bar

The heat propagation $T(t,\xi)$ along a 1-dimensional bar of length $\ell = 1$ is described by the heat equation

$$\frac{\partial}{\partial t}T(t,\xi) = k\frac{\partial^2}{\partial \xi^2}T(t,\xi), \qquad t \geq 0,\ \xi \in [0,1]$$

with heat conductivity $k > 0$, together with boundary conditions

$$T(t,1) = 0, \quad -\frac{d}{d\xi}T(t,0) = u(t), \quad \text{for all } t \geq 0$$

and initial condition

$$T(0,\xi) = 0 \text{ for all } \xi \in [0,1].$$

The spatial average temperature

$$y(t) = \int_0^1 T(t,\xi)\, d\xi$$

is taken as output quantity.

Semi-discretization yields a LTI system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = 0 \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

with

$$A = kN^2 \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \in \mathbb{R}^{N,N}, \quad B = kN \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{N,1},$$

$$C = \frac{1}{N}\begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \in \mathbb{R}^{1,N}.$$

We choose $N = 1000$, $k = 1$ and $u(t) = 1$ for $t \geq 0$.
The system is approximated by the reduced order model (obtained by balanced truncation)

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}u(t). \quad \tilde{x}(0) = 0 \\ \tilde{y}(t) &= \tilde{C}\tilde{x}(t) \end{aligned} \tag{2}$$

with

$$\tilde{A} = \begin{bmatrix} -2.256 & 1.775 & -0.6057 \\ -1.775 & -16.63 & 12.21 \\ -0.6057 & -12.21 & -40.66 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} -1.074 \\ -0.4136 \\ -0.1442 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} -1.074 & 0.4136 & -0.1442 \end{bmatrix}.$$

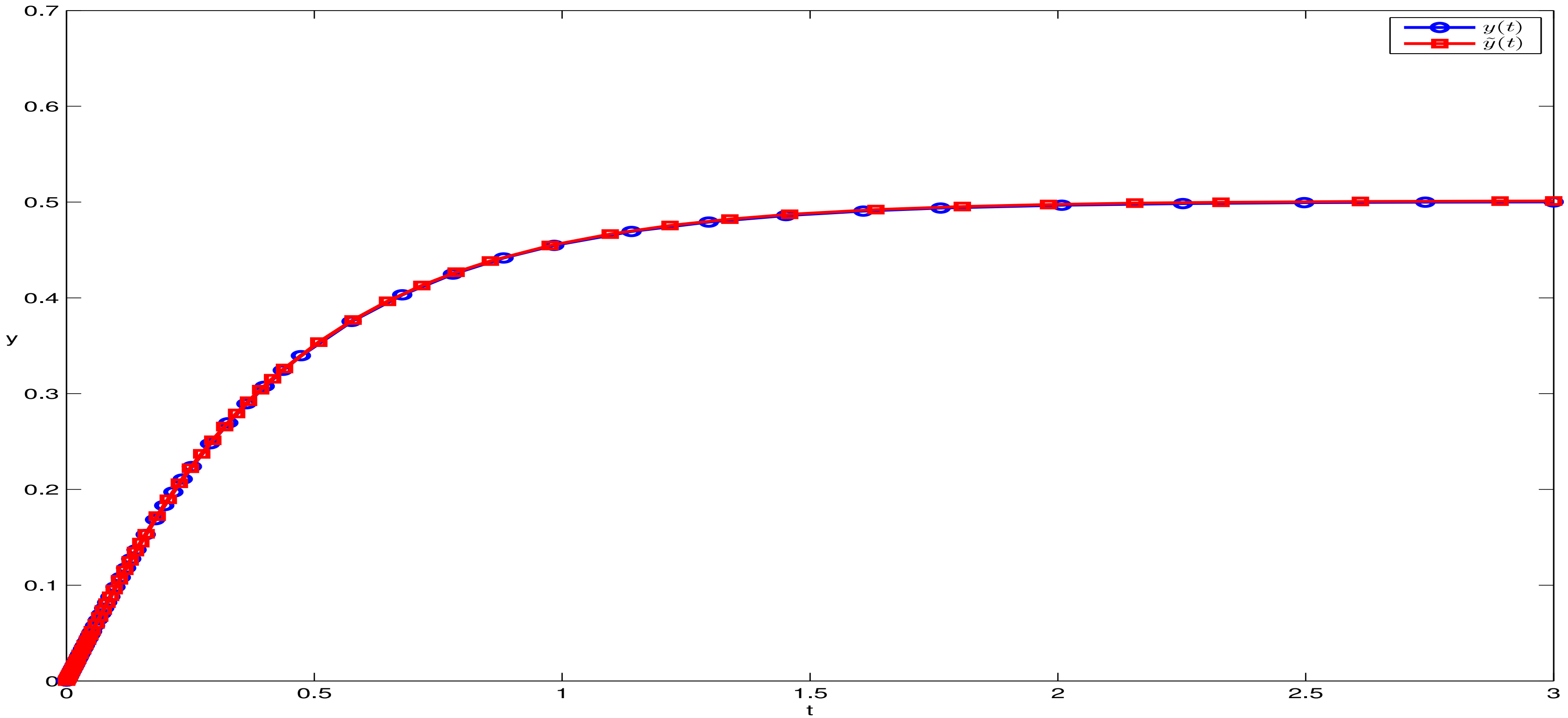

Figure 1: Output of original model (1) (blue) and reduced system model (2) (red)