Geometrische Grundlagen der Linearen Optimierung

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Contents

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1 Basic and convex facts

1.1 Notation. $\mathbb{R}^n = \{\boldsymbol{x} = (x_1, \ldots, x_n)^\intercal : x_i \in \mathbb{R}\}$ *denotes the* n*-dimensional* Euclidean space *equipped with the* Euclidean inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\intercal \boldsymbol{y} = \sum_{i=1}^n x_i\, y_i$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, *and the* Euclidean norm $\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$.

1.2 Definition [Linear, affine, positive and convex combination]. *Let* $m \in \mathbb{N}$ *and let* $\boldsymbol{x}_i \in \mathbb{R}^n$, $\lambda_i \in \mathbb{R}$, $1 \le i \le m$.

i) $\sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i$ *is called* a linear combination *of* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$.

ii) *If* $\sum_{i=1}^m \lambda_i = 1$ *then* $\sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i$ *is called an* affine combination *of* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$.

iii) *If* $\lambda_i \ge 0$ *then* $\sum_{i=1}^m \lambda_i\, x_i$ *is called* a positive combination *of* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$.

iv) *If* $\lambda_i \ge 0$ *and* $\sum_{i=1}^m \lambda_i = 1$ *then* $\sum_{i=1}^m \lambda_i\, \boldsymbol{x}_i$ *is called* a convex combination *of* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$.

v) *Let* $X \subseteq \mathbb{R}^n$. $\boldsymbol{x} \in \mathbb{R}^n$ *is called* linearly *(*affinely, positively, convexly*)* dependent *of* X, *if* $\boldsymbol{x}$ *is a linear (affine, positive, convex) combination of finitely many points of* X, *i.e., there exist* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in X$, $m \in \mathbb{N}$, *such that* $\boldsymbol{x}$ *is a linear (affine, positive, convex) combination of the points* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$.

1.3 Definition [Linearly and affinely independent points]. $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in \mathbb{R}^n$ *are called* linearly *(*affinely*)* dependent, *if one of the* $\boldsymbol{x}_i$ *is linearly (*affinely*) dependent of* $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\} \backslash \{\boldsymbol{x}_i\}$. *Otherwise* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ *are called* linearly *(*affinely*)* independent.

1.4 Proposition. *Let* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in \mathbb{R}^n$.

i) $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ *are affinely dependent if and only if* $\binom{\boldsymbol{x}_1}{1}, \ldots, \binom{\boldsymbol{x}_m}{1} \in \mathbb{R}^{n+1}$ *are linearly dependent.*

ii) $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ *are affinely dependent if and only if there exist* $\mu_i \in \mathbb{R}$, $1 \le i \le m$, *with* $(\mu_1, \ldots, \mu_m) \neq (0, \ldots, 0)$, $\sum_{i=1}^m \mu_i = 0$ *and* $\sum_{i=1}^m \mu_i\, \boldsymbol{x}_i = \boldsymbol{0}$.

iii) *If* $m \ge n+1$ *then* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ *are linearly dependent.*

iv) *If* $m \ge n+2$ *then* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ *are affinely dependent.*

1.5 Definition [Linear subspace, affine subspace, (convex) cone and convex set].

$X \subseteq \mathbb{R}^n$ *is called*

i) linear subspace (set) *if it contains all* $\boldsymbol{x} \in \mathbb{R}^n$ *which are linearly dependent of* X,

ii) affine subspace (set) *if it contains all* $\boldsymbol{x} \in \mathbb{R}^n$ *which are affinely dependent of* X,

iii) (convex) cone *if it contains all* $\boldsymbol{x} \in \mathbb{R}^n$ *which are positively dependent of* X,

iv) convex set *if it contains all* $\boldsymbol{x} \in \mathbb{R}^n$ *which are convexly dependent of* X.

1.6 Theorem. $K \subseteq \mathbb{R}^n$ *is convex if and only if*

$$\lambda \boldsymbol{x} + (1-\lambda)\,\boldsymbol{y} \in K, \qquad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in K \text{ and } 0 \leq \lambda \leq 1.$$

1.7 Example.

i) The closed n*-dimensional ball* $B_n(\boldsymbol{a}, \rho) = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{a}\| \leq \rho\}$ *with centre* a *and radius* $\rho > 0$ *is convex. The boundary of* $B_n(\boldsymbol{a}, \rho)$*, i.e.,* $\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{a}\| = \rho\}$ *is non-convex. In the case* $\boldsymbol{a} = \boldsymbol{0}$ *and* $\rho = 1$ *the ball* $B_n(\boldsymbol{0}, 1)$ *is abbreviated by* B_n *and is called* n-dimensional unit ball.

ii) Let $\boldsymbol{a} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. *The* closed halfspaces $H^+(\boldsymbol{a}, \alpha)$, $H^-(\boldsymbol{a}, \alpha) \subset \mathbb{R}^n$ *given by*

$$H^+(\boldsymbol{a}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle \geq \alpha\}, \quad H^-(\boldsymbol{a}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle \leq \alpha\}$$

are convex, as well as the hyperplane $H(\boldsymbol{a}, \alpha)$ *defined by*

$$H(\boldsymbol{a}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = \alpha\}.$$

1.8 Corollary. *Let* $K_i \subseteq \mathbb{R}^n$, $i \in I$, *be convex. Then* $\bigcap_{i \in I} K_i$ *is convex.*

1.9 Definition [Linear, affine, positive and convex hull, dimension].
Let $X \subseteq \mathbb{R}^n$.

i) *The* linear hull $\operatorname{lin} X$ *of* X *is defined by*

$$\operatorname{lin} X = \bigcap_{\substack{L \subseteq \mathbb{R}^n,\, L \text{ linear}, \\ X \subseteq L}} L.$$

ii) *The* affine hull $\operatorname{aff} X$ *of* X *is defined by*

$$\operatorname{aff} X = \bigcap_{\substack{A \subseteq \mathbb{R}^n,\, A \text{ affine}, \\ X \subseteq A}} A.$$

iii) *The* positive (conic) hull $\operatorname{pos} X$ *of* X *is defined by*

$$\operatorname{pos} X = \bigcap_{\substack{C \subseteq \mathbb{R}^n,\, C \text{ convex cone}, \\ X \subseteq C}} C.$$

iv) *The* convex hull $\operatorname{conv} X$ *of* X *is defined by*

$$\operatorname{conv} X = \bigcap_{\substack{K \subseteq \mathbb{R}^n,\, K \text{ convex}, \\ X \subseteq K}} K.$$

v) *The* dimension $\dim X$ *of* X *is the dimension of its affine hull, i.e.,* $\dim \operatorname{aff} X$.

1.10 Theorem. *Let* $X \subseteq \mathbb{R}^n$. *Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{m} \lambda_i \, \boldsymbol{x}_i : m \in \mathbb{N}, \boldsymbol{x}_i \in X, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

1.11 Remark.

i) $\operatorname{conv}\{\boldsymbol{x}, \boldsymbol{y}\} = \{\lambda \, \boldsymbol{x} + (1-\lambda) \, \boldsymbol{y} : \lambda \in [0,1]\}$.

ii) $\operatorname{lin} X = \{\sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i : \lambda_i \in \mathbb{R}, \, \boldsymbol{x}_i \in X, \, m \in \mathbb{N}\}$.

iii) $\operatorname{aff} X = \{\sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i : \lambda_i \in \mathbb{R}, \, \sum_{i=1}^{m} \lambda_i = 1, \, \boldsymbol{x}_i \in X, \, m \in \mathbb{N}\}$.

iv) $\operatorname{pos} X = \{\sum_{i=1}^{m} \lambda_i \boldsymbol{x}_i : \lambda_i \in \mathbb{R}, \, \lambda_i \geq 0, \, \boldsymbol{x}_i \in X, \, m \in \mathbb{N}\}$.

1.12 Definition [Linear, affine, convex function]. *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is called*

i) linear *if* $f(\lambda \boldsymbol{x} + \mu \boldsymbol{y}) = \lambda \, f(\boldsymbol{x}) + \mu \, f(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *and* $\lambda, \, \mu \in \mathbb{R}$.

ii) affine *if* $f(\lambda \boldsymbol{x} + (1-\lambda) \boldsymbol{y}) = \lambda \, f(\boldsymbol{x}) + (1-\lambda) \, f(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$.

iii) convex *if* $f(\lambda \boldsymbol{x} + (1-\lambda) \boldsymbol{y}) \leq \lambda \, f(\boldsymbol{x}) + (1-\lambda) \, f(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *and* $\lambda \in [0,1]$.

iv) concave *if* $f(\lambda \boldsymbol{x} + (1-\lambda) \boldsymbol{y}) \geq \lambda \, f(\boldsymbol{x}) + (1-\lambda) \, f(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ *and* $\lambda \in [0,1]$, *i.e.,* $-f$ *is convex.*

1.13 Theorem. *Let* $X \subset \mathbb{R}^n$ *be convex and let* $f : X \to \mathbb{R}$ *be a convex function. Let* $\widetilde{\boldsymbol{x}} \in X$ *be a local minimum of* f *on* X, *i.e., there exists an* $\epsilon \in \mathbb{R}_{>0}$ *with*

$$f(\boldsymbol{x}) \geq f(\widetilde{\boldsymbol{x}}) \text{ for all } \boldsymbol{x} \in X \text{ with } \|\boldsymbol{x} - \widetilde{\boldsymbol{x}}\| \leq \epsilon.$$

Then $\widetilde{\boldsymbol{x}}$ *is a global minimum, i.e.,*

$$f(\boldsymbol{x}) \geq f(\widetilde{\boldsymbol{x}}) \text{ for all } \boldsymbol{x} \in X.$$

1.14 Definition [Interior and boundary point]. *Let* $X \subseteq \mathbb{R}^n$.

i) $\boldsymbol{x} \in X$ *is called an* interior point *of* X *if there exists a* $\rho > 0$ *such that* $B_n(\boldsymbol{x}, \rho) \subseteq X$. *The set of all interior points of* X *is called the* interior *of* X *and is denoted by* $\operatorname{int} X$.

ii) $\boldsymbol{x} \in \mathbb{R}^n$ *is called* boundary point *of* X *if for all* $\rho > 0$, $B_n(\boldsymbol{x}, \rho) \cap X \neq \emptyset$ *and* $B_n(\boldsymbol{x}, \rho) \cap (\mathbb{R}^n \backslash X) \neq \emptyset$. *The set of all boundary points of* X *is called the* boundary *of* X *and is denoted by* $\operatorname{bd} X$.

1.15 Lemma. *Let $K \subseteq \mathbb{R}^n$ be convex with $\dim K = n$, and let $\boldsymbol{x} \in \operatorname{int} K$ and $\boldsymbol{y} \in K$. Then $(1-\lambda)\boldsymbol{x} + \lambda\boldsymbol{y} \in \operatorname{int} K$ for all $\lambda \in [0,1)$.*

1.16 Corollary. *Let $K \subseteq \mathbb{R}^n$ be convex, closed and $\dim K = n$. Let $\boldsymbol{x} \in \operatorname{int} K$ and $\boldsymbol{y} \in \mathbb{R}^n \setminus K$. Then the segment $\operatorname{conv}\{\boldsymbol{x}, \boldsymbol{y}\}$ intersects $\operatorname{bd} K$ in precisely one point.*

1.17 Definition [Polytope and simplex]. *Let $X \subset \mathbb{R}^n$ of finite cardinality, i.e., $\#X < \infty$.*

i) $\operatorname{conv} X$ *is called a* (convex) polytope.

ii) *A polytope $P \subset \mathbb{R}^n$ of dimension k is called a k-polytope.*

iii) *If X is affinely independent and $\dim X = k$ then $\operatorname{conv} X$ is called a k-simplex.*

1.18 Notation. *$\mathcal{P}^n = \{P \subset \mathbb{R}^n : P \text{ polytope}\}$ denotes the set of all polytopes in $\mathbb{R}^n$.*

1.19 Notation.

i) *For two sets $X, Y \subseteq \mathbb{R}^n$ the vectorial addition*

$$X + Y = \{\boldsymbol{x} + y : \boldsymbol{x} \in X,\ \boldsymbol{y} \in Y\}$$

is called the Minkowski [1] sum *of X and Y. If X is just a singleton, i.e., $X = \{\boldsymbol{x}\}$, then we write $\boldsymbol{x} + Y$ instead of $\{\boldsymbol{x}\} + Y$.*

ii) *For $\lambda \in \mathbb{R}$ and $X \subseteq \mathbb{R}^n$ we denote by $\lambda\, X$ the set*

$$\lambda\, X = \{\lambda\, \boldsymbol{x} : \boldsymbol{x} \in X\}\,.$$

For instance, $B_n(\boldsymbol{a}, \rho) = \boldsymbol{a} + \rho\, B_n$.

1.20 Theorem [Carathéodory]. [2] *Let $X \subseteq \mathbb{R}^n$. Then*

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{n+1} \lambda_i\, \boldsymbol{x}_i : \lambda_i \geq 0, \sum_{i=1}^{n+1} \lambda_i = 1, \boldsymbol{x}_i \in X, i = 1, \ldots, n+1 \right\}.$$

1.21 Corollary. *A polytope is the union of simplices.*

1.22 Corollary. *The convex hull of a compact set is compact.*

1.23 Theorem [Radon]. [3] *Let $X \subset \mathbb{R}^n$. If $\#X \geq n+2$ then there exist $X_1, X_2 \subset X$ with $X_1 \cap X_2 = \emptyset$ and $\operatorname{conv} X_1 \cap \operatorname{conv} X_2 \neq \emptyset$.*

[1] Hermann Minkowski, 1864–1909
[2] Constantin Carathéodory, 1873 - 1950
[3] Johann Karl August Radon, 1887–1956

1.24 Theorem [Helly]. [4] *Let $K_1, \dots, K_m \subseteq \mathbb{R}^n$, $m \geq n+1$, be convex. If for each $(n+1)$-index set $I \subseteq \{1, \dots, m\} = [m]$*

$$\bigcap_{i \in I} K_i \neq \emptyset,$$

then all sets K_i have a point in common, i.e., $\bigcap_{j=1}^m K_i \neq \emptyset$.

1.25 Corollary. *Let $\boldsymbol{a}_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $1 \leq i \leq m$, and let $P = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq b_i,\ 1 \leq i \leq m\}$. Then $P \neq \emptyset$ if and only if $P_I = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq b_i,\ i \in I\} \neq \emptyset$ for all $(n+1)$-index sets $I \subseteq [m]$.*

1.26 Theorem* [Doignon, Scarf, Bell]. *Let $K_1, \dots, K_m \subseteq \mathbb{R}^n$, $m \geq 2^n$, be convex. If for each 2^n-index set $I \subseteq \{1, \dots, m\} = [m]$*

$$\bigcap_{i \in I} (K_i \cap \mathbb{Z}^n) \neq \emptyset,$$

then all sets K_i have an integral point in common, i.e., $\bigcap_{j=1}^m (K_i \cap \mathbb{Z}^n) \neq \emptyset$.

1.27 Corollary. *Let $\boldsymbol{a}_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $1 \leq i \leq m$, and let $P = \{\boldsymbol{x} \in \mathbb{Z}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq b_i,\ 1 \leq i \leq m\}$. Then $P \neq \emptyset$ if and only if $P_I = \{\boldsymbol{x} \in \mathbb{Z}^n : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \leq b_i,\ i \in I\} \neq \emptyset$ for all 2^n-index sets $I \subseteq [m]$.*

[4] Eduard Helly, 1884–1943

2 Support and separate

2.1 Definition [Supporting hyperplane]. *Let $X \subset \mathbb{R}^n$. A hyperplane $H(\boldsymbol{a}, \alpha) \subset \mathbb{R}^n$ is called* supporting hyperplane *of X if:*

$$\text{i) } H(\boldsymbol{a}, \alpha) \cap X \neq \emptyset \quad \text{and} \quad \text{ii) } X \subseteq H^-(\boldsymbol{a}, \alpha).$$

$\boldsymbol{a}$ is called outer normal vector *of X and if, in addition, $\|\boldsymbol{a}\| = 1$ then it is called* outer unit normal vector *of X.*

2.2 Proposition. *Let $X \subset \mathbb{R}^n$ and let $H(\boldsymbol{a}, \alpha)$ be a supporting hyperplane of X. Then $H(\boldsymbol{a}, \alpha)$ is a supporting hyperpalne of* conv X *and*

$$H(\boldsymbol{a}, \alpha) \cap \operatorname{conv} X = \operatorname{conv}\big(H(\boldsymbol{a}, \alpha) \cap X\big).$$

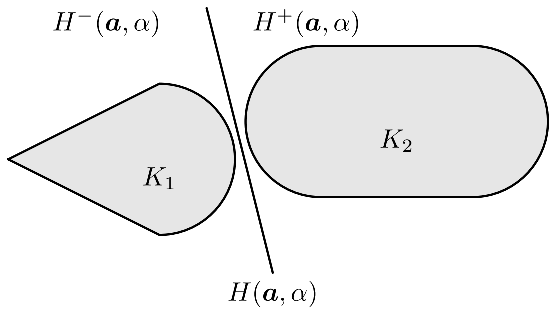


Figure 1: A strictly separating hyperplane of two compact convex sets

2.3 Theorem [Separation theorem]. *Let $K_1, K_2 \subset \mathbb{R}^n$ be convex with $K_1 \cap K_2 = \emptyset$. Then there exists* a separating hyperplane $H(\boldsymbol{a}, \alpha)$ *of K_1 and K_2, i.e., $K_1 \subseteq H^+(\boldsymbol{a}, \alpha)$ and $K_2 \subseteq H^-(\boldsymbol{a}, \alpha)$.*

If K_1 is closed and K_2 is compact, then there exists even a strictly separating hyperplane $H(\boldsymbol{a}, \alpha)$ *of K_1 and K_2, i.e., $K_1 \subset \operatorname{int} H^+(\boldsymbol{a}, \alpha)$ and $K_2 \subset \operatorname{int} H^-(\boldsymbol{a}, \alpha)$.*

2.4 Corollary [Farkas' Lemma]. [5]*Let $A \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. There exists a non-negative $\boldsymbol{x} \in \mathbb{R}^n_{\geq \boldsymbol{0}}$ with $A\boldsymbol{x} = \boldsymbol{b}$ if and only if $\langle \boldsymbol{b}, \boldsymbol{y} \rangle \geq 0$ for all $\boldsymbol{y} \in \mathbb{R}^m$ satisfying $A^\intercal \boldsymbol{y} \geq \boldsymbol{0}$, i.e.,*

$$P = \{\boldsymbol{x} \in \mathbb{R}^n_{\geq \boldsymbol{0}} : A\boldsymbol{x} = \boldsymbol{b}\} \neq \emptyset \Leftrightarrow \inf\{\langle \boldsymbol{b}, \boldsymbol{y} \rangle : \boldsymbol{y} \in \mathbb{R}^m \text{ with } A^\intercal \boldsymbol{y} \geq \boldsymbol{0}\} \geq 0.$$

2.5 Corollary. *Let $K \subset \mathbb{R}^n$ be convex and closed,* $\dim K = n$, *and let $\boldsymbol{x} \in \operatorname{bd} K$. Then there exists a supporting hyperplane $H(\boldsymbol{a}, \alpha)$ of K containing $\boldsymbol{x}$.*

2.6 Theorem. *Let $K \subset \mathbb{R}^n$, $K \neq \mathbb{R}^n$, be convex and closed,* $\dim K = n$. *Then*

$$K = \bigcap_{\substack{H(\boldsymbol{a},\alpha) \text{ supporting} \\ \text{hyperplane of } K}} H^-(\boldsymbol{a}, \alpha),$$

i.e., K is the intersection of all its "supporting halfspaces".

[5] Gyula Farkas, 1847–1930

2.7 Definition [Support function]. *Let $K \subset \mathbb{R}^n$ be convex, $K \neq \emptyset$. The function $h(K, \cdot) : \mathbb{R}^n \to \mathbb{R}$ given by*

$$h(K, \boldsymbol{u}) = \sup\big\{\langle \boldsymbol{u}, \boldsymbol{x}\rangle : \boldsymbol{x} \in K\big\}$$

is called support function *of K.*

2.8 Proposition. *Let $K \subset \mathbb{R}^n$, $K \neq \emptyset$ be convex and compact. Then*

$$K = \bigcap_{\boldsymbol{u} \in B_n, \|\boldsymbol{u}\|=1} \big\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}, \boldsymbol{x}\rangle \leq h(K, \boldsymbol{u})\big\}.$$

2.9 Definition. *Let $K \subseteq \mathbb{R}^n$ be convex and closed. The set*

$$\operatorname{rec} K = \{\boldsymbol{u} \in \mathbb{R}^n : K + \boldsymbol{u} \subseteq K\}$$

is called the recession cone *of K.*

2.10 Proposition. *Let $K \subseteq \mathbb{R}^n$ be convex and closed, and let $\boldsymbol{x} \in K$. Then*

$$\operatorname{rec}(K) = \{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{x} + \lambda \boldsymbol{u} \in K \text{ for all } \lambda \in \mathbb{R}_{\geq 0}\}$$

In particular, $\operatorname{rec}(K)$ *is a closed convex cone.*

2.11 Theorem. *Let $K \subseteq \mathbb{R}^n$ be convex and closed. Then K can be represented as*

$$K = \overline{K} \oplus L,$$

where $L \subseteq \mathbb{R}^n$ is a linear subspace and $\overline{K} \subset \overline{L}$ is a line-free *convex set contained in a complementary linear subspace $\overline{L}$ of L.*

2.12 Definition [Polar set]. *Let $X \subseteq \mathbb{R}^n$.*

$$X^\star = \big\{\boldsymbol{y} \in \mathbb{R}^n : \langle \boldsymbol{x}, \boldsymbol{y}\rangle \leq 1 \text{ for all } \boldsymbol{x} \in X\big\}$$

is called the polar set *of X.*

2.13 Proposition.

i) *$X^\star$ is a convex and closed set and $\mathbf{0} \in X^\star$.*

ii) *If $X_1 \subseteq X_2$ then $X_2^\star \subseteq X_1^\star$.*

iii) *Let M be a regular $n \times n$ matrix. Then $(MX)^\star = M^{-\intercal} X^\star$.*

iv) *Let $X_i \subseteq \mathbb{R}^n$, $i \in I$. Then $\big(\bigcup_{i \in I} X_i\big)^\star = \bigcap_{i \in I} X_i^\star$.*

v) *$X \subseteq (X^\star)^\star$.*

vi) *Let $X \subset \mathbb{R}^n$. Then $X = X^\star$ if and only if $X = B_n$.*

2.14 Lemma. *Let $K \subset \mathbb{R}^n$ be convex and closed with $\mathbf{0} \in K$. Then $(K^\star)^\star = K$.*

2.15 Proposition.

i) *Let $P = \operatorname{conv}\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\} \subset \mathbb{R}^n$. Then*

$$P^\star = \big\{\boldsymbol{y} \in \mathbb{R}^n : \langle \boldsymbol{x}_i, \boldsymbol{y}\rangle \leq 1,\ 1 \leq i \leq m\big\}.$$

ii) *Let $P = \big\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{a}_i, \boldsymbol{x}\rangle \leq 1,\ 1 \leq i \leq m\big\}$ with $\boldsymbol{a}_i \in \mathbb{R}^n$. Then*

$$P^\star = \operatorname{conv}\{\mathbf{0}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}.$$

2.16 Proposition. *Let $K \subseteq \mathbb{R}^n$ be a convex cone. Then*

$$K^\star = \{\boldsymbol{y} \in \mathbb{R}^n : \langle \boldsymbol{x}, \boldsymbol{y}\rangle \leq 0 \text{ for all } \boldsymbol{x} \in K\}.$$

3 A rough guide to polytopes

3.1 Definition [Polyhedron]. *The intersection of finitely many closed halfspaces is called a* polyhedron.

3.2 Theorem [Minkowski, Weyl]. [6,7]

i) *A bounded polyhedron is a polytope.*

ii) *A polytope is a bounded polyhedron.*

3.3 Notation [$\mathcal{V}$-Polytope, $\mathcal{H}$-Polytope]. *A polytope given as the convex hull of finitely many points is called a* $\mathcal{V}$-polytope. *If it is given as the bounded intersection of finitely many closed halfspaces, then it is called an* $\mathcal{H}$-polytope.

3.4 Corollary. *Let $P \in \mathcal{P}^n$.*

i) *Let $A \in \mathbb{R}^{m\times n}$ and $\boldsymbol{t} \in \mathbb{R}^m$. Then $AP + \boldsymbol{t}$ is a polytope.*

ii) *Let $U \subset \mathbb{R}^n$ be an affine subspace. Then $P \cap U$ is a polytope.*

3.5 Definition [Faces]. *Let $K \in \mathcal{C}^n$ be closed and let H be a supporting hyperplane of K. If $j = \dim(K \cap H)$, then $K \cap H$ is called a j-*face *of K. Moreover, K itself is regarded as a $(\dim K)$-face and the empty set $\emptyset$ as (-1)-face of K.*

3.6 Notation [Vertices, exposed points, edges, facets]. *A 0-face of $K \in \mathcal{C}^n$, K closed, is called an* exposed point *or in the case of polyhedra a* vertex, *a 1-face of a polytope is called* edge *and a $(\dim K - 1)$-face of K is called* facet *of K. K itself and the empty set are called* improper faces, *whereas the remaining faces are called* proper faces *of K.*

The set of all vertices of a polyhedra P is denoted by $\operatorname{vert} P$.

3.7 Remark.

i) *Let $K \in \mathcal{C}^n$ be closed. Every (relative) boundary point of K lies in a suitable j-face, $0 \le j \le \dim K - 1$ (cf. Corollary 2.5).*

ii) *Let $K \in \mathcal{C}^n$, $\dim K = n$. Let F be a facet of K and H a supporting hyperplane of K with $F = K \cap H$. Then $H = \operatorname{aff} F$.*

iii) *A point $\boldsymbol{v} \in K$, $K \in \mathcal{C}^n$, is called an* extreme point *of K, if it can not be written as a convex combination of two other points in K.*

iv) *Exposed points are extreme: Let $\boldsymbol{v} \in K$ be expsoed, i.e., let $H(\boldsymbol{a}, \alpha)$ be a supporting plane of K at $\boldsymbol{v}$. Then we have $\langle \boldsymbol{a}, \boldsymbol{v}\rangle = \alpha$ and $\langle \boldsymbol{a}, \boldsymbol{x}\rangle < \alpha$ for all $\boldsymbol{x} \in K \setminus \{\boldsymbol{v}\}$. Hence we cannot write $\boldsymbol{v} = \lambda\,\boldsymbol{x}_1 + (1-\lambda)\,\boldsymbol{x}_2$ with $\boldsymbol{x}_i \in K \setminus \{\boldsymbol{v}\}$, $\lambda \in [0,1]$.*

[6] Hermann Minkowski, 1864–1909
[7] Hermann Klaus Hugo Weyl, 1885 – 1955

3.8 Proposition. *Every face of a polytope is a polytope, and a polytope has only finitely many faces.*

3.9 Definition [f-vector]. *For $P \in \mathcal{P}^n$ let $f_i(P)$ be the number of i-faces of P, $-1 \leq i \leq \dim P$. Furthermore, let $f_i(P) = 0$ for $\dim P + 1 \leq i \leq n$. The vector $f(P)$ with entries $f_i(P)$, $-1 \leq i \leq n$, is called the f-*vector *of P.*

3.10 Remark. *Let T_n be an n-dimensional simplex. Then $f_i(T_n) = \binom{n+1}{i+1}$, i.e., any $(i+1)$ subset of the vertices are the vertices of an i-face.*

3.11 Lemma. *Let $P \in \mathcal{P}^n$. Then $P = \operatorname{conv}(\operatorname{vert} P)$ and for any set $W \subset \mathbb{R}^n$ with $P = \operatorname{conv} W$ it is $\operatorname{vert} P \subseteq W$.*

3.12 Lemma. *Let $P \in \mathcal{P}^n$ be an n-polytope with $\mathbf{0} \in \operatorname{int} P$. For a proper face F of P let*

$$F^\diamond = \{\boldsymbol{y} \in P^\star : \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 1 \text{ for all } \boldsymbol{x} \in F\}.$$

Then

i) *$F^\diamond$ is a face of $P^\star$.*

ii) *$F = (F^\diamond)^\diamond$.*

iii) *If G is a face of P and $F \subseteq G$, then $G^\diamond \subseteq F^\diamond$.*

iv) *$\dim F^\diamond = n - 1 - \dim F$.*

3.13 Theorem. *Let $P \in \mathcal{P}^n$ be an n-polytope with $\mathbf{0} \in \operatorname{int} P$. Then*

$$f_{n-1-i}(P^\star) = f_i(P), \quad -1 \leq i \leq n.$$

3.14 Theorem. *Let $P \in \mathcal{P}^n$ be an n-polytope with facets $F_1, \ldots, F_m$ and let $H(\boldsymbol{a}_i, \alpha_i)$, $1 \leq i \leq m$, be the supporting hyperplanes of F_i, $1 \leq i \leq m$. Then*

$$P = \bigcap_{i=1}^{m} H^-(\boldsymbol{a}_i, \alpha_i).$$

3.15 Theorem. *Let $P \in \mathcal{P}^n$ be an n-polytope.*

i) *The boundary of P is the union of all its facets.*

ii) *A k-face is the intersection of (at least) $(n-k)$ facets.*

iii) *An $(n-2)$-face is contained in exactly two facets.*

iv) *If F, G are faces of P with $F \subseteq G$, then F is a face of G.*

v) *A face of P is also a face of a facet of P.*

3.16 Theorem. *Let $P \in \mathcal{P}^n$ be an n-polytope.*

i) *Let G be a face of P and let F be a face of G. Then F is a face of P.*

ii) *Let F_j be a j-face of P and let F_k be a k-face of P with $F_j \subset F_k$. There exist i-faces F_i of P, $j < i < k$, such that*

$$F_j \subset F_{j+1} \subset \cdots \subset F_{k-1} \subset F_k.$$

3.17 Remark. *For any n-polytope $P \in \mathcal{P}^n$ we have $f_i(P) \geq \binom{n+1}{i+1}$, $1 \leq i \leq n-1$, with equality if and only if P is an n-simplex. In particular, we have $\sum_{i=-1}^{n} f_i(P) \geq 2^{n+1}$ with equality if and onyl if P is an n-simplex.*

3.18 Proposition. *Let $\boldsymbol{v}_0$ be a vertex of an n-polytope P and let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ be all* adjacent *vertices of $\boldsymbol{v}_0$, i.e., $\operatorname{conv}\{\boldsymbol{v}_0, \boldsymbol{v}_i\}$ is an edge of P. In other words, $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$ are the* neighbours *of $\boldsymbol{v}_0$. Then*

i) *$P \subset \boldsymbol{v}_0 + \operatorname{pos}\{\boldsymbol{v}_1 - \boldsymbol{v}_0, \ldots, \boldsymbol{v}_r - \boldsymbol{v}_0\}$.*

ii) *Let $\boldsymbol{c} \in \mathbb{R}^n$ with $\langle \boldsymbol{c}, \boldsymbol{v}_0 \rangle \geq \langle \boldsymbol{c}, \boldsymbol{v}_i \rangle$, $1 \leq i \leq r$. Then*

$$\max\{\langle \boldsymbol{c}, \boldsymbol{x} \rangle : \boldsymbol{x} \in P\} = \langle \boldsymbol{c}, \boldsymbol{v}_0 \rangle.$$

3.19 Theorem [Euler-Poincaré formula]. [8] [9] *Let $P \in \mathcal{P}^n$. Then*

$$\sum_{i=-1}^{n} (-1)^i f_i(P) = 0. \tag{3.19.1}$$

In particular, in the 3-dimensional case, i.e., $\dim P = 3$, it holds $f_0 - f_1 + f_2 = 2$.

3.20 Proposition. *The Euler-Poincaré formula is the only linear equation satisfied by the f-vector, i.e., let $\lambda_i \in \mathbb{R}$, such that $\sum_{i=-1}^{n} \lambda_i f_i(P) = 0$ for all $P \in \mathcal{P}^n$. Then there exists a constant $\gamma \in \mathbb{R}$, such that $\lambda_i = \gamma (-1)^i$.*

3.21 Definition [Simple and simplicial polytopes]. *Let $P \in \mathcal{P}^n$.*

i) *P is called* simplicial *if all proper faces are simplices.*

ii) *P is called* simple *if every vertex is contained in exactly $\dim P$ many facets.*

3.22 Lemma. *Let $P \in \mathcal{P}^n$ be an n-polytope with $\mathbf{0} \in \operatorname{int} P$. The following statements are equivalent:*

i) *P is simplicial.*

[8] Leonhard Euler,1707–1783
[9] Henri Poincaré, 1854–1912

ii) *All facets of P are simplices.*

iii) *$P^\star$ is simple.*

iv) *Every k-face of $P^\star$ is contained in exactly $n-k$ facets for $k = 0, \dots, n-1$.*

3.23 Theorem. *Let $P \in \mathcal{P}^n$ be a simple n-polytope. Then*

i) *Every vertex is contained in exactly $\binom{n}{k}$ k-faces of P, $k = 0, \dots, n-1$.*

ii) *The intersection of k facets containing a common vertex is an $(n-k)$-face of P.*

iii) *Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ be the neighbours of a vertex $\boldsymbol{v}_0$ of P. For each subset of k neighbours $\boldsymbol{v}_{i_1}, \dots, \boldsymbol{v}_{i_k}$ there exists a unique k-face F of P containing $\boldsymbol{v}_0, \boldsymbol{v}_{i_1}, \dots, \boldsymbol{v}_{i_k}$.*

iv) *A face of a simple polytope is simple.*

v) *Every j-face of P is contained in exactly $\binom{n-j}{k-j}$ k-faces of P.*

3.24 Theorem. *Let $P \in \mathcal{P}^n$ be a simple n-polytope.*

i) $n\, f_0(P) = 2\, f_1(P)$.

ii) $\sum_{k=0}^{n} f_k(P) \le 2^n f_0(P)$.

iii) $f_0(P) \le 2 f_{\lceil n/2 \rceil}(P)$.

3.25 Corollary. *Let P be a simple n-polytope with m facets. Then*

$$f_0(P) \le 2\binom{m}{\lfloor n/2 \rfloor} \quad \text{and} \quad \sum_{k=0}^{n} f_k(P) \le 2^{n+1}\binom{m}{\lfloor n/2 \rfloor}.$$

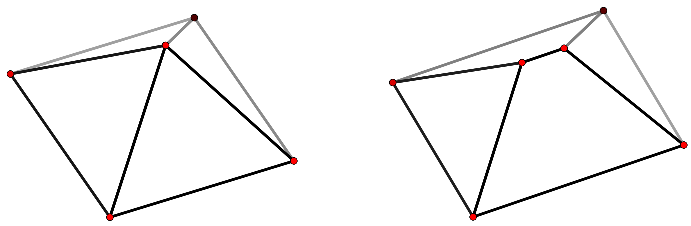
Or equivalently: Let P be a simplicial n-polytope with m vertices. Then

$$f_{n-1}(P) \le 2\binom{m}{\lfloor n/2 \rfloor} \quad \text{and} \quad \sum_{k=0}^{n} f_k(P) \le 2^{n+1}\binom{m}{\lfloor n/2 \rfloor}.$$

3.26 Lemma*. *Let P be an n-polytope.*

i) *There exists a simple n-polytope Q with the same number of facets as P and $f_i(P) \le f_i(Q)$, $0 \le i \le n-2$.*

ii) *There exists a simplical n-polytope $Q^\star$ with the same number of vertices as P and $f_i(P) \le f_i(Q^\star)$, $1 \le i \le n-1$.*



3.27 Corollary. *Let P be an n-polytope with m facets. Then*

$$f_0(P) \leq 2\binom{m}{\lfloor n/2 \rfloor}.$$

Or equivalently: Let P be an n-polytope with m vertices. Then

$$f_{n-1}(P) \leq 2\binom{m}{\lfloor n/2 \rfloor}.$$

3.28 Definition [Cyclic polytopes]. *The curve $\gamma : \mathbb{R} \to \mathbb{R}^n$ given by $\gamma(t) = (t, t^2, t^3, \dots, t^n)^\intercal$ is called* moment curve. *The convex hull of m points on the moment curve is called a* cyclic polytope *with m vertices and is denoted by $C(n, m)$.*

3.29 Proposition. *Any $n + 1$ points on the moment curve are affinely independent. In particular, cyclic polytopes are simplicial polytopes.*

3.30 Proposition [Gale's evenness condition]. *Let $t_i \in \mathbb{R}$, $1 \leq i \leq m$, $t_1 < t_2 < \cdots < t_m$, $\gamma(t_i) = (t_i, t_i^2, t_i^3, \dots, t_i^n)^\intercal$, $1 \leq i \leq m$, and let $S \subset \{1, \dots, m\}$ be a subset of cardinality n. $F_S = \operatorname{conv}\{\gamma(t_s) : s \in S\}$ is a facet of $C(n, m)$ if and only if $\#\{s \in S : i < s < j\}$ is even for all $i, j \in \{1, \dots, m\} \setminus S$.*

3.31 Remark. *All points $\gamma(t_i)$ are vertices of $C(n, m)$ and the number of i-faces of $C(n, m)$ is independent of the choice of the m-points on the moment curve. In fact, for any choice of m points on $\gamma(t)$ the cyclic polytopes are combinatorial isomorphic.*

3.32 Proposition. *The cyclic polytope $C(n, m)$ is $\lfloor n/2 \rfloor$-neighborly, i.e., the convex hull of any subset of the vertices of cardinality less than or equal $n/2$ is a face.*

3.33 Theorem* [McMullen's Upper Bound Theorem, 1971]. [10] *Let P be an n-polytope with m vertices. Then*

$$f_i(P) \leq f_i(C(n, m)) = \begin{cases} \sum_{j=0}^{(n-1)/2} \frac{i+2}{m-j}\binom{m-j}{j+1}\binom{j+1}{i+1-j}, & n \text{ odd}, \\ \sum_{j=1}^{n/2} \frac{m}{m-j}\binom{m-j}{j}\binom{j}{i+1-j}, & n \text{ even}. \end{cases}$$

In particular,

$$f_{n-1}(P) \leq f_{n-1}(C(n, m)) = \begin{cases} 2\binom{m-\lfloor n/2 \rfloor-1}{\lfloor n/2 \rfloor}, & n \text{ odd}, \\ \binom{m-\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor} + \binom{m-\lfloor n/2 \rfloor-1}{\lfloor n/2 \rfloor-1}, & n \text{ even}. \end{cases}$$

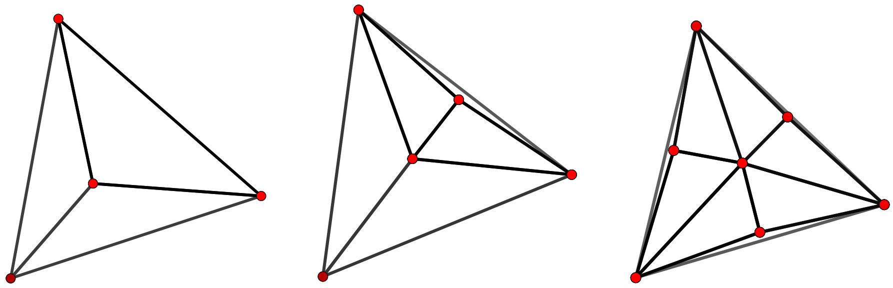
For fixed n the right hand sides are of order $m^{\lfloor n/2 \rfloor}$.

[10] Peter McMullen, born 1942

3.34 Theorem* [Barnette's Lower Bound Theorem, 1973]. [11] *Let P be a simplicial n-polytope with m vertices. P has at least as many i-faces as the so called* stacked polytopes *$P(n,m)$ with m vertices for which*

$$f_i(P(n,m)) = \begin{cases} m\binom{n}{i} - i\binom{n+1}{i+1}, & 0 \le i \le n-2, \\ n+1+(m-(n+1))(n-1), & i = n-1. \end{cases}$$

$P(n,n+1)$ is an n-simplex, and for $m \ge n+2$ an m-vertex stacked n-polytope $P(n,m)$ is the convex hull of an $(m-1)$-vertex stacked polytope with an additional point that is beyond exactly one facet.



3.35 Theorem [Dehn-Sommerville equations, 1905, 1927]. *Let P be a simple n-polytope. Then*

$$f_i(P) = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} f_j(P), \quad i = 0, \ldots, n,$$

Or equivalently: Let P be a simplicial n-polytope. Then

$$f_{i-1}(P) = \sum_{j=i}^{n} (-1)^{n-j} \binom{j}{i} f_{j-1}(P), \quad i = 0, \ldots, n.$$

3.36 Remark. *For any n-polytope $P \in \mathcal{P}^n$ we have $n f_0(P) \le 2\, f_1(P)$ with equality iff P simple and $n\, f_{n-1}(P) \le 2\, f_{n-2}(P)$ with equality iff P simplicial.*

3.37 Theorem [Steinitz, 1906]. [12] *A non-negative integral vector (f_0, f_1, f_2) is the f-vector of a 3-polytope if and only if* i) $f_0 - f_1 + f_2 = 2$, ii) $3\, f_0 \le 2\, f_1$, *and* iii) $3\, f_2 \le 2 f_1$.

3.38 Conjecture [Kalai, 1989]. [13] *Let $P \in \mathcal{P}^n$ be a 0-symmetric n-polytope. Then*

$$\sum_{i=0}^{n} f_i(P) \ge 3^n.$$

Here we have equality, for instance, for the cube C_n and its polar, the cross-poyltope $C_n^\star$, or, more generally, for the class of Hanner-polytopes. In 2007 the conjecture has been verified for all $n \le 4$ (see `http://front.math.ucdavis.edu/0708.3661`).

[11] David W. Barnette
[12] Ernst Steinitz, 1871 – 1928
[13] Gil Kalai, born 1955

4 Walking on polytopes

4.1 Definition [Graph, combinatorial diameter]. *Let $P \subset \mathbb{R}^n$ be a polyhedron.*

i) *The* graph *(1-skeleton) $G(P)$ of P consists of the vertices and edges of P.*

ii) *The distances $\delta_P(\boldsymbol{v}, \boldsymbol{w})$ between two vertices $\boldsymbol{v}, \boldsymbol{w} \in P$ (or in $G(P)$) is the minimum length of an "edge" path connecting $\boldsymbol{v}$ and $\boldsymbol{w}$ in $G(P)$.*

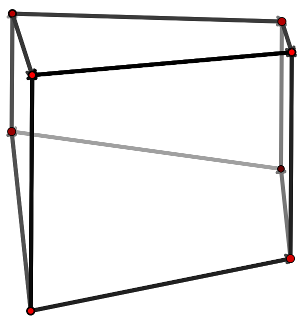
iii) *$\delta(P) = \max\{\delta_P(\boldsymbol{v}, \boldsymbol{w}) : \boldsymbol{v}, \boldsymbol{w} \in \operatorname{vert} P\}$ is called the* (combinatorial) diameter *of P.*

4.2 Theorem [Balinski, 1961]. *Let $P \in \mathcal{P}^n$ be an n-polytope. The graph $G(P)$ of P is n-connected, i.e., the graph is still connected if $n-1$ vertices and their incident edges are removed.*

4.3 Example. *$\delta(T_n) = 1 = (n+1) - n$, $\delta(C_n) = n = 2\,n - n$ and $\delta(C_n^\star) = 2 \leq 2^n - n$.*

4.4 Definition. *For integers n, m let*

$$\Delta(n, m) = \max\left\{\delta(P) : P \subset \mathbb{R}^n \text{ polyhedron, } \dim P = n \text{ and } f_{n-1}(P) = m\right\}.$$



4.5 Remark. *In 1957 Hirsch [14]conjectured $\Delta(n, m) \leq m - n$. It is known that*

i) *the conjecture is true if $n \leq 3$ or $m \leq n+5$. For unbounded polyhedra the conjecture is false, namely, for $m \geq 2n$ it is $\Delta(n, m) \geq m - n + \lfloor n/4 \rfloor$. (Klee&Walkup, 1961/1965),*

ii) *$\Delta(n, m) \leq m\, 2^{n-3}$, (Barnette, 1969; Larman, 1970),*

iii) *Disproof of the Hirsch conjecture for polytopes by Francisco Santos, 2010, see* `http://front.math.ucdavis.edu/1006.2814`

4.6 Theorem [Kalai, 1992; Kalai&Kleitman, 1992]. [15]

$$\Delta(n, m) \leq m^{\log n + 2}.$$

[14] Warren M. Hirsch

[15] Daniel J. Kleitman, born 1934

4.7 Definition [0/1**-polytope**]. *Let* $[0,1]^n$ *be the* n*-dimensional unit cube with vertices* $\{0,1\}^n = \{(x_1,\dots,x_n)^\intercal : x_i \in \{0,1\}\}$. $P \in \mathcal{P}^n$ *is called* 0/1-polytope *if* $\operatorname{vert} P \subset \{0,1\}^n$.

4.8 Lemma. *Let* $P \in \mathcal{P}^n$ *be a* 0/1*-polytope and let* $\dim P \leq n-1$. *Then there exists a* 0/1*-polytope* $\widetilde{P} \in \mathcal{P}^{n-1}$ affinely isomorphic *to* P, *i.e., there exists a bijective affine map between* P *and* $\widetilde{P}$.

4.9 Theorem [Naddef, 1989].

i) *Let* P *be a* 0/1*-polytope. Then* $\delta(P) \leq \dim P$.

ii) *Let* $P \in \mathcal{P}^n$ *be an* n*-dimensional* 0/1*-polytope with* m *facets. Then* $\delta(P) \leq m-n$.

4.10 Corollary. *Let* $P \in \mathcal{P}^n$ *be an* n*-dimensional* 0/1*-polytope with* m *facets. Then* $\delta(P) \leq m-n$.

4.11 Remark.

i) $f_{n-1}(P) \leq 2\,n!$ *for a* 0/1*-polytope* $P \in \mathcal{P}^n$.

ii) *There exist* 0/1*-polytopes* $P \in \mathcal{P}^n$ *with*

$$f_{n-1}(P) \geq \left(\frac{c\,n}{\log^2 n}\right)^{\frac{n}{2}},$$

where c *is a universal constant (Gatzouras, Giannopoulos, Markoulakis, 2004).*

Index