Theory of Stellar Oscillations

Lecture 1
Basic Equations

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Theory of Stellar Oscillations Bibliography

- C. Aerts, J. Christensen-Dalsgaard, D.W. Kurtz, 2010, ‘Asteroseismology’, Astronomy and Astrophysics Library,, Springer, ISBN: 978-1-4020-5178-4 e-ISBN: 978-1-4020-5803-5 DOI: 10.1007/978-1-4020-5803-5.

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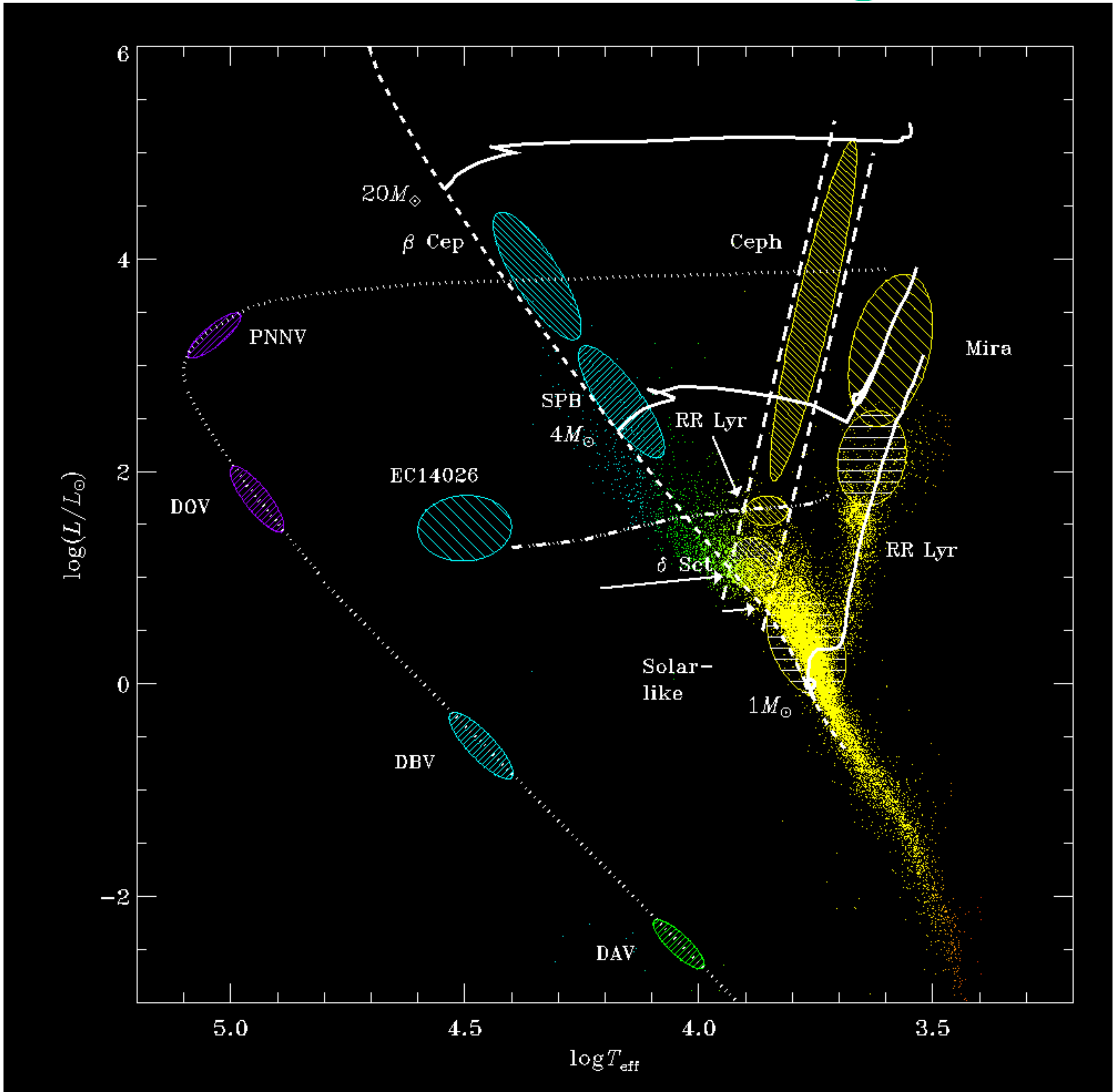
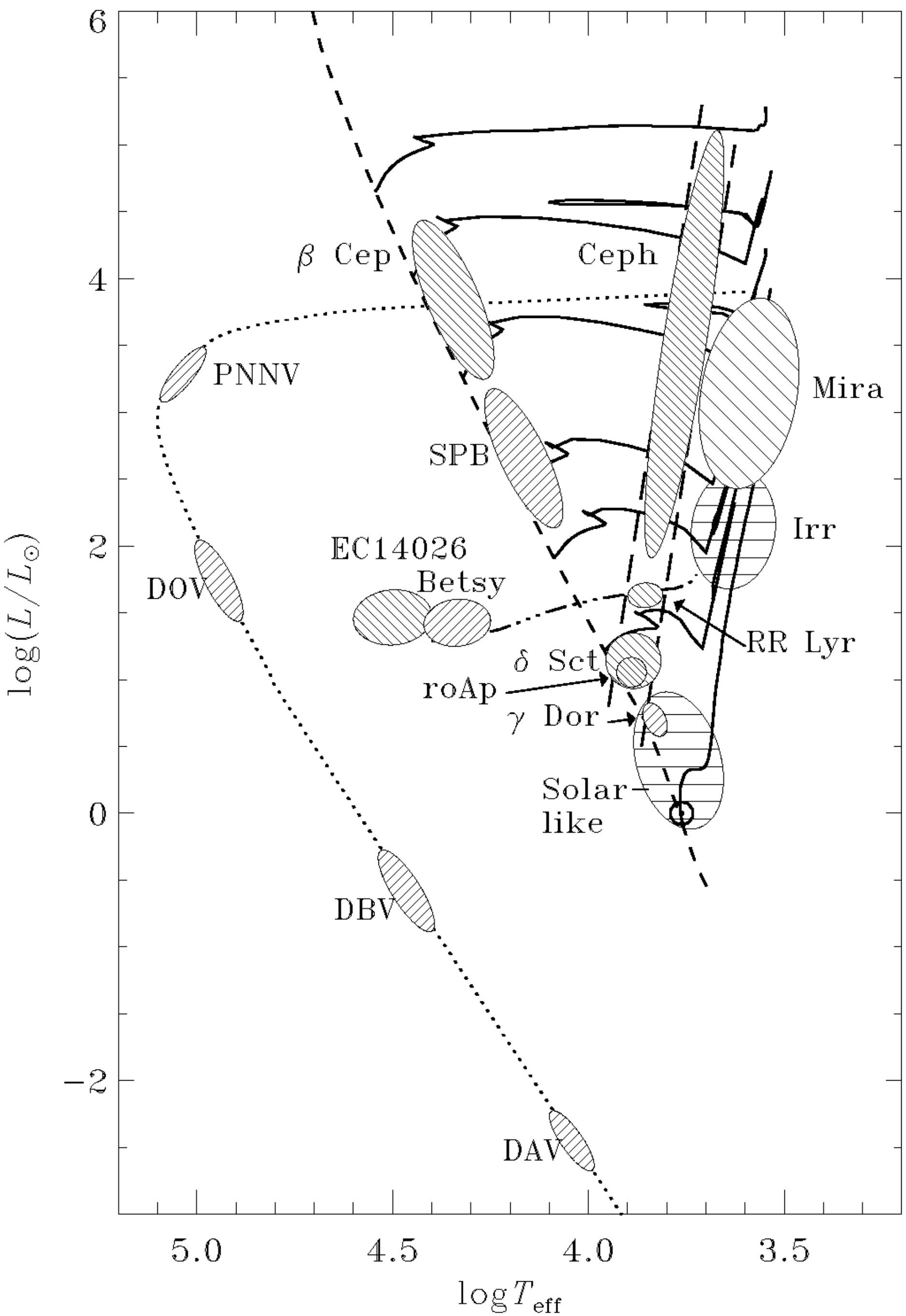
Overview – Pulsating Stars





Pulsating stars in the HR diagram

$$\tau_{\rm dyn} \simeq \sqrt{\frac{R^3}{GM}} \simeq \sqrt{\frac{1}{G\,\overline{\rho}}},$$

$$\tau_{\rm th} \simeq \frac{GM^2}{RL},$$

Excitation mechanisms

Heat engine (κ mechanism, etc)

• Critical layer in the star is heated at compression

• Mode is intrinsically unstable and grows exponentially

•Amplitude limitation mechanism, mode selection

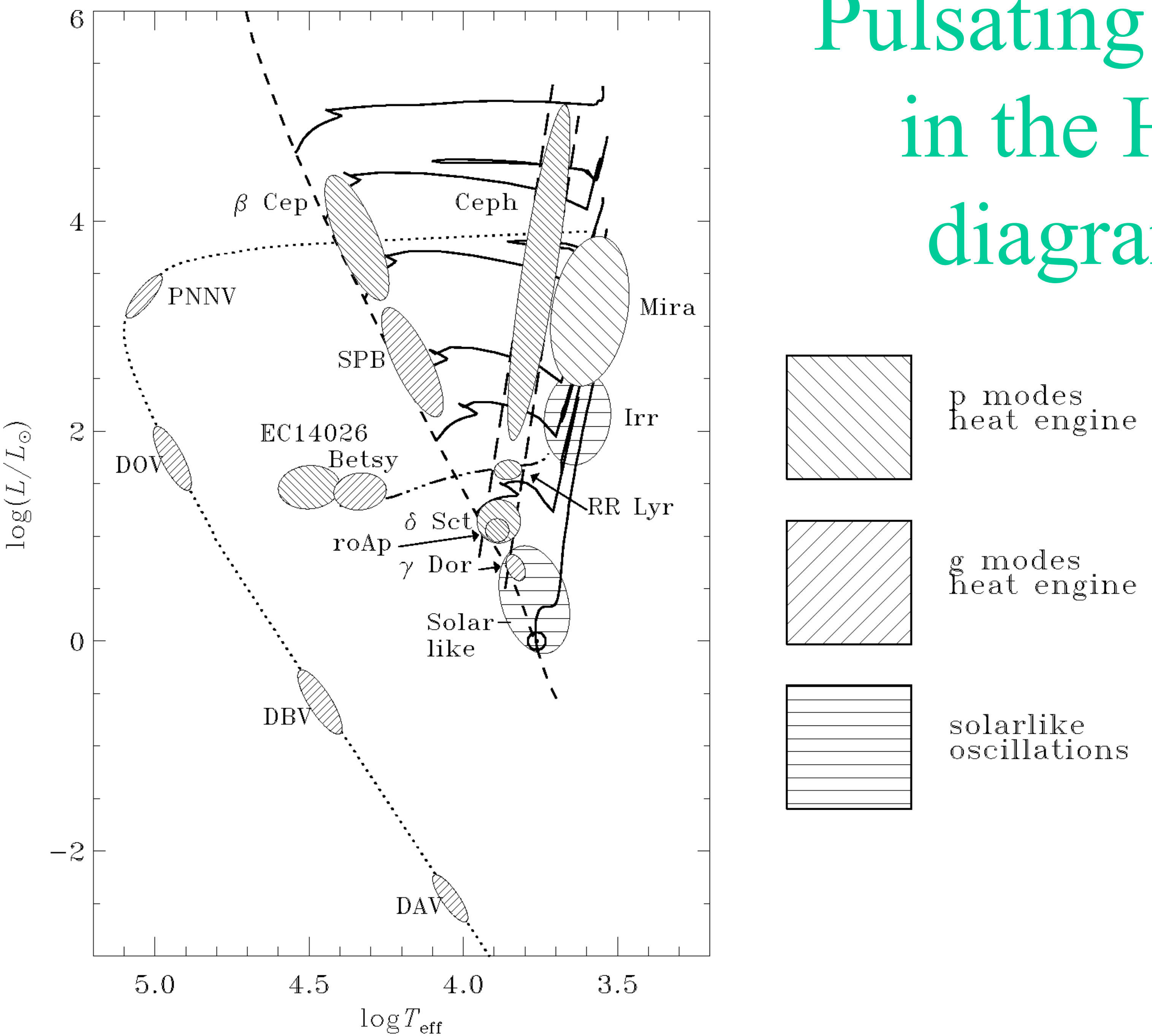
Stochastic excitation

• Mode is intrinsically damped

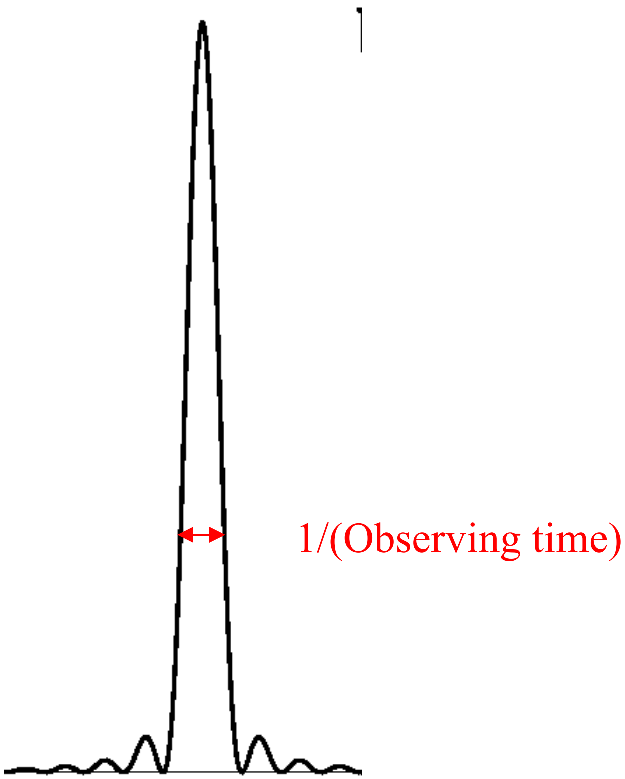
• Excitation through stochastic driving by convection (compare church bell in sandstorm)

• Resulting amplitudes from balance between forcing and damping

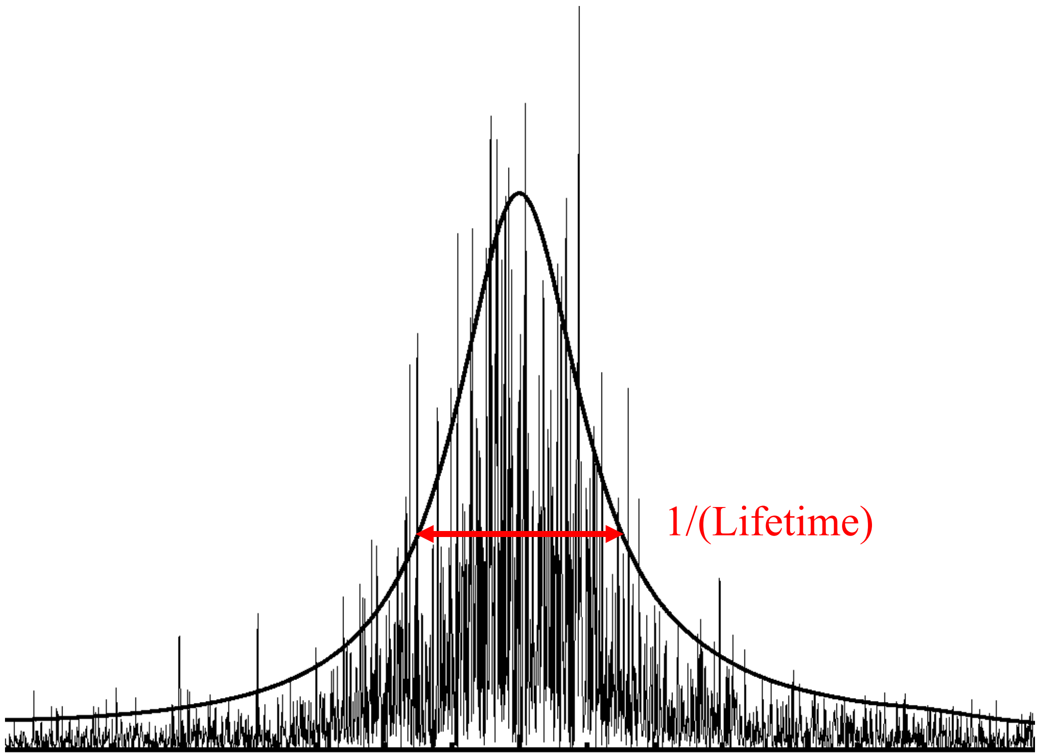
Pulsating stars in the HR diagram



Observational differences



Heat engine mode



Stochastically excited mode

Basic equations of hydrodynamics

Reasons for studying stellar ‘intrinsic’ pulsations:

- to understand why, and how, certain types of stars pulsate;
- se the pulsations to learn about the general properties (and internal structure) of stars.

The ‘*variations*’ of stars can be understood in terms of pulsations in the fundamental non-radial mode, where the star expands and contracts, while preserving spherical symmetry.

$$t_{\rm dyn} \simeq \left(\frac{R^3}{GM}\right)^{1/2} \simeq (G\bar{\rho})^{-1/2}$$

The star (‘gas’) is treated as a continuum, so that its properties ($\rho(\mathbf{r},t)$, $p(\mathbf{r},t)$,$\mathbf{v}(\mathbf{r},t)$) can be specified as functions of position **r** and time t.

Eulerian description: **r** denotes the position vector to a given point in space, the description corresponds to what is seen by a stationary observer.

***Lagrangian* description**: A given element of gas can be labeled, e.g. by its initial position the observer $\mathbf{r}_0$ follows the motion of the gas, and its motion is specified by giving its position $\mathbf{r}(t,\mathbf{r}_0)$ as a function of time. Its velocity

$$\mathbf{v}(\mathbf{r},t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \qquad \text{at fixed } \mathbf{r}_0$$

The time derivative of a quantity , observed when following the motion is

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \left(\frac{\partial\phi}{\partial t}\right)_{\mathbf{r}} + \nabla\phi\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\partial\phi}{\partial t} + \mathbf{v}\cdot\nabla\phi$$

Basic equations of hydrodynamics

Equation of continuity:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{v}) = 0 \,;$$

This is a typical conservation equation, balancing the rate of change of a quantity in a volume with the flux of the quantity into the volume.

Equation of motion:

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla p + \rho \mathbf{f} \,, \qquad \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \rho \mathbf{f}$$

Under stellar conditions we can ignore the internal friction (or viscosity) in the gas. The forces on a volume of gas therefore consist of surface forces (such as the pressure on the surface of the volume) and body forces. The force per unit mass from gravity is the gravitational acceleration **g**, which can be written as the gradient of the gravitational potentΦ:

$$\mathbf{g} = -\nabla \Phi$$

Poisson Equation:

$$\nabla^2 \Phi = 4\pi G \rho$$

It is often convenient to use also the integral solution to Poisson's equation

$$\Phi\,(\mathbf{r}, t) = -G \int_V \frac{\rho(\mathbf{r}', t)\mathrm{d}V}{|\mathbf{r} - \mathbf{r}'|}$$

Basic equations of hydrodynamics

Energy equation:

The First law of thermodynamics

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} + p\,\frac{\mathrm{d}V}{\mathrm{d}t}\,,$$

Alternative formulations of this equation , using the equation of continuity

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} - \frac{p}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} + \frac{p}{\rho}\,\mathrm{div}\,\mathbf{v}$$

Using thermodynamic identities the energy equation can be expressed in terms of other, and more convenient, variables.

$$\begin{aligned}
\frac{\mathrm{d}q}{\mathrm{d}t} &= \frac{1}{\rho(\Gamma_3 - 1)}\boxed{\left(\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\Gamma_1 p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)} \\
&= c_p \boxed{\left(\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}t}\right)} \\
&= c_V \left[\frac{\mathrm{d}T}{\mathrm{d}t} - (\Gamma_3 - 1)\frac{T}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}\right].
\end{aligned}$$

$$\Gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_{\mathrm{ad}}, \quad \frac{\Gamma_2 - 1}{\Gamma_2} = \left(\frac{\partial \ln T}{\partial \ln p}\right)_{\mathrm{ad}}, \quad \Gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_{\mathrm{ad}}$$

Basic equations of hydrodynamics

Energy equation (cont.):

We need to consider the *heat gain* in more detail,

$$\rho \frac{\mathrm{d}q}{\mathrm{d}t} = \rho\,\epsilon - \mathrm{div}\,\mathbf{F}$$

epsilon is the rate of energy generation per unit mass (e.g. from nuclear reactions), and **F** is the flux of energy.

Adiabatic approximation:

The complications of the energy equation can be avoided to a high degree of precision, by neglecting the heating term in the energy equation

$$\boxed{\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}t}} = \frac{1}{c_p}\left[\epsilon + \frac{1}{\rho}\mathrm{div}\left(\frac{4a\tilde{c}T^3}{3\kappa\rho}\nabla T\right)\right]$$

$$\frac{1}{\rho c_p}\mathrm{div}\left(\frac{4a\tilde{c}T^3}{3\kappa\rho}\nabla T\right) \sim \frac{4a\tilde{c}T^4}{3\kappa\rho^2 c_p \mathcal{L}^2} = \frac{T}{\tau_F}, \qquad \tau_F = \frac{3\kappa\rho^2 c_p \mathcal{L}^2}{4a\tilde{c}T^3} \simeq 10^{12}\frac{\kappa\rho^2\mathcal{L}^2}{T^3}, \qquad \text{in cgs units}\,.$$

$\tau_F \sim 10^7$ years (Kelvin-Helmhotz time scale) and tau_epsilon ~ cp T/epsilon ((Kelvin-Helmhotz time scale).

Where the heating can be neglected, the motion occurs adiabatically. Then p and rho are related by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\Gamma_1 p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}\,.$$

Equilibrium states and perturbation analysis

The observed solar oscillations have very small amplitudes compared with the characteristic scales of the Sun, and so it can be treated as a small perturbation around a static equilibrium state.

The equilibrium structure:

- The equations of motion reduce to the equation of hydrostatic support,

$$\nabla p_0 = \rho_0 \mathbf{g}_0 = -\rho_0 \nabla \Phi_0 \, ;$$

- Poisson's equation is unchanged,

$$\nabla^2 \Phi_0 = 4\pi G \rho_0$$

- the energy equation is

$$0 = \frac{\mathrm{d}q}{\mathrm{d}t} = \epsilon_0 - \frac{1}{\rho_0} \mathrm{div}\, \mathbf{F}_0$$

- *For the present purpose the most important example of equilibrium is clearly a spherically symmetric state, where the structure depends only on the distance r to the centre.*

Equilibrium states and perturbation analysis

The equilibrium structure (cont.):
Example of equilibrium is clearly a spherically symmetric state, where the structure depends only on the distance r to the centre.

- The equations of motion reduce to the equation of hydrostatic support,

$$\mathbf{g}_0 = -g_0 \mathbf{a}_r , \qquad \frac{\mathrm{d}p_0}{\mathrm{d}r} = -g_0 \rho_0$$

- Poisson's equation

$$g_0 = \frac{G}{r^2} \int_0^r 4\pi \rho_0 r'^2 \, \mathrm{d}r' = \frac{G\, m_0}{r^2}$$

- the energy equation is

$$\mathbf{F} = F_{r,0} \mathbf{a}_r \qquad \rho_0 \epsilon_0 = \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 F_{r,0} \right) = \frac{1}{4\pi r^2} \frac{\mathrm{d}L_0}{\mathrm{d}r}$$

$$L_0 = 4\pi r^2 F_{r,0} \qquad \frac{\mathrm{d}L_0}{\mathrm{d}r} = 4\pi r^2 \rho_0 \epsilon_0$$

Equilibrium states and perturbation analysis

Perturbation analysis:

We consider small perturbations around the equilibrium state. Thus, e.g., the pressure is written as

$$p(\mathbf{r}, t) = p_0(\mathbf{r}) + p'(\mathbf{r}, t) \qquad \textbf{(1)}$$

Where p' is a small perturbation; this is the so-called *Eulerian perturbation*, i.e., the perturbation at a given point.

Certain times, it is convenient to use also a description involving a reference frame following the motion; the perturbation in this frame is called the *Lagrangian perturbation*. If an element of gas is moved from **r**0 to $\mathbf{r}_0 + \boldsymbol{\delta}\mathbf{r}$ due to the perturbation, the Lagrangian perturbation to pressure may be calculated as

$$\begin{aligned} \delta p(\mathbf{r}) &= p(\mathbf{r}_0 + \boldsymbol{\delta}\mathbf{r}) - p_0(\mathbf{r}_0) = p(\mathbf{r}_0) + \boldsymbol{\delta}\mathbf{r} \cdot \nabla p_0 - p_0(\mathbf{r}_0) \\ &= p'(\mathbf{r}_0) + \boldsymbol{\delta}\mathbf{r} \cdot \nabla p_0 \, . \end{aligned} \qquad \textbf{(2)}$$

These equations are equivalent to the relation between the local and the material time derivative. Note also that the velocity is given by the time derivative of the displacement **r**,

$$\mathbf{v} = \frac{\partial \boldsymbol{\delta}\mathbf{r}}{\partial t} \qquad \textbf{(3)}$$

Equilibrium states and perturbation analysis

Perturbation analysis:
The 'structure' equations are then linearized in the perturbations, by expanding them in the perturbations retaining only terms that do not contain products of the perturbations. Equations for the perturbations are obtained by inserting previous expressions like **(1)** in the full equations, subtracting equilibrium equations and neglecting quantities of order higher than one in p', ρ', $\mathbf{v}$; etc.

- **continuity equation:**

$$\frac{\partial \rho'}{\partial t} + \mathrm{div}\,(\rho_0 \mathbf{v}) = 0$$

- Using (3) and integrating with respect to time

$$\rho' + \mathrm{div}\,(\rho_0 \boldsymbol{\delta}\mathbf{r}) = 0$$

Using (2) the analogue to equation

$$\delta\rho + \rho_0 \mathrm{div}\,(\boldsymbol{\delta}\mathbf{r}) = 0$$

- **Equation of motion:**

$$\rho_0 \frac{\partial^2 \boldsymbol{\delta}\mathbf{r}}{\partial t^2} = \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0$$

where, obviously, $\mathbf{g}' = -\nabla \Phi'$

- **Poisson equation:**

$$\nabla^2 \Phi' = 4\pi G \rho' \qquad \Phi' = -G \int_V \frac{\rho'(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}V$$

Equilibrium states and perturbation analysis

Energy Equation (perturbation analysis):

We need to calculate

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p = \frac{\partial p'}{\partial t} + \mathbf{v}\cdot\nabla p_0 = \frac{\partial p'}{\partial t} + \frac{\partial \boldsymbol{\delta}\mathbf{r}}{\partial t}\cdot\nabla p_0 = \frac{\partial}{\partial t}(\delta p)\,,$$

to first order in the perturbations. Note that to this order there is no difference between the local and the material time derivative of the *perturbations.* Thus we have for the energy equation,

$$\frac{\partial \delta q}{\partial t} = \frac{1}{\rho_0(\Gamma_{3,0}-1)}\left(\frac{\partial \delta p}{\partial t} - \frac{\Gamma_{1,0}p_0}{\rho_0}\frac{\partial \delta \rho}{\partial t}\right)\,.$$

This equation is most simply expressed *in Lagrangian perturbations*, but it may be transformed into *Eulerian perturbations* by using equation (2). From equation energy(heat) equation the perturbation to the heating rate is given by

$$\rho_0\frac{\partial \delta q}{\partial t} = \delta(\rho\epsilon - \operatorname{div}\mathbf{F}) = (\rho\epsilon - \operatorname{div}\mathbf{F})'$$

if equation energy (equilibrium) is used. Finally it is straightforward to obtain the perturbation to the Radiative flux, in the diffusion approximation, from equation. For adiabatic motion (adiabatic approximation) we neglect the heatin

$$\frac{\partial \delta p}{\partial t} - \frac{\Gamma_{1,0}p_0}{\rho_0}\frac{\partial \delta \rho}{\partial t} = 0\,;$$

by integrating over time

$$\delta p = \frac{\Gamma_{1,0}p_0}{\rho_0}\delta\rho \qquad p' + \boldsymbol{\delta}\mathbf{r}\cdot\nabla p_0 = \frac{\Gamma_{1,0}p_0}{\rho_0}(\rho' + \boldsymbol{\delta}\mathbf{r}\cdot\nabla\rho_0)$$

Equilibrium states and perturbation analysis

Simple waves:

It is instructive to consider simple examples of wave motion. This provides an introduction to the techniques needed to handle the perturbations. In addition, general stellar oscillations can in many cases be approximated by simple waves, which therefore give physical insight into the behaviour of the oscillations.

Acoustic waves:

As the simplest possible equilibrium situation, we may consider the spatially homogeneous case. Here all derivatives of equilibrium quantities vanish. According to equation for hydrostatic equilibrium the gravity must then be negligible. Such a situation clearly cannot be realized exactly. However, if the equilibrium structure varies slowly compared with the oscillations, this may be a reasonable approximation. I also neglect the perturbation to the gravitational potential; for rapidly varying perturbations regions with positive and negative ρ' nearly cancel in linearized Poisson equation (integral form) , and hence Φ' is small. Finally, I assume the *adiabatic approximation* The linearized equation of motion give

$$\rho_0 \frac{\partial^2 \boldsymbol{\delta}\mathbf{r}}{\partial t^2} = -\nabla p' \, ,$$

by taking the divergence

$$\rho_0 \frac{\partial^2}{\partial t^2} (\mathrm{div}\, \boldsymbol{\delta}\mathbf{r}) = -\nabla^2 p'$$

Equilibrium states and perturbation analysis

Acoustic waves (cont.):

However, $\mathrm{div}\,\boldsymbol{\delta}\mathbf{r}$ can be eliminated by using the linearized continuity equation , and p' can be expressed in terms of ρ' from the adiabatic relation. The result is

$$\frac{\partial^2 \rho'}{\partial t^2} = \frac{\Gamma_{1,0} p_0}{\rho_0} \nabla^2 \rho' = c_0^2 \nabla^2 \rho' \qquad c_0^2 \equiv \frac{\Gamma_{1,0} p_0}{\rho_0}$$

This equation has the form of the wave equation. Thus it has solutions in the form of plane waves $\rho' = a \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$

By substituting this wave-like equation into wave equation, we obtain

$$-\omega^2 \rho' = c_0^2 \,\mathrm{div}\,(i\mathbf{k}\rho') = -c_0^2 |\mathbf{k}|^2 \rho'$$

Thus this is a solution, provided ω satisfies the *dispersion relation*

$$\omega^2 = c_0^2\, |\mathbf{k}|^2 \qquad c_0^2 = \frac{\Gamma_{1,0} k_{\mathrm{B}} T_0}{\mu m_{\mathrm{u}}} .$$

With a suitable choice of phases the real solution can be written as

$$\begin{aligned} \rho' &= a \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) , \\ p' &= c_0^2 a \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) , \\ \boldsymbol{\delta}\mathbf{r} &= \frac{c_0^2}{\rho_0 \omega^2} a \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \frac{\pi}{2})\, \mathbf{k} \end{aligned}$$

Thus the displacement $\boldsymbol{\delta}\mathbf{r}$, and hence the velocity $\mathbf{v}$, is in the direction of the wave vector $\mathbf{k}$.

Equations of linear stellar oscillations

Notation:
Explicit use of the spherical symmetry. It is convenient to introduce the horizontal (or, properly speaking, tangential) component of the vector **F**:

$$\mathbf{F} = F_r \mathbf{a}_r + F_\theta \mathbf{a}_\theta + F_\phi \mathbf{a}_\phi \qquad \mathbf{F}_\mathrm{h} = F_\theta \mathbf{a}_\theta + F_\phi \mathbf{a}_\phi$$

The horizontal components of the gradient as

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

$$\nabla_\mathrm{h} V = \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi ,$$

Similarly the horizontal components of divergence and Laplacian as

$$\nabla_\mathrm{h} \cdot \mathbf{F} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi} ,$$

$$\nabla_\mathrm{h}^2 V = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial V}{\partial \theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

Equations of linear stellar oscillations

Explicit use of the spherical symmetry. These previous equations describe the general, so-called non-radial oscillations, where spherical symmetry of the perturbations is not assumed. The more familiar case of radial, or spherically symmetric, oscillations, is contained as a special case.

The Oscillation Equations:

The displacement $\boldsymbol{\delta}\mathbf{r}$ is separated into radial and horizontal components as

$$\boldsymbol{\delta}\mathbf{r} = \xi_r \mathbf{a}_r + \boldsymbol{\xi}_{\mathrm{h}}$$

$$\rho_0 \frac{\partial^2 \boldsymbol{\delta}\mathbf{r}}{\partial t^2} = \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0$$

- The horizontal component of the equations of motion, is

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}_{\mathrm{h}}}{\partial t^2} = -\nabla_{\mathrm{h}} p' - \rho_0 \nabla_{\mathrm{h}} \Phi'$$

As the horizontal gradient of equilibrium quantities is zero, the horizontal divergence of last Equation give

$$\rho_0 \frac{\partial^2}{\partial t^2} \nabla_{\mathrm{h}} \cdot \boldsymbol{\xi}_{\mathrm{h}} = -\nabla_{\mathrm{h}}^2 p' - \rho_0 \nabla_{\mathrm{h}}^2 \Phi' \, .$$

Equations of linear stellar oscillations

The equation of continuity, can be written as

$$\rho' + \mathrm{div}\,(\rho_0 \boldsymbol{\delta}\mathbf{r}) = 0 \qquad \rho' = -\frac{1}{r^2}\frac{\partial}{\partial r}(\rho_0 r^2 \xi_r) - \rho_0 \nabla_{\mathrm{h}} \cdot \boldsymbol{\xi}_{\mathrm{h}}$$

This can be used to eliminate $\nabla_{\mathrm{h}} \cdot \boldsymbol{\xi}_{\mathrm{h}}$ from linearized motion equation,

$$\rho_0 \frac{\partial^2}{\partial t^2} \nabla_{\mathrm{h}} \cdot \boldsymbol{\xi}_{\mathrm{h}} = -\nabla_{\mathrm{h}}^2 p' - \rho_0 \nabla_{\mathrm{h}}^2 \Phi' .$$

which becomes

$$-\frac{\partial^2}{\partial t^2}\left[\rho' + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho_0 \xi_r)\right] = -\nabla_{\mathrm{h}}^2 p' - \rho_0 \nabla_{\mathrm{h}}^2 \Phi'$$

The radial component of the linearized motion equation

$$\rho_0 \frac{\partial^2 \boldsymbol{\delta}\mathbf{r}}{\partial t^2} = \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0$$

is

$$\rho_0 \frac{\partial^2 \xi_r}{\partial t^2} = -\frac{\partial p'}{\partial r} - \rho' g_0 - \rho_0 \frac{\partial \Phi'}{\partial r}$$

Equations of linear stellar oscillations

The linearized Poisson's equation may be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi'}{\partial r}\right) + \nabla_{\mathrm{h}}^2\Phi' = 4\pi G\rho'$$

It should be noticed that in linearized equations (motion equation and Poisson equation), derivatives with respect to the angular variables and only appear in the combination ∇_{h}^2.

Separation of variables:

We may now address the separation of the angular variables. The object is to factor out the variation of the perturbations with theta and phi and as a function $f(\theta, \phi)$. From the form of the equations this is clearly possible, if f is an eigen-function of the horizontal Laplace operator,

$$\nabla_{\mathrm{h}}^2 f = -\frac{1}{r^2}\Lambda f \qquad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} = -\Lambda\, f\,.$$

where Lambda is a constant.

Equations of linear stellar oscillations

As the coeficientes in the previous equation are independent of phi, the solution can be further separated, as $f(\theta, \phi) = f_1(\theta) f_2(\phi)$.

It follows from equation the last equation that **f**2 satisfies an equation of the form

$$\frac{d^2 f_2}{d\phi^2} = \alpha f_2$$

Where alpha is another constant; this has the solution: $f_2 = \exp(\pm\alpha^{1/2}\phi)$

However, the solution has to be continuous and hence periodic, $f_2(0) = f_2(2\pi)$

Consequently we must demand that $\alpha^{1/2} = im$, where **m** is an integer.

When used in equation, $$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} = -\Lambda f .$$

this gives the following diferential equation for **f1:**

$$\frac{d}{dx}\left[(1-x^2)\frac{df_1}{dx}\right] + \left(\Lambda - \frac{m^2}{1-x^2}\right) f_1 = 0 , \qquad x = \cos\theta .$$

It can be shown that this equation has a regular solution only when $\Lambda = l(l+1)$

Where l is a non-negative integer and $|m| \le l$

The regular solution is the Legendre function: $f_1(\theta) = P_l^m(\cos\theta)$

Equations of linear stellar oscillations

By including an appropriate scaling factor we may finally write

$$f(\theta,\phi) = (-1)^m c_{lm} P_l^m(\cos\theta)\exp(im\phi) \equiv Y_l^m(\theta,\phi)$$

where Y^m_l is a spherical harmonic; here c_lm is a normalization constant Y^m_l is characterized by its degree l and its azimuthal order m; the properties of spherical harmonics.

From equations $\nabla_{\mathrm{h}}^2 f = -\frac{1}{r^2}\Lambda f$ and $\Lambda = l(l+1)$

We also have that $$\nabla_{\mathrm{h}}^2 f = -\frac{l(l+1)}{r^2} f$$

The dependent variables in linearized equations can now be written as

$$\xi_r(r,\theta,\phi,t) = \sqrt{4\pi}\,\tilde{\xi}_r(r) Y_l^m(\theta,\phi)\exp(-i\omega t)$$

$$p'(r,\theta,\phi,t) = \sqrt{4\pi}\,\tilde{p}'(r) Y_l^m(\theta,\phi)\exp(-i\omega t)$$

……

Equations of linear stellar oscillations

It follows from equation that relates Lagrangian and Euler Perturbation, that if the Eulerian perturbations are on the form given in the previous equations, so are the Lagrangian perturbations. Then the equations contain $Y_l^m(\theta,\phi)\exp(-i\omega t)$ as a common factor. After dividing by it, the following ordinary differential equations for the amplitude functions $\tilde{\xi}_r, \tilde{p}', \cdots,$ result:

$$\omega^2\left[\tilde{\rho}' + \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\rho_0\tilde{\xi}_r)\right] = \frac{l(l+1)}{r^2}(\tilde{p}' + \rho_0\tilde{\Phi}') \; ,$$

$$-\omega^2\rho_0\tilde{\xi}_r = -\frac{\mathrm{d}\tilde{p}'}{\mathrm{d}r} - \tilde{\rho}' g_0 - \rho_0\frac{\mathrm{d}\tilde{\Phi}'}{\mathrm{d}r} \; ,$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\tilde{\Phi}'}{\mathrm{d}r}\right) - \frac{l(l+1)}{r^2}\tilde{\Phi}' = 4\pi G\tilde{\rho}' \; ,$$

together with the energy equation $\left(\delta\tilde{p} - \frac{\Gamma_{1,0}p_0}{\rho_0}\delta\tilde{\rho}\right) = \rho_0(\Gamma_{3,0}-1)\delta\tilde{q}$

It should be noted that these equations do not depend on the azimuthal order **m**. This is a consequence of the assumed spherical symmetry of the equilibrium state, which demands that the results should be independent of the choice of polar axis for the coordinate system.

Equations of linear stellar oscillations

From equation

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}_{\mathrm{h}}}{\partial t^2} = -\nabla_{\mathrm{h}} p' - \rho_0 \nabla_{\mathrm{h}} \Phi'$$

the horizontal component of the displacement is given by

$$\boldsymbol{\xi}_{\mathrm{h}} = \sqrt{4\pi}\, \tilde{\xi}_{\mathrm{h}}(r) \left(\frac{\partial Y_l^m}{\partial \theta} \mathbf{a}_\theta + \frac{1}{\sin\theta} \frac{\partial Y_l^m}{\partial \phi} \mathbf{a}_\phi \right) \exp(-i\omega t)$$

Where

$$\tilde{\xi}_{\mathrm{h}}(r) = \frac{1}{r\omega^2} \left(\frac{1}{\rho_0} \tilde{p}' + \tilde{\Phi}' \right)$$

Thus the (physical) displacement vector can be written as

$$\begin{aligned} \boldsymbol{\delta}\mathbf{r} \;=\; & \sqrt{4\pi}\Re\left\{ \left[\tilde{\xi}_r(r) Y_l^m(\theta,\phi)\mathbf{a}_r \right.\right. \\ & \left.\left. + \tilde{\xi}_{\mathrm{h}}(r) \left(\frac{\partial Y_l^m}{\partial \theta} \mathbf{a}_\theta + \frac{1}{\sin\theta} \frac{\partial Y_l^m}{\partial \phi} \mathbf{a}_\phi \right) \right] \exp(-i\omega t) \right\} . \end{aligned}$$

Equations of linear stellar oscillations

Linear, adiabatic oscillations:

To simplify the notation, from now on I drop the tilde on the amplitude functions, and the "0" on equilibrium quantities. This should not cause any confusion.

For adiabatic oscillations, $\delta q = 0$ and linearized equation can be written

$$\rho' = \frac{\rho}{\Gamma_1 p} p' + \rho \xi_r \left(\frac{1}{\Gamma_1 p} \frac{\mathrm{d}p}{\mathrm{d}r} - \frac{1}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}r} \right)$$

This may be used to eliminate ρ' from equations

$$\omega^2 \left[\tilde{\rho}' + \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} (r^2 \rho_0 \tilde{\xi}_r) \right] = \frac{l(l+1)}{r^2} (\tilde{p}' + \rho_0 \tilde{\Phi}') ,$$

$$-\omega^2 \rho_0 \tilde{\xi}_r = - \frac{\mathrm{d}\tilde{p}'}{\mathrm{d}r} - \tilde{\rho}' g_0 - \rho_0 \frac{\mathrm{d}\tilde{\Phi}'}{\mathrm{d}r} ,$$

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{\mathrm{d}\tilde{\Phi}'}{\mathrm{d}r} \right) - \frac{l(l+1)}{r^2} \tilde{\Phi}' = 4\pi G \tilde{\rho}' ,$$

Equations of linear stellar oscillations

Linear, adiabatic oscillations:

From the first equation we obtain

$$\frac{\mathrm{d}\xi_r}{\mathrm{d}r} = -\left(\frac{2}{r} + \frac{1}{\Gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}\right)\xi_r + \frac{1}{\rho c^2}\left(\frac{S_l^2}{\omega^2} - 1\right)p' + \frac{l(l+1)}{\omega^2 r^2}\Phi'$$

And from the second equation

$$\frac{\mathrm{d}p'}{\mathrm{d}r} = \rho(\omega^2 - N^2)\xi_r + \frac{1}{\Gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}p' - \rho\frac{\mathrm{d}\Phi'}{\mathrm{d}r}$$

And last equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi'}{\mathrm{d}r}\right) = 4\pi G\left(\frac{p'}{c^2} + \frac{\rho\xi_r}{g}N^2\right) + \frac{l(l+1)}{r^2}\Phi'$$

Where c, N and S_l is the adiabatic sound speed, buoyancy frequency and acoustic Lamb frequency:

$$c^2 = \Gamma_1 p/\rho \qquad N^2 = g\left(\frac{1}{\Gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) \qquad S_l^2 = \frac{l(l+1)c^2}{r^2} = k_\mathrm{h}^2 c^2$$

These three equations constitute a fourth-order system of ordinary differential equations for the four dependent variables ξ_r, p', Φ' and $d\Phi'/\mathrm{d}r$. Thus it is a complete set of differential equations.

Equations of linear stellar oscillations

Linear, adiabatic oscillations:

It should be noticed that all coefficients in previous equations are real.
The same is true of the boundary conditions. Since the frequency only appears in the form **omega^2**, we may expect that the solution is such that **omega^2** is real, in which case the eigen-functions may also be chosen to be real. This may be proved to be true in general.

Thus the frequency is either purely real, in which case the motion is an undamped oscillator, or purely imaginary, so that the motion grows or decays exponentially.

From a physical point of view this results from the adiabatic approximation, which ensures that energy cannot be fed into the motion, except from the gravitational field; thus the only possible type of instability is a dynamical instability. I shall almost always consider the oscillatory case, with **omega**^2 > 0.

Separated equations

Separation of time as exp(- i ω t)

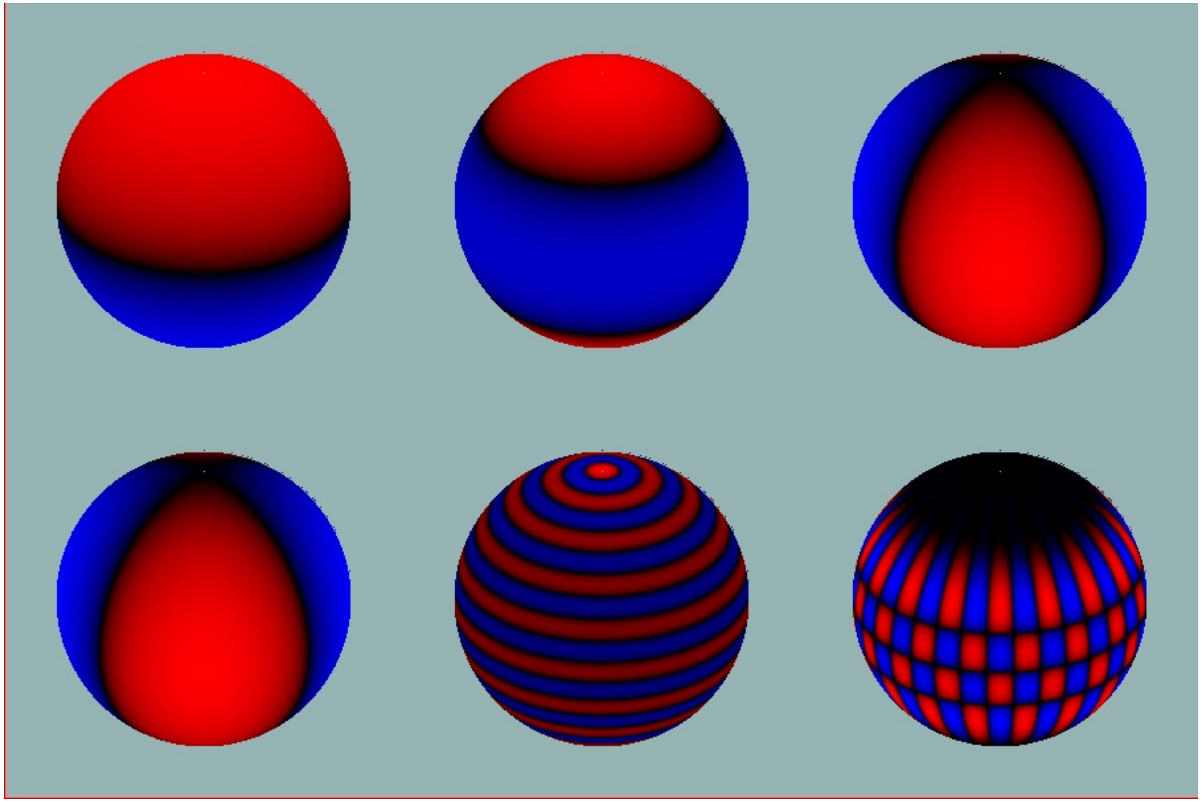
$$\frac{\mathrm{d}\xi_r}{\mathrm{d}r} = -\left(\frac{2}{r} + \frac{1}{\gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}\right)\xi_r + \frac{1}{\rho c^2}\left(\frac{S_l^2}{\omega^2} - 1\right)p' + \frac{l(l+1)}{\omega^2 r^2}\Phi' \, .$$

$$\frac{\mathrm{d}p'}{\mathrm{d}r} = \rho(\omega^2 - N^2)\xi_r + \frac{1}{\gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}p' - \rho\frac{\mathrm{d}\Phi'}{\mathrm{d}r} \, ,$$

$$S_l^2 = \frac{l(l+1)c^2}{r^2} = k_{\mathrm{h}}^2 c^2 \, , \quad c^2 = \frac{\gamma_1 p}{\rho} \, , \qquad N^2 = g\left(\frac{1}{\gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) \, .$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi'}{\mathrm{d}r}\right) = 4\pi G\left(\frac{p'}{c^2} + \frac{\rho\xi_r}{g}N^2\right) + \frac{l(l+1)}{r^2}\Phi' \, .$$

Spherical harmonics



Note: local horizontal wavenumber k_{h}:

$$k_{\mathrm{h}}^2 = \frac{l(l+1)}{r^2}$$

Local horizontal wavelength on surface:

$\lambda_{\mathrm{h}} = 2\pi/k_{\mathrm{h}} \simeq 2\pi R/(l+1/2)$

Frequency dependence on stellar structure

Frequencies depend on dynamical quantities:

$$p(r)\ , \quad \rho(r)\ , \quad g(r)\ , \quad \gamma_1(r)$$

However, from hydrostatic equilibrium and Poisson's equation p and g can be determined from ρ

Hence adiabatic oscillations are fully characterized by

$$\rho(r)\ , \quad \gamma_1(r)$$

or, equivalently

$$\rho(r)\ , \quad c^2(r)$$

Characteristic frequencies

Acoustic frequency

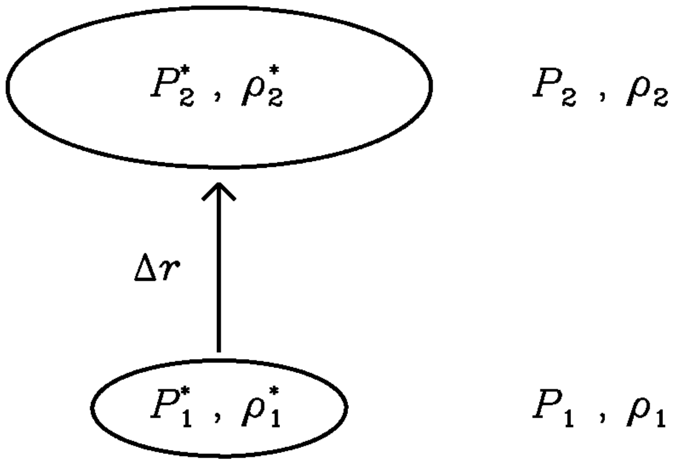
$$S_l^2 = \frac{l(l+1)c^2}{r^2}$$

Buoyancy frequency:

$$N^2 = g\left(\frac{1}{\Gamma_1}\frac{\mathrm{d}\ln p}{\mathrm{d}r} - \frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right) \simeq \frac{g^2\rho}{p}(\nabla_{\mathrm{ad}} - \nabla + \nabla_\mu)$$

$$\nabla = \frac{\mathrm{d}\ln T}{\mathrm{d}\ln p}\,, \qquad \nabla_{\mathrm{ad}} = \left(\frac{\partial\ln T}{\partial\ln p}\right)_{\mathrm{ad}}\,, \qquad \nabla_\mu = \frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln p}\,.$$

Internal gravity waves



Buoyancy force:

$$\rho \frac{\mathrm{d}^2 \Delta r}{\mathrm{d}t^2} = f_{\mathrm{buoy}} = -g(\rho_2^* - \rho_2)$$

$$= -g\rho \left(\frac{1}{\gamma_1} \frac{\mathrm{d} \ln p}{\mathrm{d}r} - \frac{\mathrm{d} \ln \rho}{\mathrm{d}r} \right) \Delta r$$

$$\equiv -\rho N^2 \Delta r$$

Oscillation with frequency $\omega = N$ if

$N^2 > 0$ or $\frac{1}{\gamma_1} \frac{\mathrm{d} \ln p}{\mathrm{d}r} > \frac{\mathrm{d} \ln \rho}{\mathrm{d}r}$ *i.e.*, for convective stability.

In reality increased inertia owing to horizontal motion

If perturbation $\propto \exp[i \mathbf{k} \cdot \mathbf{r}]$ $\mathbf{k} = k_r \mathbf{a}_r + \mathbf{k}_h$

$$\omega^2 = \left(1 + \frac{k_r^2}{|\mathbf{k}_{\mathrm{h}}|^2} \right)^{-1} N^2 = \left(1 + \frac{\lambda_{\mathrm{h}}^2}{\lambda_r^2} \right)^{-1} N^2$$

Boundary conditions

At centre

$$\xi_r \simeq l \xi_{\mathrm{h}} \,, \qquad \text{for } r \to 0 \,.$$

At surface

$$\Phi' = A\, r^{-l-1} \,, \qquad \frac{\mathrm{d}\Phi'}{\mathrm{d}r} + \frac{l+1}{r} \Phi' = 0 \qquad \text{at } r = R \,.$$

$$\delta p = p' + \xi_r \frac{\mathrm{d}p}{\mathrm{d}r} = 0 \qquad \text{at } r = R \,.$$

Equations and boundary conditions determine frequencies ω_{nl}

$$\frac{\delta h_{\mathrm{rms}}}{\delta r_{\mathrm{rms}}} = \frac{\sqrt{l(l+1)}}{\sigma^2} \quad \text{at } r = R \,, \qquad \sigma^2 = \frac{R^3}{GM} \omega^2$$

Approximated equations

$$\frac{\mathrm{d}\xi_r}{\mathrm{d}r} = -\cancel{\left(\frac{2}{r} + \frac{1}{\gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}\right)}\xi_r + \frac{1}{\rho c^2}\left(\frac{S_l^2}{\omega^2} - 1\right)p' + \cancel{\frac{l(l+1)}{\omega^2 r^2}\Phi'}\,.$$

$$\frac{\mathrm{d}p'}{\mathrm{d}r} = \rho(\omega^2 - N^2)\xi_r + \cancel{\frac{1}{\gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}}p' - \cancel{\rho\frac{\mathrm{d}\Phi'}{\mathrm{d}r}}\,,$$

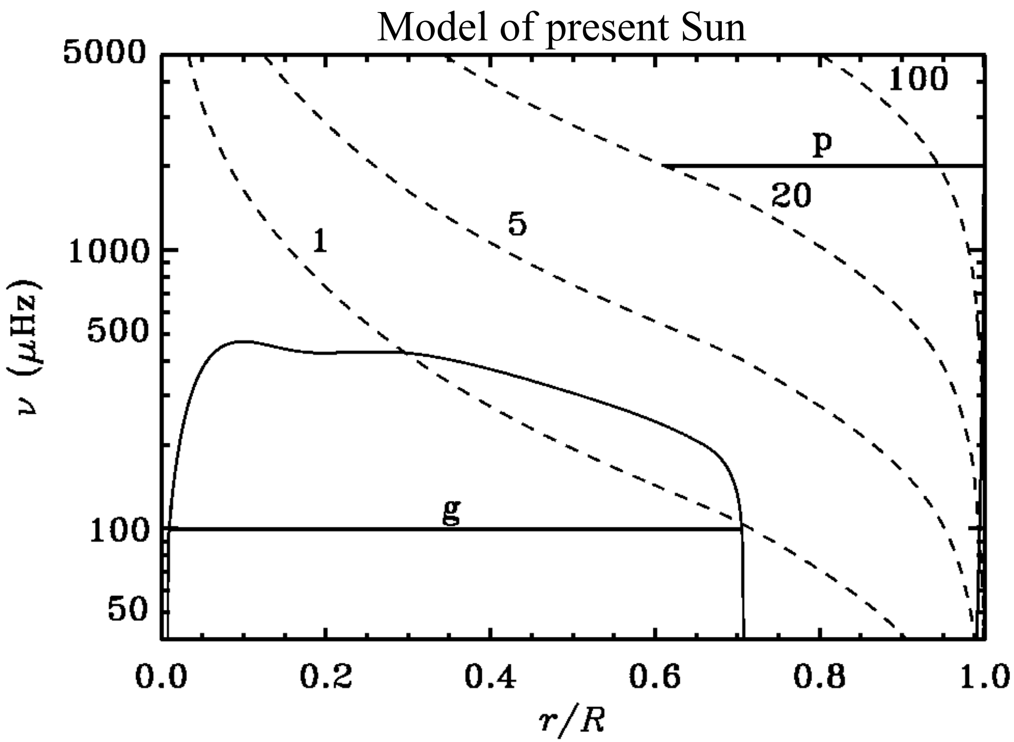
Cowling approximation

$$\cancel{\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi'}{\mathrm{d}r}\right) = 4\pi G\left(\frac{p'}{c^2} + \frac{\rho\xi_r}{g}N^2\right) + \frac{l(l+1)}{r^2}\Phi'}\,.$$

High radial order

$$\frac{\mathrm{d}^2\xi_r}{\mathrm{d}r^2} \simeq -\frac{\omega^2}{c^2}\left(\frac{S_l^2}{\omega^2} - 1\right)\left(\frac{N^2}{\omega^2} - 1\right)\xi_r$$

Mode trapping



$$\frac{\mathrm{d}^2\xi_r}{\mathrm{d}r^2} \simeq -\frac{\omega^2}{c^2}\left(\frac{S_l^2}{\omega^2}-1\right)\left(\frac{N^2}{\omega^2}-1\right)\xi_r$$

Eigenfunction oscillates as function of *r* when

$\omega^2 > S_l^2, N^2$ **p modes**

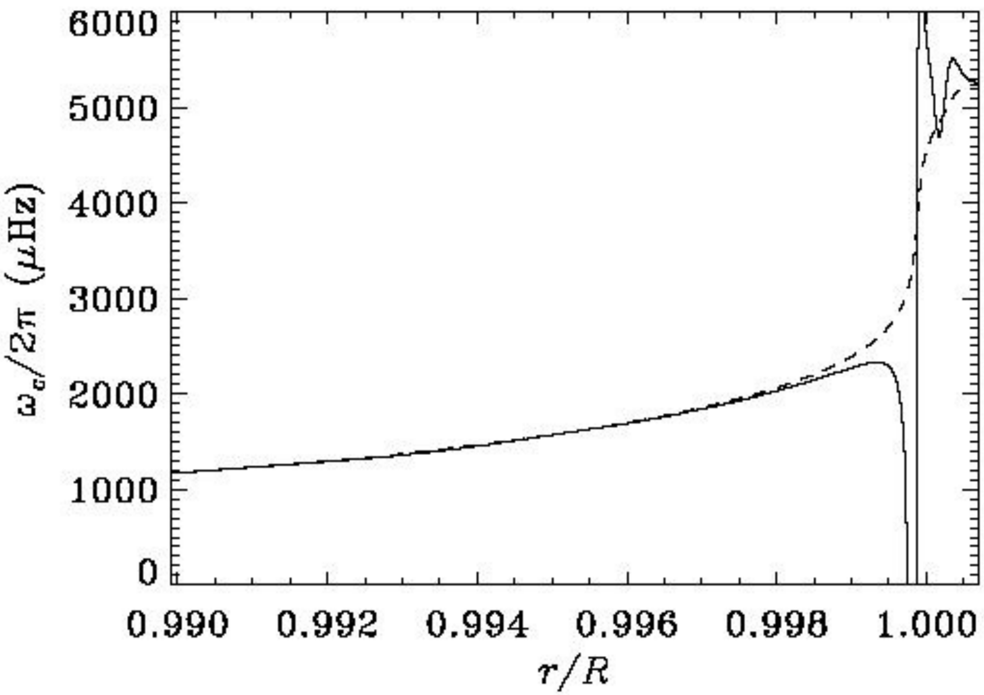
$\omega^2 < S_l^2, N^2$ **g modes**

The essential frequencies

$$N^2 = g\left(\frac{1}{\Gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) .$$

$$S_l^2 = \frac{l(l+1)c^2}{r^2} = k_\mathrm{h}^2 c^2 .$$

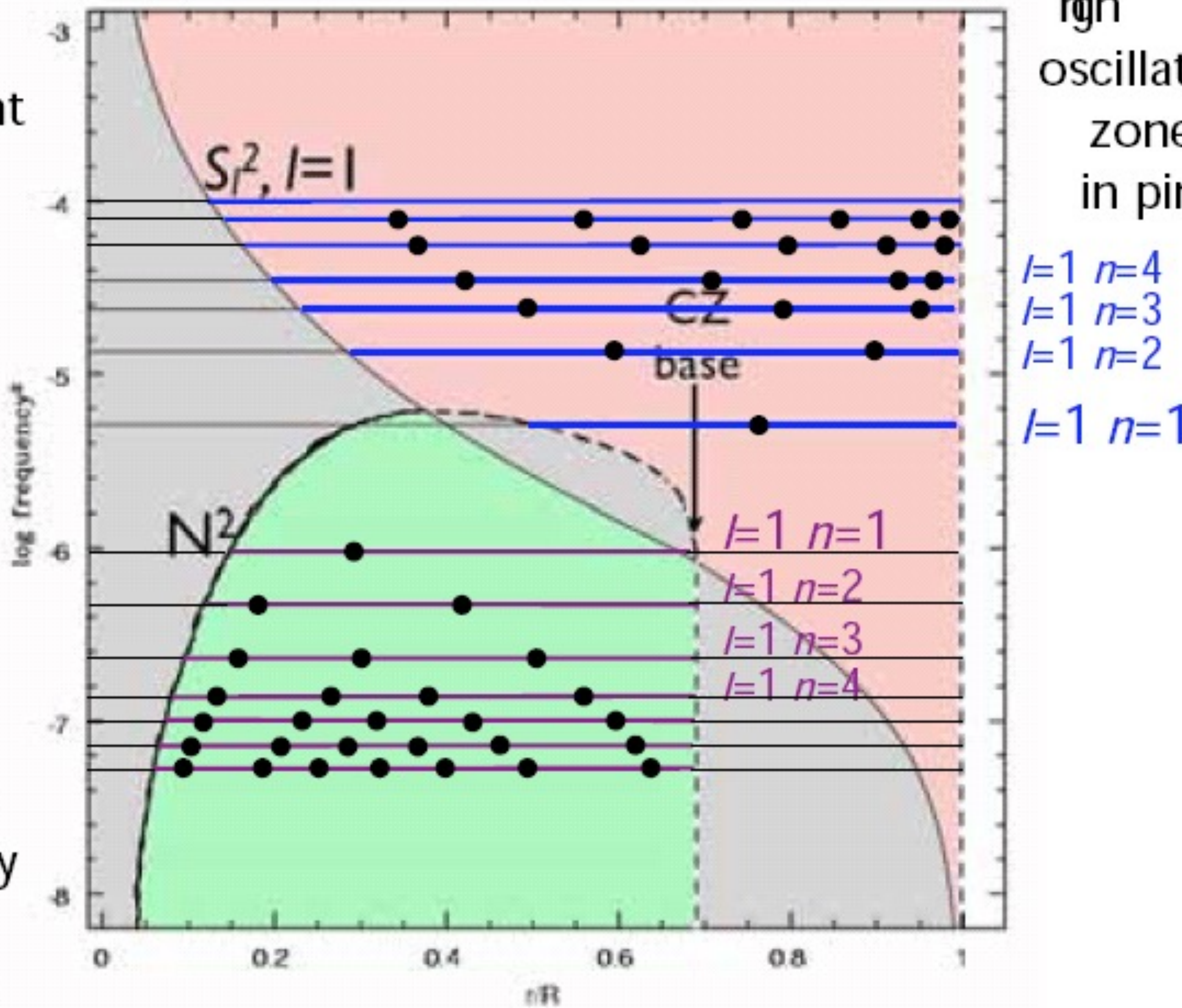
$$\omega_\mathrm{c}^2 = \frac{c^2}{4H^2}\left(1 - 2\frac{\mathrm{d}H}{\mathrm{d}r}\right) ,$$



Propagation diagram, ZAMS solar model

evanescent zones in gray

high σ oscillatory zones in pink



low σ oscillatory zones in green

Lecture 1

END

Theory of Stellar Oscillations and Planetary Formation

Lecture 2
Basic Equations

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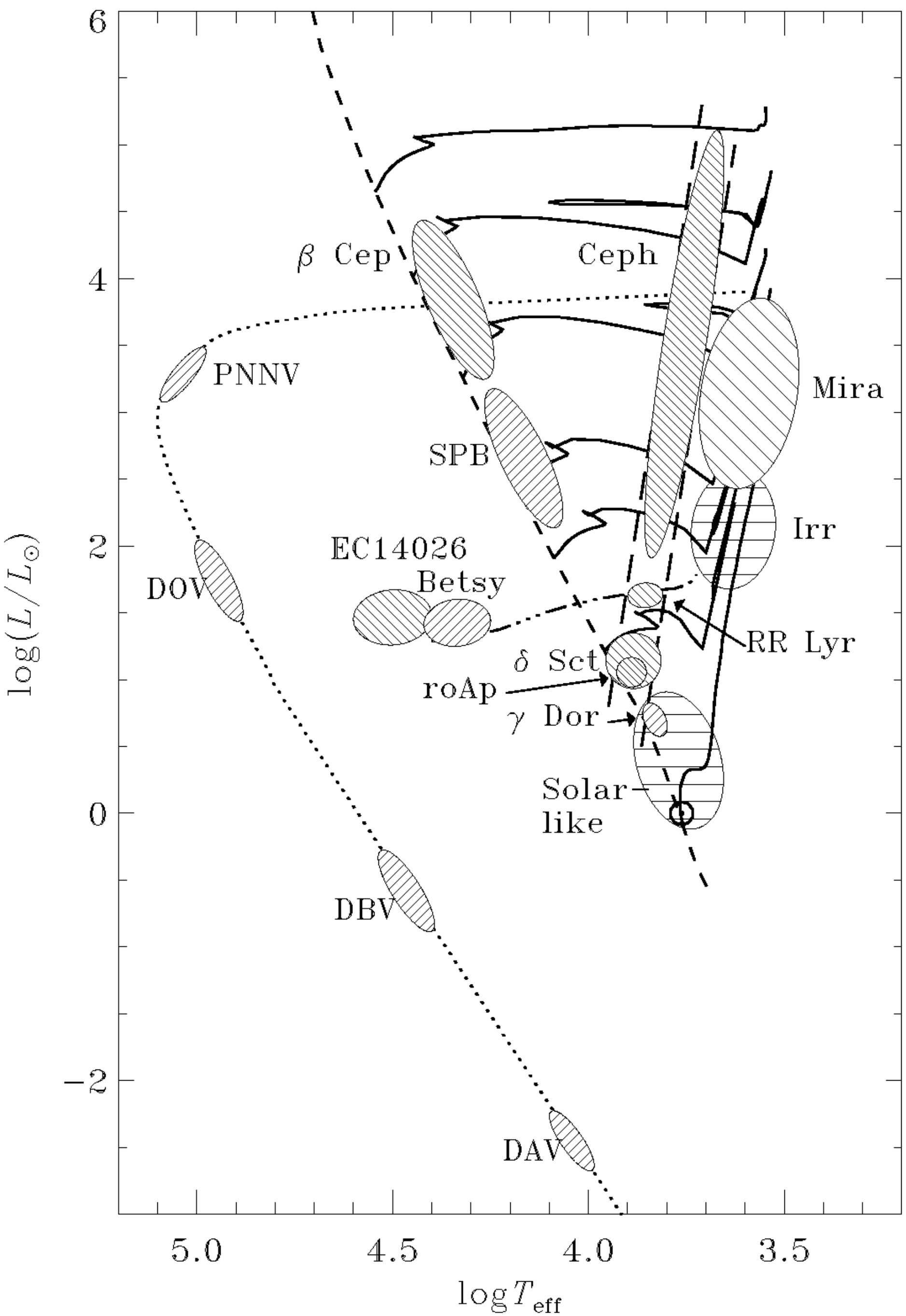
Pulsating stars in the HR diagram



$$\tau_{\rm dyn} \simeq \sqrt{\frac{R^3}{GM}} \simeq \sqrt{\frac{1}{G\,\overline{\rho}}},$$

$$\tau_{\rm th} \simeq \frac{GM^2}{RL},$$

Basic equations of hydrodynamics

Equation of continuity:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{v}) = 0\,;$$

This is a typical conservation equation, balancing the rate of change of a quantity in a volume with the flux of the quantity into the volume.

Equation of motion:

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla p + \rho\,\mathbf{f}\,, \qquad \rho\frac{\partial\,\mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} = -\,\nabla p + \rho\,\mathbf{f}$$

Under stellar conditions we can ignore the internal friction (or viscosity) in the gas. The forces on a volume of gas therefore consist of surface forces (such as the pressure on the surface of the volume) and body forces. The force per unit mass from gravity is the gravitational acceleration **g**, which can be written as the gradient of the gravitational potentΦ:

$$\mathbf{g} = -\nabla\Phi$$

Poisson Equation:

$$\nabla^2\Phi = 4\pi G\rho$$

Energy equation – Adiabatic Approximation:

$$\rho\frac{\mathrm{d}q}{\mathrm{d}t} = \rho\,\epsilon - \mathrm{div}\,\mathbf{F} \qquad\qquad \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\Gamma_1 p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}\,.$$

Equilibrium states and perturbation analysis

Perturbation analysis:
The 'structure' equations are then linearized in the perturbations, by expanding them in the perturbations retaining only terms that do not contain products of the perturbations. Equations for the perturbations are obtained by inserting previous expressions like **(1)** in the full equations, subtracting equilibrium equations and neglecting quantities of order higher than one in $p', \rho', \mathbf{v}$; etc.

- **continuity equation:**

$$\frac{\partial \rho'}{\partial t} + \mathrm{div}\,(\rho_0 \mathbf{v}) = 0$$

- Using (3) and integrating with respect to time

$$\rho' + \mathrm{div}\,(\rho_0 \boldsymbol{\delta}\mathbf{r}) = 0$$

Using (2) the analogue to equation

$$\delta\rho + \rho_0 \mathrm{div}\,(\boldsymbol{\delta}\mathbf{r}) = 0$$

- **Equation of motion:**

$$\rho_0 \frac{\partial^2 \boldsymbol{\delta}\mathbf{r}}{\partial t^2} = \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0$$

- **Poisson equation:**

$$\nabla^2 \Phi' = 4\pi G \rho'$$

- **Adiabatic Approximation:**

$$\delta p = \frac{\Gamma_{1,0} p_0}{\rho_0} \delta\rho$$

Equations of linear stellar oscillations

By including an appropriate scaling factor we may finally write

$$f(\theta, \phi) = (-1)^m c_{lm} P_l^m(\cos\theta)\exp(im\phi) \equiv Y_l^m(\theta, \phi)$$

where Y^m_l is a spherical harmonic; here c_lm is a normalization constant Y^m_l is characterized by its degree l and its azimuthal order m; the properties of spherical harmonics.

The dependent variables in linearized equations can now be written as

$$\xi_r(r, \theta, \phi, t) = \sqrt{4\pi}\,\tilde{\xi}_r(r) Y_l^m(\theta, \phi)\exp(-i\omega t)$$

$$\cdots\cdots$$

$$p'(r, \theta, \phi, t) = \sqrt{4\pi}\,\tilde{p}'(r) Y_l^m(\theta, \phi)\exp(-i\omega t)$$

Separated equations

Separation of time as exp(- i ω t)

$$\frac{\mathrm{d}\xi_r}{\mathrm{d}r} = -\left(\frac{2}{r} + \frac{1}{\gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}\right)\xi_r + \frac{1}{\rho c^2}\left(\frac{S_l^2}{\omega^2} - 1\right)p' + \frac{l(l+1)}{\omega^2 r^2}\Phi' \; .$$

$$\frac{\mathrm{d}p'}{\mathrm{d}r} = \rho(\omega^2 - N^2)\xi_r + \frac{1}{\gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}p' - \rho\frac{\mathrm{d}\Phi'}{\mathrm{d}r} \; ,$$

$$S_l^2 = \frac{l(l+1)c^2}{r^2} = k_\mathrm{h}^2 c^2 \; , \qquad c^2 = \frac{\gamma_1 p}{\rho} \; , \qquad N^2 = g\left(\frac{1}{\gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) \; .$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi'}{\mathrm{d}r}\right) = 4\pi G\left(\frac{p'}{c^2} + \frac{\rho\xi_r}{g}N^2\right) + \frac{l(l+1)}{r^2}\Phi' \; .$$

THE GOAL OF STELLAR SEISMOLOGY IS TO USE SURFACE STELLAR VIBRATIONS TO INFER THE INTERNAL STRUCTURE OF STARS

Physics of pulsations : Basics

The **observed stellar oscillations** in the surface of a star are the result of ***perturbations in the equilibrium structure*** occurring in the star’s interior.

Without loss of generality, these perturbations can be studied as a solution to a ***wave-like equation.***

Physics of pulsations : Basics

Let's remind about the wave equation (... of sound propagating inside a sphere):

$$\boxed{\psi(\vec{r},t) = \delta p(\vec{r},t)} \qquad \frac{\partial^2 \psi}{\partial t^2} = c^2 \nabla^2 \psi \qquad \boxed{c^2 = \gamma \frac{P}{\rho} \propto \frac{T}{\mu}}$$

Separation of variables:

$$\psi(\vec{r},t) \propto \Re\left[\Psi(\vec{r}) e^{i\omega t}\right]$$

$$\nabla^2 \Psi + \frac{\omega^2}{c^2} \Psi = 0$$

ω **- frequency**

k **- wave number**

$$\nu = \frac{\omega}{2\pi}$$ **- linear frequency**

$$k^2 = \frac{\omega^2}{c^2}$$

$$\lambda = \frac{2\pi}{k}$$ **- wavelength**

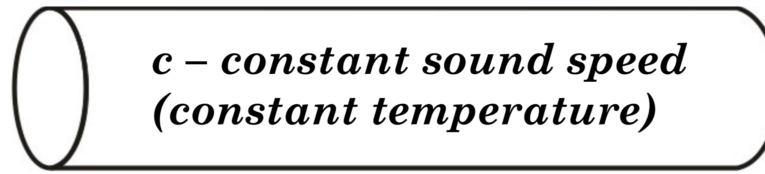
Dispersion relation

Physics of pulsations : Basics

Stellar Oscillations: The basic principles of Stellar seismology can be understood in a very simple case, such as the oscillations of an open tube.

Toy Model: Air Perturbation in an open tube

$$(x,t)$$



$$L$$

$$\frac{d^2\Psi(x)}{dx^2} + \frac{\omega^2}{c^2}\Psi(x) = 0$$

$$\Psi(L) = \Psi(0) = 0$$

Boundary conditions (Open ends)

Normal modes of oscillations:

$$\Psi_n(x) \propto \sin(k_n x) \qquad n = 1,2,3,4,\cdots$$

Eigen-frequencies (and wavelengths):

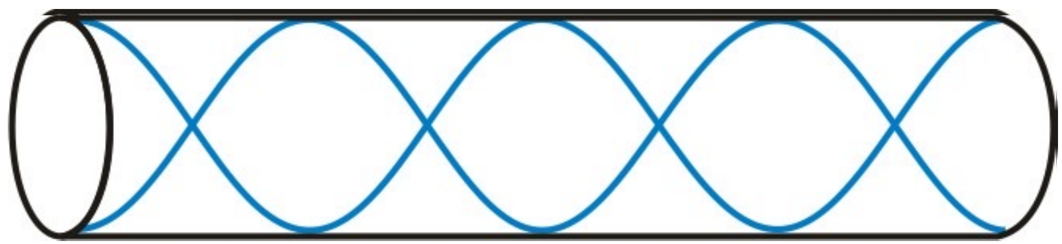
$$k_n L = n\pi \qquad \lambda_n = \frac{2L}{n} \qquad \nu_n = \frac{nc}{2L}$$

A generic perturbation: (in space and time)

$$\boxed{\psi(x,t) \propto \sum_n \sin(k_n x)\cos(\omega_n t)}$$

$$\omega_n = 2\pi\nu_n$$

Physics of pulsations : Basics



Fourth Harmonic n=4

First Harmonic (fundamental mode),
<u>depends</u> of the <u>*Length of the tube* **L** </u>and
the <u>*speed of propagation of sound waves* **c**</u>

$$\nu_o = \frac{c}{2L}$$

Acoustic mode eigen-frequency equation:

$$\nu_n = n\,\nu_o \qquad n = 1,2,3,4,\cdots$$

n – number of nodes of the eigenfunction $\Psi_n(x)$

Physics of pulsations : Stars

Spherical Coordinates

$$\nabla^2\psi + \left(\frac{\omega^2}{c^2} - \frac{\omega_c^2}{c^2} - \frac{N^2}{\omega^2}\nabla_h^2\right)\psi = 0$$

∇_h^2 Horizontal Laplacian operator

c- sound speed,
acoustic waves (**p-modes**)

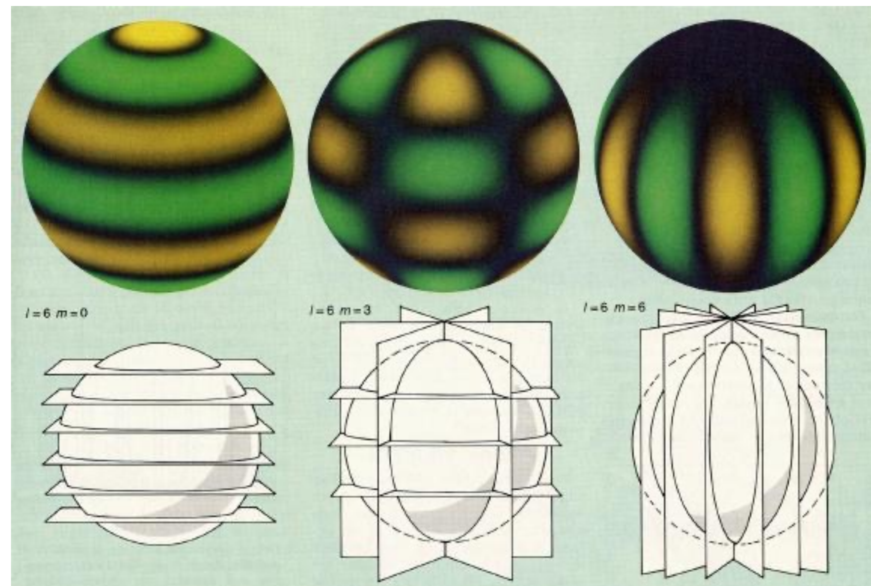
N- Buoyancy frequency,
gravity waves (**g-modes**)

Quantification (Numbers-integers):
1D wave equation – (n)
3D wave equation – (n,l,m)

n,l,m integers

Stellar Oscillations (stratified sphere):
3D wave-like equation (Gough 2001)

$$\psi(r,\theta,\phi,t) \propto \left[r^{-1}\Psi_r(r)Y_l^m(\theta,\phi)e^{-i\omega t}\right]$$



Quantification (Numbers):
radial eigen-fuction, n- number of nodes
Spherical harmonics – (l,m)

Physics of pulsations : Stars

$$\frac{d^2\Psi_r}{dr^2} + k_r^2\Psi_r = 0 \qquad k_r = \frac{\omega^2 - \omega_c^2}{c^2} - \frac{l(l+1)}{r^2}\left(1 - \frac{N^2}{\omega^2}\right)$$

$$0 \leq r \leq R$$

$k_r^2 > 0$ oscillatory solution

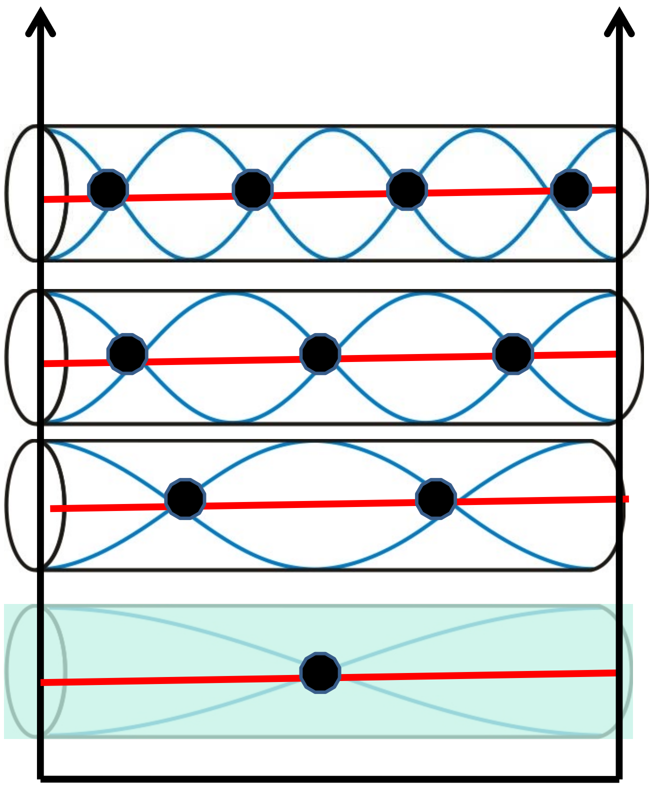
$k_r^2 < 0$ exponential decay solution

The local properties inside the star will determine regions of wave propagation and wave decay.

In particular, the speed propagation of sound waves increases dramatically towards the centre of the star.

What are the different regions where different modes form ?

Physics of pulsations : Basics



ν_4

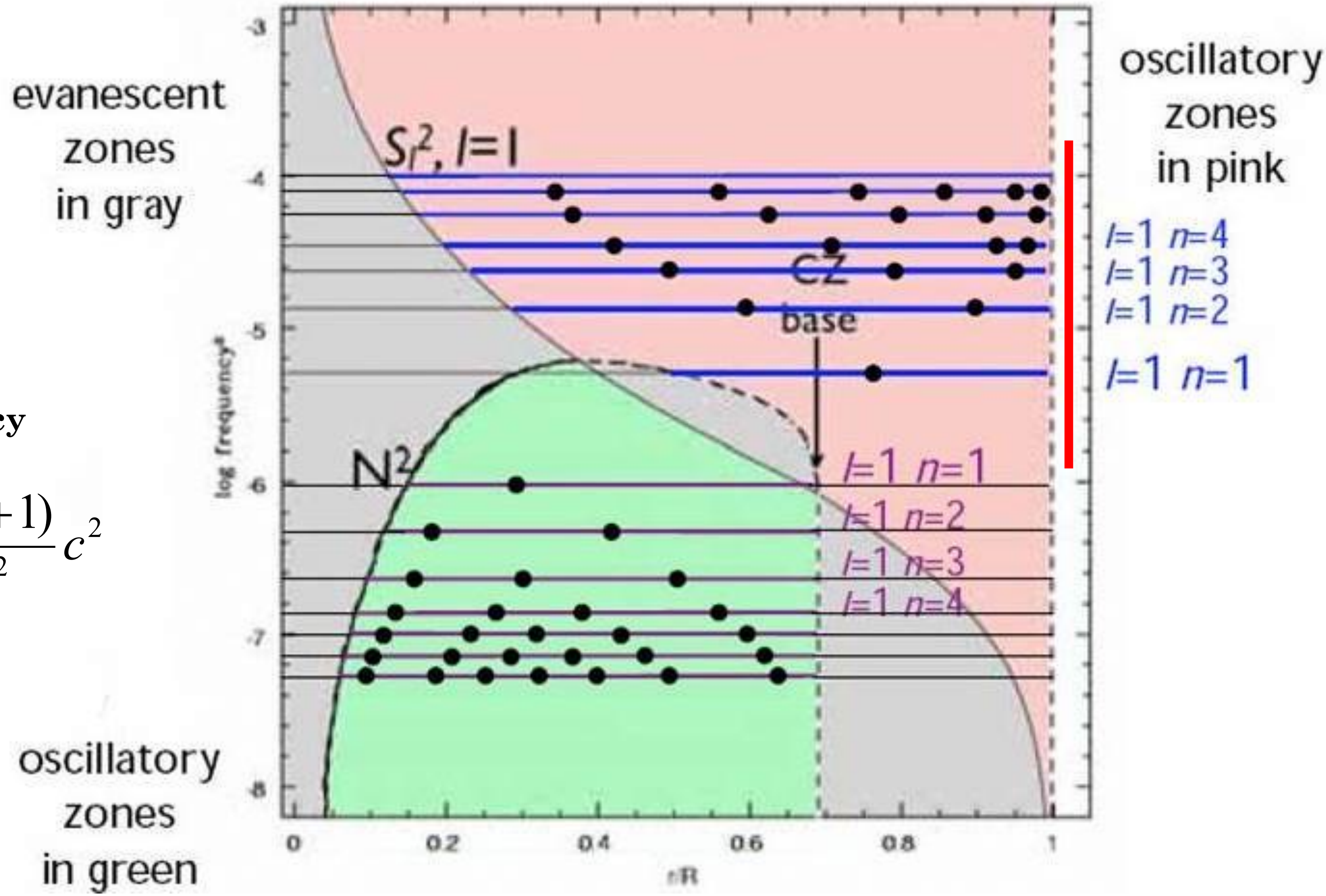
ν_3

ν_2

ν_1

Propagation Diagram (Finite potential well)

Physics of pulsations :Stars



Acoustic frequency

$$S_l^2 = \frac{l(l+1)}{r^2} c^2$$

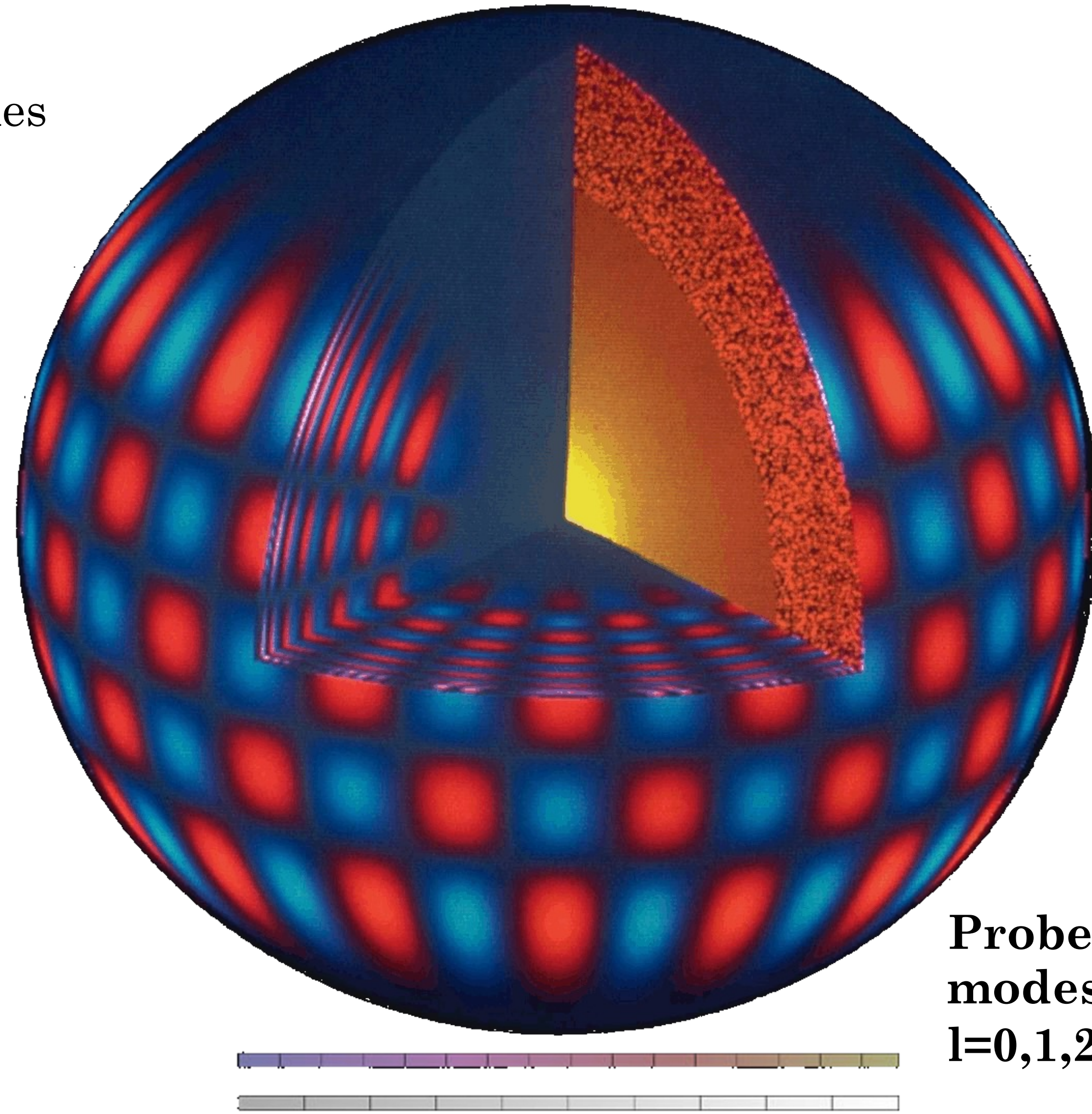
A wave of frequency ω can propagate only if

$$\boxed{k_r^2 > 0, \omega^2 > S_l^2 \text{ and } N^2 \text{ (p modes)}}$$

$$k_r^2 > 0, \omega^2 < S_l^2 \text{ and } N^2 \text{ (g modes)}$$

PHYSICS OF PULSATIONS: STARS

Acoustic modes
l=3, n=25
l=10,n=5,
l=70,n=3



Probe the solar core, modes with degree l=0,1,2,3,....

Some of these waves move through the outer portions of the Sun, but others dive deep towards the core. The waves travel through the interior of the Sun at the local sound speed. Where the solar material is very dense, they move quickly; where the density is low, they move slowly.

Physics of pulsations : Basics

Any **perturbation (oscillation) of equilibrium structure** (air in a tube, equilibrium structure inside a star) can be expressed as a finite sum of harmonic functions.

A generic perturbation (in space and time):

$$\psi(x,t) \propto \sum_n \sin(k_n x)\cos(\omega_n t)$$

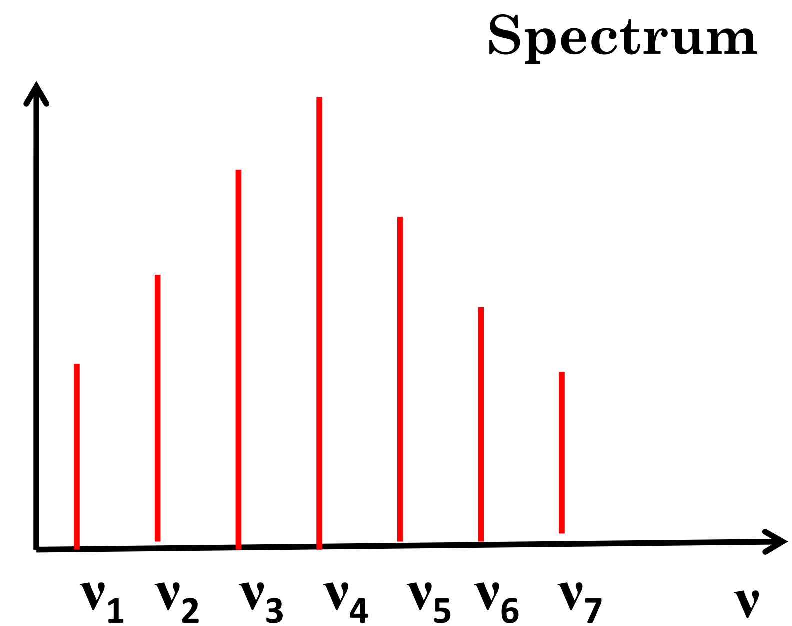
$$\boxed{\psi(r,\theta,\phi,t) \propto \sum_{n,l.m} r^{-1}\Psi_{r,n}(r)Y_l^m(\theta,\phi)e^{-i\omega_{n,l,m}t}}$$

Therefore, the **oscillation spectrum** characterizes the perturbations of the equilibrium structure of the star.

Physics of pulsations : Basics

Stellar Seismology : open tube (flute)

$$\boxed{\nu_n = n\,\nu_o}$$



$$\nu_o = \frac{c}{2L}$$

$$\boxed{\Delta\nu_n = \nu_n - \nu_{n-1} = \nu_o}$$

$$n = 1,2,3,4,\cdots$$

Large separation

Physics of pulsations : Stars

p-mode eigenfrequency (low l and $l << n$) :

$$\boxed{\nu_{l,n} = \left(n + \frac{1}{2} l + \varepsilon\right)\nu_o + \left[A l(l+1) - B\right]\frac{\nu_o^2}{\nu_{l,n}} + \cdots}$$

(Tassoul 1980)

Large separation

$$\boxed{\Delta \nu_{l,n} = \nu_{l,n} - \nu_{l,n-1} = \nu_o}$$

sensitive to sound speed in the surface

$$\nu_o = \left(2\int_0^R \frac{dr}{c}\right)^{-1}$$

$$c = \mathrm{const.} \qquad \nu_o = \frac{c}{2R}$$

Small separation

$$\boxed{\delta\nu_{l,n} = \nu_{l,n} - \nu_{l+2,n-1} \approx \frac{6\nu_o^2}{(n + l/2 + \varepsilon)} A}$$

sensitive to sound speed gradients in the interior

$$A = \frac{1}{2\pi\omega_o}\left[\frac{c(R)}{R} - \int_0^R \frac{1}{2}\frac{dc}{dr}dr\right]$$

PHYSICS OF PULSATIONS: STARS

A flute is essentially a tube that is open at both ends:

$$\Delta \nu_n = \nu_n - \nu_{n-1} = \nu_o$$

$$\Delta \nu_n = 800\,\text{Hz} - 400\,\text{Hz}$$

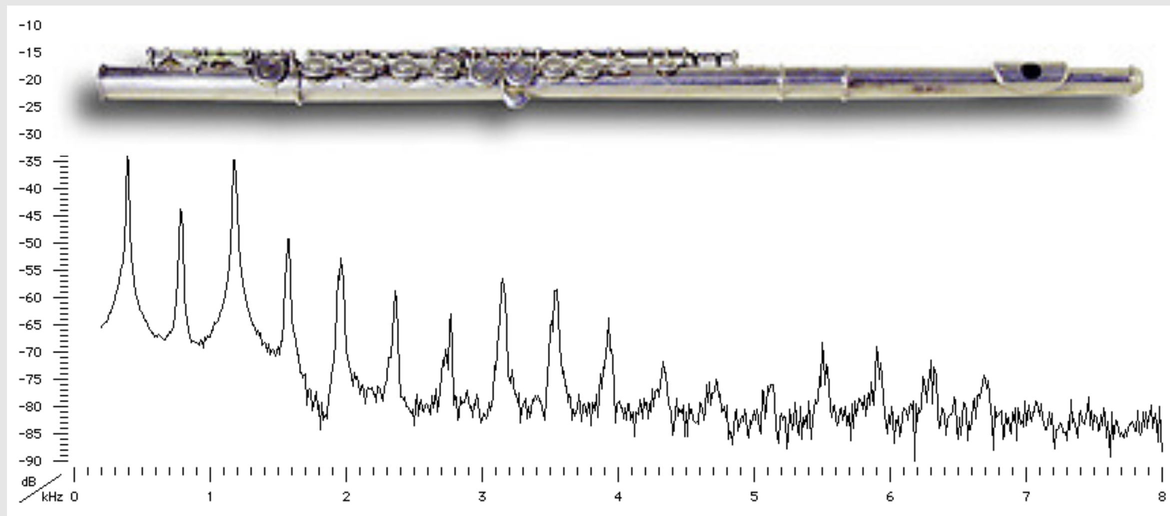
$$\Delta \nu_n(flute) = 400\,\text{Hz}$$

$$\text{L} = \frac{c_{air}}{2\nu_\text{o}}$$

$$c_{air}(T = 20\ {}^0C) = 343\ \text{m/s}$$

$$\text{L(flute)} \sim 43\ \text{cm}$$

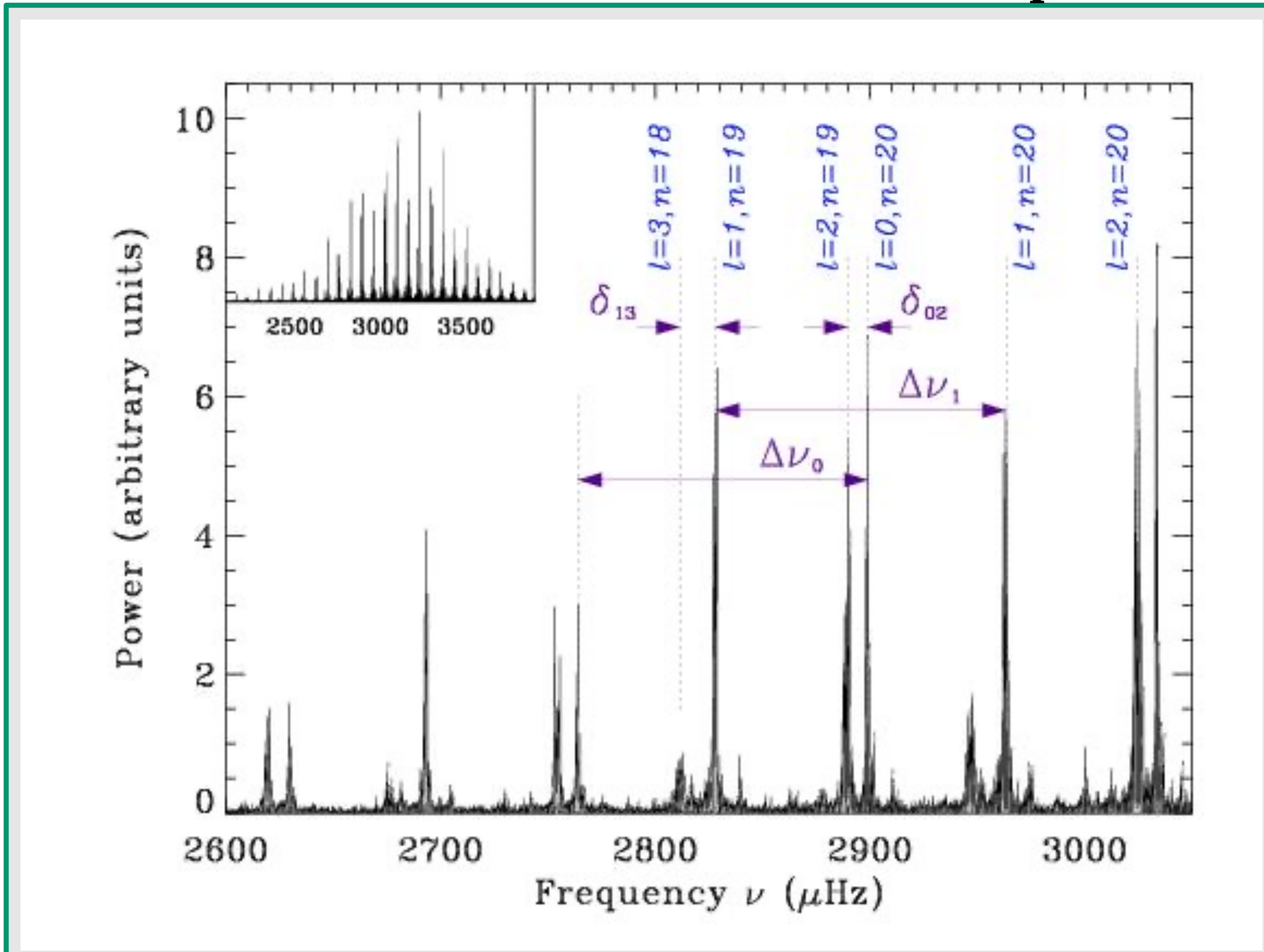
Acoustic Spectrum



The sound spectrum for a note G4 played by a flute. The spectrum has the following series of peaks at frequencies of 400 Hz 800 Hz 1200 Hz 1600 Hz 2000 Hz...,
which we can write as:
$1\nu_o$ $2\ \nu_o$ $3\ \nu_o$ $4\ \nu_o$ ... $n\ \nu_o$ ..., with ν_o =400Hz.

PHYSICS OF PULSATIONS: STARS

Acoustic Spectrum



$$\Delta\nu_{l,n} = \nu_{l,n} - \nu_{l,n-1} = \nu_o$$

$$\boxed{\delta\nu_{l,n} = \nu_{l,n} - \nu_{l+2,n-1} \approx \frac{6\nu_o^2}{(n + l/2 + \varepsilon)} A}$$

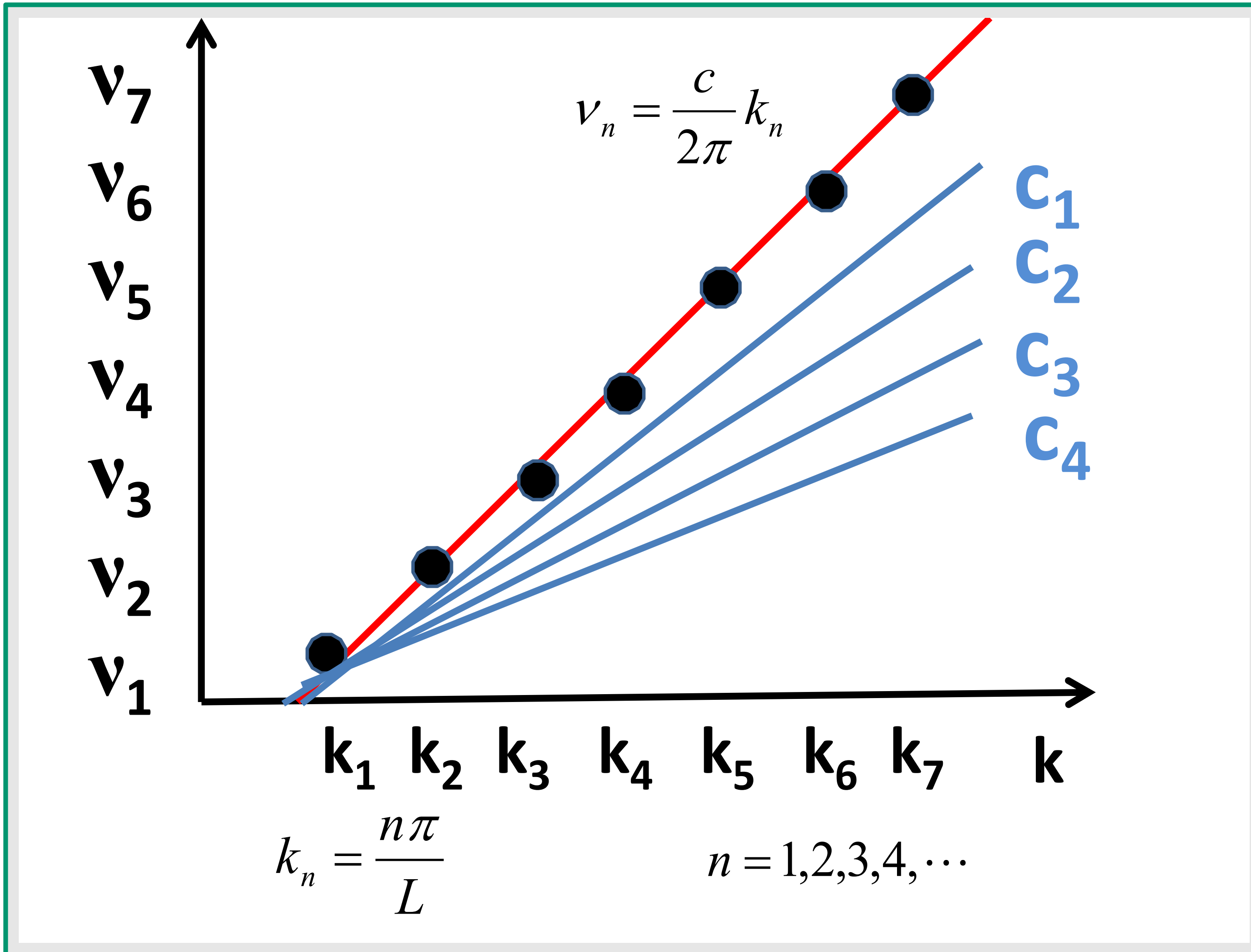
Physics of pulsations : Basics

An alternative information can be computed from the dispersion relation for stationary waves.

$$\boxed{\omega = \omega(k)}$$

The dispersion relation, i.e., the frequency dispersion relation, tells us how waves of different wavelength (or wave number) travel inside the star.

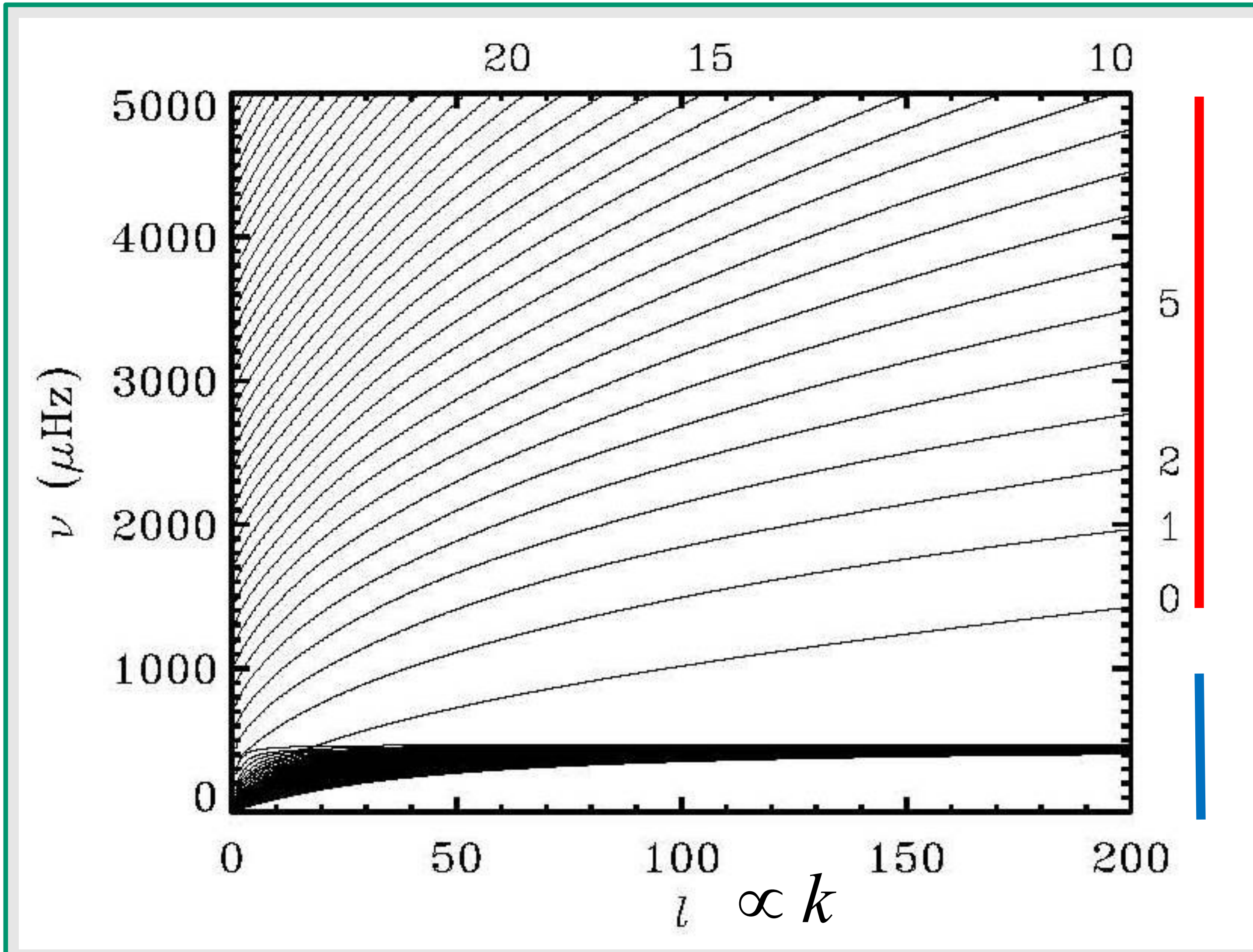
PHYSICS OF PULSATIONS : BASICS



Dispersion relation (open tube)

PHYSICS OF PULSATIONS: STARS

Dispersion relation



Numerical computation of oscillations for the solar standard model

Helioseismology:
Discovery of the Pulsating Sun

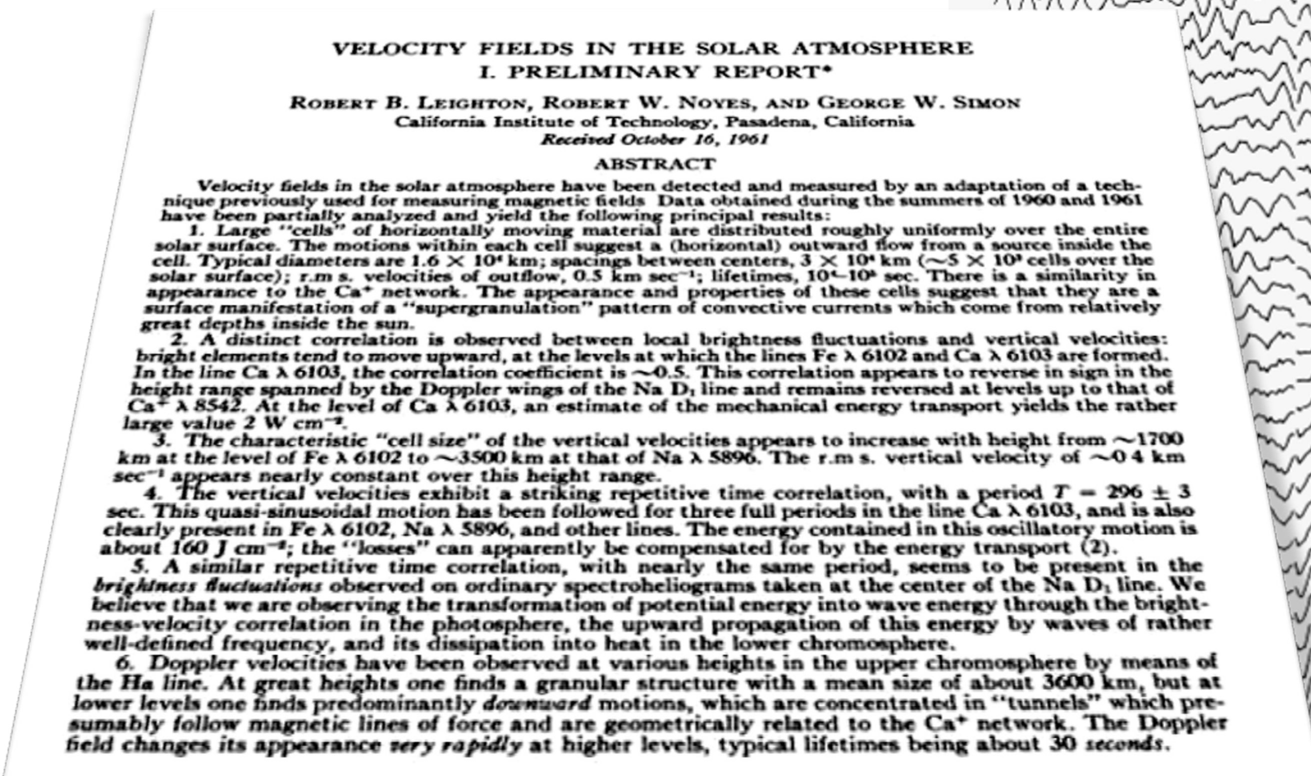
HELIOSEISMOLOGY: DISCOVERING A PULSATING SUN

Many theories of wave physics postulated:
- gravity-acoustic waves or MHD waves?

A puzzle ?

Where was the region of propagation?

Discovery in 1960 by Leighton et al. that the solar surface is rising and falling with a 5 minute period.



VELOCITY FIELDS IN THE SOLAR ATMOSPHERE
I. PRELIMINARY REPORT*

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Received October 16, 1961

ABSTRACT

Velocity fields in the solar atmosphere have been detected and measured by an adaptation of a technique previously used for measuring magnetic fields Data obtained during the summers of 1960 and 1961 have been partially analyzed and yield the following principal results:

1. Large "cells" of horizontally moving material are distributed roughly uniformly over the entire solar surface. The motions within each cell suggest a (horizontal) outward flow from a source inside the cell. Typical diameters are 1.6×10^4 km; spacings between centers, 3×10^4 km ($\sim 5 \times 10^3$ cells over the solar surface); r.m s. velocities of outflow, 0.5 km sec^{-1}; lifetimes, 10^4–10^5 sec. There is a similarity in appearance to the Ca^+ network. The appearance and properties of these cells suggest that they are a surface manifestation of a "supergranulation" pattern of convective currents which come from relatively great depths inside the sun.

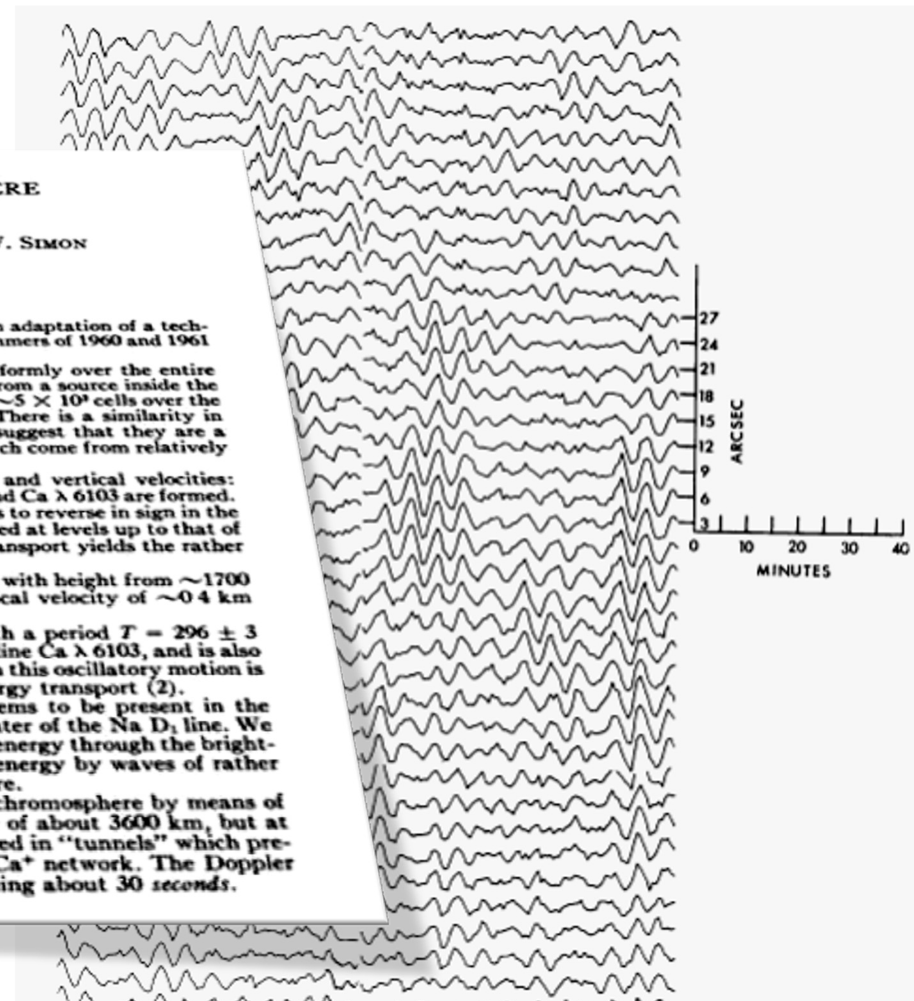
2. A distinct correlation is observed between local brightness fluctuations and vertical velocities: bright elements tend to move upward, at the levels at which the lines Fe λ 6102 and Ca λ 6103 are formed. In the line Ca λ 6103, the correlation coefficient is $\sim$0.5. This correlation appears to reverse in sign in the height range spanned by the Doppler wings of the Na D_1 line and remains reversed at levels up to that of Ca^+ λ 8542. At the level of Ca λ 6103, an estimate of the mechanical energy transport yields the rather large value 2 W cm^{-2}.

3. The characteristic "cell size" of the vertical velocities appears to increase with height from $\sim$1700 km at the level of Fe λ 6102 to $\sim$3500 km at that of Na λ 5896. The r.m s. vertical velocity of $\sim$0 4 km sec^{-1} appears nearly constant over this height range.

4. The vertical velocities exhibit a striking repetitive time correlation, with a period $T = 296 \pm 3$ sec. This quasi-sinusoidal motion has been followed for three full periods in the line Ca λ 6103, and is also clearly present in Fe λ 6102, Na λ 5896, and other lines. The energy contained in this oscillatory motion is about 160 J cm^{-2}; the "losses" can apparently be compensated for by the energy transport (2).

5. A similar repetitive time correlation, with nearly the same period, seems to be present in the *brightness fluctuations* observed on ordinary spectroheliograms taken at the center of the Na D_1 line. We believe that we are observing the transformation of potential energy into wave energy through the brightness-velocity correlation in the photosphere, the upward propagation of this energy by waves of rather well-defined frequency, and its dissipation into heat in the lower chromosphere.

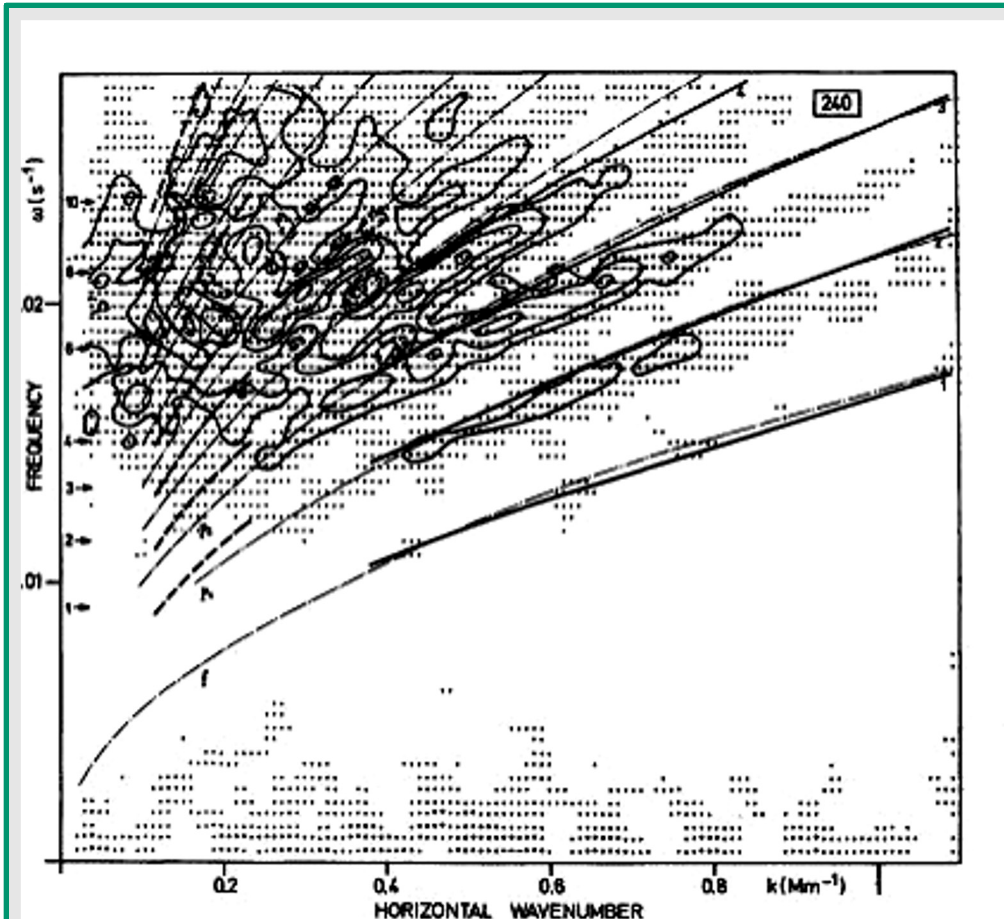
6. Doppler velocities have been observed at various heights in the upper chromosphere by means of the Hα line. At great heights one finds a granular structure with a mean size of about 3600 km, but at lower levels one finds predominantly *downward* motions, which are concentrated in "tunnels" which presumably follow magnetic lines of force and are geometrically related to the Ca^+ network. The Doppler field changes its appearance *very rapidly* at higher levels, typical lifetimes being about 30 *seconds*.



HELIOSEISMOLOGY: 70'S -ORIGIN OF SOLAR OSCILLATIONS

- Acoustic waves trapped within the internal temperature gradient.
- Dispersion relation between frequency and wavelength : $\omega = \omega(\kappa)$
- Max amplitude 20 cm/s

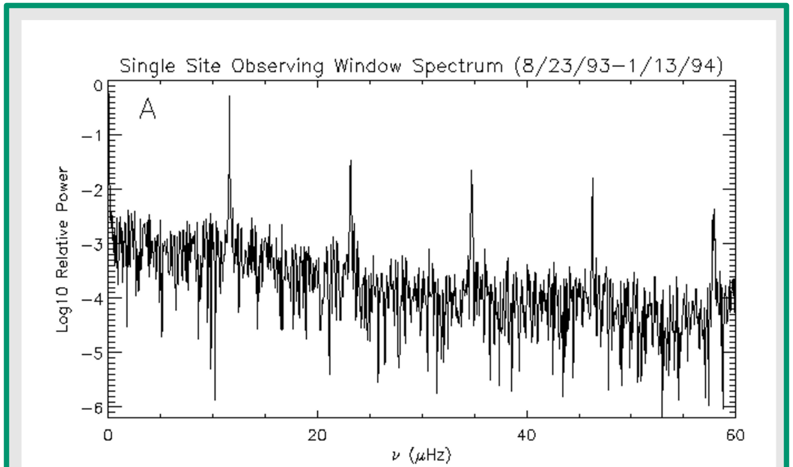
R. Ulrich 1975



<u>An Observational Problem</u>:

The Sun sets at a single terrestrial site, producing periodic time series gaps.

The solar acoustic spectrum is convolved with the temporal window spectrum, contaminating the solar spectrum with many spurious peaks.



Solution:
Antarctica – max 6 month duration
Network – BiSON, IRIS, GONG
Space – SoHO: MDI, GOLF,VIRGO

Lecture 2

END