Combinatorial Optimization: Polyhedra and Efficiency

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Combinatorial Optimization

Polyhedra and Efficiency

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Preface

The book by Gene Lawler from 1976 was the first of a series of books all entitled 'Combinatorial Optimization', some embellished with a subtitle: 'Networks and Matroids', 'Algorithms and Complexity', 'Theory and Algorithms'. Why adding another book to this illustrious series? The justification is contained in the subtitle of the present book, 'Polyhedra and Efficiency'. This is shorthand for Polyhedral Combinatorics and Efficient Algorithms.

Pioneered by the work of Jack Edmonds, polyhedral combinatorics has proved to be a most powerful, coherent, and unifying tool throughout combinatorial optimization. Not only it has led to efficient (that is, polynomial-time) algorithms, but also, conversely, efficient algorithms often imply polyhedral characterizations and related min-max relations. It makes the two sides closely intertwined.

We aim at offering both an introduction to and an in-depth survey of polyhedral combinatorics and efficient algorithms. Within the span of polyhedral methods, we try to present a broad picture of polynomial-time solvable combinatorial optimization problems — more precisely, of those problems that have been proved to be polynomial-time solvable. Next to that, we go into a few prominent NP-complete problems where polyhedral methods were succesful in obtaining good bounds and approximations, like the stable set and the traveling salesman problem. Nevertheless, while we obviously hope that the question "NP=P?" will be settled soon one way or the other, we realize that in the astonishing event that NP=P will be proved, this book will be highly incomplete.

By definition, being in P means being solvable by a 'deterministic sequential polynomial-time' algorithm, and in our discussions of algorithms and complexity we restrict ourselves mainly to this characteristic. As a consequence, we do not cover (but yet occasionally touch or outline) the important work on approximative, randomized, and parallel algorithms and complexity, areas that are recently in exciting motion. We also neglect applications, modelling, and computational methods for NP-complete problems. Advanced data structures are treated only moderately. Other underexposed areas include semidefinite programming and graph decomposition. 'This all just to keep size under control.'

Although most problems that come up in practice are NP-complete or worse, recognizing those problems that are polynomial-time solvable can be very helpful: polynomial-time (and polyhedral) methods may be used in preprocessing, in obtaining approximative solutions, or as a subroutine, for instance to calculate bounds in a branch-and-bound method. A good understanding of what is in the polynomial-time tool box is essential also for the NP-hard problem solver.

* * *

This book is divided into eight main parts, each discussing an area where polyhedral methods apply:

I. Paths and Flows
II. Bipartite Matching and Covering
III. Nonbipartite Matching and Covering
IV. Matroids and Submodular Functions
V. Trees, Branchings, and Connectors
VI. Cliques, Stable Sets, and Colouring
VII. Multiflows and Disjoint Paths
VIII. Hypergraphs

Each part starts with an elementary exposition of the basic results in the area, and gradually evolves to the more elevated regions. Subsections in smaller print go into more specialized topics. We also offer several references for further exploration of the area.

Although we give elementary introductions to the various areas, this book might be less satisfactory as an introduction to combinatorial optimization. Some mathematical maturity is required, and the general level is that of graduate students and researchers. Yet, parts of the book may serve for undergraduate teaching.

The book does not offer exercises, but, to stimulate research, we collect open problems, questions, and conjectures that are mentioned throughout this book, in a separate section entitled ‘Survey of Problems, Questions, and Conjectures’. It is not meant as a complete list of all open problems that may live in the field, but only of those mentioned in the text.

We assume elementary knowledge of and familiarity with graph theory, with polyhedra and linear and integer programming, and with algorithms and complexity. To support the reader, we survey the knowledge assumed in the introductory chapters, where we also give additional background references. These chapters are meant mainly just for consultation, and might be less attractive to read from front to back. Some less standard notation and terminology are given on the inside back cover of this book.

For background on polyhedra and linear and integer programming, we also refer to our earlier book *Theory of Linear and Integer Programming* (Wiley, Chichester, 1986). This might seem a biased recommendation, but this 1986 book was partly written as a preliminary to the present book, and it covers anyway the author's knowledge on polyhedra and linear and integer programming.

Incidentally, the reader of this book will encounter a number of concepts and techniques that regularly crop up: total unimodularity, total dual integrality, duality, blocking and antiblocking polyhedra, matroids, submodularity, hypergraphs, uncrossing. It makes that the meaning of 'elementary' is not unambiguous. Especially for the basic results, several methods apply, and it is not in all cases obvious which method and level of generality should be chosen to give a proof. In several cases we therefore will give several proofs of one and the same theorem, just to open the perspective.

* * *

While I have pursued great carefulness and precision in composing this book, I am quite sure that much room for corrections and additions has remained. To inform the reader about them, I have opened a website at the address

`www.cwi.nl/~lex/co`

Any corrections (including typos) and other comments and suggestions from the side of the reader are most welcome at

`lex@cwi.nl`

I plan to provide those who have contributed most to this, with a complimentary copy of a potential revised edition.

* * *

In preparing this book I have profited greatly from the support and help of many friends and colleagues, to whom I would like to express my gratitude.

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As it has turned out, it was only by gravely neglecting my family that I was able to complete this project. I am extremely grateful to Monique, Nella, and Juliette for their perpetual understanding and devoted support. Now comes the time for the pleasant fulfilment of all promises I made for 'when my book will be finished'.

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